GLOBAL SOLUTIONS OF HIGHER-ORDER SEMILINEAR PARABOLIC EQUATIONS IN THE SUPERCRITICAL RANGE

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ABSTRACT. We study the asymptotic behaviour of global bounded solutions of the Cauchy problem for the semilinear $2m$-th order parabolic equation

$$u_t = -(-\Delta)^m u + |u|^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+,$$

where $m > 1$, $p > 1$, with bounded integrable initial data u_0. We prove that in the supercritical Fujita range $p > p_F = 1+2m/N$ any small global solution with nonnegative initial mass, $\int u_0 dx \geq 0$, exhibits as $t \to \infty$ the asymptotic behaviour given by the fundamental solution of the linear parabolic operator (unlike the case $p \in (1, p_F]$ where solutions can blow-up for any arbitrarily small initial data). A discrete spectrum of other possible asymptotic patterns and the corresponding monotone sequence of critical exponents $\{p_l = 1 + 2m/(l + N),\ l = 0, 1, 2, ...\}$, where $p_0 = p_F$, are discussed.

1. **Introduction and main results**

We study the asymptotic behaviour of global solutions of higher-order semilinear evolution equations of parabolic type. Our basic example is the Cauchy problem for the semilinear $2m$-th order ($m > 1$) parabolic equation

$$u_t = -(-\Delta)^m u + |u|^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad u(x, 0) = u_0 \in X = L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N), \tag{1.1}$$

which is a natural generalization of the classical semilinear heat equation ($m = 1$) from combustion theory. Higher-order semilinear and quasilinear parabolic equations occur in applications in thin film theory, nonlinear diffusion, lubrication theory, flame and wave propagation (the Kuramoto-Sivashinsky equation and the extended Fisher-Kolmogorov equation), phase transition at critical Lifschitz points and bi-stable systems; see a number of models and a list of references in the book [21].

It is known that

$$p = p_F = 1 + 2m/N$$

is the *critical Fujita exponent* for this problem in the following sense:
(I) if $p \in (1, p_F]$, then any solution $u(x, t) \not\equiv 0$ with

$$\int_{\mathbf{R}^N} u_0(x)dx \geq 0, \tag{1.2}$$

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blows up in finite time [6] (i.e., there exist arbitrarily small initial data $u_0 \in X$ leading to blow-up), and
(II) for $p > p_F$, solutions are global for any sufficiently small initial data; see [4] and [11].

These blow-up and global existence results are classical for the second-order semilinear heat equation with $m = 1$ established by H. Fujita in the 1960's. Later on, using the Maximum Principle order-preserving properties of parabolic flows, they were extended to a wide class of quasilinear equations with different types of nonlinear reaction-diffusion operators; see a list of references in Chapt. 4 in [22].

In this paper we study the asymptotic behaviour of global solutions in the supercritical range in the case of higher-order diffusion operators, where the semigroup is not order-preserving. Let $b(x,t)$ be the fundamental solution of the parabolic operator $\partial/\partial t + (-\Delta)^m$ satisfying

$$u_t = -(-\Delta)^m u \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \tag{1.3}$$

with the initial function $u(x,0) = \delta(x)$ in $\mathbf{R}^N$, δ being Dirac's delta function. By the scaling invariance of the problem and uniqueness of the fundamental solution, it has the self-similar structure

$$b(x,t) = t^{-N/2m} f(y), \quad y = x/t^{1/2m}. \tag{1.4}$$

Substituting $b(x,t)$ into (1.3), one obtains the radially symmetric profile f as a unique solution of a linear ordinary differential equation (ODE) which is the radial restriction of the elliptic equation

$$\mathbf{B}f \equiv -(-\Delta_y)^m f + \frac{1}{2m}\nabla_y f \cdot y + \frac{N}{2m} f = 0 \quad \text{in } \mathbf{R}^N, \quad \int_{\mathbf{R}^N} f(y)dy = 1. \tag{1.5}$$

Applying the Fourier transform to equation (1.3) and performing the rescaling, we have

$$\mathcal{F}(b(\cdot,t))(\omega) = e^{-|\omega|^{2m}t} \quad \text{and} \quad \mathcal{F}(f(\cdot))(\xi) = e^{-|\xi|^{2m}}. \tag{1.6}$$

Then [8]

$$f(y) = (2\pi)^{-N/2} \int_0^\infty e^{-s^{2m}} (s|y|)^{N/2} J_{(N-2)/2}(s|y|)ds \quad \text{in } \mathbf{R}^N, \tag{1.7}$$

where J_ν denotes Bessel's function. The rescaled kernel satisfies a standard pointwise estimate [7], which is convenient to present in the form

$$|f(y)| \le DF(y), \quad F(y) = \omega_1 e^{-d|y|^\alpha} > 0 \quad \text{in } \mathbf{R}^N, \quad \omega_1 = \left(\int e^{-d|y|^\alpha} dy\right)^{-1} \tag{1.8}$$

(hence, $\int F(y)dy = 1$), where $\alpha = 2m/(2m-1) \in (1,2)$ and D, d are positive constants depending on m, N. Then F is the rescaled kernel of the majorizing order-preserving integral evolution equation, whose solutions can be compared with those of (1.1) [11]. Let us state the first result.

Theorem 1.1. *Let $p > p_F$. Then for initial data satisfying* (1.2) *from the class*

$$X_B = \{u_0 \in X : \ |u_0(x)| \le Be^{-d|x|^\alpha} \ \textit{in } \mathbf{R}^N\}, \tag{1.9}$$

where $B > 0$ is a sufficiently small constant, there exists a global solution $u(x,t)$ of the Cauchy problem (1.1) *such that the rescaled function*

$$v(y,t) = (1+t)^{N/2m} u(y(1+t)^{1/2m}, t) \tag{1.10}$$

has the following asymptotic behaviour:

$$v(y,t) \to C_0 \, f(y) \quad as \;\; t \to \infty \;\; uniformly\ in\ \mathbf{R}^N, \tag{1.11}$$

where constant $C_0 > 0$ depends on initial data.

Such results are well known for the case $m = 1$ established first in [13] and [18], where the convergence to the rescaled Gaussian kernel $f(y) = (4\pi)^{-N/2} e^{-|y|^2/4}$ was proved for the semilinear heat equation

$$u_t = \Delta u - u^p \quad (u \geq 0) \tag{1.12}$$

with $p > 1 + 2/N$; see also Section 3 in [10], where the stability of other self-similar profiles $f \notin L^1(\mathbf{R}^N)$ was studied. A similar result is known for the quasilinear parabolic equations like

$$u_t = \nabla \cdot (u^\sigma \nabla u) \pm u^p, \quad \sigma > 0, \tag{1.13}$$

in the supercritical range $p > \sigma + 1 + 2/N$; see p. 236 in [22].

In Sections 2 and 3 we describe complete subsets of eigenfunctions of the linear non self-adjoint operator $\mathbf{B}$ and of the adjoint one $\mathbf{B}^*$. In Section 4 we prove weighted L^∞-estimates on global solutions for $p > p_F$, which are necessary for the further asymptotic analysis. Theorem 1.1 is proved in Section 5. It establishes the asymptotic behaviour of any solution with a nontrivial final mass when

$$\int_{\mathbf{R}^N} u(x,t)dx \to C_0 \neq 0 \quad \text{as} \;\; t \to \infty. \tag{1.14}$$

If $C_0 = 0$, we need extra rescaling to derive the asymptotic shape of the solution. This is done in Section 6, where we describe a countable spectrum of different solution asymptotics of equations (1.3) and (1.1). We then derive a monotone decreasing sequence of critical exponents

$$p_l = 1 + 2m/(l+N), \quad l = 0, 1, 2, \ldots,$$

where the first one $p_0 = p_F$ gives the critical Fujita exponent.

2. Discrete real spectrum of a non self-adjoint operator

For any integer $m > 1$, the linear operator $\mathbf{B}$ is not symmetric. As a differential operator with smooth coefficients, it is closable [14]. We calculate its spectrum $\sigma(\mathbf{B})$ in the weighted space $L^2_\rho(\mathbf{R}^N)$ with the exponential weight

$$\rho(y) = e^{a|y|^\alpha} > 0 \quad \text{in} \;\; \mathbf{R}^N, \tag{2.1}$$

where $a \in (0, d)$ is a sufficiently small positive constant. In the classical self-adjoint case $m = 1$, where $\alpha = 2$ and $a = 1/4$, the weight is given by the inverse Gaussian kernel $\rho(y) = e^{|y|^2/4}$.

2.1. **Domain of the operator.** We next introduce a Hilbert space of functions $H^{2m}_\rho(\mathbf{R}^N)$ with the inner product

$$\langle v, w\rangle_\rho = \int_{\mathbf{R}^N} \rho(y) \sum_{k=0}^{2m} D^k v(y)\, \overline{D^k w(y)}\, dy,$$

where $D^k v$ stays for the vector $\{D^\beta v,\ |\beta| = k\}$, and the norm

$$\|v\|^2_\rho = \int_{\mathbf{R}^N} \rho(y) \sum_{k=0}^{2m} |D^k v(y)|^2\, dy. \tag{2.2}$$

Then $H^{2m}_\rho(\mathbf{R}^N) \subset L^2_\rho(\mathbf{R}^N) \subset L^2(\mathbf{R}^N)$.

Proposition 2.1. $\mathbf{B}$ *is a bounded linear operator from* $H^{2m}_\rho(\mathbf{R}^N)$ *to* $L^2_\rho(\mathbf{R}^N)$.

Proof. In the calculations below, one can scale out the constant a in (2.1) and therefore we just set $a = 1$. It follows from (1.5) that $\mathbf{B}v \in L^2_\rho(\mathbf{R}^N)$ for any $v \in H^{2m}_\rho(\mathbf{R}^N)$ provided that

$$\int_{\mathbf{R}^N} \rho(y)|y\cdot\nabla v|^2 dy \le C\|v\|^2_\rho \quad \text{for any } \ v \in H^{2m}_\rho(\mathbf{R}^N), \tag{2.3}$$

where $C > 0$ is a constant. Let $h \in C^\infty(\mathbf{R}^N)$ be a function such that $h(y) = 1$ for $|y| \ge 2$ and $h(y) = 0$ for $|y| \le 1$. Since the inequality

$$\int_{\mathbf{R}^N} \rho(y)|y\cdot\nabla((1-h(y))v)|^2 dy \le C_1\|v\|^2_\rho$$

is obvious, it suffices to show that

$$\int_{\mathbf{R}^N} \rho(y)|y\cdot\nabla(hv)|^2 dy \le C\|hv\|^2_\rho,$$

i.e., proving (2.3), we can suppose that $v \in C_0^\infty(\mathbf{R}^N)$ vanishes for all $|y| \le 1$.

