Generalized Sampled-Data Stabilization of Well-Posed Linear Infinite-Dimensional Systems*

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Abstract: We consider well-posed linear infinite-dimensional systems, the outputs of which are sampled in a generalized sense using a suitable weighting function. Under certain natural assumptions on the system, the weighting function and the sampling period, we show that there exists a generalized hold function such that unity sampled-data feedback renders the closed-loop system exponentially stable (in the state-space sense) as well as L^2-stable (in the input-output sense). To illustrate our main result, we describe an application to a structurally damped Euler-Bernoulli beam.

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1. Introduction

This paper continues the line of work on sampled-data control of infinite-dimensional systems which the authors have been involved in during the last eight years, see [10, 11, 12, 13, 16, 17, 28]. We consider a well-posed linear continuous-time infinite-dimensional system Σ with generating operators (A, B, C), where A is the generator of a strongly continuous semigroup governing the state evolution of the uncontrolled system, B is the control operator and C the observation operator, see, for example, [4, 21, 23, 25, 29]. Given a sampling period $\tau > 0$, the aim is to choose a generalized hold function H defined on $[0, \tau]$, such that unity sampled-data feedback renders the closed-loop system stable. Specifically, denoting the input and output of Σ by u and y, respectively, we wish to find stabilizing sampled-data feedback controls of the form

$$u(t) = v(t) - H(t - k\tau)y_k\,, \qquad t \in [k\tau, (k+1)\tau), \quad k = 0, 1, 2, \ldots . \tag{1.1}$$

Here v denotes the input of the sampled-data feedback system and y_k is the k-th sample of the output y given by the generalized sampling operation

$$y_k := \int_0^{\tau-\delta} w(s)y((k-1)\tau + \delta + s)\,ds\,,$$

where $\delta \in (0, \tau)$ and w is a suitable scalar-valued weighting function defined on $[0, \tau - \delta]$. We emphasize that for well-posed systems instantaneous sampling of the form $y_k = y(k\tau)$ is in general not well-defined due to the potentially high irregularity of y and therefore generalized sampling is unavoidable. Furthermore, we point out that the continuous-time signal y does not need to be measured on the whole time-interval $[(k-1)\tau, k\tau]$, but only on $[(k-1)\tau+\delta, k\tau]$ (in particular, δ can be chosen arbitrarily close to τ). Our main result (Theorem 4.4) states that stabilization by a sampled-data control of the form (1.1) is possible, provided that, loosely speaking,

(i) the unstable portion of the spectrum of A consists of at most finitely many eigenvalues with finite algebraic multiplicities,

(ii) the semigroup generated by the stable part of A is exponentially stable,

(iii) the unstable (finite-dimensional) part of Σ is observable,

(iv) the unstable subspace of Σ is contained in the closure of its reachable subspace,

(v) the sampling period τ is "non-pathological" in a certain standard sense.

We prove that under these conditions the sampled-data control (1.1) achieves exponential stability in the state-space sense (if $v = 0$) as well as L^2-stability in the input-output sense. Furthermore, we show that if v and $\dot{v}$ are both square-integrable and the initial state satisfies a certain natural smoothness condition, then the output $y(t)$ of the sampled-data feedback system converges to 0 as $t \to \infty$. Finally, we prove that the conditions (i)-(iii) (and (iv) if the semigroup generated by A is analytic) are necessary for stabilization by sampled-data feedback of the form (3.1), a result which is closely related to the work in [16, 17, 28]. Condition (v) is not necessary for stabilization by sampled-data feedback, but it is "generically" satisfied. We emphasize that we do not assume that the sampling period τ is small. This is in contrast to [11], where it is proved that certain "indirect" sampled-data control designs are stabilizing for all sufficiently small τ.

Our main result extends, generalizes and improves the basic result in [27] in a number of ways. The results in [27] are proved for systems with bounded operators B and C, and then stated without proof for a class of systems with unbounded B and C satisfying the conditions of the set-up developed in [3]. This class of systems is quite limited, and includes few systems described by partial differential equations with boundary control and observation. The results in [27] were further developed in [26] to encompass a class of neutral systems. In this paper, we work in the context of the theory of well-posed systems, which is the largest class of infinite-dimensional systems for which there exists a well-developed state-space and frequency-domain theory, see, for example, [4, 20, 21, 23, 24, 25, 29, 30]. Well-posed systems allow for considerable unboundedness of the control and observation operators B and C. In particular, the class of well-posed systems encompasses many of the most commonly studied partial differential equations with boundary control and observation and all functional differential equations of retarded and neutral type with delays in the inputs and outputs. For well-posed systems instantaneous sampling of the form $y_k = y(k\tau)$ is in general not possible, due to the potentially high irregularity of the output y, so we use a generalized sampling operation instead of the instantaneous sampling used in [26, 27]. Moreover, in contrast to [26, 27], we not only prove results on exponential stability, but also obtain results on input-output stability.

The paper is organised as follows: In Section 2 we describe in detail various results relevant to the sampled-data control of well-posed systems. In Section 3 we discuss issues relating to sampled-data feedback stabilization. In Section 4 we present our main result. In Section 5 we illustrate our results by applying them to a structurally damped Euler-Bernoulli beam.

Notation: $\mathbb{N}$ denotes the set of positive integers; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{R}_+ := [0, \infty)$; for $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$; for a real or complex Banach space Z, $\alpha \in \mathbb{R}$ and $0 < p \leq \infty$, we define the exponentially weighted spaces $L^p_\alpha(\mathbb{R}_+, Z) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, Z) : f(\cdot)\exp(-\alpha\,\cdot) \in L^p(\mathbb{R}_+, Z)\}$ and $W^{1,p}_\alpha(\mathbb{R}_+, Z) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, Z) : f(\cdot)\exp(-\alpha\,\cdot) \in W^{1,p}(\mathbb{R}_+, Z)\}$; we endow $L^p_\alpha(\mathbb{R}_+, Z)$ with the norm $\|f\|_{L^p_\alpha} := \|e^{-\alpha\,\cdot} f(\cdot)\|_{L^p}$; $W^{1,p}_c([a, b], Z)$ denotes the subspace of all functions in $W^{1,p}([a, b], Z)$ with support contained in the open interval (a, b); $\mathcal{B}(Z_1, Z_2)$ denotes the space of bounded linear operators from a Banach space Z_1 to a Banach space Z_2; we write $\mathcal{B}(Z)$ for $\mathcal{B}(Z, Z)$; let $A : \mathrm{dom}(A) \subset Z \to Z$ be a linear operator, where $\mathrm{dom}(A)$ denotes the domain of A; the resolvent set of A and the spectrum of A is denoted by $\varrho(A)$and $\sigma(A)$, respectively; if $A \in \mathcal{B}(Z)$, then $r(A)$ denotes the spectral radius of A.

2. Preliminaries on well-posed systems

There are a number of equivalent definitions of well-posed systems, see [4, 20, 21, 23, 24, 25, 29, 30]. We will be brief in the following and refer the reader to the above references for more details. Throughout this section, we shall be considering a well-posed system Σ with state-space X, input space $\mathbb{R}^m$ and output space $\mathbb{R}^p$, generating operators (A, B, C), input-output operator G and transfer function $\mathbf{G}$. Here X is a real Hilbert space with norm denoted by $\|\cdot\|$, A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ on X, $B \in \mathcal{B}(\mathbb{R}^m, X_{-1})$ and $C \in \mathcal{B}(X_1, \mathbb{R}^p)$, where X_1 denotes the space $\mathrm{dom}(A)$ endowed with the norm $\|z\|_1 := \|(s_0 I - A)z\|$, whilst X_{-1} denotes the completion of X with respect to the norm $\|z\|_{-1} = \|(s_0 I - A)^{-1} z\|$, where $s_0 \in \varrho(A)$ (different choices of s_0 lead to equivalent norms). Clearly, the norm $\|\cdot\|_1$ is equivalent to the graph norm of A. Moreover, $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1} with the exponential growth constant being the same on all three spaces; the generator of the restriction (extension) of $\mathbf{T}$ is a restriction (extension) of A; we shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, generator) and the associated restrictions and extensions: with this convention, we may write $A \in \mathcal{B}(X, X_{-1})$ (considered as a generator on X_{-1}, the domain of A is X). The spectra of A and its extension coincide. For $s_0 \in \varrho(A)$, $s_0 I - A$, considered as an operator in $\mathcal{B}(X, X_{-1})$, provides an isometric isomorphism from X to X_{-1} (we refer the reader to [6] for more details on the extrapolation space X_{-1}). The operator B is an *admissible control operator* for $\mathbf{T}$, i.e., for each $t \in \mathbb{R}_+$ there exists $\beta_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}_{t-s} B u(s)\, ds \right\| \leq \beta_t \|u\|_{L^2([0,t],\mathbb{R}^m)}, \quad \forall\, u \in L^2([0,t], \mathbb{R}^m).$$

The operator C is an *admissble observation operator* for $\mathbf{T}$, i.e., for each $t \in \mathbb{R}_+$ there exists $\gamma_t \geq 0$ such that

$$\left(\int_0^t \|C\mathbf{T}_s z\|^2 ds \right)^{1/2} \leq \gamma_t \|z\| \,, \quad \forall\, z \in X_1 \,.$$

The control operator B is said to be *bounded* if it is so as a map from the input space $\mathbb{R}^m$ to the state space X, otherwise is said to be *unbounded*; the observation operator C is said to be *bounded* if it can be extended continuously to X, otherwise, C is said to be *unbounded*.

The so-called Λ-*extension* C_Λ of C is defined by

$$C_\Lambda z = \lim_{s \to \infty,\, s \in \mathbb{R}} Cs(sI - A)^{-1} z \,,$$

with $\mathrm{dom}(C_\Lambda)$ consisting of all $z \in X$ for which the above limit exists. For every $z \in X$, $\mathbf{T}_t z \in \mathrm{dom}(C_\Lambda)$ for a.a. $t \in \mathbb{R}_+$ and, if $\alpha > \omega(\mathbf{T})$, then $C_\Lambda \mathbf{T} z \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^p)$, where

$$\omega(\mathbf{T}) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbf{T}_t\| \,,$$

denotes the exponential growth constant of $\mathbf{T}$. The transfer function $\mathbf{G}$ satisfies

$$\frac{1}{s - s_0} \left(\mathbf{G}(s) - \mathbf{G}(s_0) \right) = -C(sI - A)^{-1}(s_0 I - A)^{-1} B \,, \;\; \forall\, s, s_0 \in \mathbb{C}_{\omega(\mathbf{T})} \,, \;\; s \neq s_0 \,, \qquad (2.1)$$

and, for every $\alpha > \omega(\mathbf{T})$, $\mathbf{G}$ is analytic and bounded on $\mathbb{C}_\alpha$. Moreover, the input-ouptput operator $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p)$ is continuous and right-shift invariant; for every $\alpha > \omega(\mathbf{T})$, $G \in \mathcal{B}(L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m), L^2_\alpha(\mathbb{R}_+, \mathbb{R}^p))$ and

$$(\mathfrak{L}(Gu))(s) = \mathbf{G}(s)(\mathfrak{L}(u))(s) \,, \quad \forall\, s \in \mathbb{C}_\alpha \,, \;\; \forall\, u \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m) \,,$$

where $\mathfrak{L}$ denotes the Laplace transform. It follows from (2.1) that if two well-posed systems have the same generating operators, then the difference of their transfer functions is constant: roughly speaking, the generating operators determine the input-output behaviour of a well-posed system up to a constant.

