EVOLUTION COMPLETENESS OF SEPARABLE SOLUTIONS OF NONLINEAR DIFFUSION EQUATIONS IN BOUNDED DOMAINS

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Abstract. As a basic example, we consider the porous medium equation $(m > 1)$

$$u_t = \Delta(|u|^{m-1}u) \quad \text{in } \Omega \times \mathbb{R}_+, \quad u = 0 \text{ on } \partial\Omega \times \mathbb{R}_+,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$, and initial data $u_0 \in L^\infty \cap L^1$. It is well-known from the 1970's that the PME admits separable solutions $u_k(x,t) = t^{-1/(m-1)}\psi_k(x)$, $k = 0, 1, 2, ...$, where each $\psi_k \neq 0$ satisfies a nonlinear elliptic equation $\Delta(|\psi_k|^{m-1}\psi_k) + \frac{1}{m-1}\psi_k = 0$ in Ω, $\psi_k = 0$ on $\partial\Omega$. Existence of at least a countable subset $\Phi = \{\psi_k\}$ of such nonlinear eigenfunctions follows from the Lusternik-Schnirel'man variational theory from the 1930's. The first similarity pattern $t^{-1/(m-1)}\psi_0(x)$, where $\psi_0 > 0$ in Ω, is known to be asymptotically stable as $t \to \infty$ and attracts all nontrivial solutions with $u_0 \geq 0$ (Aronson and Peletier, 1981).

We show that if Φ is discrete, then it is *evolutionary complete*, i.e., describes the asymptotics of arbitrary global solutions of the PME. For $m = 1$ (the heat equation), the evolution completeness follows from the completeness-closure of the orthonormal subset $\Phi = \{\psi_k\}$ of eigenfunctions of the Laplacian Δ in L^2. The analysis applies to the perturbed PME and to the p-Laplacian equations of second and higher order.

To H.A. Levine on his 60th birthday

1. **Introduction**

Completeness and closure of countable subsets of eigenfunctions is a common feature of classes of linear self-adjoint operators in Hilbert spaces, see e.g., Birman-Solomjak's book [6]. The corresponding eigenfunction expansions are used for solving various linear partial differential equations (PDEs) and describe asymptotic properties of general solutions. These classical techniques do not apply to *essentially nonlinear* PDEs (not just small nonlinear perturbations of the dominant linear ones), where linear eigenfunction expansions cannot be used, though eigenfunctions of nonlinear operators occurring in elliptic and parabolic PDEs have been studied for a long period; see e.g., Krasnosel'skii-Zabreiko book [16]. Therefore, it is natural to ask if there exists a kind of *completeness* property of eigenfunctions for some related nonlinear PDEs.

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On the other hand, it is well known that many nonlinear evolution equations of mathematical physics including well-understood reaction-diffusion PDEs admit various particular self-similar or other group-invariant solutions (explicit or given by lower-order ODEs or PDEs); see e.g. examples and references in Barenblatt's book [5] and in [12, 15, 24]. Rather often, such particular solutions are asymptotically stable as $t \to \infty$ or as $t \to T < \infty$ (if e.g., a generic blow-up occurs), and therefore describe the asymptotic behaviour of wide classes of general solutions; see list of references in [12, 24].

In this paper we consider some classical nonlinear diffusion equations admitting countable families of global similarity solutions characterized by eigenfunction subsets Φ. We show that such subsets can be *evolutionary complete*, i.e., any appropriately rescaled solution of the parabolic PDE takes, as $t \to \infty$, the form given by a single function from Φ. For the corresponding "limit" linear models, the evolution completeness reduces to completeness (closure) of subsets of eigenfunctions of linear self-adjoint operators. We begin with a typical quasilinear example explaining main features of the analysis. Actually, the evolution completeness for a given evolution PDE just means that an optimal description of all possible asymptotic properties is available. A few of such exceptional examples are known for some reaction-diffusion PDEs, where perturbation theory of linear operators applies; see references below.

1.1. **The basic model: PME in a bounded domain.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the smooth boundary $\partial\Omega$. Consider the *porous medium equation* (PME)

$$u_t = \mathbf{A}(u) \equiv \Delta(|u|^{m-1}u) \;\text{ in } \Omega \times \mathbb{R}_+, \;\; u = 0 \;\text{ on } \partial\Omega \times \mathbb{R}_+, \;\; u(x, 0) = u_0 \;\text{ in } \Omega, \tag{1.1}$$

where $m > 1$ and initial data $u_0 \in X = L^\infty(\Omega) \cap L^1(\Omega)$, $u_0 \neq 0$. It is known from the 1950's that there exists a unique global weak continuous bounded solution $u = u(x, t)$ decaying as $t \to \infty$; see books by Lions [18], DiBenedetto [9] and Kalashnikov's survey [15]. Exponent $m > 1$ is the degree of homogenuity of operator $\mathbf{A}$,

$$\mathbf{A}(su) = |s|^{m-1}s\mathbf{A}(u) \quad \text{for all } \; s \in \mathbb{R}. \tag{1.2}$$

We are interested in the asymptotic behaviour of global solutions as $t \to \infty$ and in a description of all possible asymptotic patterns that can occur for arbitrary initial data $u_0 \in X$. Particular results have been well known for more than twenty years. For *nonnegative* u_0 and bounded Ω, the first result is due to Aronson and Peletier [4] establishing that as $t \to \infty$ uniformly in Ω,

$$u(x, t) = t^{-1/(m-1)}[\psi(x) + o(1)], \tag{1.3}$$

where $\psi(x) > 0$ in Ω is the unique bounded solution of the elliptic equation

$$\mathbf{A}(\psi) + \kappa\psi = 0 \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega, \quad \text{where } \kappa = \tfrac{1}{m-1}. \tag{1.4}$$

If u_0 changes sign in Ω, then the result does not apply. By the Lusternik-Schnirel'man theory [16], the variational elliptic problem (1.4) admits at least a countable subset of different nontrivial solutions $\Phi = \{\psi_k(x),\, k = 0, 1, 2, ...\}$; see Section 2. Then Aronson-Peletier's profile in (1.3) is the first one in Φ, $\psi = \psi_0$, and, actually, is the only stable one. Other solutions of (1.4) change sign in Ω.