Let $(r, \theta_1, ..., \theta_{N-1})$ be the spherical coordinates in $\mathbf{R}^N$. Since

$$|y\cdot\nabla v| \le r|v_r|,$$

it suffices to verify that

$$\int_0^\infty r^{N+1}|w_1(r)|^2 e^{r^\alpha}\, dr \le C_2 \int_0^\infty r^{N-1}|w_1^{(2m-1)}(r)|^2 e^{r^\alpha}\, dr, \tag{2.4}$$

if the left-hand side in bounded, and apply this estimate with $w_1 = v_r$.

Let $q = N-1$ or $q = N-3$, $\gamma = \alpha - 1 = 1/(2m-1)$ and $k \in \{0, 1, ..., 2m-2\}$. Then using the inequality

$$\int_0^\infty r^{q+2k\gamma}|w'(r) + r^\gamma w(r)|^2 e^{r^\alpha}\, dr \ge 0,$$

integrating by parts again provided that the right-hand side converges, we obtain that

$$\int_0^\infty r^{q+2k\gamma}|w'(r)|^2 e^{r^\alpha}\, dr \ge \gamma \int_0^\infty r^{q+2k\gamma+2\gamma}|w(r)|^2 e^{r^\alpha}\, dr.$$

Simple iteration implies (2.4) with $C_2 = \gamma^{1-2m}$ completing the proof. $\square$

The result also follows from a general estimate in Lemma 2.1 in [17] which goes back to the classical Hardy inequality [16]. In the second-order case, where $\mathbf{B}$ admits a symmetric representation

$$\mathbf{B} = \rho^{-1}\nabla \cdot (\rho \nabla) + \frac{N}{2} I, \quad \rho(y) = e^{|y|^2/4},$$

the operator is self-adjoint in $L^2_\rho(\mathbf{R}^N)$ with the domain $H^2_\rho(\mathbf{R}^N)$; see [2], p. 48.

2.2. **Explicit semigroup representation.** Let $u(x,t)$ be the solutions of the Cauchy problem for the linear parabolic equation (1.3) with initial data $u_0 \in L^2_\rho(\mathbf{R}^N)$ given by the convolution Poisson-type integral

$$u(x,t) = b(t) * u_0 \equiv t^{-N/2m} \int_{\mathbf{R}^N} f((x-z)t^{-1/2m}) u_0(z) dz. \tag{2.5}$$

As in the fundamental solution (1.4), we introduce the rescaled variables

$$u(x,t) = t^{-N/2m} w(y,\tau), \quad y = x/t^{1/2m}, \quad \tau = \ln t : \mathbf{R}_+ \to \mathbf{R}. \tag{2.6}$$

Then the rescaled solution w satisfies the evolution equation

$$w_\tau = \mathbf{B} w. \tag{2.7}$$

Note that $w(y,\tau)$ satisfies the Cauchy problem for (2.7) in $\mathbf{R}^N \times \mathbf{R}_+$ with initial data at $\tau = 0$ (hence, $t = 1$)

$$w_0(y) = u(y,1) \equiv b(1) * u_0 = f * u_0. \tag{2.8}$$

Thus, the linear operator $\partial/\partial\tau - \mathbf{B}$ is the rescaled version of the standard parabolic one $\partial/\partial t + (-\Delta)^m$. Therefore, the corresponding semigroup $e^{\mathbf{B}\tau}$ admits an explicit integral representation. This helps to establish some properties of $\mathbf{B}$ and describes other evolution features of the linear flow. Rescaling convolution (2.5) yields the following explicit representation of the semigroup:

$$w(y,\tau) = \int_{\mathbf{R}^N} f(y - ze^{-\tau/2m}) u_0(z) dz \equiv e^{\mathbf{B}\tau} w_0, \quad \tau \geq 0. \tag{2.9}$$

Consider Taylor's expansion of the rescaled kernel. It follows from the ODE (1.5) for radial solutions $f = f(|y|)$, that the higher-order derivatives of f are estimated as follows

$$|D^\beta f(y)| \leq C_\beta e^{-d|y|^\alpha}, \quad \text{where } C_\beta \leq c^{|\beta|} |\beta|^{(\alpha-1)|\beta|/\alpha} \text{ for } |\beta| \gg 1 \tag{2.10}$$

and $c > 1$ is independent of $|\beta|$. The sharpness of thes estimates follows from the well-known asymptotics of the profile $f(|y|)$ as $|y| \to \infty$. Indeed, by setting $|y|^\alpha = z$, the ODE (1.5) reduces to an ODE with asymptotically perturbed constant coeffficients for $z \gg 1$ to which the classical perturbation theory [3] applies; see e.g. [1].

For any $y \in \mathbf{R}^N$, Taylor's power series of the analytic kernel

$$f(y - ze^{-\tau/2m}) = \sum_{(\beta)} e^{-|\beta|\tau/2m} \frac{(-1)^{|\beta|}}{\beta!} D^\beta f(y) z^\beta \equiv \sum_{(\beta)} e^{-|\beta|\tau/2m} \frac{1}{\sqrt{\beta!}} \psi_\beta(y) z^\beta \tag{2.11}$$

(here $z^\beta \equiv z_1^{\beta_1}...z_N^{\beta_N}$ and ψ_β are normalized eigenfunctions of $\mathbf{B}$, see below) converges uniformly on compact subsets in $z \in \mathbf{R}^N$. Indeed, denoting $|\beta| = l$ and estimating the coefficients

$$\left| \sum_{|\beta|=l} \frac{(-1)^l}{\beta!} D^\beta f(y) z_1^{\beta_1}...z_N^{\beta_N} \right| \le b_l |z|^l,$$

by Stirling's formula we have that for $l \gg 1$,

$$b_l = \frac{N^l}{l!} \sup_{y \in \mathbf{R}^N, |\beta|=l} |D^\beta f(y)| \sim \frac{N^l}{l!} l^{(\alpha-1)l/\alpha} e^{-(\alpha-1)l/\alpha} \sim l^{-l/\alpha} c^l = e^{-l \ln l/\alpha + l \ln c}. \tag{2.12}$$

The series $\sum b_l |z|^l$ has the radius of convergence $R = \infty$.

Finally, we arrive at the following representation of the solution:

$$w(y,\tau) = \sum_{(\beta)} e^{-|\beta|\tau/2m} M_\beta(u_0) \psi_\beta(y), \tag{2.13}$$

where $\lambda_\beta = -|\beta|/2m$ and $\psi_\beta(y)$ are the eigenvalues and eigenfunctions of $\mathbf{B}$ and

$$M_\beta(u_0) = \frac{1}{\sqrt{\beta!}} \int_{\mathbf{R}^N} z_1^{\beta_1}...z_N^{\beta_N} u_0(z) dz \tag{2.14}$$

are the corresponding momenta of the initial datum w_0 (recall the relation (2.8) between w_0 and u_0). We will show that in terms of the dual inner product $\langle \cdot, \cdot \rangle$ in $L^2(\mathbf{R}^N)$,

$$M_\beta(u_0) = \langle w_0, \psi_\beta^* \rangle, \tag{2.15}$$

where $\{\psi_\beta^*\}$ are polynomial eigenfunctions of the adjoint operator $\mathbf{B}^*$ to be described in the next section.

Let us introduce an equivalent explicit representation of the semigroup actually determining eigenfunctions of $\mathbf{B}^*$. In order to exclude the relation like (2.8), we perform another rescaling

$$u = (1+t)^{-N/2m} w, \quad y = x/(1+t)^{1/2m}, \quad \tau = \ln(1+t) : \mathbf{R}_+ \to \mathbf{R}_+.$$

Then $w(y,\tau)$ solves the Cauchy problem for equation (2.7) with initial data $w_0(y) \equiv u_0(y)$. Rescaling the convolution (2.5), we deduce

$$w(y,\tau) = (1-e^{-\tau})^{-N/2m} \int_{\mathbf{R}^N} f((y - ze^{-\tau/2m})(1-e^{-\tau})^{-1/2m}) w_0(z) dz. \tag{2.16}$$

This gives the corresponding semigroup $\{T(\tau) = e^{\mathbf{B}\tau}\}_{\tau \ge 0}$. We can apply the same expansion analysis as above. First, by Taylor expansion in the f-term we obtain

$$w(y,\tau) = \sum_{(\mu)} \frac{(-1)^{|\mu|}}{\mu!} D^\mu f(y(1-e^{-\tau})^{-1/2m})(e^\tau - 1)^{-|\mu|/2m} \int_{\mathbf{R}^N} z^\mu w_0(z) dz. \tag{2.17}$$

Second, applying Taylor expansions for the f-terms in (2.17),

$$f(y(1-e^{-\tau})^{-1/2m}) = \sum_{(\nu)} \frac{1}{\nu!} D^\nu f(0) y^\nu \, (1-e^{-\tau})^{-|\nu|/2m}$$

and for functions $(1-e^{-\tau})^{-1/2m}$ and $(1-e^{-\tau})^{-N/2m}$ in terms of $e^{-k\tau/2m}$, $k=0,1,...$, we arrive at a similar representation

$$w(y,\tau) = e^{\mathbf{B}\tau} w_0 \equiv \sum e^{-|\beta|\tau/2m} \tilde{M}_\beta(w_0)\psi_\beta(y), \tag{2.18}$$

where the expansion coefficients $\tilde{M}_\beta(w_0)$ are the dual products $\langle w_0, \psi^*_\beta\rangle$ with the polynomial eigenfunctions ψ^*_β of the adjoint operator $\mathbf{B}^*$. Expansion (2.18) determines the adjoint eigenfunctions orthogonal to $\{\psi_\beta\}$ in $L^2(\mathbf{R}^N)$. In the next section devoted to the adjoint operator $\mathbf{B}^*$ we perform a simpler derivation of explicit formulas for polynomials $\{\psi^*_\beta\}$.