In the following, let $s_0 \in \mathbb{C}_{\omega(\mathbf{T})}$ be fixed, but arbitrary. For $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, let x and y denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function u. Then $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} Bu(s)\, ds$ for all $t \in \mathbb{R}_+$ and $y(t) = C_\Lambda \mathbf{T}_t x^0 + (Gu)(t)$ for a.a. $t \in \mathbb{R}$. Moreover, $x(t) - (s_0 I - A)^{-1} Bu(t) \in \mathrm{dom}(C_\Lambda)$ for a.a. $t \in \mathbb{R}_+$ and

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \,, \quad x(0) = x^0 \,, \quad \text{a.a. } t \in \mathbb{R}_+ \,, && (2.2\text{a}) \\ y(t) &= C_\Lambda \big(x(t) - (s_0 I - A)^{-1} Bu(t) \big) + \mathbf{G}(s_0) u(t) \,, \quad \text{a.a. } t \geq 0 \,. && (2.2\text{b}) \end{aligned}$$

Of course, the differential equation (2.2a) has to be interpreted in X_{-1}. In the following, we identify Σ and (2.2) and refer to (2.2) as a well-posed system. We say that the well-posed system (2.2) is *exponentially stable* if $\omega(\mathbf{T}) < 0$. If the well-posed system (2.2) is *regular*, i.e., the following limit

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s) = D$$

exists, then $x(t) \in \mathrm{dom}(C_\Lambda)$ for a.a. $t \in \mathbb{R}_+$ and the output equation (2.2b) simplifies to

$$y(t) = C_\Lambda x(t) + Du(t)\,, \quad \text{a.a. } t \geq 0\,.$$

Moreover, in the regular case, we have that $(sI - A)^{-1}B\mathbb{R}^m \subset \mathrm{dom}(C_\Lambda)$ for all $s \in \varrho(A)$ and

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D\,, \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}\,.$$

The matrix $D \in \mathbb{R}^{p\times m}$ is called the *feedthrough matrix* of (2.2). We mention that if the control operator B or the observation operator C is bounded, then (2.2) is regular.

The following result relates to the asymptotic behaviour of the output y of the well-posed system (2.2) under the assumption that x^0 and u satisfy certain "smoothness" conditions.

Proposition 2.1 *Let $\alpha > \omega(\mathbf{T})$, $x^0 \in X$ and $u \in W^{1,2}_\alpha(\mathbb{R}_+, \mathbb{R}^m)$. If there exists $t_0 \in \mathbb{R}_+$ such that $\mathbf{T}_{t_0}(Ax^0 + Bu(0)) \in X$, then the output y of the well-posed system (2.2) is continuous on $[t_0, \infty)$ † and satisfies*

$$\lim_{t\to\infty} y(t)e^{-\alpha t} = 0\,.$$

Proof: Let $x^0 \in X$, $t_0 \in \mathbb{R}_+$ and $u \in W^{1,2}_\alpha(\mathbb{R}_+, \mathbb{R}^m)$ be such that $\mathbf{T}_{t_0}(Ax^0 + Bu(0)) \in X$. The output y of the well-posed system (2.2) is given by

$$y(t) = C_\Lambda \mathbf{T}_t x^0 + (Gu)(t)\,, \quad \text{a.a. } t \in \mathbb{R}_+\,. \tag{2.3}$$

Let us first assume that $\alpha = 0$. Then, by hypothesis, $0 = \alpha > \omega(\mathbf{T})$, that is, the well-posed system (2.2) is exponentially stable. Define a right-shift-invariant operator $F : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p)$ by setting

$$(Ff)(t) := \int_0^t \left((Gf)(\zeta) - \mathbf{G}(0)f(\zeta)\right) d\zeta\,, \quad \forall\, f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m),\ \forall\, t \in \mathbb{R}_+\,.$$

The transfer function $\mathbf{F}$ of F is given by $\mathbf{F}(s) = (\mathbf{G}(s) - \mathbf{G}(0))/s$. Clearly, $\mathbf{F}$ is analytic and bounded on $\mathbb{C}_0$ and so, $F \in \mathcal{B}(L^2(\mathbb{R}_+, \mathbb{R}^m), L^2(\mathbb{R}_+, \mathbb{R}^p))$. Using that G commutes with the integration operator (by right-shift invariance), a routine calculation gives

$$Gu = F\dot{u} + \mathbf{G}(0)u + G(u(0)\theta) - \mathbf{G}(0)u(0)\,,$$

† The output y of the well-posed system (2.2) is an element in $L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ and so, strictly speaking y is not a function, but an equivalence class of functions coinciding almost everywhere in $\mathbb{R}_+$. We say that y is continuous on $[t_0, \infty)$ if there exists a representative in the equivalence class which is continuous on $[t_0, \infty)$.

where θ denotes the unit-step function. Setting

$$y_1 := F\dot{u} + \mathbf{G}(0)u \qquad \text{and} \qquad y_2 := C_\Lambda \mathbf{T}x^0 + G(u(0)\theta) - \mathbf{G}(0)u(0)\,,$$

it follows from (2.3) that

$$y(t) = y_1(t) + y_2(t)\,, \quad \text{a.a. } t \in \mathbb{R}_+\,. \tag{2.4}$$

It is clear that y_1 is continuous. Since $u, \dot{u} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, we may conclude that $\lim_{t\to\infty} u(t) = 0$. Using again that $\dot{u} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it follows from the boundedness of F and G that $F\dot{u}$ and $(d/dt)(F\dot{u})$ are in $L^2(\mathbb{R}_+, \mathbb{R}^p)$, showing that $\lim_{t\to\infty}(F\dot{u})(t) = 0$. Thus, $\lim_{t\to\infty} y_1(t) = 0$. Taking the Laplace transform of y_2 gives

$$(\mathfrak{L}y_2)(s) = C(sI - A)^{-1}x^0 + \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0))u(0)\,, \quad \forall\, s \in \mathbb{C}_0\,.$$

Invoking (2.1) we obtain that, for all $s \in \mathbb{C}_0$,

$$(\mathfrak{L}y_2)(s) = C(sI - A)^{-1}x^0 + C(sI - A)^{-1}A^{-1}Bu(0) = C(sI - A)^{-1}A^{-1}(Ax^0 + Bu(0))\,,$$

implying that $y_2(t) = C_\Lambda \mathbf{T}_t A^{-1}(Ax^0 + Bu(0))$ for a.a. $t \in \mathbb{R}_+$. Hence, since $\mathbf{T}_{t_0}(Ax^0 + Bu(0)) \in X$,

$$y_2(t) = C\mathbf{T}_{t-t_0}A^{-1}\mathbf{T}_{t_0}(Ax^0 + Bu(0))\,, \quad \text{for a.a. } t \in [t_0\infty)\,. \tag{2.5}$$

Obviously, the right-hand side of (2.5) is continuous on $[t_0, \infty)$ and converges to 0 as $t \to \infty$. The claim now follows from (2.4).

Let us now assume that $\alpha \neq 0$. Define the operator $G_\alpha : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p)$ by setting $G_\alpha(u) := e^{-\alpha\cdot}G(e^{\alpha\cdot}u)$. It is trivial that there exists a well-posed system Σ_α with generating operators $(A - \alpha I, B, C)$ and input-output operator G_α (the exponentially weighted version of the well-posed system (2.2)). Since $\alpha > \omega(\mathbf{T})$, it is clear that Σ_α is exponentially stable. If y is the output of the well-posed system (2.2), then

$$y(t)e^{-\alpha t} = C_\Lambda \mathbf{T}_t e^{-\alpha t}x^0 + (G_\alpha(e^{-\alpha\cdot}u))(t)\,, \quad \text{a.a. } t \in \mathbb{R}_+\,. \tag{2.6}$$

The right-hand side of (2.6) is the output of the exponentially stable well-posed system Σ_α corresponding to the initial value x^0 and the control function $e^{-\alpha\cdot}u \in W^{1,2}(\mathbb{R}_+, \mathbb{R}^m)$. Moreover, since $\mathbf{T}_{t_0}(Ax^0 + Bu(0)) \in X$,

$$\mathbf{T}_{t_0}e^{-\alpha t_0}\big((A - \alpha I)x^0 + B(e^{-\alpha\cdot}u)(0)\big) = e^{-\alpha t_0}\mathbf{T}_{t_0}\big(Ax^0 + Bu(0) - \alpha x^0\big) \in X\,.$$

Thus, by what we have already proved, it follows that the right-hand side of (2.6), and hence the function $t \mapsto y(t)e^{-\alpha t}$, is continuous on $[t_0, \infty)$ and converges to 0 as $t \to \infty$. $\square$

We close this section with a simple sufficient condition for a triple of operators (A, B, C) to be the generating operators of a well-posed system. Here $A : \mathrm{dom}(A) \subset X \to X$ generates a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ and $B \in \mathcal{B}(\mathbb{R}^m, X_{-1})$ and $C \in \mathcal{B}(X_1, \mathbb{R}^p)$ are admissible control and observation operators for $\mathbf{T}$, respectively. Assume that the semigroup $\mathbf{T}$ is analytic, let $s_0 \in \varrho(A)$ and let $\alpha \geq 0$. Then the fractional powers $(s_0 I - A)^{-\alpha}$ and $(s_0 I - A)^{\alpha}$ are well-defined (where $(s_0 I - A)^0 := I$), $(s_0 I - A)^{\alpha}$ is closed and $(s_0 I - A)^{-\alpha} \in \mathcal{B}(X)$. We endow the domain of $(s_0 I - A)^{\alpha}$ with the norm

$$\|z\|_{\alpha} := \|(s_0 I - A)^{\alpha} z\|$$

and denote the resulting Hilbert space by X_{α}. Let $X_{-\alpha}$ be the completion of X with respect to the norm

$$\|z\|_{-\alpha} := \|(s_0 I - A)^{-\alpha} z\| .$$

It is trivial that $X_0 = X$ and $(s_0 I - A)^{-\alpha} \in \mathcal{B}(X, X_{\alpha})$. If $\alpha \in (0,1)$, then X_{α} and $X_{-\alpha}$ can be interpreted as interpolation spaces: between X and X_1 in case of the former and between X and X_{-1} in case of the latter. The operator $(s_0 I - A)^{\alpha}$ extends to an operator in $\mathcal{B}(X, X_{-\alpha})$ and similarly, $(s_0 I - A)^{-\alpha}$ extends to an operator in $\mathcal{B}(X_{-\alpha}, X)$; we shall use the same symbol $(s_0 I - A)^{\alpha}$ (respectively, $(s_0 I - A)^{-\alpha}$) to denote the extensions.

Proposition 2.2 *Assume that the semigroup $\mathbf{T}$ generated by A is analytic and that $B \in \mathcal{B}(\mathbb{R}^m, X_{-1})$ and $C \in \mathcal{B}(X_1, \mathbb{R}^p)$ are admissible control and observation operators for $\mathbf{T}$, respectively. If there exist $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ and such that $B \in \mathcal{B}(\mathbb{R}^m, X_{-\alpha})$ and $C \in \mathcal{B}(X_{\beta}, \mathbb{R}^p)$, then there exists a regular well-posed system with generating operators (A, B, C).*

Proof: Fix $\lambda \in \varrho(A)$. It follows from the hypothesis that $\tilde{B} := (\lambda I - A)^{-\alpha} B \in \mathcal{B}(\mathbb{R}^m, X)$ and $\tilde{C} = C(\lambda I - A)^{-\beta} \in \mathcal{B}(X, \mathbb{R}^p)$. Since $\alpha + \beta \leq 1$, the operator $(\lambda I - A)^{\alpha+\beta}(sI - A)^{-1}$ is in $\mathcal{B}(X)$ for all $s \in \varrho(A)$. Consequently, the function $\mathbf{G}$ defined by

$$\mathbf{G}(s) := \tilde{C}(\lambda I - A)^{\alpha+\beta}(sI - A)^{-1}\tilde{B}$$

is analytic on $\varrho(A)$. Moreover,

$$\begin{aligned}(\lambda I - A)^{\alpha+\beta}(sI - A)^{-1} &= (\lambda I - A)^{\alpha+\beta-1}(\lambda I - A)(sI - A)^{-1} \\ &= (\lambda I - A)^{\alpha+\beta-1}[(\lambda + s)(sI - A)^{-1} + I]\,, \quad \forall\, s \in \varrho(A). \end{aligned} \tag{2.7}$$