Our main goal is to clarify the role of the whole subset Φ of non-zero solutions of the stationary elliptic equation (1.4). Discussions on this subject were initiated by S.I. Pohozaev at the Seminar on Nonlinear Equations of the Moscow Energy University at the beginning of the 1980's in connection with the fibering method of calculus of variation [22] establishing existence of countable subsets of eigenvectors of nonlinear variational operators.

1.2. **Evolution completeness.** We assume that the whole subset Φ of arbitrary non-trivial solutions of (1.4) is discrete, which is a key hypothesis (see Section 2 for some comments on continuous subsets). The definition is easily extended to continuous subsets Φ (but no asymptotic results are available for $N > 1$).

Definition 1.1. We say that Φ is *evolutionary complete* in X if for any initial data $u_0 \in X$, $u_0 \neq 0$, there exists a *finite* $k = k(u_0) \geq 0$ such that (1.3) with $\psi = \psi_k$ holds in a suitable topology.

It follows from the Lusternik-Schnirel'man theory (see Section 2) that, by continuity, $k = \infty$ is associated with the trivial solution $\psi_\infty(x) \equiv 0$, hence $\psi_\infty \notin \Phi$.

1.3. **On a counterpart in the spectral theory of linear operators.** The evolution completeness for the PME has a natural counterpart in the linear operator theory. In the limit case $m = 1$, (1.1) is the canonical *heat equation* (HE)

$$u_t = \Delta u, \tag{1.5}$$

with the zero Dirichlet boundary condition. It admits separable solutions $u_k(x,t) = e^{\lambda_k t}\psi_k(x)$, where $\lambda_k < 0$ and $\psi_k \neq 0$ are eigenvalue-eigenfunction pairs of the Laplacian Δ,

$$\Delta\psi_k - \lambda_k\psi_k = 0 \quad \text{in } \Omega, \quad \psi_k = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

By the theory of self-adjoint operators [6], the sequence of eigenvalues $\{\lambda_k\}$ is decreasing and the subset of the corresponding orthonormal eigenfunctions $\Phi = \{\psi_k\}$ is complete and closed in $L^2(\Omega)$. Therefore, given initial data $u_0 = \sum c_i\psi_i \in L^2(\Omega)$, the unique solution $u(\cdot,t) = \sum c_i e^{\lambda_i t}\psi_i(\cdot) \in H_0^2(\Omega)$ for $t > 0$ satisfies: there exists a finite $k \geq 0$ corresponding to the first non-zero coefficient c_k in the eigenfunction expansion of u_0 such that, uniformly in Ω (cf. (1.3)),

$$u(x,t) = c_k e^{\lambda_k t}[\psi_k(x) + o(1)] \quad \text{as } t \to \infty \quad (c_k \neq 0). \tag{1.7}$$

Thus, for the HE, the evolution completeness of Φ is an easy consequence of the completeness (closure) of the subset of eigenfunctions (1.6) of the Laplacian Δ in L^2. Moreover, since $\lambda_k \to -\infty$ as $k \to \infty$, the asymptotics (1.7) assume that any solution with a "super-exponential" decay (i.e., decaying faster than e^{-Kt} for any $K > 0$) is trivial. This is quite a general principle for linear differential equations in Hilbert spaces; see Agmon's results [1] (a nonlinear analogy will be discussed in Section 2).

In the case $m > 1$ of the nonlinear operator in (1.4), the evolutionary completeness has nothing to do with the linear theory. We continue to call $\psi_k \neq 0$ from (1.4) the "nonlinear eigenfunctions" of the operator $\Delta(|\psi|^{m-1}\psi)$ corresponding to the fixed "nonlinear eigenvalue" $\lambda_0 = -\kappa$, which is a standard terminology in the theory of nonlinear operators [16]). The elliptic eigenfunction problem (1.4) should be always associated with the corresponding evolution PDE (1.1). Indeed, separately, (1.4) does not define the unique eigenvalue λ_0. Unlike the linear case $m = 1$, in view of the homogenuity law (1.2), scaling $\psi \mapsto C\psi$ with a constant $C \neq 0$ implies the change of the eigenvalue $\lambda_0 \mapsto |C|^{1-m}\lambda_0$, i.e., there exists a continuous branch of eigenvalue-eigenfunction pairs. This is a typical property for nonlinear Hammerstein operators, [16].

Note that in both linear and nonlinear cases, there exists a strictly positive eigenfunction $\psi_0 > 0$ in Ω corresponding to the first eigenvalue λ_0. On the contrary, in the nonlinear problem (1.4) the eigenvalue λ_0 has *infinite* multiplicity, while the eigenvalues λ_k of the Sturm-Liouville problem (1.6) are known to be of finite multiplicity (and λ_0 is simple), [6]. Of course, any analogy between linear and nonlinear spectra is often rough and formal. In particular, for linear operators, eigenfunctions are defined up to a non-zero multiplier thus giving a continuous family of functions (therefore, $\Phi = \{\psi_k\}$ is discrete if it is orthonormal). Note that the same happens to nonlinear operators of the algebraic homogenuity 1; see Example 3 in Section 4. In the general nonlinear non-analytic case, existence of continuous families of eigenfunctions almost always leads to very difficult stabilization problems; see comments below.

1.4. **Main result, outline of methods and generalizations.** The main results are as follows. **(I)** For the quasilinear eigenvalue problem (1.4), we show in Section 2 that

$$\Phi \quad \text{is discrete} \implies \Phi \quad \text{is evolutionary complete.} \tag{1.8}$$

Discreteness of Φ means that it consists of isolated points in a natural topology. (1.8) is a general conclusion applied to a number of other second and higher-order parabolic models. The proof relies on two ingredients:

(i) the gradient structure of the rescaled equations, and

(ii) lower estimates establishing that nontrivial solutions of (1.1) cannot decay as $t \to \infty$ faster than $t^{-1/(m-1)}$. See Alikakos and Rostamian [2], where on p. 337, an open problem on what we call here the evolution completeness is posed.

