NON-LINEAR NEUMANN'S CONDITION FOR THE HEAT EQUATION: A PROBABILISTIC REPRESENTATION USING CATALYTIC SUPER-BROWNIAN MOTION

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Abstract. Let D be a bounded domain in $\mathbb{R}^d$ with smooth boundary ∂D. We give a probabilistic representation formula for the non-negative solution of the mixed Dirichlet non-linear Neumann boundary value problem (DNP)

$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u = \varphi \quad \text{on } F_2, \\ \partial_n u + 2u^2 = 0 \quad \text{on } F_1, \end{cases}$$

where (F_1, F_2) is a non-trivial partition of ∂D, φ is a non-negative, bounded and continuous function defined on F_2, and ∂_n denotes the outward normal derivative on the boundary of D.

To solve (DNP), we consider a catalytic super-Brownian motion with underlying motion a Brownian motion reflected on ∂D, killed when it reaches F_2 and catalysed by the set F_1, i.e. the branching rate is given by the local time of the paths on F_1. Then we prove that the log-Laplace transform of φ integrated with respect to the exit measure of the catalytic process on F_2, is a non-negative weak solution of (DNP).

In a second part we show that we still have a probabilistic representation formula if the Dirichlet condition on F_2 is replaced by a Neumann condition.

1. Introduction

Super-Brownian motion are measure valued stochastic processes. Since the works of Dynkin, Kuznetsov and Le Gall (see for example the monograph [8], and the references therein), the log-Laplace transform of the super-Brownian motion appears to be a powerful tool to study the non-linear PDE $\Delta u = u^2$ in a domain D. In particular, using a probabilistic representation formula, it is possible to describe all the non-negative solutions of this non linear PDE.

Super-Brownian motion represents a cloud of infinitesimal particles which evolve according to independent Brownian motions and are subject to a critical branching mechanism. Roughly speaking the spatial motion appears in the PDE through its infinitesimal generator, which in our case is the Laplacian Δu. The branching mechanism is responsible of the non-linear term, u^2 in our case. Since the early nineties, models appeared where the branching occurs only in a subset of the space called the catalytic set. Such models are called catalytic super-Brownian motion (see for example the survey [10]). Outside the catalytic set, the catalytic super-Brownian motion has a density w.r.t. the Lebesgue measure and this density solves

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the heat equation (with random boundary condition on the catalytic set). In particular, the non-linear phenomenon is located on the catalytic set.

In 1999, Dynkin asked if one could use a catalytic super-Brownian motion to give a probabilistic representation for solutions of

$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u = \varphi \quad \text{on } F_2, \\ \partial_n u + 2u^2 = 0 \quad \text{on } F_1, \end{cases} \tag{1}$$

where D is a smooth domain, (F_1, F_2) is a non-trivial partition of ∂D, and ∂_n denotes the outward normal derivative on the boundary of D. In this paper we, indeed, give such a representation formula. Instead of building the catalytic super-Brownian motion as a limit of branching particle systems, we use the construction introduced in [13] based on collision local time. From this construction, we derive a representation formula for non-negative solutions of (1) with Dirichlet or Neumann condition on F_2.

Let us describe more precisely the content of the paper. We consider a reflected Brownian motion in D, $B = (B_t, t \geq 0)$. In Section 2, we recall some facts on excursion theory from [12], introducing the family of σ-finite measures $(H^x, x \in F_1)$ which describe the "law" of the excursion of B in D started from $x \in F_1$. If L denotes the associated capacitary local time on F_1 (see Section 2.1 for a precise definition), we prove that L has a density, say ρ, with respect to the local time of B on F_1.

In Section 3, we consider under $\mathbb{P}^X_\nu$, $(X_t, t \geq 0)$ a superprocess started at the initial measure ν, with quadratic branching mechanism and underlying motion a process $\xi = (\xi_t, t \geq 0)$. The process ξ is, up to a random time change, the trace on F_1 of B before it hits F_2. More precisely, let $l^* = (l^*_t, t \geq 0)$ be the local time on F_1 of B before it hits F_2, $l^{*,-1}$ its right-continuous inverse, and set $\xi_t = (l^{*,-1}_t, B_{l^{*,-1}_t})$. Then we consider the total occupation measure $\Gamma(dr, dx) = \int_0^\infty ds\; X_s(dr, dx)$. From this, we introduce in Section 4.1 the random measure, Z^{Dir}, on F_2 defined for any non-negative function φ on F_2 by

$$\langle Z^{\text{Dir}}, \varphi \rangle = \iint \Gamma(dr, dx)\, \rho(x) H^x[\varphi(e_{\tau_2}) \mathbf{1}_{\{\tau_2 < \infty\}}],$$

where τ_2 is the hitting time of F_2 for the excursion e under H^x. Intuitively, the measure Z^{Dir} describes the death positions of infinitesimal particles released from the catalyst at time dr and position dx according to the random measure $\rho(x)\Gamma(dr, dx)$, performing independent Brownian excursions outside F_1 killed when they first reach F_2. Let us assume the measure ν is of the form $\delta_0 \otimes \eta$, where δ_0 is the Dirac mass at 0 and η a finite measure on F_1. The random measure Z^{Dir} corresponds to the so-called exit measure of $\bar{D}$ of the catalytic superprocess with catalytic set F_1, quadratic branching mechanism and initial measure η. If the initial measure is not supported by F_1, then one has to make some slight modification to get the exit measure (see Definition 4.1). Let $\mathbb{P}^Z_{\delta_x}$ denote the law of the exit measure when the initial measure is the Dirac mass at $x \in D$, δ_x.

In Sections 4.2 and 4.3, we study the properties of the log-Laplace transform, w, of the measure Z^{Dir}, defined by

$$w(x) = -\log \mathbb{E}^Z_{\delta_x}\left[\exp -\langle Z^{\text{Dir}}, \varphi \rangle\right].$$

In particular, we prove that w is a solution to (DNP) in a weak sense, see Definition 4.8 and Corollary 4.11.