Fix $\gamma > \omega(\mathbf{T})$. The fact that A generates an analytic semigroup guarantees the existence of a constant $M > 0$ such that $\|(sI - A)^{-1}\| \leq M/|s - \gamma|$ for all $s \in \mathbb{C}_{\gamma}$. Therefore we obtain from (2.7) that the $\mathcal{B}(X)$-valued function $s \mapsto (\lambda I - A)^{\alpha+\beta}(sI - A)^{-1}$ is bounded on $\mathbb{C}_{\gamma}$. Consequently, $\mathbf{G}$ is bounded on $\mathbb{C}_{\gamma}$. Moreover, since

$$(sI - A)^{-1}(\lambda I - A)^{\alpha} z = (\lambda I - A)^{\alpha}(sI - A)^{-1} z \in X\,, \quad \forall\, z \in X,\ \forall\, s \in \varrho(A)$$

and

$$(\lambda I - A)^{\alpha}(\lambda I - A)^{\beta} z = (\lambda I - A)^{\alpha+\beta} z \in X\,, \quad \forall\, z \in X_1\,,$$

an application of the resolvent identity yields for all $s, s_0 \in \varrho(A)$ with $s \neq s_0$

$$\frac{1}{s_0 - s}(\mathbf{G}(s) - \mathbf{G}(s_0)) = \tilde{C}(\lambda I - A)^{\alpha+\beta}(sI - A)^{-1}(s_0 I - A)^{-1}\tilde{B} = C(sI - A)^{-1}(s_0 I - A)^{-1}B\,.$$

Invoking a result in [4], we may now conclude that there exists a well-posed system with generating operators (A, B, C). To show that this system is regular, it suffices to prove that $(s_0 I - A)^{-1}B\mathbb{R}^m \subset \operatorname{dom} C_\Lambda$ for $s_0 \in \varrho(A)$, see [29]. But this follows trivially from the identity

$$C(sI - A)^{-1}(s_0 I - A)^{-1}B = \tilde{C}(\lambda I - A)^{\alpha+\beta}(s_0 I - A)^{-1}(sI - A)^{-1}\tilde{B}$$

and the facts that $\tilde{C} \in \mathcal{B}(X, \mathbb{R}^p)$, $(\lambda I - A)^{\alpha+\beta}(s_0 I - A)^{-1} \in \mathcal{B}(X)$ and $\tilde{B} \in \mathcal{B}(\mathbb{R}^m, X)$. $\square$

3. The sampled-data system

Let $\tau > \delta > 0$, $H \in L^2([0,\delta], \mathbb{R}^{m\times p})$, $w \in L^2([0, \tau-\delta], \mathbb{R})$ and $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$. We apply the following sampled-data feedback control law to the well-posed system (2.2)

$$u(t) = \begin{cases} v(t) - H(t - k\tau)y_k\,, & t \in [k\tau, k\tau + \delta) \\ v(t)\,, & t \in [k\tau + \delta, (k+1)\tau) \end{cases}, \quad \forall\, k \in \mathbb{N}_0\,, \quad \text{where} \tag{3.1a}$$

$$y_0 := 0 \quad \text{and} \quad y_k := \int_0^{\tau-\delta} w(s)y((k-1)\tau + \delta + s)\,ds\,, \quad \forall\, k \in \mathbb{N}. \tag{3.1b}$$

The function v represents the input signal of the sampled-data feedback system and emphasises our input-output as well as state-space point of view.

Remark 3.1 Defining $H_\tau \in L^2([0,\tau], \mathbb{R}^{m\times p})$ by

$$H_\tau(t) := \begin{cases} H(t)\,, & t \in [0,\delta] \\ 0\,, & t \in (\delta, \tau]\,, \end{cases} \tag{3.2}$$

and setting

$$\big(\mathcal{H}((y_k))\big)(t) := H_\tau(t - k\tau)y_k\,, \quad \forall\, t \in [k\tau, (k+1)\tau),\ \forall\, k \in \mathbb{N}_0\,,$$

(3.1a) can be written in the form $u = v - \mathcal{H}((y_k))$. The operator $\mathcal{H}$ represents a generalized hold operation with hold function H_τ (see, for example, [1]). Similarly, (3.1b) describes a generalized sampling operation (see [1]). The function w is called the weighting function of the sampler (3.1b). Note that instantaneous sampling of the form $y_k = y(k\tau)$ is in general not possible due to the potential irregularity of the output y of the well-posed system (2.2). Also note that the state-space formula (2.2b) for the output does not hold for all $t \in \mathbb{R}_+$, but for almost all $t \in \mathbb{R}$: in particular, it might not hold for $t = k\tau$ for some $k \in \mathbb{N}_0$. $\diamond$

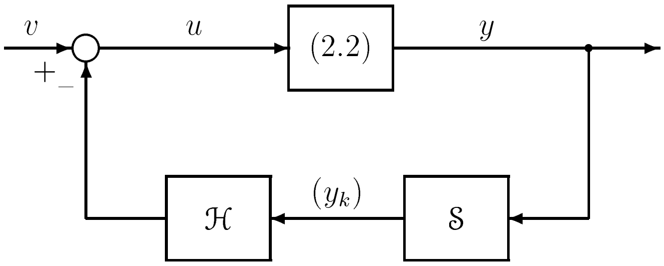

Figure 1: Feedback system with generalized sampling $\mathcal{S}$ and generalized hold $\mathcal{H}$

The sampled-data feedback system obtained by applying the control law (3.1) to the well-posed system (2.2) is illustrated in Figure 1, where $\mathcal{S}$ denotes the generalized sampling operation given by (3.1b).

It is clear that, for given initial state $x^0 \in X$ and given input function $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, the (unique) state trajectory $x(\cdot\,; x^0, v)$ of the sampled-data feedback system given by (2.2) and (3.1) can be obtained recursively from (2.2b), (3.1b) and

$$x(0; x^0, v) \;=\; x^0\,, \tag{3.3a}$$

$$x(k\tau + t; x^0, v) \;=\; \mathbf{T}_t x(k\tau; x^0, v) + \int_0^t \mathbf{T}_{t-s} B(v(k\tau + s) - H_\tau(s) y_k)\, ds\,, \quad \forall\, t \in (0, \tau],\ \forall\, k \in \mathbb{N}_0\,. \tag{3.3b}$$

Note that $x(\cdot\,; x^0, v)$ is a continuous X-valued function defined on $\mathbb{R}_+$. For simplicity, in the following we shall occasionally use the abbreviation $x := x(\cdot\,; x^0, v)$. We define

$$x_k := x(k\tau)\,, \quad x_{k,\delta} := x(k\tau + \delta)\,; \qquad \forall\, k \in \mathbb{N}_0\,.$$

For $\sigma, \tau > 0$, we define the left-shift/truncation operator $\mathbf{L}^\tau_\sigma : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m) \to L^2(\mathbb{R}_+, \mathbb{R}^m)$ by setting

$$(\mathbf{L}^\tau_\sigma f)(t) := \begin{cases} f(t + \sigma)\,, & t \in [0, \tau] \\ 0\,, & t \in (\tau, \infty)\,. \end{cases}$$

In the following lemma we establish the basic discrete-time equations (involving x_k, $x_{k,\delta}$, y_k and $\mathbf{L}^\tau_{k\tau+\delta} v$) associated with the sampled-data feedback system given by (2.2) and (3.1) .

Lemma 3.2 *Let $\tau > \delta > 0$, $H \in L^2([0, \delta], \mathbb{R}^{m \times p})$ and $w \in L^2([0, \tau - \delta], \mathbb{R})$. We assume that*

$$\int_0^{\tau - \delta} w(s) \mathbf{T}_s z\, ds \in X_1\,, \quad \forall\, z \in X\,. \tag{3.4}$$

Then the following statements hold.

(1) *The operator*

$$L_w : X \to X_1, \ z \mapsto \int_0^{\tau-\delta} w(s)\mathbf{T}_s z \, ds \tag{3.5}$$

is in $\mathcal{B}(X, X_1)$.

(2) *The sequences* (x_k), $(x_{k,\delta})$ *and* (y_k) *satisfy, for all* $k \in \mathbb{N}_0$,

$$\begin{aligned} x_{k+1} &= \mathbf{T}_{\tau-\delta} x_{k,\delta} + \int_0^{\tau-\delta} \mathbf{T}_{\tau-\delta-s} B v(k\tau+\delta+s)\, ds\,, & (3.6)\\ y_{k+1} &= C L_w x_{k,\delta} + M_w \mathbf{L}^{\tau}_{k\tau+\delta} v\,, & (3.7)\\ x_{k+1,\delta} &= (\mathbf{T}_\tau + K_H C L_w) x_{k,\delta} + M_{H,w} \mathbf{L}^{\tau}_{k\tau+\delta} v\,, & (3.8) \end{aligned}$$

where $K_H \in \mathcal{B}(\mathbb{R}^p, X)$, $M_w \in \mathcal{B}(L^2(\mathbb{R}_+, \mathbb{R}^m), \mathbb{R}^p)$ *and* $M_{H,w} \in \mathcal{B}(L^2(\mathbb{R}_+, \mathbb{R}^m), X)$ *are defined by*

$$\begin{aligned} K_H z &= -\int_0^{\delta} \mathbf{T}_{\delta-s} B H(s) z \, ds\,, \quad \forall\, z \in \mathbb{R}^p\,, & (3.9)\\ M_w f &= \int_0^{\tau-\delta} w(s)(Gf)(s)\, ds\,, \quad \forall\, f \in L^2(\mathbb{R}_+, \mathbb{R}^m)\,, & (3.10)\\ M_{H,w} f &= K_H M_w f + \int_0^{\tau} \mathbf{T}_{\tau-s} B f(s)\, ds\,, \quad \forall\, f \in L^2(\mathbb{R}_+, \mathbb{R}^m)\,, & (3.11) \end{aligned}$$

respectively.

Remark 3.3 It is easy to show, using integration by parts, that (3.4) holds for any $w \in L^2([0, \tau-\delta], \mathbb{R})$ for which there exist a partition $0 = t_0 < t_1 < \ldots < t_n = \tau - \delta$ and functions $w_j \in W^{1,1}([t_{j-1}, t_j], \mathbb{R})$ such that $w(t) = w_j(t)$ for all $t \in (t_{j-1}, t_j)$ and all $j = 1, 2, \ldots, n$. $\diamond$

Proof of Lemma 3.2: Statement (1) follows from a routine application of the closed-graph theorem. To prove statement (2), note first that (3.6) follows immediately from the variation-of-parameters formula combined with the fact that the control u given by (3.1a) satisfies

$$u(t) = v(t)\,, \quad \forall\, t \in [k\tau + \delta, (k+1)\tau)\,. \tag{3.12}$$

To derive (3.7), we use (2.2b) and (3.12), to obtain

$$\begin{aligned} y(k\tau+\delta+s) = C_\Lambda \big(x(k\tau+\delta+s) - (s_0 I - A)^{-1} B v(k\tau+\delta+s)\big)\,, \\ + \mathbf{G}(s_0) v(k\tau+\delta+s) \quad \text{for a.a. } s \in [0, \tau-\delta]\,. \end{aligned} \tag{3.13}$$

It follows from the variation-of-parameters formula that the function $\tilde{x} : s \mapsto x(k\tau + \delta + s)$ is the state-trajectory of (2.2) corresponding to the initial condition $\tilde{x}(0) = x(k\tau + \delta) = x_{k,\delta}$

and the control function $s \mapsto v(k\tau + \delta + s)$. By (3.13), the function $s \mapsto y(k\tau + \delta + s)$ is the corresponding output and thus

$$y(k\tau + \delta + s) = C_\Lambda \mathbf{T}_s x_{k,\delta} + (G\mathbf{L}^\tau_{k\tau+\delta} v)(s)\,, \quad \text{for a.a. } s \in [0, \tau - \delta]\,.$$

Combining this with (3.1b) gives

$$y_{k+1} = \int_0^{\tau-\delta} w(s) \left(C_\Lambda \mathbf{T}_s x_{k,\delta} + (G\mathbf{L}^\tau_{k\tau+\delta} v)(s) \right) ds$$

A standard argument involving the approximation of $x_{k,\delta}$ by elements in X_1, the admissibility of C and the boundedness of the operator L_w (see statement (1)) shows that

$$\int_0^{\tau-\delta} w(s) C_\Lambda \mathbf{T}_s x_{k,\delta}\, ds = C L_w x_{k,\delta}\,.$$

Hence, with M_w given by (3.10),

$$y_{k+1} = C L_w x_{k,\delta} + M_w \mathbf{L}^\tau_{k\tau+\delta} v\,,$$

which is (3.7). To prove (3.8), note that $k\tau + \delta + s \in [(k+1)\tau, (k+1)\tau + \delta]$ for all $s \in [\tau - \delta, \tau]$ and so, by (3.1a),

$$u(k\tau + \delta + s) = v(k\tau + \delta + s) - H(s + \delta - \tau) y_{k+1}\,, \quad \forall\, s \in [\tau - \delta, \tau],\ \forall\, k \in \mathbb{N}_0\,.$$