Surprisingly, conclusion (1.8) via (i) and (ii) is rather straightforward, and surely was anticipated before; see e.g., [2] and [26]. Nevertheless, for general Ω's, the nonlinear completeness theory is expected to be extremely difficult. Firstly, if, on the contrary, for an $\Omega \subset \mathbb{R}^N$ with $N \geq 2$, Φ contains a continuous branch, the evolution completeness problem remains open and seems to be very difficult. Such an analysis can be associated with an open problem on ω-limits for general essentially non-analytic and non-smooth flows induced by nonlinear degenerate equations. The analytic case is covered by well-known Simon's technique [25], where ω-limit sets of compact orbits always consist of a single point. One must admit that for nonlinear degenerate equations, the evolution completeness problem cannot be solved in a generality comparable with the linear case.

Secondly, for nonlinear problems like (1.4), the hypothesis that Φ is discrete is very hard to prove. Note that the Lusternik-Schnirel'man theory applied to the variational problem (1.4), in general, does not detect all the possible solutions of the elliptic problem.

Moreover, Φ can include continuous branches of functions, see [16] and further comments below. The discreteness (and hence the evolution completeness) of Φ is elementary in 1D and in the radial geometry for various second-order models.

The best result we can expect is that Φ is discrete for "almost all" domains Ω [11]. For the simplest shape, Ω is the unit ball $B_1 \subset \mathbb{R}^N$, problem (1.4) has non-radial solutions and hence solutions families continuous with respect to parameters of rotations, so that Φ is not discrete. Once a continuous subset of equilibria is available, stabilization to a unique point for non-analytic flows generated by the PME (1.1) in dimensions $N \geq 2$ remains an open problem. In 1D, Sturm's theorem on zero sets can play a key role in the stabilization theory (but for $N = 1$, Φ for (1.4) is discrete).

(II) It is of principal importance to show that (1.8) remains true for *nonlinear perturbations* of the PME. As a simple example, in Section 3 we consider the perturbed PME

$$u_t = \Delta(|u|^{m-1}u) + \Delta(|u|^{n-1}u), \quad \text{with any exponent } \; n > m, \tag{1.9}$$

and show that the discrete subset Φ defined by (1.4) is still evolutionary complete.

(III) The results on the evolution completeness apply to the p-Laplacian equation

$$u_t = \mathbf{A}(u) \equiv \nabla \cdot (|\nabla u|^{m-1} \nabla u), \tag{1.10}$$

and to the fully nonlinear dual PME

$$u_t = \mathbf{A}(u) \equiv |\Delta u|^{m-1} \Delta u. \tag{1.11}$$

Some results are also true for higher-order parabolic equations with coercive and monotone operators. In Section 4 we consider the $4l$-th order p-Laplacian equation

$$u_t = \mathbf{A}(u) \equiv -\Delta^l(|\Delta^l u|^{m-1} \Delta^l u), \quad l \geq 1. \tag{1.12}$$

Other higher-order models with monotone, coercive, potential, variational, homogeneous operators (see models in [18, 2]) are included. The general conclusion (1.8) remains true but the actual discreteness of Φ becomes much more difficult. This is a principal hard question occurring for higher-order degenerate nonlinear eigenvalue problems, which seems cannot be solved via a unified approach (unlike the linear case $m = 1$, where the theory of self-adjoint operators applies).

Remark 1: on the PME in $\mathbb{R}^N$. A new difficulty occurs for the case $\Omega = \mathbb{R}^N$, where another special construction of nonlinear eigenvalues and eigenfunctions is necessary. For the PME (1.1), the similarity solutions have a more complicated form

$$u_k(x,t) = t^{-\alpha_k} \psi_k(y), \quad y = x/t^{\beta_k},$$

where ψ solves a quasilinear elliptic equation in $\mathbb{R}^N$ with exponents $\{\alpha_k, \beta_k\}$ being the nonlinear eigenvalues. In view of the scaling invariance, for any $b \neq 0$, $\psi_k(y; b) = |b|^{(3-m)/(m-1)} b \psi_k(y/b)$ is also an eigenfunctions, so that the corresponding subset $\Phi = \{\psi_k(y;b),\, k \in \mathbb{N},\, b \in \mathbb{R} \setminus \{0\}\}$ consists of a countable number of continuous branches. Actually, Φ is not known in dimensions $N \geq 2$ (nonlinear eigenvalue problems are very difficult to solve). The asymptotic analysis in 1D or in the radial geometry in $\mathbb{R}^N$, where

Φ has been identified, then uses scaling techniques and Sturm's Theorem on zero sets, [10]. The linear case $m = 1$ leads to the spectral analysis of the linear self-adjoint operator

$$\mathbf{B} = \Delta + \tfrac{1}{2} y \cdot \nabla + \tfrac{N}{2} I \quad \text{in } L^2_\rho(\mathbb{R}^N), \quad \text{with weight } \rho(x) = e^{|x|^2/4}. \tag{1.13}$$

$\mathbf{B}$ is known to define Hermite polynomials $H_\beta = \rho(D^\beta \rho^{-1})$ in $\mathbb{R}^N$ ([6], p. 48, β is a multiindex), and then $\Phi = \{\rho^{-1} H_\beta\}$ is a discrete, orthogonal, complete and closed (hence, evolutionary complete) eigenfunction subset.

2. **Proof of evolution completeness of eigenfunctions**

Here and later on $\| \cdot \|_p$ denotes the norm in $L^p(\Omega)$ for any $p \in [1, \infty]$, and $\langle \cdot, \cdot \rangle$ stands for the scalar product in $L^2 = L^2(\Omega)$.