2.3. **Discrete spectrum and compact resolvent.**

Theorem 2.1. *(i) The spectrum of* $\mathbf{B}$ *comprises real eigenvalues only*

$$\sigma(\mathbf{B}) = \{\lambda_\beta = -|\beta|/2m, \ |\beta| = 0,1,2,...\}. \tag{2.19}$$

Eigenvalues λ_β *have finite multiplicity with eigenfunctions*

$$\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta f(y) \equiv \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} \left(\frac{\partial}{\partial y_1}\right)^{\beta_1} ... \left(\frac{\partial}{\partial y_N}\right)^{\beta_N} f(y). \tag{2.20}$$

(ii) The subset of eigenfunctions $\Phi = \{\psi_\beta\}$ *is complete in* $L^2(\mathbf{R}^N)$ *and in* $L^2_\rho(\mathbf{R}^N)$.

(iii) For any $\lambda \notin \sigma(\mathbf{B})$*, the resolvent* $(\mathbf{B}-\lambda I)^{-1}$ *is a compact operator in* $L^2_\rho(\mathbf{R}^N)$.

In the case $m=1$, where f is the rescaled Gaussian kernel $f(y) = (4\pi)^{-N/2} e^{-|y|^2/4}$, we have the eigenfunctions

$$\psi_\beta(y) = c_\beta e^{-|y|^2/4} H_\beta(y), \quad H_\beta(y) \equiv H_{\beta_1}(y_1)...H_{\beta_N}(y_N),$$

where H_β denote Hermite polynomials in $\mathbf{R}^N$ having the separable decomposition. Operator $\mathbf{B}$ with the domain $H^2_\rho(\mathbf{R}^N)$, $\rho = e^{|y|^2/4}$, is self-adjoint and the eigenfunctions form an orthonormal basis in $L^2_\rho(\mathbf{R}^N)$ [2], p. 48. If $m>1$, then the operator is not self-adjoint. Observe that since for $|\beta|=0$, $\psi_0 = f$,

$$\int_{\mathbf{R}^N} \psi_0(y)dy = \int_{\mathbf{R}^N} f(y)dy = 1. \tag{2.21}$$

On the other hand, it follows from (2.20) that

$$\int_{\mathbf{R}^N} \psi_\beta(y)dy = 0 \quad \text{for any } \ |\beta| \neq 0. \tag{2.22}$$

This expresses the orthogonality property to the adjoint eigenfunctions in terms of the dual inner product. The adjoint eigenfunctions are polynomials which form a complete subset in $L^2_{\rho^*}(\mathbf{R}^N)$ with decaying exponential weight $\rho^*(y) = e^{-a|y|^\alpha}$; see the next section.
Proof. (i) **Point spectrum.** Denote $l = |\beta|$. The existence of such eigenvalues and eigenfunctions follows by applying differentiation D^β to the elliptic equation (1.5)

$$D^\beta \mathbf{B} f \equiv \mathbf{B} D^\beta f + \frac{|\beta|}{2m} D^\beta f = 0. \tag{2.23}$$

It follows from the asymptotic analysis of expansions (2.13) and (2.18) as $\tau \to \infty$ that no other eigenfunctions exist, all eigenvalues are real and are given in (2.19).
(ii) **Completeness in L^2.** Let us prove that the system of the eigenfunctions $\{D^\beta f\}$ is complete in $L^2(\mathbf{R}^N)$. By the Riesz-Fischer theorem [19], we have to show that, given a function $g \in L^2(\mathbf{R}^N)$, the equalities

$$\int D^\beta f(x) g(x) dx = 0 \quad \text{for any } \beta \tag{2.24}$$

imply that $g = 0$. Let $F(\xi)$ and $G(\xi)$ be the Fourier transforms of f and g. Then

$$\int \xi^\beta F(\xi) G(-\xi) d\xi = 0 \quad \text{for any } \beta.$$

Applying the Fourier transform to equation (1.5) yields

$$|\xi|^{2m} F + \frac{1}{2m} \xi \cdot \nabla F = 0,$$

and hence $F(\xi) = e^{-|\xi|^{2m}}$ (see (1.6)). Therefore,

$$\int \xi^\beta e^{-|\xi|^{2m}} G(-\xi) d\xi = 0 \quad \text{for any } \beta. \tag{2.25}$$

The function

$$M(z) = \int e^{-|\xi|^{2m}} G(-\xi) e^{iz\xi} d\xi$$

is entire analytic in $\mathbb{C}^N$ (since $|e^{iz\xi}| \leq e^{|\mathrm{Im}\, z||\xi|}$). Equality (2.25) means that $D^\beta M(0) = 0$ for any β. Therefore, $M(z) \equiv 0$. Thus, $G(\xi) = 0$ almost everywhere and $g = 0$.
Completeness in L^2_ρ. Suppose that a function $g \in L^2_\rho$ is orthogonal relative to the inner product in L^2_ρ to all eigenfunctions, i.e.,

$$\int D^\alpha f(y) g(y) \rho(y) dy = 0$$

for all α. Since f is analytic it implies that

$$\int f(y - x) g(y) \rho(y) dy = 0$$

for all $x \in \mathbf{R}^N$. Consider the Cauchy problem for the linear parabolic equation (1.3) with initial data $u_0(x) = g(x)\rho(x)$. One can see from the Poisson-type integral (2.5) and the kernel estimate (1.8) that the solution exists for all $0 < t < 2$, provided that the exponent $a > 0$ in the weight (2.1) is sufficiently small. Then $u(x,t)$ is analytic in x. We have

$$u(x, 1) = \int f(x - y) g(y) \rho(y) dy.$$

Therefore, $u(x, 1) = 0$. It follows by the uniqueness theorem for the inverse parabolic equation [9], p. 181, that $u(x, 0) = 0$, and $g = 0$.

(iii) **Compact resolvent.** Let us prove that the resolvent is a compact operator. In what follows, $\|\cdot\|$ is the norm in $L^2(\mathbf{R}^N)$, $V(\xi)$ denotes the Fourier transform of $v(y)$, $V(\xi) = \int v(y) e^{-iy\cdot\xi} dy$, and C stand for various positive constants.

Lemma 2.1. *There holds*

$$\int (1+|y|^2)^\rho |D^\beta v(y)|^2 dy \le C \int \left[|D^{2m} v(y)|^2 + (1+|y|^2)^\alpha |v(y)|^2 \right] dy,$$

if $v \in C_0^\infty(\mathbf{R}^N)$ *and*

$$\rho/\alpha + |\beta|/2m \le 1.$$

Proof. Since

$$\int |D^\gamma v(y)|^2 dy \le C \int (|D^{2m} v(y)|^2 + |v(y)|^2) dy$$

for all γ with $|\gamma| < 2m$, it suffices to verify that

$$\int |y|^{2\rho} |D^\beta v(y)|^2 dy \le C \int (|D^{2m} v(y)|^2 + |y|^{2\alpha} |v(y)|^2) dy.$$

Put

$$I = \int (|D^{2m} v(y)|^2 + |y|^{2\alpha}|v(y)|^2) dy,$$

$$I(\lambda) = \int_{\lambda<|y|<2\lambda} (|D^{2m} v(y)|^2 + |y|^{2\alpha}|v(y)|^2) dy,$$

$$J = \int |y|^{2\rho} |D^\beta v(y)|^2 dy, \quad J(\lambda) = \int_{\lambda<|y|<2\lambda} |y|^{2\rho}|D^\beta v(y)|^2 dy.$$

We have to prove that $J \le CI$, or

$$\sum_{k=-\infty}^{\infty} J(2^{-k}) \le C \sum_{k=-\infty}^{\infty} I(2^{-k}).$$

It suffices to show that for any integer k, the inequality $J(2^{-k}) \le CI(2^{-k})$ holds. We are checking the inequality $J(\lambda) \le CI(\lambda)$ for any $\lambda > 0$. Put $y = \lambda z$. Then our inequality takes the form

$$\int_{1<|z|<2} |z|^{2\rho} |D^\beta v(z)|^2 dz \le C[\lambda^{2|\beta|-4m-2\rho} \int_{1<|z|<2} |D^{2m} v(z)|^2 dz$$
$$+ \lambda^{2\alpha-2\rho+2|\beta|} \int_{1<|z|<2} |z|^{2\alpha} |v(z)|^2 dz].$$

Since here $1 < |z| < 2$, it is reduced to

$$\int_{1<|z|<2} |D^\beta v(z)|^2 dz \le C[\lambda^{2|\beta|-4m-2\rho} \int_{1<|z|<2} |D^{2m} v(z)|^2 dz + \lambda^{2\alpha-2\rho+2|\beta|} \int_{1<|z|<2} |v(z)|^2 dz].$$

However, for any $\varepsilon > 0$, the inequality

$$\int_{1<|z|<2} |D^\beta v(z)|^2 dz \le C\left[\varepsilon \int_{1<|z|<2} |D^{2m} v(z)|^2 dz + \varepsilon^{|\beta|/(|\beta|-2m)} \int_{1<|z|<2} |v(z)|^2\right] dz$$

holds. Setting $\varepsilon = \lambda^{2|\beta|-4m-2\rho}$, we see that our inequality holds if $|\beta|/2m + \rho/\alpha = 1$. $\square$

Theorem 2.2. *There exist positive numbers a_0, k_0 such that for all $k \ge k_0$, $a \le a_0$, the operator*

$$(\mathbf{B} - kI)^{-1} : L^2_\rho(\mathbf{R}^N) \to L^2_\rho(\mathbf{R}^N) \tag{2.26}$$

is compact.

Proof. *Step 1.* It suffices to prove (2.26) for the equivalent smooth weight $\rho_1(y) = \exp\{a(1+|y|^2)^{\alpha/2}\}$ since $\rho(y) \le \rho_1(y) \le e^a \rho(y)$ in $\mathbf{R}^N$.