Combining this with (3.12), we may conclude that

$$x_{k+1,\delta} = \mathbf{T}_\tau x_{k,\delta} + \int_0^\tau \mathbf{T}_{\tau-s} B v(k\tau + \delta + s)\, ds - \int_{\tau-\delta}^\tau \mathbf{T}_{\tau-s} B H(s + \delta - \tau) y_{k+1}\, ds\,.$$

Changing the integration variable s in the second integral to $\zeta = s + \delta - \tau$ gives

$$\begin{aligned} x_{k+1,\delta} &= \mathbf{T}_\tau x_{k,\delta} + \int_0^\tau \mathbf{T}_{\tau-s} B v(k\tau + \delta + s)\, ds - \int_0^\delta \mathbf{T}_{\delta-\zeta} B H(\zeta) y_{k+1}\, d\zeta \\ &= \mathbf{T}_\tau x_{k,\delta} + K_H y_{k+1} + \int_0^\tau \mathbf{T}_{\tau-s} B v(k\tau + \delta + s)\, ds\,, \quad \forall\, k \in \mathbb{N}_0\,, \end{aligned}$$

where K_H is given by (3.9). Together with (3.7) and (3.11) this yields (3.8). □

The sampled-data feedback system given by (2.2) and (3.1) is called *exponentially bounded* if there exist constants $N \geq 1$ and $\nu \in \mathbb{R}$ such that

$$\|x(t; x^0, 0)\| \leq N e^{\nu t} \|x^0\|\,, \quad \forall\, t \in \mathbb{R}_+\,,\ \forall\, x^0 \in X\,, \tag{3.14}$$

where $x(t; x^0, 0)$ is given by (3.3) (with $v = 0$). The number ν is called an *exponential bound* of the sampled-data feedback system. Obviously any bounded operator $\Delta \in \mathcal{B}(X)$ satisfies $\|\Delta^k\| \leq \|\Delta\|^k$, i.e. Δ is *power bounded.* If $q > 0$ is such that there exists $M \geq 1$ so that

$$\|\Delta^k\| \leq M q^k\,, \quad \forall\, k \in \mathbb{N}_0\,, \tag{3.15}$$

then q is a *power bound* for Δ.

Lemma 3.4 *Let $\tau > \delta > 0$, $H \in L^2([0,\delta], \mathbb{R}^{m\times p})$, $w \in L^2([0,\tau-\delta], \mathbb{R})$, let $L_w \in \mathcal{B}(X, X_1)$ and $K_H \in \mathcal{B}(\mathbb{R}^p, X)$ be given by* (3.5) *and* (3.9)*, respectively, and assume that* (3.4) *holds. Furthermore, let $\nu \in \mathbb{R}$. Then the following statements hold.*

(1) *If $e^{\nu\tau}$ is a power bound for the operator $\mathbf{T}_\tau + K_H C L_w$, then $\nu \in \mathbb{R}$ is an exponential bound for the sampled-data feedback system given by* (2.2) *and* (3.1).

(2) *Under the additional assumption that $\mathbf{T}$ is a group, the converse of statement* (1) *holds, that is, if $\nu \in \mathbb{R}$ is an exponential bound for the sampled-data feedback system given by* (2.2) *and* (3.1)*, then $e^{\nu\tau}$ is a power bound for $\mathbf{T}_\tau + K_H C L_w$.*

The lemma shows in particular that the sampled-data feedback system is exponentially bounded. We define the *exponential growth* $\omega_{\rm sd}$ of the sampled-data feedback system to be the infimum of all $\nu \in \mathbb{R}$ for which there exists $N \geq 1$ such that (3.14) holds. Note that $-\infty \leq \omega_{\rm sd} < \infty$. If $\omega_{\rm sd} < 0$, then we say that the sampled-data feedback system is *exponentially stable*. Similarly, the infimum of all $q > 0$ for which there exists $M \geq 1$ such that (3.15) holds is called the *power growth* of Δ. If the power growth is smaller than 1, we say that Δ is *power stable*. It follows from Gelfand's spectral radius formula

$$r(\Delta) = \lim_{k\to\infty} \|\Delta^k\|^{1/k}$$

that the power growth of Δ coincides with $r(\Delta)$. As a consequence, Lemma 3.4 has the following corollary.

Corollary 3.5 *Let $\tau > \delta > 0$, $H \in L^2([0,\delta], \mathbb{R}^{m\times p})$, $w \in L^2([0,\tau-\delta], \mathbb{R})$, let $L_w \in \mathcal{B}(X, X_1)$ and $K_H \in \mathcal{B}(\mathbb{R}^p, X)$ be given by* (3.5) *and* (3.9)*, respectively, and assume that* (3.4) *holds. Then $r(\mathbf{T}_\tau + K_H C L_w) \geq e^{\omega_{\rm sd}\tau}$; under the additional assumption that $\mathbf{T}$ is a group, we have $r(\mathbf{T}_\tau + K_H C L_w) = e^{\omega_{\rm sd}\tau}$ (we adopt the convention $e^{-\infty\tau} := 0$).*

Proof of Lemma 3.4: We define $\Delta \in \mathcal{B}(X)$ by setting

$$\Delta := \mathbf{T}_\tau + K_H C L_w\,.$$

To prove statement (1), let $\nu \in \mathbb{R}$ and assume that $e^{\nu\tau}$ is a power bound for Δ. By the variation-of-parameter formula we obtain for the state trajectory $x(\cdot\,; x^0, 0)$ of the sampled-data feedback system

$$x(k\tau + t; x^0, 0) = \mathbf{T}_t x_k - \int_0^t \mathbf{T}_{t-s} B H_\tau(s) y_k \, ds\,, \quad \forall\, t \in [0,\tau),\ \forall\, k \in \mathbb{N}_0\,,$$

where H_τ is given by (3.2). Using (3.6) and (3.7), we obtain

$$x(k\tau + t; x^0, 0) = \mathbf{T}_{t+\tau-\delta} x_{k-1,\delta} - \int_0^t \mathbf{T}_{t-s} B H_\tau(s) C L_w x_{k-1,\delta} \, ds\,, \quad \forall\, t \in [0,\tau),\ \forall\, k \in \mathbb{N}.$$

Invoking the admissibility of B, equation (3.8) and the hypothesis, we may conclude that there exist $N_1, N_2 \geq 0$ such that

$$\|x(k\tau + t; x^0, 0)\| \leq N_1 \|x_{k-1,\delta}\| \leq N_2 (e^{\nu\tau})^{k-1} \|x_{0,\delta}\|\,, \quad \forall\, t \in [0, \tau),\ \forall\, k \in \mathbb{N}.$$

Noting that $x(t; x^0, 0) = \mathbf{T}_t x^0$ for all $t \in [0, \tau]$ and setting

$$N := \left(\max\{ \sup_{0 \leq s \leq \tau} \|\mathbf{T}_s\|, N_2 \|\mathbf{T}_\delta\| e^{-\nu\tau} \} \right) \sup_{0 \leq s \leq \tau} e^{-\nu s}$$

it follows that

$$\|x(k\tau + t; x^0, 0)\| \leq N e^{\nu(k\tau + t)} \|x^0\|\,, \quad \forall\, t \in [0, \tau),\ \forall\, k \in \mathbb{N}_0\,.$$

This holds for all $x^0 \in X$, showing that ν is an exponential bound for the sampled-data feedback system.

To prove statement (2), assume that $\mathbf{T}$ is a group and let $\nu \in \mathbb{R}$ be an exponential bound for the sampled-data feedback system. Then there exists $N \geq 1$ such that (3.14) holds and therefore

$$\|x_{k,\delta}\| \leq N e^{\nu(k\tau + \delta)} \|x^0\| = N e^{\nu\delta} (e^{\nu\tau})^k \|x^0\|\,, \quad \forall\, k \in \mathbb{N}_0\,.$$

Since $x_{0,\delta} = \mathbf{T}_\delta x^0$, it follows from (3.8) that $x_{k,\delta} = \Delta^k \mathbf{T}_\delta x^0$. Hence, using the group property of $\mathbf{T}$, we obtain

$$\|\Delta^k \mathbf{T}_\delta x^0\| \leq N \|\mathbf{T}_{-\delta}\| e^{\nu\delta} (e^{\nu\tau})^k \|\mathbf{T}_\delta x^0\|\,, \quad \forall\, k \in \mathbb{N}_0\,.$$

Since this holds for all $x^0 \in X$, it follows that $e^{\nu\tau}$ is a power bound for Δ. □

4. Main result

We first state and prove a technical lemma.

Lemma 4.1 *Let $S \in \mathbb{R}^{n\times n}$, $a > 0$ and $f \in L^1([0, a], \mathbb{R})$. The matrix $\int_0^a f(t) e^{St}\, dt$ is invertible if and only if $\int_0^a f(t) e^{\lambda t}\, dt \neq 0$ for all $\lambda \in \sigma(S)$.*

Proof: Using the Jordan form of S, it is easy to show that a complex number μ is an eigenvalue of the matrix $\int_0^a f(t) e^{St} dt$ if and only if $\mu = \int_0^a f(t) e^{\lambda t}\, dt$ for some $\lambda \in \sigma(S)$. □

In the following we shall impose a number of assumptions on the well-posed system (2.2), the weighting function w and the sampling constants $\tau > \delta > 0$.

A1 There exists $\beta < 0$ such that $\sigma(A) \cap \overline{\mathbb{C}}_\beta$ consists of finitely many isolated eigenvalues of A with finite algebraic multiplicities.

If A1 holds, then there exists a simple closed curve Γ in the complex plane not intersecting $\sigma(A)$, enclosing $\sigma(A) \cap \overline{\mathbb{C}}_\beta$ in its interior and having $\sigma(A) \cap (\mathbb{C} \setminus \overline{\mathbb{C}}_\beta)$ in its exterior. The operator

$$\Pi := \frac{1}{2\pi i} \int_\Gamma (sI - A)^{-1} \, ds \tag{4.1}$$

is a projection operator and we have

$$X = X^+ \oplus X^- , \qquad \text{where} \quad X^+ := \Pi X , \quad X^- := (I - \Pi) X . \tag{4.2}$$

It follows from a standard result (see, for example, Lemma 2.5.7 in [5]) that $\dim X^+ < \infty$, $X^+ \subset X_1$, X^+ and X^- are $\mathbf{T}_t$-invariant for all $t \geq 0$ and

$$\sigma(A|_{X^+}) = \sigma(A) \cap \overline{\mathbb{C}}_\beta , \qquad \sigma(A|_{X^-}) = \sigma(A) \cap (\mathbb{C} \setminus \overline{\mathbb{C}}_\beta) .$$

It is useful to introduce the following notation

$$A^+ := A|_{X^+} , \quad A^- := A|_{X_1 \cap X^-} , \qquad \mathbf{T}_t^+ := \mathbf{T}_t|_{X^+} , \quad \mathbf{T}_t^- := \mathbf{T}_t|_{X^-} . \tag{4.3}$$

Clearly, $\mathbf{T}_t^+$ is a semigroup on the finite-dimensional space X^+ with generator A^+, i.e. $\mathbf{T}_t^+ = e^{A^+ t}$, and $\mathbf{T}_t^-$ is a strongly continuous semigroup on X^- with generator A^-. Since the spectrum of A considered as an operator on X coincides with the spectrum of A considered as an operator on X_{-1}, the projection operator Π on X defined in (4.1) extends to a projection on X_{-1}. We will use the same symbol Π for the original projection and its associated extension. Obviously, the operator A^- extends to an operator in $\mathcal{B}(X^-, (X_{-1})^-)$ and the same symbol A^- will be used to denote this extension. The decomposition (4.2) induces decompositions of the control operator $B \in \mathcal{B}(\mathbb{R}^m, X_{-1})$ and the observation operator $C \in \mathcal{B}(X_1, \mathbb{R}^p)$:

$$B^+ := \Pi B , \quad B^- := (I - \Pi) B , \quad C^+ := C|_{X^+} , \quad C^- := C|_{X_1 \cap X^-} . \tag{4.4}$$

The following simple lemma will be useful in the proof of Theorem 4.4.