2.1. **Countable subset of nonlinear eigenfunctions.** First, we need to study the elliptic equation (1.4). Setting $|\psi|^{m-1}\psi = \varphi$, we obtain a standard elliptic problem

$$\mathbf{D}(\varphi) \equiv \Delta\varphi + \kappa|\varphi|^{\alpha-1}\varphi = 0 \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

with the exponent $\alpha = \frac{1}{m} \in (0, 1)$. The corresponding variational formulation leads to a functional in the Sobolev space H^1_0,

$$V(\varphi) = \tfrac{1}{2}\|\nabla\varphi\|_2^2 - a\|\varphi\|_{\alpha+1}^{\alpha+1}, \quad a = \tfrac{\kappa}{\alpha+1}, \tag{2.2}$$

which is not positive for all $\varphi \neq 0$. Existence of an, at least, countable subset of critical points follows from the setting developed by Clark [8]; see also [2], Sect. 2, and applications of the fibering method [22, 23]. Let us show that the classical theory implies that

$$\psi_k \to \psi_\infty = 0 \quad \text{as } k \to \infty, \tag{2.3}$$

i.e., 0 is the *limit point* Φ, which is an important property, and we will use (2.3) as the definition of ψ_∞. Consider the even positive uniformly differentiable functional

$$W(v) = a\|v\|_{\alpha+1}^{\alpha+1} \tag{2.4}$$

on the unit sphere S_1 in H^1_0. By Lusternik-Schnirel'man's theorem ([16], p. 381), $W'(v)$ has a countable subset of eigenvectors $\{v_k,\ k = 0, 1, 2, ...\}$ and

$$c_k = W(v_k) \to 0 \quad \text{as } k \to \infty. \tag{2.5}$$

Then each v_k solves $\mu_k \Delta v_k + \kappa|v_k|^{\alpha-1}v_k = 0$ on S_1, where $\mu_k > 0$ is the corresponding Lagrange multiplier. Hence, $\mu_k = (\alpha+1)c_k \to 0$. Setting $v_k = \mu_k^{-1/(1-\alpha)}\varphi_k$ yields the subset $\{\varphi_k\}$ of solutions to (2.1) satisfying $\|\nabla\varphi_k\|_2^2 \equiv \kappa\|\varphi_k\|_{(m+1)/m}^{(m+1)/m} \sim c_k^{2m/(m-1)}$. Therefore, (2.3) holds for $\psi_k = |\varphi_k|^{(1-m)/m}\varphi_k$,

$$\|\psi_k\|_{m+1}^{m+1} \equiv \|\varphi_k\|_{(m+1)/m}^{(m+1)/m} \sim c_k^{2m/(m-1)} \to 0 \quad \text{as } k \to \infty. \tag{2.6}$$

2.2. **Rescaled equation is a gradient system.** We introduce the rescaled variables

$$u(x,t) = (1+t)^{-1/(m-1)} w(x,\tau), \quad \tau = \ln(1+t) : \mathbb{R}_+ \to \mathbb{R}_+, \tag{2.7}$$

where $w(x,\tau)$ satisfies the rescaled equation

$$w_\tau = \mathbf{B}(w) \equiv \mathbf{A}(w) + \kappa w \tag{2.8}$$

with the same initial data $w_0 = u_0$. First, the forward orbit $O^+(w_0) = \{w(\tau), \tau > 0\}$ is globally bounded (cf. [4]).

Proposition 2.1. *There exists a constant $C > 0$ such that*

$$|w(x,\tau)| \le C \quad \textit{in } \Omega \times \mathbb{R}_+. \tag{2.9}$$

Proof. Global estimates in Sobolev spaces are straightforward; see general abstract results in [18, 2]. The proof of the L^∞-bound (2.9) by comparison of weak solutions is elementary. Fix a sufficiently large constant $R > 0$ such that $\bar\Omega \subset B_R = \{|x| < R\}$ and denote by $\psi_R(x)$ the unique positive solution of (1.4) for $\Omega = B_R$, see [4]. Then, $|u_0| \le \psi_R$ in Ω if $R \gg 1$, and hence by comparison $|u(x,t)| \le (1+t)^{-1/(m-1)}\psi_R(x)$ in $\Omega \times \mathbb{R}_+$. Therefore, after scaling (2.7) we arrive at (2.9) with $C = \sup \psi_R$. $\square$

By the parabolic regularity theory (see [18], Chapt. 2 and also [9, 15]), the solution satisfies the following "strong" estimates

$$w \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2), \quad |w|^{(m-1)/2} w \in L^2_{\rm loc}(\mathbb{R}_+; H^1_0), \quad w_\tau \in L^{(m+1)/m}_{\rm loc}(\mathbb{R}_+; W^{-1,m+1}). \tag{2.10}$$

Optimal regularity and other special results for the PME can be found in the PhD Course [26], see also Chapt. 2 in [12].

We now focus on the crucial property that (2.8) is a *gradient system* in $H^{-1}(\Omega)$ denoted also by H. In view of the uniform bounds, (2.8) is well posed in L^{m+1}. Note the compact embedding $L^{m+1} \subset H^{-1}$. By $\langle \cdot,\cdot \rangle_{-1} = \langle w, (-\Delta)^{-1} v \rangle$, where $\tilde v = (-\Delta)^{-1} v$ satisfies $\Delta \tilde v = -v$ in Ω, $\tilde v = 0$ on $\partial\Omega$, we denote the scalar product in H^{-1} and by $\|\cdot\|_{-1}$ the induced norm.

We will need other weaker estimates and identities; see Chapt. 2 in [18] on monotonicity methods. First, we multiplying equation (2.8) by w in H^{-1} to get

$$\tfrac{1}{2} \tfrac{d}{d\tau} \|w\|^2_{-1} = -\|w\|^{m+1}_{m+1} + \tfrac{1}{m-1} \|w\|^2_{-1}. \tag{2.11}$$

See [2] and [26] for justification. Second, we have that (2.8) is a gradient system [2].