In Section 5, using techniques developed in [2], we replace the Dirichlet condition on F_2 by a Neumann condition. In particular, we are able to give in Proposition 5.18 a similar representation formula for solutions to the PDE

$$\begin{cases} \Delta u = 0 \text{ in } D, \\ \partial_n u - 2\varphi = 0 \text{ on } F_2, \\ \partial_n u + 2u^2 = 0 \text{ on } F_1. \end{cases}$$

Eventually, we collect in the appendix some results on reflected Brownian motion in D.

2. Notations

If E is a polish space, let $\mathcal{B}(E)$ denote its Borel σ-field as well as the set of real measurable functions defined on E. Let $\mathcal{B}_+(E)$ (resp. $\mathcal{C}(E)$) be the subset of $\mathcal{B}(E)$ of non-negative (resp. continuous) functions. For $\varphi \in \mathcal{B}(E)$ bounded, we write $\| \varphi \|_\infty = \sup_{x\in E} |\varphi(x)|$. Let $\mathcal{M}_f(E)$ be the set of finite measures on E, endowed with the topology of weak convergence. For $\nu \in \mathcal{M}_f(E)$ and $\varphi \in \mathcal{B}(E)$ bounded or non-negative, we write $\langle \nu, \varphi \rangle$ for $\int_E \nu(dx)\, \varphi(x)$. If A is a Borel subset of $\mathbb{R}^d$, let $\bar{A}$ denote its closure.

Let D be a bounded domain, i.e. a connected open subset of $\mathbb{R}^d$, $d \geq 2$, with $\mathcal{C}^3$-boundary ∂D. Let $\mathcal{C}^p(D)$ (resp. $\mathcal{C}^p(\bar{D})$) be the set of continuous functions defined on D (resp. $\bar{D}$) of class $\mathcal{C}^p$. Let $(n_x, x \in \partial D)$ be the outward unit normal vector field and $\partial_n f(x) := \langle \nabla f, n_x \rangle$ denote the outward unit normal derivative on ∂D at x of a function $f \in \mathcal{C}^1(\bar{D})$. Let F_1 and F_2 two relatively open subsets of ∂D. We assume that F_1 and F_2 are non empty, disjoint and that $\bar{F}_1 \cup \bar{F}_2 = \partial D$. We also assume that the relative boundary of F_1 is equal to the relative boundary of F_2, and that it is either empty or a $\mathcal{C}^2$-manifold of codimension 2. We shall denote it by ∂F.

Let $B = (B_t, t \geq 0)$ be a reflecting Brownian motion in D, with normal reflection, started at $x \in \bar{D}$ under $\mathbb{P}_x$. Let $(\mathcal{F}_t, t \geq 0)$ be the filtration generated by B completed the usual way. See Section 6.1 in the appendix for some properties of B. We say a property holds a.s. if it holds $\mathbb{P}_x$-a.s. for all $x \in \bar{D}$. For $t > 0$, let $p_t(x, y)$ denote the transition density of B. There exists a unique continuous additive functional $\ell = (\ell_t, t \geq 0)$ of B called the local time on ∂D, such that for every $\varphi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$ and $x \in \bar{D}$,

$$\mathbb{E}_x \left[\int_0^\infty d\ell_s\, \varphi(s, B_s) \right] = \int_0^\infty ds \int_{\partial D} \sigma(dy)\, \varphi(s, y) p_s(x, y), \tag{2}$$

where σ is the surface measure on ∂D. In other words, σ is the Revuz-measure of the continuous additive functional ℓ. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$ and for $x \in \bar{D}$, let $d(x, \partial D) = \inf\{|x - y| : y \in \partial D\}$. The continuous additive functional ℓ can be constructed explicitly as

$$\ell_t = \lim_{n\to\infty} \frac{1}{\varepsilon_n} \int_0^t ds\, \mathbf{1}_{\{d(B_s, \partial D) \leq \varepsilon_n\}}, \tag{3}$$

where the limit exists for all $t \geq 0$, $\mathbb{P}_x$-a.s., for a positive sequence $(\varepsilon_n, n \geq 1)$ decreasing to zero which does not depend on $x \in \bar{D}$ (see Theorem 7.2 in [15]).

2.1. Local times on F_1.

A key-rôle is played by the exit systems, introduced by Maisonneuve in [12]. In particular, we shall need the last exit decomposition of B out of F_1.

For $i = 1, 2$, let $\tau_i = \inf\{t > 0 : B_t \in F_i\}$ be the first hitting time of F_i, with the convention that $\inf \emptyset = +\infty$. Notice the stopping times τ_i are finite a.s. (see Lemma 6.3). Let F_1^r be the set of regular points of F_1, i.e. $F_1^r := \{x \in \bar{D} : \mathbb{P}_x(\tau_1 = 0) = 1\}$. Since ∂D and ∂F are smooth, we have $F_1^r = \bar{F}_1$. We set

$$M := \{t > 0, B_t \in \bar{F}_1\}.$$

So, M is almost surely a closed subset of $(0,\infty)$. Furthermore the set M is optional and time homogeneous. Following [12], we set

$$\begin{aligned} R &:= \inf\{s > 0 : s \in M\}, \\ R_t &:= \inf\{s > 0 : s + t \in M\}, \\ G &:= \{t > 0 : R_{t-} = 0, R_t > 0\}. \end{aligned}$$

Notice that $R = \tau_1$ a.s. The set G, is the set of left endpoints in $(0,\infty)$ of the intervals contiguous to M. Notice, G is countable and $G \subset M$ a.s. Since F_1^r is regular for itself, we deduce that $G = \{t \in G, \mathbb{P}_{B_t}(R = 0) = 1\}$. Following [12], there exists a continuous additive functional $L = (L_t, t \geq 0)$ of B, such that for all $x \in \bar{D}$,

$$\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-t}\ dL_t\right] = \mathbb{E}_x[\mathrm{e}^{-\tau_1}].$$

The Revuz measure μ associated with L by: for any function $\varphi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$,

$$\mathbb{E}_x\Big[\int_0^\infty \varphi(s, B_s)\ dL_s\Big] = \int_0^\infty ds\ \int \mu(dz)\ p_s(x,z)\varphi(s,z),$$

is the 1-capacitary measure of the set F_1. Hence, we call the additive functional L the "capacitary local time" on F_1.