Lemma 4.2 *Assume that* A1 *holds. There exists a well-posed system Σ^- with generating operators (A^-, B^-, C^-)* [‡] *and input-output operator $G^- := G - G^+$, where G^+ denotes the input-output operator of the (finite-dimensional) system (A^+, B^+, C^+), that is, $(G^+ u)(t) = \int_0^t C^+ e^{A^+(t-s)} B^+ u(s) \, ds$ for all $t \in \mathbb{R}_+$ and all $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$. Moreover, for any $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, the output y of the well-posed system* (2.2) *can be written in the form*

$$y(t) = (C^-)_\Lambda \mathbf{T}_t^- (I - \Pi) x^0 + (G^- u)(t) + C^+ \Pi x(t) , \quad \text{a.a. } t \in \mathbb{R}_+ , \tag{4.5}$$

[‡] For (A^-, B^-, C^-) to be the generating operators of a well-posed system it is of course necessary that B^- maps into $(X^-)_{-1} = ((I - \Pi)X)_{-1}$, the extrapolation space associated with A^-. Since, by definition, B^- maps into $(I - \Pi)X_{-1} =: (X_{-1})^-$, their seems to be a difficulty. However, it is clear that the spaces $(X^-)_{-1}$ and $(X_{-1})^-$ are both completions of X^- endowed with the norm $\| \cdot \|_{-1}$. Hence there exists an isometric isomorphism $(X^-)_{-1} \to (X_{-1})^-$ whose restriction to X^- is the identity, and so we can safely identify $(X^-)_{-1}$ and $(X_{-1})^-$.

where $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} Bu(s)\, ds$ *for all* $t \in \mathbb{R}_+$*. The* Λ*-extension of* C^- *satisfies*

$$(C^-)_\Lambda z = C_\Lambda z\,, \quad \forall\, z \in \mathrm{dom}((C^-)_\Lambda) = \mathrm{dom}(C_\Lambda) \cap X^-\,. \tag{4.6}$$

Proof: It is trivial that the Λ-extension of C^- satisfies (4.6). The admissibility of B and C immediately implies that B^- and C^- are admissible control and observation operators for $\mathbf{T}^-$, respectively. Defining $\mathbf{G}^+(s) := C^+(sI - A^+)^{-1}B^+$ it follows from (2.1) that

$$\frac{1}{s-s_0}(\mathbf{G}(s) - \mathbf{G}(s_0)) - \frac{1}{s-s_0}(\mathbf{G}^+(s) - \mathbf{G}^+(s_0)) = \\ - C^-(sI - A^-)^{-1}(s_0 I - A^-)^{-1}B^-\,, \quad \forall\, s, s_0 \in \mathbb{C}_{\omega(\mathbf{T})},\; s \neq s_0\,.$$

Choosing $\alpha > \omega(\mathbf{T})$ and setting $\mathbf{G}^-(s) := \mathbf{G}(s) - \mathbf{G}^+(s)$ for all $s \in \mathbb{C}_\alpha$, it is clear that $\mathbf{G}^-$ is analytic and bounded on $\mathbb{C}_\alpha$ and $\mathbf{G}^-$ satisfies

$$\frac{1}{s-s_0}\big(\mathbf{G}^-(s) - \mathbf{G}^-(s_0)\big) = -C^-(sI - A^-)^{-1}(s_0 I - A^-)^{-1}B^-\,,\ \forall\, s, s_0 \in \mathbb{C}_\alpha\,,\ s \neq s_0\,.$$

Invoking a result in [4], we may now conclude that there exists a well-posed system Σ^- with generating operators (A^-, B^-, C^-) and input-output operator G^- (or, equivalently, transfer function $\mathbf{G}^-$).[§] To prove (4.5), let $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ and note that

$$\Pi \mathbf{T}_t x^0 \in X^+ \subset X_1 \subset \mathrm{dom}(C_\Lambda)\,, \quad \forall\, t \in \mathbb{R}_+$$

and

$$(I - \Pi)\mathbf{T}_t x^0 = \mathbf{T}_t^-(I - \Pi)x^0 \in \mathrm{dom}(C_\Lambda) \cap X^- = \mathrm{dom}((C^-)_\Lambda)\,, \quad \text{a.a. } t \in \mathbb{R}_+\,.$$

Thus, by (4.6), we may write the output $y = C_\Lambda \mathbf{T} x^0 + Gu$ in the form

$$y = (C^-)_\Lambda \mathbf{T}^-(I - \Pi)x^0 + G^- u + C^+\mathbf{T}^+\Pi x^0 + G^+ u \tag{4.7}$$

With x given by $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} Bu(s)\, ds$, it is clear that Πx is the state trajectory of the finite-dimensional system given by (A^+, B^+, C^+) corresponding to the initial state Πx^0 and the input function u. Therefore, $C^+\mathbf{T}^+\Pi x^0 + G^+ u = C^+\Pi x$, and (4.5) follows from (4.7). □

Let $t_0 > 0$. We recall that the linear bounded map

$$R_{t_0} : L^2([0, t_0], \mathbb{R}^m) \to X\,,\ \ f \mapsto \int_0^{t_0} \mathbf{T}_{t_0 - s} Bf(s)\, ds \tag{4.8}$$

is called the reachability operator of the well-posed system (2.2) at time t_0.

In addition to A1 we assume that the following conditions are satisfied.

[§] Alternatively, the claim that there exists a well-posed system Σ^- with generating operators (A^-, B^-, C^-) and input-output operator G^- can be proved by direct verification of the defining properties of a well-posed system as given in, for example, [23, 25, 29].

A2 The semigroup $\mathbf{T}^-$ is exponentially stable, that is $\omega(\mathbf{T}^-) < 0$.

A3 The pair (C^+, A^+) is observable.

A4 There exists $t_0 > 0$ such that $\overline{\operatorname{im} R_{t_0}} \supset X^+$.

A5 The constants τ and δ and the function $w \in L^2([0, \tau - \delta], \mathbb{R})$ are such that $\tau > \delta \geq t_0$, (3.4) holds,

$$\tau(\lambda - \mu) \neq 2\pi i k\,, \quad \forall\, \lambda, \mu \in \sigma(A^+),\ \forall\, k \in \mathbb{Z} \setminus \{0\} \tag{4.9}$$

and

$$\int_0^{\tau-\delta} w(s) e^{\lambda s}\, ds \neq 0\,, \quad \forall\, \lambda \in \sigma(A^+)\,. \tag{4.10}$$

Remark 4.3 Of course, A2 holds if the generator A^- satisfies the spectrum determined growth assumption. Trivially, for A4 to hold it is sufficient that the well-posed system (2.2) is approximately controllable in time t_0. If the function w is a non-zero constant, then it is clear that (4.10) holds if and only if

$$(\tau - \delta)\lambda \neq 2\pi i k\,, \quad \forall\, \lambda \in \sigma(A^+),\ \forall\, k \in \mathbb{Z} \setminus \{0\}\,.$$

Conditions (4.9) and (4.10) are "generically" satisfied in the following sense: the set of all $\tau > t_0$ for which (4.9) holds is open and dense in (t_0, ∞) and, for given $\tau > \delta \geq t_0$, the set of all $w \in L^2([0, \tau - \delta], \mathbb{R})$ for which (4.10) holds is open and dense in $L^2([0, \tau - \delta], \mathbb{R})$. $\diamond$

The control function u generated by the sampled-data control law (3.1) depends on the initial value $x^0 \in X$ and the input function $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$. We express this dependence by writing $u = u(\cdot\,; x^0, v)$. It is natural to define the output $y(\cdot\,; x^0, v)$ of the sampled-data feedback system given by (2.2) and (3.1) to be the output of (2.2) corresponding to the initial condition x^0 and the control $u(\cdot\,; x^0, v)$. We are now in the position to formulate the main result of this paper.

Theorem 4.4 *Assume that* A1–A5 *are satisfied. For every $\varepsilon \in (0, -\omega(\mathbf{T}^-))$ there exists $H \in L^2([0, \delta], \mathbb{R}^{m \times p})$ such that the following statements hold.*

(1) *The sampled-data feedback system given by* (2.2) *and* (3.1) *is exponentially stable with exponential growth $\omega_{\mathrm{sd}} < \omega(\mathbf{T}^-) + \varepsilon < 0$.*

(2) *For every $\alpha \in [\omega(\mathbf{T}^-) + \varepsilon, 0]$ there exists $N \geq 1$ such that*

$$\|y(\cdot\,; x^0, v)\|_{L^2_\alpha} \leq N(\|x^0\| + \|v\|_{L^2_\alpha})\,, \quad \forall\, x^0 \in X,\ \forall\, v \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)\,;$$

(3) *If $\alpha \in [\omega(\mathbf{T}^-) + \varepsilon, 0]$, $x^0 \in X$ and $v \in W^{1,2}_\alpha(\mathbb{R}_+, \mathbb{R}^m)$ and there exists $t_0 \in \mathbb{R}_+$ such that $\mathbf{T}_{t_0}(Ax^0 + Bv(0)) \in X$, then $y(\cdot\,; x^0, v)$ is continuous on $[t_0, \infty)$ and*

$$\lim_{t \to \infty} y(t; x^0, v) e^{-\alpha t} = 0\,.$$

Statement (2) shows in particular that there exists $H \in L^2([0,\delta],\mathbb{R}^{m\times p})$ such that the sampled-data feedback system given by (2.2) and (3.1) is L^2_α-input-output stable.

Proof of Theorem 4.4: We define $\Delta_H \in \mathcal{B}(X)$ by setting

$$\Delta_H := \mathbf{T}_\tau + K_H C L_w\,, \tag{4.11}$$

where the operators $L_w \in \mathcal{B}(X, X_1)$ and $K_H \in \mathcal{B}(\mathbb{R}^p, X)$ are given by (3.5) and (3.9), respectively. It is convenient to set $\omega^- := \omega(\mathbf{T}^-)$. Let $\varepsilon \in (0, -\omega^-)$.

(1) To prove that, for a suitable hold function H, $\omega_{\rm sd} < \omega^- + \varepsilon$, we note that, by Corollary 3.5, it is sufficient to show the existence of a function $H \in L^2([0,\delta],\mathbb{R}^{m\times p})$ such that $r(\Delta_H) < e^{(\omega^-+\varepsilon)\tau}$. Defining the operators

$$K_H^+ := \Pi K_H\,, \quad K_H^- := (I-\Pi)K_H\,, \quad L_w^\pm := L_w|_{X^\pm} = \int_0^{\tau-\delta} w(s)\mathbf{T}_s^\pm\,ds\,, \tag{4.12}$$

we have $K_H^\pm \in \mathcal{B}(\mathbb{R}^p, X^\pm)$, $L_w^+ \in \mathcal{B}(X^+)$ and $L_w^- \in \mathcal{B}(X^-, X_1\cap X^-)$, where $X_1 \cap X^-$ is endowed with the norm $\|\cdot\|_1$. The operator Δ_H can then be written in the form

$$\Delta_H = \begin{pmatrix} \mathbf{T}_\tau^+ + K_H^+C^+L_w^+ & K_H^+C^-L_w^- \\ K_H^-C^+L_w^+ & \mathbf{T}_\tau^- + K_H^-C^-L_w^- \end{pmatrix}. \tag{4.13}$$

Using the observability of the (continuous-time) pair (C^+, A^+) (guaranteed by A3) and the fact that $\tau(\lambda-\mu) \neq 2\pi i k$ for all $\lambda, \mu \in \sigma(A^+)$ and all $k \in \mathbb{Z}\setminus\{0\}$ (guaranteed by A5), it follows from a well-known result on sampled observability (see Proposition 6.2.11 in [22]) that the (discrete-time) pair $(C^+, \mathbf{T}_\tau^+)$ is observable. Furthermore, by (A5), $\int_0^{\tau-\delta} w(s)e^{\lambda s}\,ds \neq 0$ for all $\lambda \in \sigma(A^+)$ and hence an application of Lemma 4.1 shows that the matrix $L_w^+ = \int_0^{\tau-\delta} w(s)e^{A^+s}\,ds$ is invertible. Since L_w^+ and $\mathbf{T}_\tau^+ = e^{A^+\tau}$ commute we have that

$$(C^+L_w^+, (L_w^+)^{-1}\mathbf{T}_\tau^+L_w^+) = (C^+L_w^+, \mathbf{T}_\tau^+)\,.$$