Proposition 2.2. *Equation* (2.8) *admits a decreasing in τ Lyapunov function*

$$L(w)(\tau) = \tfrac{1}{m+1} \|w(\tau)\|^{m+1}_{m+1} - \tfrac{1}{2(m-1)} \|w(\tau)\|^2_{-1}. \tag{2.12}$$

Proof. Formally multiplying (2.8) by w_τ and integrating over (τ_*, τ) yields the result,

$$L(w)(\tau) - L(w)(\tau_*) = -\int_{\tau_*}^{\tau} \|w_\tau(s)\|^2_{-1} ds \le 0, \quad \text{so that} \tag{2.13}$$

$$L(w)(\tau) \le L(w)(\tau_*) \quad \text{for all } \tau \ge \tau_* > 0. \tag{2.14}$$

Justification of monotonicity (2.14) is done by Galerkin's method; see [18, 2]. $\square$

We define the ω-limit set of the global forward orbit $O^+(w_0)$,

$$\omega(w_0) = \{\psi \in H : \ \exists \text{ a sequence } \{\tau_j\} \to \infty \text{ such that } w(\cdot, \tau_j) \to \psi\}, \tag{2.15}$$

with the convergence in H^{-1}. The next estimate is proved by regularization of (2.8) via uniformly parabolic equations, see [17].

Proposition 2.3. *The following integral converges:*

$$\int_1^\infty \|w_\tau(s)\|_H^2 ds < \infty. \tag{2.16}$$

Note another stronger estimate $\int^\infty \|(|w|^{(m-1)/2}w)_\tau(s)\|_2^2 ds < \infty$ [17] implying better continuity properties. As a consequence of (2.16), we arrive at the following conclusions which are typical for general theory of dynamical systems, [13]. These two results are sufficient for our further analysis.

Proposition 2.4. *For any $w_0 \in X$, (i) $\omega(w_0) \subseteq \Phi$ and (ii) $\omega(w_0)$ is connected in H.*

Proof. (i) For given $\tau > 0$ and $s \in \mathbb{R}$, one can see from the definition of $\langle \cdot, \cdot \rangle_{-1}$ that by the Hölder inequality and Fubini's theorem,

$$\|w(\tau + s) - w(\tau)\|_{-1}^2 = \left\| \int_\tau^{\tau+s} w_\tau(s) ds \right\|_{-1}^2 \le |s| \int_\tau^{\tau+s} \|w_\tau(s)\|_{-1}^2 ds. \tag{2.17}$$

Given a sequence $\{\tau_j\} \to \infty$ such that $w(\tau_j) \to \psi \in \omega(w_0)$, setting $\tau = \tau_j$ and passing to the limit $\tau_j \to \infty$, in view of (2.16), we obtain that the right-hand side of (2.17) tends to zero meaning that the limit does not depend on s. This is the key conclusion from [17] establishing that $\psi \in \Phi$ is a stationary solution of (2.8). (ii) (2.17) also yields that $w \in C((1, \infty); H^{-1})$. Then, as in the classical theory [13], (ii) follows (of course, there is continuity of the orbit in stronger topology). $\square$

It is important that in proving (i), we do not use the monotonic properties of the Lyapunov functions with, possibly, unknown continuity. This general idea of [17] makes it possible to extend (i) to small non-autonomous perturbations of parabolic PDEs with the porous medium and other operators; see perturbation analysis in Section 3. Then in this case, a Lyapunov function, in general, does not exist, but the convergence (2.16) still true. In a similar way, (i) can be proved by using another Lyapunov construction performed by multiplication of (2.8) by $(|w|^{m-1}w)_\tau$ in L^2; see [26] (this corresponds to "stronger" Lyapunov function with different continuity properties).

2.3. **Evolution completeness.**

Theorem 2.1. *Under the stated assumptions,* (1.8) *holds.*

Proof. Fix any $w_0 \in X$, $w_0 \neq 0$. It follows from Proposition 2.4, (ii) that $\omega(w_0)$ consists of a single stationary profile ψ satisfying (1.4). Let us prove that $\psi \neq 0$, i.e.,

$$0 \notin \omega(w_0) \quad \text{for } \ w_0 \neq 0. \tag{2.18}$$

This follows from the following result, see [2], p. 342.

Proposition 2.5. *For any $w_0 \neq 0$, there exists a constant $C_1 = C_1(w_0) > 0$ such that*

$$\|w(\cdot,\tau)\|_H \geq C_1 \quad \textit{for } \tau \gg 1. \tag{2.19}$$

Proof. STEP 1. By slightly shifting the origin in time, we may set $\tau_* = 0$ in the Lyapunov monotonicity property (2.14). By the homogenuity (1.2), given the solution $u(x,t)$ of the original PDE (1.1), one can define a family of solutions

$$u_R(x,t) = R^{-\kappa} u(x, t/R) \quad \text{for any } \; R > 0. \tag{2.20}$$

The corresponding rescaled function (2.7) has initial data $w_{R0}(x) = R^{-\kappa} w_0(x)$ so that

$$L(w_{R0}) \equiv R^{-2\kappa}[-\tfrac{\kappa}{2}\|w_0\|_{-1}^2 + \tfrac{1}{m+1} R^{-1} \|w_0\|_{m+1}^{m+1}] < 0 \quad \text{for all } \; R \gg 1. \tag{2.21}$$

STEP 2. We now fix an arbitrary R large enough, and substitute estimate from (2.14) into (2.11) for $w = w_R$, $w_0 = w_{R0}$,

$$-\|w\|_{m+1}^{m+1} \geq -\tfrac{m+1}{2(m-1)} \|w\|_{-1}^2 - (m+1)L(w_0), \tag{2.22}$$

to get the following inequality:

$$\tfrac{d}{d\tau}\|w\|_{-1}^2 \geq -\|w\|_{-1}^2 + A_0, \quad \text{with } \; A_0 = -2(m+1)L(w_0) \quad \text{for } \; \tau \gg 1. \tag{2.23}$$

Since $A_0 > 0$ by (2.21), (2.23) yields that $\|w(\tau)\|_{-1}^2 \geq C_1^2 = \frac{1}{2}A_0 = (m+1)|L(w_0)| > 0$ for $\tau \gg 1$. $\square$

Remark 2: analogy with perturbation theory of linear operators. In terms of the original variable $u(x,t)$, estimate (2.19) implies that

$$\|u(\cdot,t)\|_H \geq C_1 t^{-1/(m-1)} \quad \text{for } \; t \gg 1. \tag{2.24}$$

In other words, $u(x,t) \not\equiv 0$ cannot decay faster than $O(t^{-1/(m-1)})$ and hence

$$u(x,t) = o(t^{-1/(m-1)}) \quad \text{in } \; \Omega \; \text{ as } \; t \to \infty \quad \Longrightarrow \quad u(x,t) \equiv 0. \tag{2.25}$$