The capacitary local time L is called in [12] the local time on F_1. However, the so called local time on F_1, ℓ_s^1, is defined by $d\ell_s^1 = \mathbf{1}_{F_1}(B_s) d\ell_s$ (this correspond to ∂D replaced by F_1 in (3)). In fact L and ℓ^1 do not coincide in general. However, in our setting, the next Lemma implies that L is absolutely continuous with respect to ℓ^1. Recall that σ, the Revuz measure of ℓ, is also the surface measure on ∂D.

Lemma 2.1. *There exists $\rho \in \mathcal{B}_+(\mathbb{R}^d)$, such that*

$$\mu(dx) = \rho(x)\mathbf{1}_{F_1}(x)\sigma(dx).$$

The proof of this lemma is postponed to Section 6.4 of the appendix.

In the particular case, where $D \subseteq \mathbb{R}^d$, $d \geq 2$, is an open ball of radius r, and $F_1 = \partial D$, we deduce from Proposition 1.9 in [14] that

$$\mu(dy) = \frac{2\pi^{d/2} r^{d-2}}{\Gamma(\frac{d}{2} - 1)}\, \sigma(dy),$$

where Γ denotes the Gamma-function. Notice that the density of μ with respect to σ depends on the curvature of ∂D.

2.2. **Exit formula out of F_1 and applications.** Let δ be a cemetery point added to $\mathbb{R}^d$ and let $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d \cup \{\delta\})$ be the set of càdlàg functions defined on $\mathbb{R}_+$. For $s > 0$, let $i_s : \mathbb{D} \to \mathbb{D}$ be the family of translation operators defined by,

$$\begin{aligned} i_s(e)(t) &= e(t+s) \quad \text{for } 0 \leq t < R_s, \\ i_s(e)(t) &= \delta \quad \text{for } t \geq R_s. \end{aligned}$$

Moreover, let $(Q_t^1, t \geq 0)$ be the transition kernels of the reflected Brownian motion killed on F_1. We recall the exit formula.

Theorem 2.2 (Maisonneuve). *There exists a family of universally measurable σ-finite measures $(H^x, x \in F_1)$, on $(\mathbb{D}, \mathcal{B}(\mathbb{D}))$, such that for any non-negative predictable process $Z = (Z_s, s \geq 0)$, w.r.t. the filtration generated by B, and for any function $f \in \mathcal{B}_+(\mathbb{D})$, such that $f(\delta) = 0$, we have the exit formula:*

$$\mathbb{E}_x\left[\sum_{s\in G} Z_s f \circ i_s\right] = \mathbb{E}_x\left[\int_0^\infty Z_s H^{B_s}[f] dL_s\right].$$

Furthermore, H^x is strong Markov with respect to $(Q_t^1, t \geq 0)$.

For $i = 1, 2$ and $e \in \mathbb{D}$, let $\tau_i(e)$ be the first hitting time of F_i:

$$\tau_i(e) = \inf\{s > 0 : e(s) \in F_i\}.$$

We use the convention that $e_{+\infty} = \delta$ and we always write τ_i for $\tau_i(e)$ as well as e_s for $e(s)$, when there is no ambiguity. We now give particular applications, we shall use later. Let $\varphi \in \mathcal{B}_+(\mathbb{R}^d)$. For $\theta \geq 0$, set $f(e) = \mathrm{e}^{-\theta\tau_2} \varphi(e_{\tau_2}) \mathbf{1}_{\{\tau_2<\infty\}}$ and $Z_s(e) = \mathrm{e}^{-\theta s} \mathbf{1}_{\{\tau_2>s\}}$. From Theorem 2.2, we have

$$\begin{aligned} \mathbb{E}_x\Big[\int_0^{\tau_2} \mathrm{e}^{-\theta s} H^{B_s}\big[\mathrm{e}^{-\theta\tau_2} \varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}\big]\, dL_s\Big] &= \mathbb{E}_x\Big[\sum_{s\in G} \mathrm{e}^{-\theta s} \mathbf{1}_{\{\tau_2>s\}} f \circ i_s\Big] \\ &= \mathbb{E}_x[\mathbf{1}_{\{\tau_2>\tau_1\}} \mathrm{e}^{-\theta\tau_2} \varphi(B_{\tau_2})], \end{aligned} \tag{4}$$

since $\tau_2 \circ i_s + s = \tau_2$ on $\{\tau_2 > s\}$. With $\theta = 0$, we get, as $\tau_2 < \infty$ a.s.,

$$\mathbb{E}_x\Big[\int_0^{\tau_2} H^{B_s}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\, dL_s\Big] = \mathbb{E}_x[\mathbf{1}_{\{\tau_2>\tau_1\}}\varphi(B_{\tau_2})]. \tag{5}$$

With $Z_s = \mathrm{e}^{-\theta s}$ and f defined by $f(e) = \displaystyle\int_0^\infty d\ell_t\ \mathrm{e}^{-\theta t} \varphi(e_t)$, where $\ell = \ell(e)$ is defined by (3) with B replaced by e, we obtain

$$\mathbb{E}_x\Big[\int_0^\infty dL_s\ \mathrm{e}^{-\theta s} H^{B_s}\Big[\int_0^\infty d\ell_t\ \mathrm{e}^{-\theta t} \varphi(e_t)\Big]\Big] = \mathbb{E}_x\Big[\int_{\tau_1}^\infty d\ell_t\ \mathrm{e}^{-\theta t} \varphi(B_t)\mathbf{1}_{F_2}(B_t)\Big]. \tag{6}$$