Denoting $\mathbf{C} = \sqrt{\rho_1}(\mathbf{B} - kI)\sqrt{1/\rho_1}$, (2.26) means that $\mathbf{C}^{-1} : L^2(\mathbf{R}^N) \to L^2(\mathbf{R}^N)$ is compact. Since the subset

$$\{v \in C_0^\infty(\mathbf{R}^N);\ \int |D^{2m} v(y)|^2 dy \le 1,\ \int (1+|y|^2)^\alpha |v(y)|^2 dy \le 1\} \tag{2.27}$$

is compact in $L^2(\mathbf{R}^N)$, it suffices to prove that for some constant $C > 0$,

$$\int (1+|y|^2)^\alpha v^2 dy + \int |D^{2m} v|^2 dy \le C \int |\mathbf{C} v|^2 dy \quad \text{for } v \in C_0^\infty(\mathbf{R}^N).$$

There holds

$$\mathbf{C}v = -(-\Delta)^m v - kv - \frac{a\alpha N}{2m} |y|^2 (1+|y|^2)^{(\alpha-2)/2} v + \frac{N}{2m} y D_y v + \sum_{|\beta|=0}^{2m-1} P_\beta(y) D^\beta v,$$

where the smooth functions P_β are such that

$$|P_\beta(y)| \le C_\beta[|y|^{(\alpha-1)(2m-|\beta|)} a^{2m-|\beta|} + C_{a\beta}]. \tag{2.28}$$

Step 2. Since

$$\int |D^\beta v P_\beta(y)|^2 dy \le C \int |D^\beta v|^2 [|y|^{(\alpha-1)(2m-|\beta|)} a^{2m-|\beta|} + C_{a,\beta}] dy$$

and $(\alpha-1)(2m-|\beta|)/\alpha + |\beta|/2m = 1$, we can apply Lemma 2.1 and conclude that for any $a > 0$, there exists $k > 0$ such that

$$\int |D^\beta v P_\beta(y)|^2 dy \le Ca^2 \int |D^{2m} v|^2 dy + Ca^2 \int (1+|y|^2)^\alpha |v|^2 dy + k \int |v|^2 dy.$$

Therefore, the integrals which are obtained by multiplication of the last sum in the definition of $\mathbf{C}$ with other terms, can be neglected.

Step 3. We have

$$\int |(-\Delta)^m v|^2 dy \ge \int |D^{2m} v|^2 dy,$$

$$\int (-\Delta)^m v\ |y|^2 (1+|y|^2)^{(\alpha-2)/2} v dy = \int |y|^2 (1+|y|^2)^{(\alpha-2)/2} |\nabla^m v|^2 dy$$

$$+\int \sum_{|\beta|\le m, |\gamma|\le m, |\beta+\gamma|\le 2m-1} R_{\beta,\gamma}(y) D^\beta v(y) D^\gamma v(y) dy,$$

where $|R_{\beta\gamma}(y)| \le C_{\beta\gamma}[|y|^{(\alpha-1)} + 1]$. Integrating by parts again, we obtain that

$$\int \sum_{|\beta|\le m, |\gamma|\le m, |\beta+\gamma|\le 2m-1} R_{\beta\gamma}(y) D^\beta v(y) D^\gamma v(y) dy = \int \sum_{|\beta|\le 2m-1} D^\beta v(y)\ v(y) S_\beta(y) dy,$$

where $|S_\beta(y)| \le C_\beta[|y|^{(\alpha-1)} + 1]$. Therefore,

$$\int (-\Delta)^m v\ |y|^2 (1+|y|^2)^{(\alpha-2)/2} v dy \ge -C \int [(1+|y|^{\alpha-1})v^2 + |D^{2m-1}v|^2] dy$$

$$\ge -\varepsilon \int [(1+|y|^2)^\alpha v^2 + |D^{2m}v|^2] dy - C_\varepsilon \int v^2 dy.$$

Step 4. It remains to estimate the integrals of functions formed by multiplication of $yD_y v$ with other terms. It is easy to see that

$$\int yD_y v(-\Delta)^m v dy = \int (\partial(\xi V)|\xi|^{2m}/\partial\xi) V d\xi$$

$$= \int |V|^2 |\xi|^{2m} d\xi + \int (\xi \partial V/\partial\xi)|\xi|^{2m} V d\xi$$

$$\ge -\frac{1}{2}\int |V(\xi)|^2 (\partial(\xi|\xi|^{2m})/\partial\xi) d\xi \ge -C\int |D^m v|^2 dy,$$

$$-\int v y D_y v dx = \frac{N}{2}\int v^2 dy \ge 0,$$

$$-\int yD_y v|y|^2(1+|y|^2)^{(\alpha-2)/2} v dy = \frac{1}{2}\int v^2 (\partial(y|y|^2(1+|y|^2)^{(\alpha-2)/2})/\partial y) dy$$

$$\ge -C\int |y|^\alpha v^2 dy.$$

Therefore, fixing $\varepsilon > 0$ small enough, we obtain

$$\|\mathbf{C}v\|^2 \ge C\ \left[\int |D^{2m}v|^2 dx + \int (k^2 + a^2|x|^{2\alpha}) v^2 dx\right]$$

if k is sufficiently large. We then have that for large k, $\mathbf{C}^{-1}$ maps bounded subsets in $L^2(\mathbf{R}^N)$ onto compact subsets of the form (2.27). □

End of Proof of Theorem 2.1. Given a $\lambda \notin \sigma_p(\mathbf{B})$ and a $k \ge k_0$, $k \gg |\lambda|$, the resolvent $(\mathbf{B}-\lambda I)^{-1}$ satisfies

$$(\mathbf{B}-\lambda I)^{-1} = [(\mathbf{B}-kI)(I+(k-\lambda)(\mathbf{B}-kI)^{-1})]^{-1} = (I+(k-\lambda)(\mathbf{B}-kI)^{-1})^{-1}(\mathbf{B}-kI)^{-1}$$

and hence is compact in $L^2_\rho(\mathbf{R}^N)$ as the product of bounded and compact operators. Therefore, $\mathbf{B}$ has the discrete spectrum (2.19) only. □

For convenience, let us derive the explicit representation of the resolvent using the descent method [7] of construction of fundamental solutions of elliptic operators from the parabolic ones. Consider the auxiliary evolution equation

$$w_\tau = \mathbf{B}w - e^{\lambda\tau} g, \quad w(0) = 0, \quad g \in L^2_\rho(\mathbf{R}^N).$$

Setting $w(\tau) = e^{\lambda\tau} v$ yields $v_\tau = (\mathbf{B} - \lambda I)v - g$ with $v(0) = 0$, and hence

$$v(\tau) = -\int_0^\tau e^{\mathbf{B}-\lambda I(\tau-s)} g\, ds,$$

where the semigroup is given in (2.16). Setting $\tau - s = \eta$ and passing to the limit $\tau \to \infty$ yields that there exists a limit

$$v(\infty) = -\int_0^\infty e^{(\mathbf{B}-\lambda I)\eta} g\, d\eta \equiv (\mathbf{B} - \lambda I)^{-1} g$$

provided the integral converges. Using the semigroup representation (2.16) and performing the change $e^{-\eta} = z \in (0,1)$ we arrive at the following integral operator:

$$(\mathbf{B} - \lambda I)^{-1} g(y) = \int_{\mathbf{R}^N} K(y,\zeta) g(\zeta) d\zeta \quad \text{with the kernel} \tag{2.29}$$

$$K(y,\zeta) = \int_0^1 z^{\lambda-1}(1-z)^{-N/2m} f[(y - \zeta z^{1/2m})(1-z)^{-1/2m}] dz. \tag{2.30}$$

By the semigroup representation (2.16), we have the following auxiliary result.

Proposition 2.2. $\mathbf{B}$ *is the infinitezimal generator of the strong continuous semigroup* $\{T(\tau)\}_{\tau \ge 0}$ *given by* (2.16).

Proof. We need to prove that

$$\mathbf{B}_\tau w_0 \equiv \frac{1}{\tau}(T(\tau)w_0 - w_0) \to \mathbf{B}w_0 \quad \text{as } \tau \to 0$$

in $L^2_\rho(\mathbf{R}^N)$ for any $w_0 \in H^{2m}_\rho(\mathbf{R}^N)$; see [9], Part 2. By expansion (2.18) we have

$$(\mathbf{B}_\tau - \mathbf{B})w_0 = \sum H_\beta(\tau) \tilde{M}(w_0) \tilde{\psi}_\beta \to 0 \quad \text{as } \tau \to 0, \quad H_\beta(\tau) \equiv \frac{e^{\lambda_\beta \tau} - 1}{\tau} - \lambda_\beta,$$

since $|H_\beta(\tau)| = O(\tau)$ as $\tau \to 0$ for any fixed $|\beta| \ge 0$, and for any $\tau > 0$, $|H_\beta(\tau)| \le C|\lambda_\beta|$ for $|\beta| \gg 1$, and $\mathbf{B}w_0 \in L^2_\rho(\mathbf{R}^N)$. $\square$

3. Spectrum and polynomial eigenfunctions of the adjoint operator

We now describe the eigenfunctions of the adjoint operator

$$\mathbf{B}^* = -(-\Delta)^m - \frac{1}{2m} y \cdot \nabla\,. \tag{3.1}$$

In the second-order case $m = 1$ it admits a symmetric representation

$$\mathbf{B}^* = (\rho^*)^{-1} \nabla \cdot (\rho^* \nabla), \quad \rho^*(y) = e^{-|y|^2/4}, \tag{3.2}$$

$-\mathbf{B}^* \geq 0$ is semibounded from below, and there exists its unique Friedrichs extension, which is a self-adjoint operator in the weighted space $L^2_{\rho^*}(\mathbf{R}^N)$ with the domain $H^2_{\rho^*}(\mathbf{R}^N)$ and a discrete spectrum. The eigenfunctions form an orthonormal basis in $L^2_{\rho^*}(\mathbf{R}^N)$ and the classical Hilbert-Schmidt theory applies; see [19] and [2].

Let $m > 1$. We consider $\mathbf{B}^*$ in the weighted space $L^2_{\rho^*}(\mathbf{R}^N)$ with the exponentially decaying weight function

$$\rho^*(y) \equiv 1/\rho(y) = e^{-a|y|^\alpha} > 0, \tag{3.3}$$

and ascribe to $\mathbf{B}^*$ the domain $H^{2m}_{\rho^*}(\mathbf{R}^N)$ dense in $L^2_{\rho^*}(\mathbf{R}^N)$. As in the previous section, we show that $\mathbf{B}^*: \ H^{2m}_{\rho^*}(\mathbf{R}^N) \to L^2_{\rho^*}(\mathbf{R}^N)$ is a bounded linear operator and

$$\langle \mathbf{B}v, w\rangle = \langle v, \mathbf{B}^* w\rangle \quad \text{for any } \ v \in H^{2m}_{\rho}(\mathbf{R}^N), \ \ w \in H^{2m}_{\rho^*}(\mathbf{R}^N). \tag{3.4}$$

3.1. **Discrete spectrum, polynomial eigenfunctions and compact resolvent.** We next establish spectral properties of the adjoint operator which are quite similar to those in Theorem 2.1.