As a further link to the linear theory, it is worth mentioning that this conclusion for abstract nonlinear equations with coercive, monotone, potential operators [2] is to be compared with the classical Agmon and Ogawa-type estimates from the 1960's for a wide class of linear operators equations guaranteeing that any solution with a "super-exponential" decay as $t \to \infty$ (i.e., faster than any exponent) is zero; see [1, 21] and [14, 7], where further earlier references can be found. For instance, Agmon's result [1] (see also [14]) applies to the equations with small non-autonomous integrable perturbations

$$w_t = \mathbf{B}(t)w + \mathbf{C}(t)w, \tag{2.26}$$

where $\mathbf{B}(t)$ is symmetric and negative and

$$|\langle \dot{\mathbf{B}}(t)w, w\rangle| \leq \beta(t)\|w\|_{H^1}^2, \;\; \|\mathbf{C}(t)w\|_2 \leq \gamma(t)\|w\|_2, \;\; |\langle \mathbf{B}(t)w, w\rangle| \geq \|w\|_{H^1}^2, \tag{2.27}$$

$$\text{with the integrability condition} \quad \int^\infty [\beta(t) + \gamma(t)]dt < \infty. \tag{2.28}$$

Then for a $w(\tau_0) \neq 0$, there exists a $\sigma = \sigma(\tau_0) > 0$ such that

$$\|w(\tau)\|_2 \geq e^{-\sigma(\tau - \tau_0)} \|w(\tau_0)\|_2 \quad \text{for all } \; \tau \geq \tau_0. \tag{2.29}$$

Hence, if $w(\tau)$ has a super-exponential decay as $\tau \to \infty$, then $w = 0$. Such estimates played a significant role in Henry's time-dependent Sturm-Liouville theory [14] establishing a complete classification of multiple zeros of solutions of semilinear heat equations in 1D related to the evolution completeness of eigenfunctions of the perturbed linear operator adjoint to (1.13) (cf. Angenent's approach [3]).

Remark 3: on the uniform lower estimate. Indeed, the crucial lower estimate (2.19) is related to the special structure of the operator $\mathbf{B}$ in (2.8). Its derivative $\mathbf{B}'(0) = \kappa I$ has the unique positive eigenvalue $\sigma(\mathbf{B}'(0)) = \{\kappa\}$ of infinite multiplicity. This establishes a formal relation to the invariant manifold theory (see e.g., [19]), where $\mathbf{B}(w)$ is not expected to have any local stable or centre manifold and the unstable one $W^u(0)$ comprises the whole space H^{-1}. This means that no nontrivial orbits can approach 0 and hence (2.18) holds, i.e., $W^u(0) = H^{-1} \setminus \{0\}$ implies a uniform estimate like (2.19).

Remark 4: on discreteness of Φ as a main hypothesis. Let us return to the two main ingredients of the proof: (i) there exists a $k \in [0, \infty]$ such that $\omega(w_0) = \{\psi_k\}$, and (ii) $k < \infty$ (recall that $\psi_\infty = 0$ is defined by continuity in (2.3)).

We have used an easy fact that (i) always holds for sufficiently smooth gradient systems if Φ is discrete. On the other hand, it is known that $\omega(w_0)$ consists of a unique equilibria *without* the discreteness assumption on Φ for general 1D quasilinear uniformly parabolic equations with smooth coefficients (but not analytic so that [25] does not apply)

$$w_\tau = \mathbf{B}(w) \equiv a(x, w, w_x) w_{xx} + b(x, w, w_x) \quad \text{in } (a, b) \times \mathbb{R}_+ \tag{2.30}$$

and nonlinear boundary conditions at the end point $x = a, b$, even if the corresponding ODE problem, $a(x, \psi, \psi')\psi'' + b(x, \psi, \psi') = 0$ on (a, b) plus boundary conditions, contains continuous branches of solutions. We refer to [27], where the proof is based on the construction of a classical integral Lyapunov function and delicate exponential estimates of the rate of convergence, and [20], where the analysis uses Sturm's Theorem on zero sets for linear parabolic equations. For the present case of the PME, these results are not crucial since the corresponding ODE always has a discrete subset of stationary solution Φ; see examples below.

Thus, [27] implies that for the uniformly parabolic 1D equations like (2.30), or for the smooth fully nonlinear ones $w_\tau = \mathbf{B}(w) \equiv F(x, w, w_x, w_{xx})$, where the Sturmian argument [20] is to be used, for classes of global orbits, Φ is evolutionary complete provided that (2.18) holds (possibly, because $\mathbf{B}'(0) = \kappa I$ with a constant $\kappa > 0$). This is an exceptional 1D situation, and the results for multi-dimensional parabolic equations similar to those in [27, 20] are not known.

2.4. **Evolutionary complete subsets of nonlinear eigenfunctions.**

We are not aware of general results on discreteness of Φ for general domains Ω except that this is expected to be true for a.a. smooth Ω, [11]. It seems that this is a hard part of the nonlinear eigenfunction analysis. Below we present two easy examples where Φ is discrete, and hence is evolutionary complete.

Example 1: the one-dimensional problem. We take $\Omega = (0, 1)$, and then (1.4) is the ODE problem

$$(|\psi|^{m-1}\psi)'' + \kappa\psi = 0, \quad \psi(0) = \psi(1) = 0, \tag{2.31}$$

which can be integrated in quadratures. Let $\psi_k(x) \not\equiv 0$ be a solution of (2.31). Note that $\psi_k(x)$ has precisely k isolated zeros on $(0, 1)$ (the well known Sturmian property for ODEs like that). Let us show that ψ_k is isolated in H^{-1} or in stronger topologies. We use the obvious rescaling property of the ODE (2.31): if ψ is a solution, then

$$\psi_a(x) = a^{2/(m-1)}\psi(x/a) \quad \text{is a solution for any } a > 0. \tag{2.32}$$