Using a monotone class argument, Theorem 2.2 implies that for all predictable processes $Z = (Z_s, s \geq 0)$ and for any function $f \in \mathcal{B}_+(\mathbb{R}_+ \times \mathbb{D})$ such that $f(\cdot, \delta) = 0$,

$$\mathbb{E}_x\left[\sum_{s\in G} Z_s f(s, \cdot) \circ i_s\right] = \mathbb{E}_x\left[\int_0^\infty Z_s H^{B_s}[f(s, \cdot)] dL_s\right].$$

Setting $Z_s = \mathbf{1}_{\{\tau_2>s\}}$ and for fixed $t > 0$, $f(s, e) = \mathbf{1}_{\{0<t-s<\tau_2\}}\phi \circ i_{t-s}(e)$, where $\phi(e) := \mathbf{1}_{\{\tau_2<+\infty\}}\varphi(e_0)$, we deduce that

$$\mathbb{E}_x\Big[\int_0^{\tau_2} \mathbf{1}_{\{s<t\}} H^{B_s}[\mathbf{1}_{\{t-s<\tau_2<+\infty\}}\varphi(e_{t-s})]\, dL_s\Big] = \mathbb{E}_x[\mathbf{1}_{\{\tau_1<t<\tau_2\}}\varphi(B_t)]. \tag{7}$$

3. F_1-catalytic super-Brownian motion

In this section we contruct a catalytic super-Brownian motion in D with catalytic set F_1 and underlying motion a reflected Brownian motion B, killed when it first hits F_2. Even if the construction of this catalytic superprocess is not explicitly needed to solve the boundary value problem, it gives insights to the underlying ideas. Our construction is motivated by the methods developed in [13].

Recall that τ_2 denotes the first hitting time of F_2 by the reflected Brownian motion B. Consider the local time $\ell^* = (\ell^*_t, t \geq 0)$ on F_1 of B killed on F_2. It is defined by

$$d\ell^*_t = \mathbf{1}_{\{t<\tau_2\}} d\ell_t.$$

Let $\ell^{*,-1}$ denote the right continuous inverse of the continuous additive functional ℓ^*, i.e.

$$\ell^{*,-1}_t := \inf\{s \geq 0 : \ell^*_s > t\},$$

with the convention that $\inf \emptyset = +\infty$.

Let $E = (\mathbb{R}_+ \times F_1) \cup \{\delta\}$, where δ is a cemetery point. We define the E-valued time-homogeneous Markov process $\xi = (\xi_t, t \geq 0)$ by

$$\xi_t := \begin{cases} (\ell^{*,-1}_t, B \circ \ell^{*,-1}_t) & \text{if } \ell^{*,-1}_t < \infty, \\ \delta & \text{otherwise,} \end{cases}$$

and denote by $\mathbb{P}^\xi_{t,\hat{x}}$ its law started at $\hat{x} \in E$ at time $t \geq 0$. We also write $\mathbb{P}^\xi_{\hat{x}}$ for $\mathbb{P}^\xi_{0,\hat{x}}$. For $\nu \in \mathcal{M}_f(E)$ and $t \geq 0$, let $\mathbb{P}^X_{t,\nu}$ denote the law of the quadratic (non-catalytic) superprocess $X = (X_{s'}, s' \geq t)$ with spatial motion ξ, starting at ν at time t. We shall write $\mathbb{P}^X_\nu$ for $\mathbb{P}^X_{0,\nu}$. Recall that X is an $\mathcal{M}_f(E)$-valued Markov process. Its total occupation measure Γ, defined under $\mathbb{P}^X_{t,\nu}$, by

$$\Gamma(dr, dx) := \int_t^\infty ds'\, X_{s'}(dr, dx),$$

plays the key-rôle in the construction of the F_1-catalytic super-Brownian motion.

Lemma 3.1. *Let $\phi \in \mathcal{B}_+(E)$. The function v defined on E by*

$$\mathbb{E}^X_\nu \Big[\exp -\langle \Gamma, \phi \rangle \Big] = \exp -\langle \nu, v \rangle, \tag{8}$$

is a non-negative solution of the integral equation

$$v(s,x) + \mathbb{E}_x \Big[\int_0^{\tau_2} d\ell_r \; v^2(r+s, B_r) \Big] = \mathbb{E}_x \Big[\int_0^{\tau_2} d\ell_r \; \phi(r+s, B_r) \Big], \tag{9}$$

where $s \geq 0$ and $x \in F_1$. If $\phi(\cdot, x) = \tilde{\phi}(x)$ does not depend on time, we get that for $s \geq 0$, $v(s,x) = \tilde{v}(x)$, where $\tilde{v}$ is a non-negative solution on F_1 of

$$\tilde{v}(x) + \mathbb{E}_x \Big[\int_0^{\tau_2} d\ell_r \; \tilde{v}^2(B_r) \Big] = \mathbb{E}_x \Big[\int_0^{\tau_2} d\ell_r \; \tilde{\phi}(B_r) \Big]. \tag{10}$$

Remark 3.2. It is not clear that the integral equations (9) or (10) have a unique solution. From the previous Lemma, we can compute the first moment of Γ:

$$\mathbb{E}^X_\nu [\langle \Gamma, \phi \rangle] = \int \nu(ds, dx)\; \mathbb{E}_x \Big[\int_0^{\tau_2} d\ell_r \; \phi(r+s, B_r) \Big]. \tag{11}$$

Proof of Lemma 3.1. As a special case of the weighted occupation time formula (see e.g. [11], II.3) we have for all non-negative, bounded and measurable functions ϕ and h on $(\mathbb{R}_+ \times F_1) \cup \{\delta\}$ and $\mathbb{R}_+$ respectively, with $\phi(\delta) = 0$ and such that h has compact support,

$$\mathbb{E}^X_{t,\nu} \Big[\exp - \int_t^\infty ds'\, h(s') \langle X_{s'}, \phi \rangle \Big] = \exp -\langle \nu, v_t \rangle,$$

where v is the unique, non-negative solution of the integral equation for $t \geq 0$ and $\hat{x} \in E$,

$$v_t(\hat{x}) + \mathbb{E}^\xi_{t,\hat{x}} \Big[\int_t^\infty ds'\, v^2_{s'}(\xi_{s'}) \Big] = \mathbb{E}^\xi_{t,\hat{x}} \Big[\int_t^\infty ds'\, h(s') \phi(\xi_{s'}) \Big].$$