Theorem 3.1. *(i) The spectrum of $\mathbf{B}^*$ consists of eigenvalues of finite multiplicity*

$$\sigma(\mathbf{B}^*) = \sigma(\mathbf{B}) = \{\lambda_\beta = -|\beta|/2m, \ \ |\beta| = 0, 1, 2, ...\}, \tag{3.5}$$

and eigenfunctions $\{\psi^*_\beta(y)\}$ are polynomials of order $|\beta|$.

(ii) The subset of eigenfunctions $\Phi^* = \{\psi^*_\beta\}$ is complete and closed in $L^2_{\rho^*}(\mathbf{R}^N)$.

(iii) For any $\lambda \notin \sigma(\mathbf{B}^*)$, the resolvent $(\mathbf{B}^* - \lambda I)^{-1}$ is a compact operator in $L^2_{\rho^*}(\mathbf{R}^N)$.

Proof. (i) **Spectrum and polynomial eigenfunctions.** We have $\sigma(\mathbf{B}) = \sigma(\mathbf{B}^*)$ [19]. Let us derive polynomials $\{\psi^*_\beta\}$. Applying the Fourier transform to the eigenvalue problem $\mathbf{B}^* u = \lambda u$ yields the following first-order equation:

$$\frac{1}{2m}\xi \cdot \nabla V + (\frac{N}{2m} - |\xi|^{2m})V = \lambda V \quad \text{in } \ \mathbf{R}^N. \tag{3.6}$$

The general solution is given by

$$V(\xi) = \Phi |\xi|^{2m\lambda - N} e^{|\xi|^{2m}} \quad \text{in } \quad \mathbf{R}^N \setminus \{0\}, \tag{3.7}$$

where $\Phi = \Phi(\xi/|\xi|)$ is an arbitrary smooth function on the unit sphere $S_1 = \{|\xi| = 1\}$ in $\mathbf{R}^N$. In view of the exponentially growing factor $e^{|\xi|^{2m}}$ in (3.7), the only distributions satisfying equation (3.6) correspond to $\Phi \equiv 0$ on S_1, i.e., those having supports concentrated at the origin $\xi = 0$. Therefore, $u(y)$ must be a polynomial. If its degree is k, then

$$u(y) = \sum_{j=0}^{s} P_j(y),$$

where $P_j(y)$ is a homogeneous polynomial of degree $k - 2mj$ with $s = [k/2m]$. Since by the Euler identity

$$-\frac{1}{2m}\sum_{j=1}^{N} y_j \frac{\partial P_0(y)}{\partial y_j} = -\frac{k}{2m} P_0(y) = \lambda P_0(y),$$

we see that $\lambda = -k/2m$ and $P_0(y)$ may be an arbitrary homogeneous polynomial of degree k. Other polynomials $P_j(y)$ are then defined as follows:

$$P_j(y) = \frac{1}{j!}(-\Delta)^{mj} P_0(y), \quad j = 1, ..., s.$$

This structure of $u(y)$ implies the completeness of the set of eigenfunctions in $L^2_{\rho^*}(\mathbf{R}^N)$. In the second-order case $m = 1$ this construction leads to the Hermite polynomials. Note that the polynomial structure of adjoint eigenfunctions follows from the expansion (2.18), where the coefficients of initial data $\tilde{M}_\beta(w_0)$ in (2.14) are the dual products of $w_0 \in L^2_\rho(\mathbf{R}^N)$ and $\tilde{\psi}^*_\beta \in L^2_{1/\rho}(\mathbf{R}^N)$. This implies that $\psi^*_\beta(y)$ are composed from a finite number of elementary polynomials y^γ.

We now fix $P_0(y) = y^\beta$ so that for eigenfunctions $\{\psi_\beta\}$ of $\mathbf{B}$ in (2.20) the corresponding adjoint eigenfunctions take the form

$$\psi^*_\beta(y) = \frac{1}{\sqrt{\beta!}} \left[y^\beta + \sum_{j=1}^{[|\beta|/2m]} \frac{1}{j!}(-\Delta)^{mj} y^\beta \right]. \tag{3.8}$$

Then the orthonormality condition holds

$$\langle \psi_\beta, \psi^*_\gamma \rangle = \delta_{\beta,\gamma} \quad \text{for any } \beta,\ \gamma,$$

where $\langle \cdot, \cdot \rangle$ is the duality product in $L^2(\mathbf{R}^N)$ and $\delta_{\beta,\gamma}$ is the Kronecker delta. Operators $\mathbf{B}$ and $\mathbf{B}^*$ have zero Morse index (no eigenvalues with positive real parts are available).

(ii) **Completeness and closure.** It follows from the well-known fact that polynomials $\{y^\beta\}$, which are higher-order terms in any eigenfunction ψ^*_β, are complete in suitable weighted L^p-spaces; see [19], p. 431. Closure is seen via orthogonalization, see Section 2. Recall that for $m = 1$, these are well-known properties of the Hermite polynomials generated by a singular self-adjoint Sturm-Liouville problem, [2], Chapt. 2.

(iii) **Compact resolvent.** Since $(\cdot)^*$ and $(\cdot)^{-1}$ commute for operators in Banach spaces and adjoint operator of a compact operator is compact [19], we have from Theorem 2.2 that for any $\lambda \notin \sigma(\mathbf{B}^*)$, $(\mathbf{B}^* - \lambda I)^{-1}$ is compact with the point spectrum only. $\square$

3.2. **Semigroup of the adjoint operator.** Let us find an explicit representation of the semigroup $e^{\mathbf{B}^*\tau}$. We introduce the rescaled variables corresponding to blow-up as $t \to 1^-$

$$u(x,t) = w(y,\tau), \quad y = x/(1-t)^{1/2m}, \quad \tau = -\ln(1-t) : (0,1) \to \mathbf{R}_+.$$

Then w solves the problem

$$w_\tau = \mathbf{B}^* w \quad \text{for } \tau > 0, \quad w(0) = w_0 = u_0. \tag{3.9}$$

Rescaling solution (2.5) of the linear parabolic equation (1.3), we obtain the following explicit representation of the semigroup with the infinitesimal generator $\mathbf{B}^*$:

$$w(y,\tau) = \phi^N(\tau) \int_{\mathbf{R}^N} f((z - ye^{-\tau/2m})\phi(\tau)) w_0(z)\,dz, \quad \phi(\tau) = (1 - e^{-\tau})^{-1/2m}. \tag{3.10}$$

As in the previous section, we use Taylor expansions of the exponential terms $(1-e^{-\tau})^{-1/2m}$ and $(1-e^{-\tau})^{-N/2m}$ in terms of integer powers of $e^{-\tau}<1$ for $\tau>0$ in the uniformly converging kernel expansion

$$(3.11)\qquad f((z-ye^{-\tau/2m})\phi(\tau))=\sum \frac{1}{\beta!}D^\beta f(z\phi(\tau))(-1)^{|\beta|}(1-e^{-\tau})^{-|\beta|/2m}e^{-|\beta|\tau/2m}\,y^\beta.$$

The dependence on y in each term is purely polynomial. We also perform Taylor expansions in the terms

$$D^\beta f(z\phi(\tau))=\sum \frac{1}{\mu!}D^\mu f(0)\phi^{|\mu|}(\tau)D^\beta(z^\mu)\equiv \sum \frac{1}{(\mu-\beta)!}D^\mu f(0)\phi^{|\mu|}(\tau)z^{\mu-\beta}.$$

Substituting all such expressions into (3.10), we arrive at the eigenfunctions expansion of the solution

$$(3.12)\qquad w(y,\tau)=\sum e^{\lambda_\beta\tau}M^*_\beta(w_0)\psi^*_\beta(y),\quad \text{where}\;\; M^*_\beta(w_0)=\langle w_0,\psi_\beta\rangle.$$

3.3. **Eigenfunction expansions and little Hilbert spaces.** Given the complete eigenfunction subset $\Phi=\{\psi_\beta\}$ of the non self-adjoint operator $\mathbf{B}$, we define the linear subspace $\tilde L^2_\rho$ of eigenfunction expansions

$$(3.13)\qquad v\in \tilde L^2_\rho \quad \text{iff}\quad v=\sum c_\beta\psi_\beta \;\text{ with convergence in }\; L^2_\rho(\mathbf{R}^N)$$

as the closure of the subset of finite sums $\{\sum_{|\beta|\le K}c_\beta\psi_\beta,\ K\ge 0\}$ in the L^2_ρ-norm. An extra characterization of $\tilde L^2_\rho$ can be done by introducing little Hilbert spaces.

Let l^2_ρ be a Hilbert space of functions $v=\sum a_\beta\psi_\beta\in\tilde L^2_\rho$ with coefficients satisfying

$$(3.14)\qquad \sum |a_\beta|^2<\infty,$$

where the scalar product and the induced norm are given by

$$(3.15)\qquad (v,w)_0=\sum a_\beta c_\beta,\;\; w=\sum c_\beta\psi_\beta\in l^2_\rho,\quad \text{and}\;\; \|v\|_0^2=(v,v)_0.$$

Then we arrive at a number of straightforward consequences (this list can be easily extended).

Proposition 3.1. *(i) The Hilbert space l^2_ρ is a dense subspace of $\tilde L^2_\rho$ in $L^2_\rho(\mathbf{R}^N)$,*
(ii) $\Phi=\{\psi_\beta\}$ is complete and closed in l^2_ρ in the topology of $L^2_\rho(\mathbf{R}^N)$,
(iii) the resolvent $(\mathbf{B}-\lambda I)^{-1}$ for $\lambda\notin\sigma(\mathbf{B})$ is compact in l^2_ρ,
(iv) $\mathbf{B}$ is sectorial in l^2_ρ.

Proof. (i) l^2_ρ is separable and complete since the same is true for the isomorphic Hilbert space l^2 of sequences. Let us show that $l^2_\rho\subseteq\tilde L^2_\rho$. For any $v\in l^2_\rho$,

$$\int\rho|v|^2dy=\int|\sum a_\beta\sqrt{\rho}\psi_\beta|^2dy\le\sum|a_\beta|^2\sum\int e^{a|y|^\alpha}|\frac{1}{\sqrt{\beta!}}D^\beta f(y)|^2dy,$$

and by the same estimates as in (2.12) we conclude that for and $l = |\beta| \gg 1$,

$$\int e^{a|y|^\alpha} |\frac{1}{\sqrt{\beta!}} D^\beta f(y)|^2 dy \le l^{l(\nu+\varepsilon)}, \tag{3.16}$$

where $\nu = (\alpha - 2)/\alpha < 0$, $\varepsilon > 0$ is small. Therefore, $l^2_\rho \subset \tilde{L}^2_\rho$. Concerning the density of l^2_ρ, we note that given a $v = \sum a_\beta \psi_\beta \in \tilde{L}^2_\rho$, the sequence of truncations $\{\sum_{|\beta| \le K} a_\beta \psi_\beta, K = 1, 2, ...\} \subset l^2_\rho$ converges to v in the topology of $L^2_\rho(\mathbf{R}^N)$ as $K \to \infty$ by completeness and closure of $\{\psi_\beta\}$.