Actually, $\pm\psi_a$ in (2.32) gives all non-zero solutions of equation (2.31) satisfying $\psi(0) = 0$. Taking $\psi = \psi_k$, it follows that for all $a \approx 1$, $a \neq 1$, $\psi_a(1) = a^{2/(m-1)}\psi_k(1/a) \neq 0$, i.e., problem (2.31) does not have another solution in a sufficiently small neighbourhood of ψ_k (cf. another proof in [26]). By Theorem 2.1, $\Phi = \{\psi_k\}$ is evolutionary complete.
Example 2: the radial N-dimensional problem. Then the ODE (1.4) with the variable $r = |x| \geq 0$ and $\Omega = \{r < 1\}$ takes the form

$$\tfrac{1}{r^{N-1}}(r^{N-1}(|\psi|^{m-1}\psi)')' + \kappa\psi = 0 \quad \text{on } (0, 1), \quad \psi(1) = 0, \tag{2.33}$$

with the symmetry condition at the origin $(|\psi|^{m-1}\psi)'(0) = 0$. Each $\psi_k(r)$ has exactly k isolated zeros on $(0, 1)$. By the invariance property, $\psi_a(r) = a^{2/(m-1)}\psi_k(r/a)$ describes all symmetric solutions and then $\psi_a(1) \neq 0$ for all $a \approx 1$, $a \neq 1$, i.e., any ψ_k is isolated. Hence, Theorem 2.1 applies.

3. Evolution completeness for perturbed PME

Now, the scheme of our analysis applies to the perturbed PME (1.9).

Theorem 3.1. (1.8) *holds for equation* (1.9).

Proof. We perform both rescalings, (2.7) and (2.20) at once, by setting

$$u = R^\kappa(1+s)^{-\kappa}w(x, \tau), \quad \text{where } \tau = \ln(1+s), \ \ s = Rt, \tag{3.1}$$

to get an exponentially small non-autonomous perturbation of equation (2.8):

$$w_\tau = \mathbf{A}(w) + \kappa w + R^\gamma e^{-\gamma\tau}\Delta|w|^{n-1}w, \quad \text{with } \gamma = (n-m)/(m-1) > 0. \tag{3.2}$$

We follow the lines of our analysis in Section 2.
(I) L^∞-BOUND. (2.9) is proved by construction of a large weak supersolution of (1.9) of the standard form

$$\bar{u}(x, t) = (T+t)^{-\kappa}\bar{\psi}(x), \quad \bar{\psi}(x) = A(1 - |x|^2/a^2)_+^{1/m}, \tag{3.3}$$

with large a such that $\bar{\Omega} \subset B_a$, $A \gg 1$ and T sufficiently small. This gives an upper estimate and the lower one follows by comparison with the subsolution $-\bar{u}$ and finally

$$|w(x, \tau)| \leq C[(1+s)(R+s)^{-1}]^\kappa. \tag{3.4}$$

(II) Estimate of $w_\tau(\tau)$ as $\tau \to \infty$. We use approximation as in [18, 2], and as above, multiply (3.2) by w_τ in H^{-1}. Estimating the perturbation term yields

$$R^\gamma e^{-\gamma\tau} \langle \Delta |w|^{n-1} w, w_\tau \rangle_{-1} = R^\gamma e^{-\gamma\tau} \tfrac{1}{n+1} \tfrac{d}{d\tau} \|w\|_{n+1}^{n+1}$$
$$= R^\gamma \tfrac{d}{d\tau}(e^{-\gamma\tau} \|w\|_{n+1}^{n+1}) + R^\gamma \gamma e^{-\gamma\tau} \|w\|_{n+1}^{n+1}.$$

Therefore, integrating over (τ_*, ∞) and using the uniform bound (2.9) yields (2.16).
(III) A lower bound: $0 \notin \omega(w_0)$. Firstly, using (3.4) , instead of monotonicity (2.14), we obtain a perturbed version

$$L(w)(\tau) \le L(w)(\tau_*) + O(R^{-(m+1)\kappa} e^{-\gamma\tau_*}) \quad \text{for } \tau \ge \tau_* \gg 1. \tag{3.5}$$

Secondly, one can see that adding to the right-hand side of (2.11) the perturbation

$$R^\gamma e^{-\gamma\tau} \langle \Delta |w|^{n-1} w, w \rangle_{-1} = R^\gamma e^{-\gamma\tau} \|w\|_{n+1}^{n+1} = O(R^{-(m+1)\kappa} e^{-\gamma\tau}), \quad \tau \gg 1,$$

and using instead of (2.22) a similar estimate following from (3.5), does not essentially change inequality (2.23), where instead of A_0 we have $A_0 + O(R^{-(m+1)\kappa} e^{-\gamma\tau_*})$. Then we first choose $R \gg 1$ such that $A_0 > 0$ and next take $\tau_* = \tau_*(R) \gg 1$. $\square$

For more general perturbation of the PME

$$u_t = \Delta(|u|^{m-1} u) + g(x, t, u, Du, D^2 u), \tag{3.6}$$

the main assumption on $g(\cdot)$ follows from rescaling (2.7),

$$s^m g(x, s^{m-1}, s^{-1} w, s^{-1} Dw, s^{-1} D^2 w) \to 0 \quad \text{as } s \to \infty, \tag{3.7}$$

for sufficiently smooth functions w. Extra hypotheses are necessary for solvability and global existence of solutions $u(\cdot, t) \to 0$ as well as for using all the asymptotic techniques by applying suitable embedding and interpolation inequalities. These are not easy to formulate in a generality comparable with Agmon's and Henry's results for perturbed linear equations (2.26)-(2.28). Note that the exponential character of rescaled perturbation term in (3.2) is not necessary, but a kind of *integrability* at $\tau = \infty$ is crucial (cf. (2.28) in the linear case). We leave this to the interested reader.