By substitution ($\ell_r^* = s'$), we have with $\hat{x} = (s,x) \in E$, that

$$v_t(s,x) + \mathbb{E}^{\xi}_{t,(s,x)}\Big[\int_{\ell_t^{*,-1}=s}^{\infty} d\ell_r^*\, v^2_{\ell_r^*}(r,B_r)\Big] = \mathbb{E}^{\xi}_{t,(s,x)}\Big[\int_{\ell_t^{*,-1}=s}^{\infty} d\ell_r^*\, h(\ell_r^*)\phi(r,B_r)\Big].$$

Using the time homogeneity of ξ and B, this last equation can be written as

$$v_t(s,x) + \mathbb{E}_x\Big[\int_0^{\infty} d\ell_r^*\, v^2_{l_r^*+t}(r+s,B_r)\Big] = \mathbb{E}_x\Big[\int_0^{\infty} d\ell_r^*\, h(\ell_r^*+t)\phi(r+s,B_r)\Big]. \tag{12}$$

Using the time homogeneity of the process X, we also get that

$$\mathbb{E}^X_{t,\nu}\Big[\exp - \int_t^{\infty} ds'\, h(s')\langle X_{s'},\phi\rangle\Big] = \mathbb{E}^X_{\nu}\Big[\exp - \int_0^{\infty} ds'\, h(s'+t)\langle X_{s'},\phi\rangle\Big].$$

In particular, the function v^T defined for $t \in [0,T]$ by the equation,

$$\mathbb{E}^X_{\nu}\Big[\exp - \int_0^{T-t} ds'\langle X_{s'},\phi\rangle\Big] = \exp - \langle \nu, v_t^T\rangle,$$

is the only non-negative solution of (12), with $h(t) = \mathbf{1}_{[0,T]}(t)$. By monotone convergence, letting T tend to $+\infty$, we get that v_t^T increases point-wise to a function v, independent of t, defined by (8), and v is a non-negative solution of

$$v(s,x) + \mathbb{E}_x\Big[\int_0^{\infty} d\ell_r^*\, v^2(r+s,B_r)\Big] = \mathbb{E}_x\Big[\int_0^{\infty} d\ell_r^*\, \phi(r+s,B_r)\Big].$$

Using the definition of ℓ^*, this last integral equation can be written as (9) where $s \geq 0$ and $x \in F_1$. Hence, the lemma holds for any bounded, non-negative function ϕ. By monotone convergence it also holds for any $\phi \in \mathcal{B}_+(E)$. If $\phi(\cdot,x) = \tilde{\phi}(x)$, we get from (12) that

$$v_t(s,x) + \mathbb{E}_x\Big[\int_0^{\infty} d\ell_r^*\, v^2_{\ell_r^*+t}(r+s,B_r)\Big] = \mathbb{E}_x\Big[\int_0^{\infty} d\ell_r^*\, h(\ell_r^*+t)\tilde{\phi}(B_r)\Big]. \tag{13}$$

In particular $v_t^{(s_0)}$ defined by $v_t^{(s_0)}(s,x) = v_t(s_0+s,x)$ also solves (13). By uniqueness, we obtain $v_t^{(s_0)} = v_t$ for any $s_0 \geq 0$. Hence, we have that the function $v_t(s,x)$ does not depend on s, i.e. $v_t(s,x) = \tilde{v}_t(x)$ for any $s \geq 0$. Following the arguments after (12), we deduce that v defined by (8) does not depend on time and solves (10). □

Let $\eta \in \mathcal{M}_f(\bar{D})$ be a finite measure on $\bar{D}$. Define $\nu_\eta \in \mathcal{M}_f(\mathbb{R}_+ \times F_1)$ to be the hitting distribution of $\mathbb{R}_+ \times F_1$ by (t,B_t), starting from $\delta_0 \otimes \eta$ and killed on F_2. For any $\psi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$, we have

$$\langle \nu_\eta, \psi\rangle = \int \eta(dx)\,\mathbb{E}_x[\mathbf{1}_{\{\tau_1<\tau_2\}}\psi(\tau_1,B_{\tau_1})]. \tag{14}$$

Recall the definition of the density ρ from Lemma 2.1. We define, under $\mathbb{P}^X_{\nu_\eta}$, the $\mathcal{M}_f(\bar{D})$-valued process $Z = (Z_t : t \geq 0)$ by $Z_0 := \eta$ and for $t > 0$,

$$\langle Z_t,\varphi\rangle = \langle \eta, Q_t\varphi\rangle + \iint \Gamma(dr,dx)\,\mathbf{1}_{\{r<t\}}\rho(x)H^x[\mathbf{1}_{\{t-r<\tau_2<\infty\}}\varphi(e_{t-r})], \tag{15}$$

where $\varphi \in \mathcal{B}_+(\bar{D})$ and Q_t denotes the semi group of the Brownian motion B killed when it first hits ∂D, i.e.

$$Q_t\varphi(x) = \mathbb{E}_x[\varphi(B_t)\mathbf{1}_{\{t<\tau_1\wedge\tau_2\}}].$$

We write $\mathbb{P}^Z_\eta$ the law of Z started at η. Let us give an intuitive interpretation of the measure valued process Z defined by (15). The measure Z_t describes a cloud of infinitesimal particles at time t. The first summand in (15) corresponds to those particles which have not reached the catalyst, F_1, at time t and which are distributed according to the starting measure η at time 0. The second summand corresponds to the particles which have reached the catalyst before time t and perform a branching process. Particles are then released from the catalyst at time dr and location dx according to the random measure $\rho(x)\Gamma(dr, dx)$, and then they perform excursions outside the catalyst. As all these excursions are independent, a law of large numbers effect lets us only observe an average over all excursions.