(ii) Since Φ is orthonormal in l^2_ρ, it follows that the only element orthogonal to $\{\psi_\beta\}$ is 0, and hence completeness of $\{\psi_\beta\}$ in l^2_ρ follows from the Riesz-Fischer theorem. It is closed as an orthonormal subset in a separable Hilbert space [19].

(iii) For any $v = \sum a_\beta \psi_\beta \in l^2_\rho$ from the unit ball T_1 in l^2_ρ with $\sum |a_\beta|^2 \le 1$,

$$(\mathbf{B} - \lambda I)^{-1} v = \sum b_\beta \psi_\beta, \quad \text{where}$$

$$b_\beta = \frac{a_\beta}{\lambda_\beta - \lambda} = -\frac{a_\beta}{|\beta|/2m + \lambda} = -\frac{2m a_\beta}{|\beta|}[1 + O(\frac{1}{|\beta|})] \quad \text{for } |\beta| \gg 1. \tag{3.17}$$

Therefore, for any $\varepsilon > 0$, there exists $K = K(\varepsilon) > 0$ such that for any $v \in T_1$,

$$\sum_{|\beta| \ge K} |b_\beta|^2 \le \frac{4m^2}{K^2} \sum |a_\beta|^2 \le \frac{4m^2}{K^2} < \varepsilon.$$

By the criterion of compactness in l^2 [19], $(\mathbf{B} - \lambda I)^{-1}$ maps T_1 onto a compact subset in l^2_ρ. $\square$

(iv) Recall that $(\mathbf{B} - \lambda I)^{-1}$ is a meromorphic function having a pole $\sim 1/\lambda$ as $\lambda \to 0$ since $\lambda_0 = 0$ has multiplicity one [14]. We then need an extra estimate on the resolvent which is easy to get in l^2_ρ (one can check that it is not that easy in the big space L^2_ρ). In the sector $\Phi_\theta = \{\lambda \in \mathbb{C} : \lambda \ne 0, |\arg \lambda| < \pi/2 + \theta\}$ with a $\theta \in (0, \pi/2)$, for any $v = \sum a_\beta \psi_\beta \in l^2_\rho$, we apply (3.17) by using the fact that $1/|\lambda_\beta - \lambda| \le 1/|\lambda| \sin\theta$ in Φ_θ to get

$$\|(\mathbf{B} - \lambda I)^{-1} v\|_0 = \left(\sum |a_\beta|^2 \frac{1}{|\lambda_\beta - \lambda|^2} \right)^{1/2} \le \frac{1}{\sin\theta} \frac{1}{|\lambda|} \|v\|_0.$$

Since $\mathbf{B}$ is closed and densely defined, it is a sectorial operator in l^2_ρ, see [9]. $\square$

Similarly, for the adjoint operator $\mathbf{B}^*$ we define the subspace $\tilde{L}^2_{\rho^*} \subseteq L^2_{\rho^*}(\mathbf{R}^N)$ where the eigenfunction subset Φ^* is closed (see (3.13), and the little Hilbert space $l^2_{\rho^*}$ of functions $v^* = \sum a_\beta \psi^*_\beta \in \tilde{L}^2_{\rho^*}$ with coefficients satisfying (3.14) and the inner product (3.15). The list of the corresponding properties in Proposition 3.1 remains the same.

It is important that in the self-adjoint case $m = 1$, the little space coincides with the big one,

$$l^2_\rho = L^2_\rho(\mathbf{R}^N) \quad \text{for } m = 1 \text{ if } a = 1/4 \text{ in (2.1)}$$

(note that if $m = 1$ but $a \ne 1/4$, then $\mathbf{B}$ is not self-adjoint in $L^2_\rho(\mathbf{R}^N)$). Since the orthonormality of the eigenfunction basis is of a key importance in many applications of the linear operator theory, in a number of linear and nonlinear problems involving operators $\mathbf{B}$, $\mathbf{B}^*$, the little spaces l^2_ρ and $l^2_{\rho^*}$ can be more natural than the big ones L^2_ρ and $L^2_{\rho^*}$.

4. **Invariant region in the supercritical case**

We return to the problem (1.1) in the supercritical range $p > p_F$. We take initial data from the class (1.9), where $d > 0$ is as defined in (1.8) and $B > 0$ is a constant to be specified. Next, we introduce a subset of continuous functions

$$Y_A = \{u : \ |u(x,t)| \le A(1+t)^{-N/2m} \exp\{-k|x|^\alpha/(1+t)^{\alpha/2m}\} \ \text{in} \ \mathbf{R}^N \times \mathbf{R}_+\}, \tag{4.1}$$

where $A > 0$ is a constant depending on B. We set $k = d/p$ and show that Y_A is an invariant domain in the following sense.

Theorem 4.1. *Let $p > p_F$. Then for any sufficiently small $B > 0$, there exists an $A > 0$ such that the solution of* (1.1) *satisfies*

$$u_0 \in X_B \implies u(x,t) \in Y_A. \tag{4.2}$$

Our analysis is based on the following typical estimate from S.D. Eidel'man's book [7], Lemma 5.1.

Proposition 4.1. *For the integral*

$$I_1 = \int_{\mathbf{R}^N} [(t-\beta)(\beta-\tau)]^{-N/2m} \exp\left\{-d\left[\frac{|x-y|^\alpha}{(t-\beta)^{\alpha/2m}} + \frac{|y-\xi|^\alpha}{(\beta-\tau)^{\alpha/2m}}\right]\right\} dy, \tag{4.3}$$

where $\tau < \beta < t$, the following estimate holds:

$$I_1 \le M(\varepsilon)(t-\tau)^{-N/2m} \exp\left\{-(d-\varepsilon)\frac{|x-\xi|^\alpha}{(t-\tau)^{\alpha/2m}}\right\}, \tag{4.4}$$

$\varepsilon \in (0,d)$ being arbitrary, $M(\varepsilon) = M_1\varepsilon^{-N/\alpha}$ and $M_1 > 0$ is independent of ε.

Proof. We note that the inequality

$$\frac{|x-y|^\alpha}{(t-\beta)^{\alpha/2m}} + \frac{|y-\xi|^\alpha}{(\beta-\tau)^{\alpha/2m}} \ge \frac{|x-\xi|^\alpha}{(t-\tau)^{\alpha/2m}}$$

holds in $\mathbf{R}^N$. The rest of the analysis is similar to that in [7], pp. 35-36. $\square$

Proof of Theorem 4.1. Consider the equivalent integral equation in the space of continuous functions

$$u(t) = \mathcal{M}(u)(t) \equiv b(t) * u_0 + \int_0^t b(t-s) * |u(s)|^p ds \equiv J_1 + J_2 \ \text{for} \ t > 0. \tag{4.5}$$

Using estimate (1.8) of the rescaled kernel f of fundamental solution (1.4), we apply Proposition 4.1 to two terms on the right-hand side of (4.5). Define

$$d - \varepsilon = d/p = k \quad \implies \quad \varepsilon = d(p-1)/p.$$

Let us estimate J_1 in (4.5) for a $u_0 \in X_B$. We have from (1.8) and Proposition 4.1 with $\xi = 0$, $\beta = 0$ and $\tau = -1$ that

$$
\begin{aligned}
|J_1| &\le D\omega_1 B \int_{\mathbf{R}^N} t^{-N/2m} \exp\left\{-d\left[\frac{|x-y|^\alpha}{t^{\alpha/2m}} + |y|^\alpha\right]\right\} dy \\
&\le D\omega_1 B M(\varepsilon)(1+t)^{-N/2m} \exp\left\{-k\,\frac{|x|^\alpha}{(1+t)^{\alpha/2m}}\right\}. \qquad (4.6)
\end{aligned}
$$

Estimating similarly the second term in (4.5) with $u \in Y_A$, we obtain

$$
\begin{aligned}
|J_2| \le\ & D\omega_1 A^p \int_0^t (1+s)^{-N(p-1)/2m} \int_{\mathbf{R}^N} [(t-s)(1+s)]^{-N/2m} \\
&\times\ \exp\left\{-d\left[\frac{|x-y|^\alpha}{(t-s)^{\alpha/2m}} + \frac{|y|^\alpha}{(1+s)^{\alpha/2m}}\right]\right\} dy\, ds. \qquad (4.7)
\end{aligned}
$$

We put $\xi = 0$, $\beta = s$ and $\tau = -1$ in Proposition 4.1 to get

$$
(4.8)\quad |J_2| \le D\omega_1 A^p M(\varepsilon)\left(\int_0^\infty (1+s)^{-N(p-1)/2m} ds\right)(1+t)^{-N/2m} \exp\left\{-k\,\frac{|x|^\alpha}{(1+t)^{\alpha/2m}}\right\},
$$

where the integral $\int_0^\infty (\cdot) ds$ converges if $p > p_F$. Summing up estimates (4.6) and (4.8), we obtain that if $u_0 \in X_B$ and $u \in Y_A$ then

$$
(4.9)\quad |\mathcal{M}(u)| \le D\omega_1 M(\varepsilon)\left(B + \frac{2m}{N(p-p_F)} A^p\right)(1+t)^{-N/2m} \exp\left\{-k\,\frac{|x|^\alpha}{(1+t)^{\alpha/2m}}\right\}.
$$

Hence, Y_A is invariant provided that

$$
D\omega_1 M(\varepsilon)\left(B + \frac{2m}{N(p-p_F)} A^p\right) \le A.
$$

One can see that operator (4.5), $\mathcal{M} : Y_A \to Y_A$ is a contraction in $Y_A \cap \{0 \le t \le \delta\}$ in the sup-metric provided that $\delta >$ is sufficiently small. By Banach's Contraction Principle [19], there exists a unique classical solution $u(x,t) \in Y_A$ for $t \in [0,\delta]$, which is extended for $t > \delta$ by the standard parabolic theory [7], [9] and remains in Y_A for all $t > 0$ by the above invariance analysis. $\square$