4. Extension to other equations

4.1. **Second-order parabolic models.** Theorem 2.1 remains valid for the p-Laplacian equation (1.10) with the Dirichlet boundary conditions. The rescaled equation (2.8) with the coercive, monotone, homogeneous, potential operator is a gradient system in $H = L^2$, a natural functional setting for such equations [18]. The Lyapunov function takes the form

$$L(w)(\tau) = \tfrac{1}{m+1} \|\nabla w(\tau)\|_{m+1}^{m+1} - \tfrac{1}{2(m-1)} \|w(\tau)\|_2^2,$$

and the Lusternik-Schnirel'man theory applies to the corresponding problem (1.4) to guarantee that $\Phi = \{\psi_k\}$ is at least countable. Proposition 2.3 is proved by approximation as in [17], and hence Proposition 2.4 holds in the topology of L^2. Proposition 2.5 is obtained quite similarly, see an example in [2]. Examples 1 and 2 on evolution completeness remain valid with some easy changes in the proof. Theorem 3.1 applies to the p-Laplacian equation with e.g., a perturbation $\nabla \cdot (|\nabla u|^{n-1} \nabla u)$, where $n > m$.

Example 3: continuous eigenfunction family for a nonlinear operator. Given the p-Laplacian in (1.10), we formally define the operator

$$\mathbf{C}(w) = |\mathbf{A}(w)|^{(1-m)/m}\mathbf{A}(w), \tag{4.1}$$

which, obviously, is of algebraic homogenuity 1: $\mathbf{C}(sw) = sw$ for all $s \in \mathbb{R}$. If $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathbf{C}$, i.e., there exists a $\psi_1 \neq 0$ such that

$$\mathbf{C}(\psi_1) = \lambda\psi_1, \tag{4.2}$$

then by (4.1) this defines a continuous family of eigenfunctions $\psi_C = C\psi_1$ for any non-zero $C \in \mathbb{R}$. In this case, the evolution completeness problem of eigenfunction subset Φ (possibly consisting of a countable subset of continuous families) by using the corresponding evolution equation

$$w_t = \mathbf{C}(w) \quad \Longrightarrow \quad |w_t|^{m-1}w_t = \mathbf{A}(w), \quad t > 0,$$

represents a much harder open problem. Eigenvalue problems like (4.2) has been less studied in the literature and are not that well-understood as more standard variational quasilinear elliptic equations with leading-order coercive operators.

Concerning the fully nonlinear dual PME (1.11), it suffices to note that the PME (1.1) reduces to the dual PME by setting $u = \Delta\tilde{u}$ in Ω with $\tilde{u} = 0$ on $\partial\Omega$.

4.2. **On higher-order equations.** Consider the higher-order equation (1.12) with the Dirichlet boundary conditions

$$\partial^j u/\partial\nu^j = 0 \quad \text{for } j = 0, 1, ..., 2l-1 \text{ on } \partial\Omega \times \mathbb{R}_+, \tag{4.3}$$

where $\partial/\partial\nu$ denotes the derivative in the direction of outward normal to the smooth boundary $\partial\Omega$. The rescaled orbit given by (2.7) is uniformly bounded in $W_0^{l,m+1}$; see [18], Chapt. 2 for general results. The gradient system (2.8) admits the Lyapunov function

$$L(w)(\tau) = \tfrac{1}{m+1}\|\Delta^l w(\tau)\|_{m+1}^{m+1} - \tfrac{1}{2(m-1)}\|w(\tau)\|_2^2$$

and the proof by approximation follows the lines of that in [2], Proposition 1.3. Regularization techniques are known to apply to higher-order equations, see abstract settings in [18], Chapt. 2 and [2]. Therefore, (2.16) implies Proposition 2.4 with the properties defined in $H = L^2$. The operator in (1.12) satisfies hypotheses (i) - (iv) in [2], so that the lower bound (2.19) on nontrivial solutions holds; see the abstract approach in Section 1 therein and an example for a fourth-order equation on in Section 3. Operator in (1.12) is coercive and monotone, so that hypotheses (vi), (vii) in [2] hold and hence Φ is at least countable. The same follows from the fibering method [22, 23]. Hence, Theorem 2.1 is valid.

There exists an essential difference between the second and higher-order equations, and here the discreteness of Φ even in 1D is not guaranteed by groups of scalings.

Example 4. Consider the one-dimensional case for the fourth-order ODE (1.12) ($l = 1$) on $\Omega = (0, 1)$ so that the nonlinear eigenfunctions solve

$$-(|\psi''|^{m-1}\psi'')'' + \kappa\psi = 0, \quad \psi = \psi' = 0 \quad \text{at } x = 0 \text{ and } 1. \tag{4.4}$$

The equation admits a similar to (2.32) group of scaling transformations

$$\psi_a(x) = a^{2(m+1)/(m-1)}\psi(x/a), \quad a > 0. \tag{4.5}$$

Since (4.4) is a fourth-order equation, there exists a two-parameter family of solutions satisfying the boundary conditions at left-hand end point $x = 0$. Therefore, the one-parametric symmetry group (4.5) does not exclude the possibility of existence of a continuous family of solutions.

Degenerate equations like (4.4) admit essentially non-analytic solutions. This is easily seen after the transformation $|\psi''|^{m-1}\psi'' = \varphi$, which reduces (4.4) to the semilinear ODE

$$-\varphi^{(4)} + \kappa|\varphi|^{\alpha-1}\varphi = 0, \quad \alpha = \tfrac{1}{m}. \tag{4.6}$$

Since the nonlinearity is not analytic at $\varphi = 0$ (and is not Lipschitz), the analyticity of the solutions brakes down at any point x_0 with $\varphi(x_0) = 0$ (i.e., at the degeneracy points x_0 of (4.4), where $\psi''(x_0) = 0$). General results on the analytic dependence of solutions on parameters for ODEs (4.6), which would guarantee that Φ is discrete, seem unavailable. Nevertheless, we expect that the discreteness of Φ (and hence the completeness) remains valid. Degenerate ODEs like can (4.4) can admit families of solutions with analytic dependence on parameters regardless of braking the analyticity of solutions in x. Namely, the non-analytic behaviour in x can happen at, possibly, a finite number of isolated points in $[0, 1]$, and by local analysis via the equivalent integral equation, this does not affect the analytic dependence on parameters. We can show this for $m = 3$ (if φ vanishes at a finite number of points), and then such φ's are isolated. For φ having a non-isolated zero of infinite order (if any), the problem remains open.

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