Consequently, the pair $(C^+L_w^+, \mathbf{T}_\tau^+)$ is observable, and so, by the pole-placement theorem for finite-dimensional systems, there exists $Q \in \mathcal{B}(\mathbb{R}^p, X^+)$ such that

$$\sigma(\mathbf{T}_\tau^+ + QC^+L_w^+) = \{0\}\,. \tag{4.14}$$

Denoting the canonical basis of $\mathbb{R}^p$ by $(e_1, e_2 \ldots, e_p)$, it follows from the fact that $\delta \geq t_0$ (see A5) combined with assumption A4 that, for every $\eta > 0$, there exist $h_1, h_2, \ldots h_p \in L^2([0,\delta],\mathbb{R}^m)$ such that

$$\sum_{j=1}^p \|R_\delta h_j - Qe_j\|^2 \leq \eta^2\,. \tag{4.15}$$

Setting $H := -(h_1, h_2, \ldots, h_p) \in L^2([0,\delta],\mathbb{R}^{m\times p})$, it follows that

$$R_\delta h_j = K_H e_j\,, \quad \forall\, j \in \{1, 2, \ldots, p\}\,.$$

Therefore, invoking (4.15), we obtain that for all $z = (z_1, z_2, \ldots, z_p)^T \in \mathbb{R}^p$

$$\begin{aligned} \|K_H z - Qz\| &= \|\sum_{j=1}^{p} z_j (K_H e_j - Qe_j)\| \leq \sum_{j=1}^{p} \|K_H e_j - Qe_j\| \, |z_j| \\ &\leq \left(\sum_{j=1}^{p} \|K_H e_j - Qe_j\|^2 \right)^{1/2} \|z\| \leq \eta \|z\| \,. \end{aligned}$$

Thus, $\|K_H - Q\| \leq \eta$, and so, since Q maps into X^+,

$$\begin{aligned} \|K_H^+ - Q\| &= \|\Pi(K_H - Q)\| \leq \|\Pi\| \eta \,, && \text{(4.16a)} \\ \|K_H^-\| &= \|(I - \Pi)(K_H - Q)\| \leq \|I - \Pi\| \eta \,. && \text{(4.16b)} \end{aligned}$$

Using (4.13), we may write

$$\Delta_H = \begin{pmatrix} \mathbf{T}_\tau^+ + QC^+ L_w^+ & QC^- L_w^- \\ 0 & \mathbf{T}_\tau^- \end{pmatrix} + \begin{pmatrix} (K_H^+ - Q) C^+ L_w^+ & (K_H^+ - Q) C^- L_w^- \\ K_H^- C^+ L_w^+ & K_H^- C^- L_w^- \end{pmatrix} . \qquad (4.17)$$

We denote the first operator on the right-hand side of (4.17) by Δ and the second by P_H. Obviously, by (4.14), $r(\Delta) = e^{\omega^- \tau}$. By upper semi-continuity of the spectrum (see [9], pp. 208), there exists $\gamma > 0$ such that

$$r(\Delta_H) = r(\Delta + P_H) < e^{(\omega^- + \varepsilon)\tau} \,, \qquad (4.18)$$

provided that $\|P_H\| \leq \gamma$. It follows from (4.16) that the latter can be accomplished by choosing $\eta > 0$ sufficiently small.

(2) To prove statement (2), choose $H \in L^2([0, \delta], \mathbb{R}^{m \times p})$ such that (4.18) holds. Choose $\nu \in (\omega^-, \omega^- + \varepsilon)$ such that $e^{\nu\tau}$ is a power-bound for Δ_H. Let $x^0 \in X$, $\alpha \in (\nu, 0]$ and $v \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)$. Recall that the feedback control produced by the sampled-data control law (3.1) is denoted by $u(\cdot\,; x^0, v)$. With H_τ defined by (3.2) we have

$$u(t; x^0, v) e^{-\alpha t} = e^{-\alpha t} v(t) - e^{-\alpha(t - k\tau)} H_\tau(t - k\tau) y_k e^{-\alpha k \tau} \,, \quad \forall\, t \in [k\tau, (k+1)\tau), \ \forall\, k \in \mathbb{N}_0 \,,$$

In the following, the numbers $N_i > 0$ are suitable constants, depending only on α, but not on x^0 and v. It follows from the above identity that

$$\int_{k\tau}^{(k+1)\tau} \|u(t; x^0, v) e^{-\alpha t}\|^2 dt \leq N_1 \left(\|y_k e^{-\alpha k\tau}\|^2 + \int_{k\tau}^{(k+1)\tau} \|v(t) e^{-\alpha t}\|^2 dt \right) , \quad \forall\, k \in \mathbb{N}_0 \,. \qquad (4.19)$$

Using that $e^{\nu\tau}$ is a power-bound for Δ_H and that $0 \geq \alpha > \nu$, we may conclude from (3.7), (3.8) and (4.11) that

$$\sum_{k=0}^{\infty} \|x_{k,\delta} e^{-\alpha k\tau}\|^2 \leq N_2 \left(\|x_{0,\delta}\|^2 + \int_0^\infty \|v(t) e^{-\alpha t}\|^2 dt \right) . \qquad (4.20)$$

and

$$\sum_{k=0}^{\infty} \|y_k e^{-\alpha k\tau}\|^2 \leq N_3 \left(\|x_{0,\delta}\|^2 + \int_0^{\infty} \|v(t)e^{-\alpha t}\|^2 dt \right) . \tag{4.21}$$

Now $y_0 = 0$, and so $u(t) = v(t)$ for all $t \in [0, \tau)$. Hence,

$$x(t; x^0, v) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} B v(s)\, ds\,, \quad \forall\, t \in [0, \tau)\,, \tag{4.22}$$

showing that

$$\|x_{0,\delta}\| = \|x(\delta; x^0, v)\| \leq N_4(\|x^0\| + \|v\|_{L^2}) \leq N_4(\|x^0\| + \|v\|_{L^2_\alpha})\,. \tag{4.23}$$

Inserting this into (4.20) and (4.21) yields

$$\sum_{k=0}^{\infty} \|x_{k,\delta} e^{-\alpha k\tau}\|^2 \leq N_5(\|x^0\|^2 + \|v\|^2_{L^2_\alpha})\,, \tag{4.24}$$

and

$$\sum_{k=0}^{\infty} \|y_k e^{-\alpha k\tau}\|^2 \leq N_6(\|x^0\|^2 + \|v\|^2_{L^2_\alpha})\,, \tag{4.25}$$

It follows from (4.19) and (4.25) that

$$\|u(\cdot\,; x^0, v)\|_{L^2_\alpha} \leq N_7(\|x^0\| + \|v\|_{L^2_\alpha})\,. \tag{4.26}$$

To derive a similar estimate for $x(\cdot\,; x^0, v)$, we note that by the variations-of-parameter formula we have, for $k \in \mathbb{N}$ and $t \in [0, \tau)$,

$$x(k\tau+t; x^0, v) = \mathbf{T}_{t+\tau-\delta} x_{k-1,\delta} - \int_{k\tau}^{k\tau+t} \mathbf{T}_{k\tau+t-s} B H_\tau(s-k\tau) y_k\, ds + \int_{(k-1)\tau+\delta}^{k\tau+t} \mathbf{T}_{k\tau+t-s} B v(s)\, ds\,,$$

where H_τ is defined in (3.2). A change of variables leads to

$$x(k\tau + t; x^0, v) = \mathbf{T}_{t+\tau-\delta} x_{k-1,\delta} - \int_0^t \mathbf{T}_{t-s} B H_\tau(s) y_k\, ds + \int_{\delta-\tau}^t \mathbf{T}_{t-s} B v(k\tau + s)\, ds\,.$$

Hence,

$$\begin{aligned}\|x(k\tau + t; x^0, v)e^{-\alpha(k\tau+t)}\|^2 \leq N_8 \Bigg(& \|x_{k-1,\delta} e^{-\alpha(k-1)\tau}\|^2 + \|y_k e^{-\alpha k\tau}\|^2 \\ & + \int_{(k-1)\tau}^{(k+1)\tau} \|v(s)e^{-\alpha s}\|^2 ds \Bigg)\,,\ \forall\, k \in \mathbb{N},\ \forall\, t \in [0, \tau)\,, \end{aligned} \tag{4.27}$$

and so,

$$\int_{k\tau}^{(k+1)\tau} \|x(t;x^0,v)e^{-\alpha t}\|^2 dt \leq N_9 \Bigg(\|x_{k-1,\delta}e^{-\alpha(k-1)\tau}\|^2 + \|y_k e^{-\alpha k\tau}\|^2 + \int_{(k-1)\tau}^{(k+1)\tau} \|v(s)e^{-\alpha s}\|^2 ds \Bigg), \quad \forall\, k \in \mathbb{N} \quad (4.28)$$

Combining this with (4.22), (4.24) and (4.25) shows that

$$\|x(\cdot\,; x^0, v)\|_{L^2_\alpha} \leq N_{10}(\|x^0\| + \|v\|_{L^2_\alpha})\,. \tag{4.29}$$

Using that $\alpha > \nu > \omega^-$, we have that the weighted semigroup $t \mapsto \mathbf{T}^-_t e^{-\alpha t}$ is exponentially stable and $G^- \in \mathcal{B}(L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m), L^2_\alpha(\mathbb{R}_+, \mathbb{R}^p))$. Combining this with (4.26) and (4.29), an application of (4.5) (with $u = u(\cdot\,; x^0, v)$, $x = x(\cdot\,; x^0, v)$ and $y = y(\cdot\,; x^0, v)$) yields the claim.

(3) Since the space of all $W^{1,2}_c([0,\delta], \mathbb{R}^{m\times p})$ is dense in $L^2[0,\delta], \mathbb{R}^{m\times p})$, an inspection of the proof of statement (1) shows that there exists $H \in W^{1,2}_c([0,\delta], \mathbb{R}^{m\times p})$ such that (4.18) holds. Choose $\nu \in (\omega^-, \omega^- + \varepsilon)$ such that $e^{\nu\tau}$ is a power-bound for Δ_H. Fix $\alpha \in (\nu, 0]$. Let $x^0 \in X$ and $v \in W^{1,2}_\alpha(\mathbb{R}_+, \mathbb{R}^m)$ be such that $Ax^0 + Bv(0) \in X$. It follows from (3.1a) and (4.21) that $u(\cdot\,; x^0, v) \in W^{1,2}_\alpha(\mathbb{R}_+, \mathbb{R}^m)$. Denoting the output of the well-posed system Σ^- corresponding to the initial value $(I - \Pi)x^0$ and the control $u(\cdot\,; x^0, v)$ by y^-, we have that

$$y^- = (C^-)_\Lambda \mathbf{T}^-(I - \Pi)x^0 + G^- u(\cdot\,; x^0, v)\,. \tag{4.30}$$

Since $u(0; x^0, v) = v(0)$, we may conclude that

$$\mathbf{T}^-_{t_0}\big(A^-(I-\Pi)x^0 + B^- u(0; x^0, v)\big) = (I - \Pi)\mathbf{T}_{t_0}\big(Ax^0 + Bv(0)\big) \in (I - \Pi)X = X^-\,.$$

An application of Proposition 2.1 to Σ^- now yields that y^- is continuous on $[t_0, \infty)$ and

$$\lim_{t\to\infty} \|y^-(t)e^{-\alpha t}\| = 0\,. \tag{4.31}$$

Since $v \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}^m)$, it is clear that $\int_{(k-1)\tau}^{(k+1)\tau} \|v(s)e^{-\alpha s}\|^2 ds$ converges to 0 as $k \to \infty$. Furthermore, it follows from (4.24) and (4.25) that $x_{k,\delta}e^{-\alpha k\tau}$ and $y_k e^{-\alpha k\tau}$ converge to 0 as $k \to \infty$. Consequently, the right-hand side of (4.27) converges to 0 as $k \to \infty$ and therefore,

$$\lim_{t\to\infty} \|x(t)e^{-\alpha t}\| = 0\,. \tag{4.32}$$

Finally, by (4.30) and Lemma 4.2 (applied to the well-posed system (2.2) with control $u = u(\cdot\,; x^0, v)$),

$$y(\cdot\,; x^0, v) = y^-(t) + C^+ \Pi x(\cdot\,; x^0, v)\,.$$

Therefore, $y(\cdot\,; x^0, v)$ is continuous on $[t_0, \infty)$, and, furthermore, we may conclude from (4.31) and (4.32) that $\lim_{t\to\infty} y(t; x^0, v)e^{-\alpha t} = 0$. □

Remark 4.5 (1) If in Theorem 4.4 assumption A4 is replaced by the stronger assumption that $\operatorname{im} R_{t_0} \supset X^+$ for some $t_0 > 0$ (that is, every state in X^+ is reachable from 0 in time t_0), then an inspection of the above proof shows that there exists $H \in L^2([0,\delta], \mathbb{R}^{m\times p})$ such that (i) $\omega_{\text{sd}} \le \omega(\mathbf{T}^-)$ and (ii) the conclusions of statements (2) and (3) of Theorem 4.4 remain true for every $\alpha \in (\omega(\mathbf{T}^-), 0]$.