5. Perturbed dynamical system: proof of Theorem 1.1

5.1. Rescaling and first estimates. The rescaled solution (1.10) satisfies the following semilinear parabolic equation:

$$
(5.1)\quad v_\tau = \mathbf{C}(v,\tau) \equiv \mathbf{B}v + e^{-\gamma\tau}|v|^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+.
$$

The parameter γ in the exponential function is positive for $p > p_F$,

$$
\gamma = (p-1)\frac{N}{2m} - 1 \equiv \frac{N}{2m}(p-p_F) > 0.
$$

The initial function is the same, $v_0(y) \equiv u_0(y)$ in $\mathbf{R}^N$. Here $\mathbf{B}$ is the rescaled linear operator (1.5), so that the rescaled fundamental profile f is a stationary solution of the limit equation (5.1) with $\tau = \infty$,

$$\mathbf{B}f = 0 \quad \text{in } \mathbf{R}^N. \tag{5.2}$$

We see that after rescaling, the nonlinear term $|u|^p$ in (1.1) forms a small perturbation in (5.1) with the exponential decay rate $e^{-\gamma\tau} \to 0$ as $\tau \to \infty$ on bounded orbits. It follows from Theorem 4.1 that

$$|v(y,\tau)| \le \Phi_A(y) \equiv Ae^{-k|y|^\alpha} \quad \text{for any } v_0 \in X_B. \tag{5.3}$$

5.2. **Proof of Theorem 1.1.** We consider the dynamical system (5.1) as an asymptotically small perturbation of the autonomous dynamical system

$$v_\tau = \mathbf{B}v \quad \text{for } \tau > 0, \tag{5.4}$$

and will apply the stability result from [12], Theorem 3 in Section 3. We then need to check the following three hypotheses under which we can compare the ω-limit sets of two dynamical systems (5.1) and (5.4).

(H1) *Compactness of the orbits.* We introduce a metric space

$$Y = \{h \in L^1(\mathbf{R}^N): \ |h(y)| \le \Phi_A(y)\}$$

with the L^1-distance $d(\cdot,\cdot)$ and consider a class $\mathcal{L}$ of solutions $v \in C([0,\infty):Y)$, $v \in Y_A$ of (5.1) with initial data $v_0 \in X_B$ satisfying

$$\int v_0\, dy > 0. \tag{5.5}$$

Integrating equation (5.1) over $\mathbf{R}^N$, we derive that for solutions $v \in Y$

$$\frac{d}{d\tau}\int v(y,\tau)dy = e^{-\gamma\tau}\int |v(y,\tau)|^p dy > 0 \ \ (v(y,\tau) \not\equiv 0), \tag{5.6}$$

so that under hypothesis (1.2), by a small shifting the origin in time, we can assume that (5.5) holds. By standard regularity for uniformly parabolic equations [7], [9] the derivatives v_τ and $D_y v$ are uniformly bounded on subsets $B_R \times [\tau,\infty)$ for $R \gg 1$ and $\tau \gg 1$. Since by (5.3) $|v| \le Ae^{-kR^\alpha}$ is arbitrarily small in the complement $\mathbf{R}^N \setminus B_R$ for $R \gg 1$, by Ascoli-Arzelá theorem the sets $\{v^s(\tau) = v(\cdot,\tau+s)\}_{s>0}$ composed from solutions satisfying (5.3) are relatively compact in $L^\infty_{\rm loc}([0,\infty):Y)$.

Let $\omega(v_0)$ be the ω-limit set of the rescaled orbit $\{v(\tau),\ \tau > 0\}$ with convergence in $L^1(\mathbf{R}^N)$ (and hence with uniform convergence by parabolic regularity).

(H2) *Convergence.* We have to check that $\mathbf{C}$ is a small perturbation of $\mathbf{B}$ in the following sense: given a solution $v \in \mathcal{L}$, if for a sequence $\{\tau_j\} \to \infty$ the bounded sequence $\{v(\tau_j+s)\}$ converges in $L^\infty_{\rm loc}([0,\infty):Y)$ as $j \to \infty$ to a function $v(s)$ then v is an Y-valued solution of (5.4) (in the class $\mathcal{L}$). Using the uniform estimate on v, the passage to the limit as $\tau = \tau_j + s \to \infty$ in the perturbed equation (5.1) is straightforward. Then the limit function $v(s)$ solves equation (5.4) in the weak sense and hence it is a bounded classical solution by standard regularity results for uniformly parabolic equations.

(H3) *Uniform stability for the unperturbed equation.* Consider the reduced ω-limit set Ω_* of the autonomous equation (5.4) as the closure of the set of all possible ω-limits for initial data from X_B. Rescaling solution (2.5) by setting $u = t^{-N/2m}v$, $x = yt^{1/2m}$ for $t > 1$, we obtain that the rescaled orbit satisfies

$$v(y,t) = t^{N/2m} \int_{\mathbf{R}^N} f(y-z)u_0(zt^{1/2m})dz. \tag{5.7}$$

Hence, as $t \to \infty$,

$$v(y,t) = \|u_0\|_1 \, f(y) + o(1), \tag{5.8}$$

where $\|u_0\|_1 = \|v_0\|_1 > 0$ for $u_0 \in X_B$. Therefore, the reduced ω-limit set of equation (5.4) with suitable data satisfies

$$\Omega_* \subseteq \{c\, f(y),\ 0 \le c \le C_1\}, \tag{5.9}$$

where $C_1 > 0$ is a constant.

We now prove the main property of Ω_* establishing the uniform Lyapunov stability of Ω_*: for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for any solution $v(\tau) \in \mathcal{L}$ of (5.4),

$$d(v_0, \Omega_*) \le \delta \quad \Longrightarrow \quad d(v(\tau), \Omega_*) \le \varepsilon \quad \text{for every } \ \tau > 0.$$

Proposition 5.1. *There exists a constant $c_* = c_*(m,N) > 1$ for $m > 1$ such that, given two initial data $v_{10}, v_{20} \in X$, the solutions of* (5.4) *satisfy*

$$\|v_1(\tau) - v_2(\tau)\|_1 \le c_* \|v_{10} - v_{20}\|_1 \quad for\ \ \tau > 0. \tag{5.10}$$

Proof. Using (5.7), by Fubini's Theorem we have

$$\begin{aligned} \|v_1(\tau) - v_2(\tau)\|_1 &= t^{N/2m} \int \left| \int f(y-z)(u_{10}(zt^{1/2m}) - u_{20}(zt^{1/2m}))dz \right| dy \\ &\le t^{N/2m} \int \left| u_{10}(zt^{1/2m}) - u_{20}(zt^{1/2m}) \right| \left(\int |f(y-z)| dy \right) dz \\ &\le c_* \|u_{10} - u_{20}\|_1 \equiv c_* \|v_{10} - v_{20}\|_1, \end{aligned}$$

where $c_* = \int_{\mathbf{R}^N} |f(z)| dz > 1$, and this completes the proof. $\square$

Observe that $f(y) > 0$ for $m = 1$, so that for the corresponding order-preserving semigroup $c_* = 1$. This expresses the well-known fact that the Laplacian Δ generates a semigroup of contractions in $L^1(\mathbf{R}^N)$. If $m > 1$, then the profile f changes sign, the semigroup is not order-preserving, and we have the strict inequality $c_* > 1$, where c_* plays a role of the optimal order deficiency of the parabolic operator in (1.3) [11].

Estimate (5.10) implies the uniform stability of Ω_* in the L^1-metric, so that all the three hypotheses (H1) - (H3) from [12], Sect. 3, are valid. By stability Theorem 3 there we arrive at the following result.

Proposition 5.2. *For any orbit in $\mathcal{L}$,*

$$\omega(v_0) \subseteq \Omega_*. \tag{5.11}$$

As the last step, we establish that $\omega(v_0)$ consists of a unique nontrivial profile and this completes the proof of Theorem 1.1.

Proposition 5.3. *For any orbit in $\mathcal{L}$, there exists a constant $C_0 > 0$ such that*

$$\omega(v_0) = \{C_0 f\}. \tag{5.12}$$

Proof. By the monotonicity (5.6) of the total mass of the rescaled solutions we see that there exists a finite positive limit

$$\int v(y,\tau)dy \to C_0 \quad \text{as } \tau \to \infty. \tag{5.13}$$

Since the set $\{cf,\, c \in \mathbf{R}\}$ is strictly ordered in c relative to the mass, $\int c\, f(y)dy = c$, (5.11) and (5.13) imply (5.12). $\square$

5.3. **Eigenfunction expansion: on the exponential rate of convergence.** Let us now apply the eigenfunction expansion for functions from $\tilde{L}^2_\rho$ (or from the little Hilbert space h^2_ρ) of sufficiently smooth solutions v from the invariant set Y_A

$$v(y,\tau) = \sum a_\beta(\tau)\psi_\beta(y) \quad \text{for } \tau > 0 \tag{5.14}$$

converging pointwise by parabolic regularity theory [7], [9]. Moreover, we assume that for small and smooth initial data $u_0 \in X_B \cap \tilde{L}^2_\rho$, the series converges uniformly. Substituting the expansion into equation (5.1) and multiplying by ψ^*_β in $L^2(\mathbf{R}^N)$, we obtain the dynamical system

$$\dot{a}_\beta = \lambda_\beta a_\beta + e^{-\gamma\tau}\langle |v(\tau)|^p, \psi^*_\beta\rangle, \quad |\beta| \ge 0. \tag{5.15}$$

It follows from (5.3) that the first coefficient $a_0(\tau)$ with $|\beta| = 0$ and $\lambda_0 = 0$ satisfies $\dot{a}_0 = O(e^{-\gamma\tau})$ for $\tau \gg 1$. Then (2.21) and (2.22) imply that $a_0(\infty) = C_0$. Therefore,

$$a_0(\tau) = C_0 + O(e^{-\gamma\tau}) \quad \text{as } \tau \to \infty. \tag{5.16}$$

Solving the rest of equations in (5.15) with $\lambda_\beta < 0$, we have

$$a_\beta(\tau) = a_\beta(0)e^{\lambda_\beta\tau} + e^{\lambda_\beta\tau}\int_0^\tau e^{-(\lambda_\beta+\gamma)s}\langle |v(s)|^p, \psi^*_\beta\rangle ds. \tag{5.17}$$

Since $\langle |v(s)|^p, \psi^*_\beta\rangle$ are bounded uniformly in s by (5.3), we deduce that all such coefficients decay exponentially fast and at least,

$$a_\beta(\tau) = O(e^{-\gamma\tau} + \tau e^{\lambda_\beta\tau}) \quad \text{as } \tau \to \infty \text{ for any } |\beta| \ne 0. \tag{5.18}$$

Estimates (5.16) and (5.18) guarantee exponential convergence to the first eigenfunction.