Let $C := \sup_{x\in\bar{D}} \mathbb{E}_x[\ell_{\tau_2}] < \infty$ (see Lemma 6.3). The following proposition characterizes the finite dimensional marginals of the process Z in terms of their Laplace transform.

Proposition 3.3. *Let $0 < t_1 \leq \ldots \leq t_n$ and $\varphi_1, \ldots, \varphi_n$ elements of $\mathcal{B}_+(\bar{D})$, such that we have $2C \sum_{i=1}^n \|\varphi_i\|_\infty < 1$. Then,*

$$\mathbb{E}_\eta^Z \left[\exp - \sum_{i=1}^n \langle Z_{t_i}, \varphi_i\rangle\right] = \exp -\langle \eta, w(0,\cdot)\rangle,$$

where $(w(s,x), s \geq 0, x \in \bar{D})$ is the unique non-negative solution of

$$w(s,x) + \mathbb{E}_x\left[\int_0^{\tau_2} d\ell_r\, w^2(r+s, B_s)\right] = \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}} \mathbb{E}_x[\mathbf{1}_{\{t_i-s<\tau_2\}}\varphi_i(B_{t_i-s})]. \tag{16}$$

Remark 3.4. From this proposition, it is easy to check that Z is a time-homogeneous Markov process. However, notice that the process Z is not adapted to the filtration generated by the superprocess X.

Proof of Prop. 3.3. Using $\phi(s,x) := \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\rho(x)H^x[\mathbf{1}_{\{t_i-s<\tau_2<\infty\}}\varphi_i(e_{t_i-s})]$, we have

$$\mathbb{E}_\eta^Z \left[\exp - \sum_{i=1}^n \langle Z_{t_i}, \varphi_i\rangle\right] = \exp -\Big(\sum_{i=1}^n \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \nu_\eta, \tilde{w}\rangle\Big), \tag{17}$$

where, thanks to Lemma 3.1, $(\tilde{w}(s,x), s \geq 0, x \in F_1)$ is a non-negative solution of

$$\tilde{w}(s,x) + \mathbb{E}_x\left[\int_0^{\tau_2} d\ell_r\, \tilde{w}^2(r+s, B_r)\right] = \mathbb{E}_x\left[\int_0^{\tau_2} d\ell_r\, \phi(r+s, B_r)\right]. \tag{18}$$

By Lemma 2.1, we have a.s. for all $t \geq 0$,

$$dL_t = \rho(B_t)\mathbf{1}_{F_1}(B_t)d\ell_t. \tag{19}$$

Using the definition of ϕ and the exit-formula (7) we obtain

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^{\tau_2} d\ell_r\, \phi(r+s, B_r)\right] &= \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r \sum_{i=1}^n \mathbf{1}_{\{r+s<t_i\}}\rho(B_r)H^{B_r}[\mathbf{1}_{\{t_i-s-r<\tau_2<\infty\}}\varphi_i(e_{t_i-s-r})]\Big] \\
&= \mathbb{E}_x\Big[\int_0^{\tau_2} dL_r \sum_{i=1}^n \mathbf{1}_{\{r+s<t_i\}}H^{B_s}[\mathbf{1}_{\{t_i-s-r<\tau_2<\infty\}}\varphi_i(e_{t_i-s-r})]\Big] \\
&= \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\mathbb{E}_x[\mathbf{1}_{\{\tau_1<t_i-s<\tau_2\}}\varphi_i(B_{t_i-s})]. \qquad (20)
\end{aligned}$$

We define for $s \geq 0$, $x \in \bar{D}$,

$$w(s,x) := \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}Q_{t_i-s}\varphi_i(x) + \mathbb{E}_x\left[\mathbf{1}_{\{\tau_1<\tau_2\}}\tilde{w}(s+\tau_1, B_{\tau_1})\right].$$

Using the strong Markov property of B at time τ_1, (18) and (20), one check that w satisfies (16). Notice, that by construction, we have

$$\begin{aligned}\langle \eta, w(0,\cdot)\rangle &= \sum_{i=1}^{n} \langle \eta, Q_{t_i}\varphi_i\rangle + \int \eta(dx)\, \mathbb{E}_x\left[\mathbf{1}_{\{\tau_1<\tau_2\}}\tilde{w}(\tau_1, B_{\tau_1})\right]\\ &= \sum_{i=1}^{n} \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \nu_\eta, \tilde{w}\rangle.\end{aligned}$$

Thanks to (17), this implies the first equality of the Lemma. To prove the uniqueness, let w_1 and w_2 be non-negative solutions of equation (16). Then both, w_1 and w_2 are bounded by $\sum_{i=1}^{n} \|\varphi_i\|_\infty$. We have,

$$w_1(s,x) - w_2(s,x) = -\mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\, (w_1^2(s+r, B_r) - w_2^2(s+r, B_s))\Big].$$

Hence, we can deduce

$$\begin{aligned}\|w_1 - w_2\|_\infty &\le \sup_{x\in\bar{D}, s\ge 0} \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\, \big|w_1^2(s+r,B_r) - w_2^2(s+r,B_s)\big|\Big]\\ &\le 2C\sum_{i=1}^{\infty} \|\varphi_i\|_\infty\, \|w_1 - w_2\|_\infty.\end{aligned}$$

As $2C\sum_{i=1}^{\infty}\|\varphi_i\|_\infty < 1$, we get that $w_1 = w_2$ and (16) has a unique non-negative solution. □

4. Dirichlet condition on F_2

4.1. **The exit measure Z^{Dir}.** In this section, we define a measure Z^{Dir} on $\bar{F}_2$ and characterize it in terms of its Laplace functionals. According to Section 3, the measure Z^{Dir} can be seen as the exit-measure of the F_1-catalytic super-Brownian motion on F_2. Intuitively, Z^{Dir} describes the spatial distribution of the generic particles of a F_1-catalytic super-Brownian motion in D "frozen" when they first hit F_2.

Let us keep the same notation as in Section 3. In particular, for $\eta \in \mathcal{M}_f(\bar{D})$, the measure Γ is the total occupation measure of the (non-catalytic) superprocess X starting at $X_0 = \nu_\eta$ (see (14) for the definition of ν_η).