(2) From a practical point of view, it is important that the "structure" of the stabilizing hold function H (the existence of which is guaranteed by Theorem 4.4) is as simple as possible. In this context, we define $S([0,\delta], \mathbb{R}^{m\times p})$ to be the space of $\mathbb{R}^{m\times p}$-valued step functions on $[0,\delta]$ and $CPL_c([0,\delta], \mathbb{R}^{m\times p})$ to be the space of $\mathbb{R}^{m\times p}$-valued continuous piecewise affine-linear functions with support contained in the open interval $(0,\delta)$. We recall that $S([0,\delta], \mathbb{R}^{m\times p})$ and $CPL_c([0,\delta], \mathbb{R}^{m\times p})$ are dense in $L^2([0,\delta], \mathbb{R}^{m\times p})$. Moreover, it is clear that $CPL_c([0,\delta], \mathbb{R}^{m\times p}) \subset W_c^{1,2}([0,\delta], \mathbb{R}^{m\times p})$. Therefore an inspection of the proof of Theorem 4.4 shows that, for every $\varepsilon \in (0, -\omega(\mathbf{T}^-)$, there exists

(i) $H \in S([0,\delta], \mathbb{R}^{m\times p})$ such that statements (1) and (2) of Theorem 4.4 hold;

(ii) $H \in CPL_c([0,\delta], \mathbb{R}^{m\times p})$ such that statements (1)–(3) of Theorem 4.4 hold. $\diamond$

It follows from [16, 28] that assumptions A1 and A2 are necessary conditions for the stabilization of (2.2) by any of the commonly used sampled-data feedback designs including the control law (3.1) (see [16, 28]). In this context the following proposition is of interest.

Proposition 4.6 *Let $\tau > \delta > 0$, $H \in L^2([0,\delta], \mathbb{R}^{m\times p})$ and $w \in L^2([0,\tau-\delta], \mathbb{R})$. Assume that* (3.4) *holds. If the sampled-data feedback system given by* (2.2) *and* (3.1) *is exponentially stable, then the following statements hold.*

(1) *The conditions* A1–A3 *are satisfied.*

(2) *Under the additional assumption that the semigroup* $\mathbf{T}$ *is analytic, the conditions* A1–A4 *are satisfied.*

Proof: Assume that the sampled-data feedback system given by (2.2) and (3.1) is exponentially stable. It follows from [28] that A1 and A2 hold. To prove statement (1), it remains to show that A3 is satisfied. In the following, let $C^\pm$ be given by (4.4), Δ_H by (4.11) and $K_H^\pm$ and $L_w^\pm$ by (4.12). Seeking a contradiction, suppose that A3 does not hold, that is, the pair (C^+, A^+) is not observable. By the Hautus criterion for observability it then follows that there exist $\lambda \in \sigma(A^+)$ and $z \in X^+$, $z \neq 0$, such that $A^+z = \lambda z$ and $C^+z = 0$. Consequently,

$$\mathbf{T}_s^+ z = e^{A^+ s} z = e^{\lambda s} z\,, \ \ \forall\, s \in \mathbb{R}; \quad C^+ z = 0\,.$$

and so,

$$C^+ L_w^+ z = C^+ \int_0^{\tau-\delta} w(s) \mathbf{T}_s^+ z\, ds = \left(\int_0^{\tau-\delta} w(s) e^{\lambda s}\, ds \right) C^+ z = 0\,.$$

Combining this with (4.13) yields

$$\begin{aligned}\Delta_H \begin{pmatrix} z \\ 0 \end{pmatrix} &= \begin{pmatrix} \mathbf{T}^+_\tau + K^+_H C^+ L^+_w & K^+_H C^- L^-_w \\ K^-_H C^+ L^+_w & \mathbf{T}^-_\tau + K^-_H C^- L^-_w \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{T}^+_\tau z \\ 0 \end{pmatrix} \\ &= e^{\lambda\tau} \begin{pmatrix} z \\ 0 \end{pmatrix}, \end{aligned} \tag{4.33}$$

Choose $z^0 \in X^+$ such that $z = \mathbf{T}^+_\delta z^0$. We consider the state-trajectory $x(\cdot\,; x^0, 0)$ of the sampled-data feedback system corresponding to the initial state

$$x^0 := \begin{pmatrix} z^0 \\ 0 \end{pmatrix}$$

and the control function $v = 0$. Then

$$x(k\tau + \delta; x^0, 0) = x_{k,\delta} = \Delta^k_H x_{0,\delta} = \Delta^k_H \mathbf{T}_\delta x^0 = \Delta^k_H \begin{pmatrix} \mathbf{T}^+_\delta z^0 \\ 0 \end{pmatrix} = \Delta^k_H \begin{pmatrix} z \\ 0 \end{pmatrix} = (e^{\lambda\tau})^k \begin{pmatrix} z \\ 0 \end{pmatrix},$$

where the last equation follows from (4.33). Since $\lambda \in \sigma(A^+)$, we have that $\mathrm{Re}\,\lambda \geq 0$, and so, $|e^{\lambda\tau}| \geq 1$. Moreover, $z \neq 0$, and therefore we may conclude that $x(k\tau + \delta; x^0, 0)$ does not converge to 0 as $k \to \infty$, yielding a contradiction to the exponential stability of the sampled-data feedback system.

To prove statement (2), assume that $\mathbf{T}$ is analytic. Define the operator $B^+_\tau : \mathbb{R}^p \to X^+$ by

$$B^+_\tau z = \int_0^\tau \mathbf{T}^+_{\tau - s} B^+ H_\tau(s) z \, ds \,, \quad \forall\, z \in \mathbb{R}^p \,,$$

where H_τ is defined in (3.2). It follows from [28] that the pair $(\mathbf{T}^+_\tau, B^+_\tau)$ is controllable. A routine argument based on the Hautus criterion for controllability then shows that the pair (A^+, B^+) is also controllable. Finally, an application of Proposition 2.1 in [17] yields that condition A4 is satisfied. □

5. Example

We will illustrate Theorem 4.4 with a standard model for an Euler-Bernoulli beam with structural damping. Let $z(\xi, t)$ be the lateral deflection of a beam, where $\xi \in [0, 1]$ and $t > 0$ denote space and time, respectively. We assume that the flexural rigidity EI and the mass density per unit length m are both constant. We normalize so that $EI/m = 1$. To model structural damping, we assume that the damping term arises from a lateral force on the right end of the beam. The Euler-Bernoulli beam with structural damping is decribed by the following 4th-order partial differential equation

$$z_{tt}(\xi, t) - 2\gamma z_{t\xi\xi}(\xi, t) + z_{\xi\xi\xi\xi}(\xi, t) = 0\,, \tag{5.1}$$

where $\gamma \in (0,1)$ denotes the damping constant. We assume that the beam is hinged at $\xi = 0$ and has a freely sliding clamped end at $\xi = 1$, with shear (also known as lateral) force $u(t)$ at $\xi = 1$:

$$z(0,t) = 0\,, \qquad z_{\xi\xi}(0,t) = 0\,, \tag{5.2a}$$

$$z_\xi(1,t) = 0\,, \quad -z_{\xi\xi\xi}(1,t) + 2\gamma z_{t\xi}(1,t) = u(t)\,. \tag{5.2b}$$

For this system we consider a standard observation, the velocity at $\xi = 1$:

$$y(t) = z_t(1,t)\,. \tag{5.3}$$

A number of mathematical models for unforced linear elastic systems with structural damping, including the model given by (5.1), were discussed in [2]. The controlled system given by (5.1) and (5.2) was perhaps first considered in [19], which in particular contains a detailed discussion of the damping term. For a discussion of the boundary conditions we refer the reader to [8, 19]. The applicability of our considerations below to other boundary conditions is briefly discussed in Remark 5.1 at the end of this section.

Our first aim is to represent the controlled and observed partial differential equation given by (5.1)-(5.3) as an abstract well-posed system of the form (2.2). We write $L^2(0,1)$ and $W^{q,2}(0,1)$, respectively, in place of the more cumbersome $L^2([0,1],\mathbb{R})$ and $W^{q,2}([0,1],\mathbb{R})$. Let $A_0 : \mathrm{dom}(A_0) \subset L^2(0,1) \to L^2(0,1)$ be given by

$$A_0 f = d^4 f/d\xi^4\,, \quad \mathrm{dom}(A_0) = \{f \in W^{4,2}(0,1) : f(0) = 0,\ f''(0) = 0,\ f'(1) = 0,\ f'''(1) = 0\}\,.$$

The operator A_0 is closed, bijective, self-adjoint, coercive and has compact resolvent. The numbers $(-\pi/2 + \pi k)^4$, where $k \in \mathbb{N}$, are the eigenvalues of A_0 with associated eigenvectors e_k given by

$$e_k(\xi) = \sqrt{2}\sin((-\pi/2 + \pi k)\xi)\,, \quad k \in \mathbb{N}\,.$$

The family $(e_k)_{k\in N}$ forms an orthonormal basis of $L^2(0,1)$. Moreover,

$$A_0^{1/2} f = -f''\,, \quad \mathrm{dom}(A_0^{1/2}) = \{f \in W^{2,2}(0,1) : f(0) = 0,\ f'(1) = 0\}\,.$$

Let $X := \mathrm{dom}(A_0^{1/2}) \times L^2(0,1)$. Endowed with the inner product

$$\langle (x_1,x_2)^T, (y_1,y_2)^T\rangle := \langle A_0^{1/2}x_1, A_0^{1/2}y_1\rangle_{L^2} + \langle x_2, y_2\rangle_{L^2}\,,$$

X becomes a Hilbert space. Defining the operator

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -2\gamma A_0^{1/2} \end{pmatrix}, \quad \mathrm{dom}(A) = \mathrm{dom}(A_0) \times \mathrm{dom}(A_0^{1/2})\,, \tag{5.4}$$

(5.1) and (5.2) (with $u = 0$) can be written in the form $\dot{x} = Ax$, where $x(t) = (z(\cdot\,,t), z_t(\cdot\,,t))^T$. The eigenvalues of A are given by

$$\lambda_{\pm k} = (-\gamma \pm i\sqrt{1-\gamma^2})(-\pi/2 + \pi k)^2\,, \quad k \in \mathbb{N}, \tag{5.5}$$

with associated eigenvectors

$$f_{\pm k} = \frac{\sqrt{2}}{1 - e^{\mp 2i\varphi}} \begin{pmatrix} e_k/\lambda_{\pm k} \\ e_k \end{pmatrix}, \quad k \in \mathbb{N},$$

where $\varphi := \arccos(-\gamma)$, so that $e^{i\varphi} = -\gamma + i\sqrt{1-\gamma^2}$. It is a routine exercise to check that $(f_{\pm k})_{k\in\mathbb{N}}$ is a Riesz basis for X. For $k \in \mathbb{N}$, the unit vectors

$$g_{\pm k} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e_k/\lambda_{\mp k} \\ e_k \end{pmatrix} \in \operatorname{dom}(A^*)$$

are eigenvectors of A^* with associated eigenvalues $\bar{\lambda}_{\pm k} = \lambda_{\mp k}$. Furthermore, introducing the set $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$, we have that

$$\langle f_j, g_l \rangle = \begin{cases} 0, & j \neq l \\ 1, & j = l, \end{cases}$$

i.e., $(f_j)_{j\in\mathbb{Z}^*}$ and $(g_j)_{j\in\mathbb{Z}^*}$ are biorthogonal. Consequently, A is a Riesz spectral operator (as defined in [5]) and thus can be represented in the form

$$Ax = \sum_{j\in\mathbb{Z}^*} \lambda_j \langle x, g_j \rangle f_j, \quad \forall\, x \in \operatorname{dom}(A) = \{x \in X : \sum_{j\in\mathbb{Z}^*} |\lambda_j|^2 |\langle x, g_j\rangle|^2 < \infty\};$$

moreover, $\sigma(A) = \{\lambda_j : j \in \mathbb{Z}^*\}$ and A generates a strongly continuous semigroup $\mathbf{T}$ given by

$$\mathbf{T}_t x = \sum_{j\in\mathbb{Z}^*} e^{\lambda_j t} \langle x, g_j \rangle f_j, \quad \forall\, x \in X,$$

see, e.g., Theorem 2.3.5 in [5]. It follows from the location of $\sigma(A)$ combined with a standard result in semigroup theory (see, e.g., Theorem 5.2 on p. 61 in [14]) that the semigroup $\mathbf{T}$ is analytic.