6. Spectra of asymptotic patterns and critical exponents

We now consider arbitrary initial data $u_0 \in X_B$ which are not supposed to satisfy (1.2). Then there exists a global solution which is sufficiently small in the sense that $u(x,t) \in Y_A$. It follows from (5.6) that there exists a finite limit (5.13). If $C_0 \neq 0$, then similar to the proof of Theorem 1 we establish that

$$a_0(\tau) \to C_0 \quad \text{and} \quad v(\tau) \to C_0 f \quad \text{as } \tau \to \infty. \tag{6.1}$$

We now consider the asymptotics of small global solutions with initial data in X_B satisfying the condition of *zero eventual mass*:

$$a_0(\tau) \to 0 \ \text{ and } \int v(y,\tau)\,dy \to 0 \quad \text{as } \tau \to \infty. \tag{6.2}$$

Obviously, such patterns can be generated by the eigenfunctions of the linear operator $\mathbf{B}$ described in Section 2. In the previous section we used the fact that $\lambda_0 = 0 \in \sigma(\mathbf{B})$ and the rescaled fundamental solution $\psi_0 = f$ is the first eigenfunction. Other patterns can evolve close to the stable eigenspace of $\mathbf{B}$.

6.1. Asymptotic patterns of the linear problem, spherical polynomials in $\mathbf{R}^N$.

Let $\{P_{lk}(z), l = 0,1,\dots, k = 1,\dots,\nu_l\}$ be the set of homogeneous polynomials such that

$$P_{lk}(tz) = t^l P_{lk}(z) \quad \text{for } t \in \mathbf{R} \quad \text{with } \nu_l = \frac{(2l+N-2)!(l+N-3)!}{l!(N-2)!},$$

and the set $\{P_{lk}(z/|z|)\}$ be an orthonormal base on the unit sphere in $\mathbf{R}^N$. For example, it can be the set of harmonic polynomials. We have $z^\beta = \sum_{k=1}^{\nu_l} a_{\beta k} P_{lk}(z), \quad l = |\beta|$, and (2.13) can be rewritten in the form

$$w(y,\tau) = \sum_{l=0}^{\infty}\sum_{k=0}^{\nu_l} e^{-l\tau/2m} N_{lk}(u_0)\chi_{lk}(y), \quad \text{where} \tag{6.3}$$

$$N_{lk}(u_0) = \int_{\mathbf{R}^N} P_{lk}(z)u_0(z)\,dz, \quad \chi_{lk}(y) = \sum_{|\beta|=l} a_{\beta k}\psi_\beta(y).$$

The functions χ_{lk} are eigenfunctions of $\mathbf{B}$, $\mathbf{B}\chi_{lk} = -\frac{l}{2m}\chi_{lk}$. This formula implies that given special initial datum u_0, if there exists an l-th momentum $N_{lk} \neq 0$ and all other l-th momenta vanish as well as all j-th momenta for $j \le l-1$, the asymptotic behaviour of the solution of the linear equation (2.7) is given by the following pattern on the corresponding stable eigenspace:

$$v(y,\tau) = Ce^{-l\tau/2m}[\chi_{lk}(y) + o(1)] \quad \text{as } \tau \to \infty, \tag{6.4}$$

where $C \neq 0$. It follows from the completeness of eigenfunction that these are all possible asymptotic patterns in the linear problem for initial data in $L^2_\rho(\mathbf{R}^N)$. In the original variables, this gives the following asymptotic behaviour:

$$u(x,t) = Ct^{-(N+l)/2m}[\chi_{lk}(x/t^{1/2m}) + o(1)] \quad \text{as} \quad t \to \infty. \tag{6.5}$$

6.2. **Critical exponents.** Once we have classified the asymptotic behaviour of all decaying patterns in the linearized problem, we now in a position to compare those with similar patterns in the perturbed nonlinear equation. Namely, the following result holds where a countable spectrum of critical exponents occurs.

Proposition 6.1. *If equation* (5.1) *admits a solution with the linearized behaviour*

$$v(y,\tau) = Ce^{-l\tau/2m}[\varphi_l(y) + o(1)] \quad \text{as } \tau \to \infty \quad \text{uniformly,} \tag{6.6}$$

where $l \ge 0$ is an integer, φ_l is an eigenfunction of $\mathbf{B}$, $\mathbf{B}\varphi_l = -(l/2m)\varphi_l$, and $C \ne 0$, then $p > p_l = 1 + 2m/(l+N)$.

Proof. By the parabolic regularity theory [7], [9], estimate (6.6) admits differentiation with respect to y and τ on any subset there $|v| \ge \delta > 0$. Substituting the corresponding asymptotically sharp expressions into equation (5.1), we obtain $e^{-[\gamma+l(p-1)/2m]\tau} = o(1)$ as $\tau \to \infty$, whence the condition $\gamma + l(p-1)/2m \equiv (p-1)(N+l)/2m - 1 > 0$. $\square$

In order to construct all such linearized patterns, we perform the scaling $u = t^{-1/(p-1)}w$, $y = x/t^{1/2m}$, $\tau = \ln t$, which gives the autonomous equation (cf. (5.1)

$$w_\tau = \tilde{\mathbf{B}}w + |w|^p, \quad \tilde{\mathbf{B}} = \mathbf{B} + c_* I, \ \ c_* = 1/(p-1) - N/2m.$$

By Theorem 2.1, $s(\tilde{\mathbf{B}}) = \{\tilde{\lambda}_l = c_* - l/2m\}$, so that if $c_* \ne l/2m$ for all $l \ge 0$ (i.e., $p \ne p_l$), then $\tilde{\mathbf{B}}$ does not have a centre subspace and has a finite-dimensional unstable subspace. In view of known spectral properties of the linear operators involved, the existence of exponentially decaying patterns follows from the general invariant manifold theory, see Chap. 9 in [20]. But in the critical cases $\{p = p_l, \ l = 0, 1, ...\}$, the situation is entirely different, since now $\tilde{\mathbf{B}}$ has a centre subspace generating a non-trivial centre manifold behaviour to be discuss next.

6.3. **Critical asymptotic behaviour.** Bearing in mind the typical linearized behaviour (6.5), where $l = 0, 1, ...$, we perform in (1.1) the rescaling

$$u(x,t) = (1+t)^{-(l+N)/2m}v(y,\tau), \quad y = x/(1+t)^{1/2m}, \quad \tau = \ln(1+t).$$

The rescaled solution $v(y,\tau)$ satisfies the perturbed equation

$$v_\tau = (\mathbf{B} + \tfrac{l}{2m}I)v + e^{-\gamma_l\tau}|v|^p, \tag{6.7}$$

with the exponent $\gamma_l = (p-1)(l+N)/2m - 1$ (so that $\gamma_0 = \gamma$). It follows that $\gamma_l = 0$ iff $p = p_l$, and in these critical cases we arrive at the autonomous equation

$$v_\tau = (\mathbf{B} + \tfrac{l}{2m}I)v + |v|^p. \tag{6.8}$$

We are looking for solution $v(\cdot,\tau)$ with the behaviour for $\tau \gg 1$ close to the centre subspace of the linearized operator $\mathbf{B} + \frac{l}{2m}I$. Such a centre subspace asymptotic dominance assumes that in the eigenfunction expansion (5.14), the leading term as $\tau \to \infty$ corresponds to an eigenfunction φ_l of $\mathbf{B}$, $(\mathbf{B} + \frac{l}{2m}I)\varphi_l = 0$, i.e., φ_l belongs to the centre subspace of the linearized operator $\mathbf{B} + \frac{l}{2m}I$ in the perturbed equation (6.8). Hence, we suppose that

$$v(\tau) = a_l(\tau)[\varphi_l + o(1)] \quad \text{as } \tau \to \infty \tag{6.9}$$

and this asymptotic equality can be differentiated in y and τ.

Consider the radially symmetric case, where l is even, eigenvalues are simple and φ_l is a radial eigenfunction. Then, using the eigenfunction expansion as in the previous section we deduce that for solutions in Y_A, the corresponding equation for the leading expansion coefficient $a_l(\tau)$ takes the form

$$\dot{a}_l = |a_l|^{p_l}[c_l + o(1)] \quad \text{for } \tau \gg 1, \quad \text{where } c_l = \langle |\varphi_l|^{p_l}, \varphi_l^* \rangle. \tag{6.10}$$

Without loss of generality, we consider the case $c_l > 0$. Then, according to the asymptotic ODE (6.10), any solution with arbitrarily small initial data having positive $a_l(\tau)$ blows up in finite time, i.e., such centre subspace behaviour is strongly unstable. This is exactly the case for equation (1.1) at the first critical exponent $p = p_F = p_0$, where solutions with arbitrarily small initial data satisfying (1.2) blow up.

On the other hand, solving (6.10) for $a_l(\tau) < 0$, we obtain globally decaying negative coefficients with the behaviour

$$a_l(\tau) = -[(p_l - 1)c_l \tau]^{-1/(p_l-1)}(1 + o(1)) \quad \text{as } \tau \to \infty. \tag{6.11}$$

In terms of the original (x, t, u)-variables, such a behaviour takes a form of a logarithmically perturbed linearized pattern

$$u(x,t) = -C_l(t \ln t)^{-(l+N)/2m}[\varphi_l(x/t^{1/2m}) + o(1)] \quad \text{as } t \to \infty, \tag{6.12}$$

where the constant $C_l > 0$ does not depend on initial data, $C_l = [2mc_l/(l+N)]^{-(l+N)/2m}$.

In view of known spectral and sectorial properties of the operator $\mathbf{B}$ (see Theorem 2.1 and Section 3), this centre manifold behaviour can be justified by the stable invariant manifold theory in interpolation spaces [20], Chapter 9, in the case of sufficiently smooth, C^2, perturbations which imposes the extra condition $p \geq 2$. For the second-order semilinear heat equation (1.12) with the critical exponent $p = 1 + 2/N$, the global behaviour as $t \to \infty$ including extra $\ln t$-multipliers of the fundamental similarity structure is proved to be stable and generic [10]. Similar results are true for different quasilinear heat equations (1.13) with critical absorption; see [12] and references therein.

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