Definition 4.1. *We define the random measure Z^{Dir} on $\bar{F}_2$ by: for all $\varphi \in \mathcal{B}_+(\bar{F}_2)$,*

$$\langle Z^{\mathrm{Dir}}, \varphi\rangle = \langle \eta, Q^1(\varphi)\rangle + \iint \Gamma(dr,dx)\,\rho(x) H^x[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}],$$

with $Q^1(\varphi)(r,x) = \mathbb{E}_x[\varphi(B_{\tau_2})\mathbf{1}_{\{\tau_2\le\tau_1\}}]$. We call the measure Z^{Dir} the exit measure of the F_1-catalytic super-Brownian motion on F_2, and write $\mathbb{P}^Z_\eta$ for its law.

Remark 4.2. To check that Z^{Dir} is finite, we compute its first moment. Thanks to (11),

$$\begin{aligned}
\mathbb{E}^Z_\eta[\langle Z^{\mathrm{Dir}},\varphi\rangle] &= \langle \eta, Q^1(\varphi)\rangle + \mathbb{E}^X_{\nu_\eta}\Big[\iint \Gamma(dr,dx)\ \rho(x)H^x[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\\
&= \langle \eta, Q^1(\varphi)\rangle + \int \nu_\eta(ds,dx)\ \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\ \rho(B_r)H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\\
&= \langle \eta, Q^1(\varphi)\rangle + \int \eta(dx)\ \mathbb{E}_x\Big[\mathbf{1}_{\{\tau_1<\tau_2\}}\mathbb{E}_{B_{\tau_1}}\Big[\int_0^{\tau_2} dL_r\ H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\Big]\\
&= \int \eta(dx)\ \mathbb{E}_x[\varphi(B_{\tau_2})\mathbf{1}_{\{\tau_2\leq\tau_1\}}] + \int \eta(dx)\ \mathbb{E}_x\Big[\int_0^{\tau_2} dL_r\ H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\\
&= \int \eta(dx)\ \mathbb{E}_x[\varphi(B_{\tau_2})\mathbf{1}_{\{\tau_2\leq\tau_1\}}] + \int \eta(dx)\ \mathbb{E}_x\Big[\mathbf{1}_{\{\tau_1<\tau_2\}}\varphi(B_{\tau_2})]\Big]\\
&= \int \eta(dx)\ \mathbb{E}_x[\varphi(B_{\tau_2})],
\end{aligned}$$

where we used Lemma 2.1 (or (19)) and the definition of ν_η, (14), for the third equality, the strong Markov property for B for the fourth and (5) for the fifth.

Recall the definition of the constant $C=\sup_{x\in\bar{D}}\mathbb{E}_x[\ell_{\tau_2}]<\infty$.

Lemma 4.3. *For any $\varphi\in\mathcal{B}_+(\bar{F}_2)$,*

$$\mathbb{E}^Z_\eta\big[\exp-\langle Z^{\mathrm{Dir}},\varphi\rangle\big] \;=\; \exp-\langle\eta,w\rangle,$$

where $(w(x),x\in\bar{D})$ is a non-negative solution of the integral equation on $\bar{D}$ given by

$$w(x)+\mathbb{E}_x\left[\int_0^{\tau_2} d\ell_r\, w^2(B_r)\right]=\mathbb{E}_x[\varphi(B_{\tau_2})]. \tag{21}$$

If we additionally assume that $2C\|\varphi\|_\infty<1$, then the non-negative solution w is also unique.

Proof. Using $\phi(x,r):=\rho(x)\,H^x[\varphi(e_{\tau_2})]$, we can compute

$$\begin{aligned}
\mathbb{E}^Z_\eta\left[\exp-\langle Z^{\mathrm{Dir}},\varphi\rangle\right] &=\mathbb{E}^X_{\nu_\eta}\left[\exp-(\langle\eta,Q^1(\varphi)\rangle+\langle\Gamma,\phi\rangle)\right]\\
&=\exp-(\langle\eta,Q^1(\varphi)\rangle+\langle\nu_\eta,v\rangle),
\end{aligned}$$

where, thanks to the second part of Lemma 3.1, the function v is a non-negative solution on F_1 of the integral equation,

$$\begin{aligned}
v(x)+\mathbb{E}_x\left[\int_0^{\tau_2} d\ell_r\, v^2(B_r)\right] &=\mathbb{E}_x\left[\int_0^{\tau_2} d\ell_r\,\rho(B_r)H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\right]\\
&=\mathbb{E}_x\left[\int_0^{\tau_2} dL_r\,H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\right]\\
&=\mathbb{E}_x[\mathbf{1}_{\{\tau_1<\tau_2\}}\varphi(B_{\tau_2})],
\end{aligned} \tag{22}$$

where we used (19) for the second equality and (5) for the last equality. We define for $x\in\bar{D}$,

$$w(x):=Q^1(\varphi)(x)+\mathbb{E}_x[\mathbf{1}_{\{\tau_1<\tau_2\}}v(B_{\tau_1})].$$

Notice that $\langle\eta,w\rangle=\langle\eta,Q^1(\varphi)\rangle+\langle\nu_\eta,v\rangle$. In particular, we have

$$\mathbb{E}^Z_\eta\left[\exp-\langle Z^{\mathrm{Dir}},\varphi\rangle\right]=\exp-\langle\eta,w\rangle.$$

Using the strong Markov property of B and (22), we get that w is a non-negative solution of (21). The proof of uniqueness is similar to the one for Proposition 3.3. ∎

4.2. **Properties of the dual function** w. Fix $\varphi \in \mathcal{B}_+(\bar{F}_2)$ continuous (and of course bounded). Let w be the non-negative function defined on $\bar{D}$ by

$$w(x) := -\log \mathbb{E}^Z_{\delta_x}\left[\exp -\langle Z^{\mathrm{Dir}}, \varphi\rangle\right], \tag{23}$$

where δ_x is the Dirac mass at x. Notice that w is bounded, as (21) implies $\|w\|_\infty \le \|\varphi\|_\infty$. In this section, we establish some properties of the function w. We use techniques similar to those developed in [1].