To write the controlled partial differential equation given by (5.1) and (5.2) in the abstract form (2.2a), we need to determine the input operator B. Moreover, in order to prove admissibility of B, we need to expand B in terms of the functions f_j. To this end it is useful to recall that the inner product on X has a continuous extension to $X_{-1} \times \operatorname{dom}(A^*)$, where $\operatorname{dom}(A^*)$ is endowed with the graph norm of A^*. More precisely, there exists a bounded non-degenerate sesquilinear form $[\,\cdot\,,\cdot\,]$ on $X_{-1} \times \operatorname{dom}(A^*)$ such that $[x_1, x_2] = \langle x_1, x_2 \rangle$ for all $(x_1, x_2) \in X \times \operatorname{dom}(A^*)$. The space X_{-1} may be identified with the dual of $\operatorname{dom}(A^*)$. Following the procedure outlined in [7], we obtain that

$$B = (0, \delta_1)^T, \tag{5.6}$$

where δ_1 denotes the Dirac distribution (or unit mass) with support at $\xi = 1$.¶ Consequently, the controlled partial differential equation given by (5.1) and (5.2) can be written in the form

¶ Strictly speaking, B is the operator in $\mathcal{B}(\mathbb{R}, X_{-1})$ given by $Bv = v(0, \delta_1)^T$, but it is convenient to identify B and $B1 = (0, \delta_1)^T$.

(2.2a) with $x(t) = (z(\cdot\,,t), z_t(\cdot\,,t))^T$ and the operators A and B given by (5.4) and (5.6), respectively.

In order to verify that B is admissible, we first note that $(f_j)_{j\in\mathbb{Z}^*}$ is a Schauder basis of X_{-1}. Indeed, for arbitrary $x \in X_{-1}$, we have that

$$x = AA^{-1}x = A\sum_{j\in\mathbb{Z}^*}\langle A^{-1}x, g_j\rangle f_j = \sum_{j\in\mathbb{Z}^*}\langle A^{-1}x, g_j\rangle \lambda_j f_j\,,$$

and it is clear that the coefficients $\langle A^{-1}x, g_j\rangle\lambda_j$ in the expansion on the right-hand side are unique. It is easy to see that $\langle A^{-1}x, g_j\rangle = [x, g_j]/\lambda_j$ for $x \in X_{-1}$ and $j \in \mathbb{Z}^*$. Thus, for arbitrary $x \in X_{-1}$,

$$x = \sum_{j\in\mathbb{Z}^*}[x, g_j] f_j\,.$$

Since $[B, g_j] = \sin(-\pi/2 + \pi|j|) = (-1)^{|j|+1}$, we obtain the following expansion for B in X_{-1},

$$B = \sum_{j\in\mathbb{Z}^*}(-1)^{|j|+1} f_j\,. \tag{5.7}$$

A standard application of the Carleson measure criterion, see [7, 31], yields that B is an admissible control operator for the semigroup $\mathbf{T}$. Since the observation (5.3) is described by the operator $C := B^*$, we conclude that C is an admissible observation operator. From (5.5) and (5.7), it is easy to see that for any $\varepsilon > 0$, $B \in \mathcal{B}(\mathbb{R}, X_{-(1/4+\varepsilon)})$ and $C \in \mathcal{B}(X_{1/4+\varepsilon}, \mathbb{R})$. Hence we can apply Proposition 2.2 to conclude that (A, B, C) are the generating operators of a regular well-posed system.

The semigroup generated by A has exponential growth constant $-\gamma\pi^2/4$, the real part of the right-most eigenvalue of A. Our aim is to construct a hold function H such that the sampled-data feedback control law (3.1) with weighting $w(s) \equiv 1$ achieves closed-loop exponential growth $\omega_{\mathrm{sd}} \le -9\gamma\pi^2/4$. To this end, fix $\beta \in (-9\gamma\pi^2/4, -\gamma\pi^2/4)$. Then assumption A1 holds, the subspace X^+ of X is spanned by $\{f_{-1}, f_1\}$ and $\sigma(A^+) = \sigma(A) \cap \overline{\mathbb{C}}_\beta = \{\lambda_1, \overline{\lambda}_1\}$. It is clear that $\omega(\mathbf{T}^-) = -9\gamma\pi^2/4 < 0$, showing that A2 holds. It is straightforward to show that $(f_j)_{j\in\mathbb{Z}^*}$ is a Schauder basis of X_1, so that $(f_j)_{j\in\mathbb{Z}^*}$ is a Schauder basis of each of the three spaces X_1, X and X_{-1}. With respect to this basis we have that

$$A = \mathrm{diag}_{j\in\mathbb{Z}^*}(\lambda_j)\,, \quad \mathbf{T}_t = \mathrm{diag}_{j\in\mathbb{Z}^*}(e^{\lambda_j t})\,, \quad B = (((-1)^{|j|+1})_{j\in\mathbb{Z}^*})^T\,, \quad C = (c_j)_{j\in\mathbb{Z}^*}\,,$$

where

$$c_k = 2(-1)^{k+1}/(1 - e^{-2i\varphi})\,, \quad c_{-k} = \overline{c}_k\,, \qquad \forall\, k \in \mathbb{N}.$$

Furthermore,

$$A^+ = \mathrm{diag}(\ldots, 0, 0, \overline{\lambda}_1, \lambda_1, 0, 0, \ldots)\,, \quad \mathbf{T}^+_t = \mathrm{diag}(\ldots, 0, 0, e^{\overline{\lambda}_1 t}, e^{\lambda_1 t}, 0, 0, \ldots)\,,$$

$$B^+ = (\ldots, 0, 0, 1, 1, 0, 0, \ldots)^T, \quad C^+ = (\ldots, 0, 0, \overline{c}_1, c_1, 0, 0, \ldots).$$

It follows in particular that assumption A3 is satisfied. Furthermore, since

$$\sum_{j \in \mathbb{Z}^*} \frac{1}{|\mathrm{Re}\, \lambda_j|} = 2 \sum_{j=1}^{\infty} \frac{1}{\gamma(-\pi/2 + \pi j)^2} < \infty,$$

Theorem 4.1 in [18] implies that, for any $t > 0$, there exists a unique sequence $(p_j)_{j \in \mathbb{Z}^*}$ in $L^2([0,t], \mathbb{C})$ such that

$$\int_0^t e^{\lambda_j s}\, \overline{p}_l(s)\, ds = \begin{cases} 0, & j \neq l \\ 1, & j = l, \end{cases} \tag{5.8}$$

that is, $(e^{\lambda_j \cdot})_{j \in \mathbb{Z}^*}$ and $(p_j)_{j \in \mathbb{Z}^*}$ are biorthogonal (note that $\overline{p}_j = p_{-j}$ for all $j \in \mathbb{Z}^*$). Consequently,

$$\mathrm{im}\, R_{t_0} \supset X^+, \quad \forall\, t_0 \in (0, \infty), \tag{5.9}$$

where R_{t_0} is the reachability operator given by (4.8). The inclusion (5.9) shows in particular that A4 holds for every $t_0 > 0$. Since $\sigma(A^+) = \{\lambda_1, \overline{\lambda}_1\}$, condition (4.9) is satisfied, provided that

$$\tau \neq \frac{4k}{\pi\sqrt{1 - \gamma^2}}, \quad \forall\, k \in \mathbb{N}. \tag{5.10}$$

Furthermore, since $w(s) \equiv 1$ and $\mathrm{Re}\, \lambda_1 \neq 0$, (4.10) holds for all $\tau > \delta > 0$, and therefore, we may conclude that assumption A5 is satisfied.

Choose $\tau > 0$ such that (5.10) holds and fix $\delta \in (0, \tau)$. It follows from Theorem 4.4 (combined with Remark 4.5 and (5.9)) that there exists $H \in L^2([0,\delta], \mathbb{R})$ such that the sampled-data feedback control law (3.1) with weighting $w(s) \equiv 1$ achieves closed-loop exponential growth $\omega_{\mathrm{sd}} \leq -9\gamma\pi^2/4$. We now use the construction in the proof of Theorem 4.4 to compute such a hold function H. To this end, note that the operator L_w^+ defined in (4.12) can be represented as

$$L_w^+ = \mathrm{diag}(\ldots, 0, 0, \overline{\lambda}, \lambda, 0, 0, \ldots), \quad \text{where } \lambda := (e^{\lambda_1(\tau - \delta)} - 1)/\lambda_1.$$

We first find $Q \in \mathcal{B}(\mathbb{C}, X^+)$ such that (4.14) holds. Since Q is of the form

$$Q = (\ldots, 0, 0, q_{-1}, q_1, 0, 0, \ldots)^T,$$

we do this by computing $q_{-1}, q_1 \in \mathbb{C}$ with the property that the two eigenvalues of the matrix

$$\begin{pmatrix} e^{\overline{\lambda}_1 \tau} & 0 \\ 0 & e^{\lambda_1 \tau} \end{pmatrix} + \begin{pmatrix} q_{-1} \\ q_1 \end{pmatrix} (\overline{c}_1, c_1) \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} e^{\overline{\lambda}_1 \tau} + q_{-1}\overline{c}_1\overline{\lambda} & q_{-1} c_1 \lambda \\ q_1 \overline{c}_1 \overline{\lambda} & e^{\lambda_1 \tau} + q_1 c_1 \lambda \end{pmatrix}$$

are both equal to 0. A routine calculation leads to

$$q_1 = \frac{-e^{2\lambda_1 \tau}}{(e^{\lambda_1 \tau} - e^{\overline{\lambda}_1 \tau}) c_1 \lambda} = \frac{e^{2\lambda_1 \tau} \lambda_1}{c_1 (e^{\lambda_1 \tau} - e^{\overline{\lambda}_1 \tau})(1 - e^{\lambda_1(\tau - \delta)})}, \quad q_{-1} = \overline{q}_1. \tag{5.11}$$

We now compute $h \in L^2([0,\delta],\mathbb{R})$ such that $R_\delta h = Q$, in which case (4.15) holds for every $\eta > 0$. Using (5.8) to solve $R_\delta h = Q$ for h, we find that

$$h(t) = q_1 \overline{p}_1(\delta - t) + \overline{q}_1 p_1(\delta - t), \quad \forall\, t \in [0,\delta].$$

The control law (3.1) with $H = -h$ (and with $w(s) \equiv 1$) achieves closed-loop exponential growth $\omega_{\mathrm{sd}} \leq -9\gamma\pi^2/4$. It is shown in [19, Section 4] how to construct the functions p_j.

Remark 5.1 If we kept the same form for the boundary control in (5.2), but modified the remaining boundary conditions to other "natural" boundary conditions, identified in [8, 19], we could go through the same process to find a "stabilizing" generalized hold function H. The only difference would be that the eigenvalues and eigenvectors will be given by asymptotic formulas – see, e.g., [15] for the formulas for such a beam with one end clamped and the other end free. On the other hand, if the control appears as a bending moment force (e.g. $z_{\xi\xi}(1,t) = u(t)$), then the resulting system will not be well posed, and our theory does not apply. $\diamond$

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