Lemma 4.4. *Let $x \in \bar{D}$, and T be a finite $\mathcal{F}_t$-stopping time. Then, we have*

$$\mathbb{E}_x[w(B_{\tau_2\wedge T})] - w(x) = \mathbb{E}_x\left[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\right].$$

Proof. Applying the strong Markov property at time $\tau_2 \wedge T$, the integral equation for w yields

$$\begin{aligned}
\mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_s\, w^2(B_s)\Big] =& \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x\Big[\int_{\tau_2\wedge T}^{\tau_2} d\ell_s\, w^2(B_s)\Big]\\
=& \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x\Big[\mathbb{E}_{B_{\tau_2\wedge T}}\Big[\int_0^{\tau_2} d\ell_s\, w^2(B_s)\Big]\Big]\\
=& \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x\left[\mathbb{E}_{B_{\tau_2\wedge T}}[\varphi(B_{\tau_2})] - w(B_{\tau_2\wedge T})\right]\\
=& \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x[\varphi(B_{\tau_2})] - \mathbb{E}_x[w(B_{\tau_2\wedge T})].
\end{aligned}$$

On the other hand, the integral equation for w also gives,

$$\mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_s\, w^2(B_s)\Big] = \mathbb{E}_x[\varphi(B_{\tau_2})] - w(x),$$

which completes the proof of the lemma. ◻

Using Lemma 4.4, we can easily show that the function w is harmonic in D.

Lemma 4.5. *The function w is in $\mathcal{C}^2(D)$ and solves $\Delta u = 0$ in D.*

Proof. Let $x \in D$. As D is open, we may find an open ball around x denoted by O_x such that $O_x \subset D$. Let $T := \inf\{t > 0 : B_t \in \partial O_x\}$ be the first hitting time of the boundary, ∂O_x, of O_x. As $T < \tau_2$ a.s., Lemma 4.4 gives that $w(x) = \mathbb{E}_x[w(B_T)]$. Hence, w is harmonic in D and therefore belongs to $\mathcal{C}^2(D)$. ◻

For $A, B \subseteq \mathbb{R}^d$ let $d(A,B) := \inf\{|a-b| : a \in A, b \in B\}$ denote the Euclidean distance between the sets A and B.

Lemma 4.6. *The function w is continuous on $\bar{D}$.*

Remark 4.7. In particular, the process $M^{\mathrm{Dir}} = (M^{\mathrm{Dir}}_t, t \ge 0)$ defined by

$$M^{\mathrm{Dir}}_t := w(B_{t\wedge\tau_2}) - w(B_0) - \int_0^{t\wedge\tau_2} d\ell_s\, w^2(B_s)$$

is a continuous $\mathcal{F}_t$-martingale.

Proof of Lemma 4.6. As we already know that w is continuous in D, it remains to deal with ∂D.

First case. Let $y \in \bar{F}_2$. As w is bounded, say by M, we have

$$\mathbb{E}_x\left[\int_0^{\tau_2} d\ell_s w^2(B_s)\right] \leq M^2 \mathbb{E}_x[\ell_{\tau_2}],$$

which converges to 0 as $x \to y$ by Lemma 6.3. As φ is continuous, we have by Lemma 6.5,

$$\lim_{x\to y} \mathbb{E}_x[\varphi(B_{\tau_2})] = \varphi(y).$$

Hence by (21) w is continuous at y.

Second case. Let $y \in F_1$. As F_1 is relatively open there exists an open ball O_y around y such that $O_y \cap F_2 = \emptyset$ and $d(O_y \cap \bar{D}, F_2) > 0$. By Lemma 4.4 applied to the deterministic time $T = t > 0$, we have for all $x \in O_y \cap \bar{D}$,

$$\begin{aligned} w(x) &= \mathbb{E}_x[w(B_{\tau_2})\mathbf{1}_{\{\tau_2 \leq t\}}] + \mathbb{E}_x[w(B_t)\mathbf{1}_{\{\tau_2 > t\}}] - \mathbb{E}_x\Big[\int_0^{\tau_2 \wedge t} d\ell_s w^2(B_s)\Big] \\ &= \mathbb{E}_x[w(B_{\tau_2})\mathbf{1}_{\{\tau_2 \leq t\}}] + \mathbb{E}_x[w(B_t)] - \mathbb{E}_x[w(B_t)\mathbf{1}_{\{\tau_2 \leq t\}}] - \mathbb{E}_x\Big[\int_0^{\tau_2 \wedge t} d\ell_s w^2(B_s)\Big]. \end{aligned}$$

Now, for a fixed $t > 0$, the function $x \mapsto \mathbb{E}_x[w(B_t)]$ is continuous, and all other expressions in the right-hand-side of the last equation converge to zero, uniformly in $x \in O_y \cap \bar{D}$, as $t \downarrow 0$ using Lemma 6.8 and (40), with $n = 1$, for the last term. □

4.3. **Weak solutions of the Neumann Problem.** Notice that, by Lemmas 4.5 and 4.6, we have $w \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$ and $\Delta w = 0$ in D. Moreover, (21) implies $w = \varphi$ on F_2. Assume additionally that $w \in \mathcal{C}^1(\bar{D})$. Then it is easy to show using Lemma 4.4 that $\partial_n w + 2w^2 = 0$ on F_1. Hence, w is a strong solution of the mixed Dirichlet non-linear Neumann boundary value problem,

$$\begin{cases} \Delta u = 0 \text{ in } D, \\ u = \varphi \text{ on } F_2, \\ \partial_n u + 2\,u^2 = 0 \text{ on } F_1. \end{cases} \tag{24}$$

It is easy to check that strong solutions are also unique (see [17] for the details). In general, it is not clear that w belongs to $\mathcal{C}^1(\bar{D})$. However, w is a solution (24) in a weak sense.

Let us define a set of test functions by

$$\mathcal{S}_1 := \Big\{\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D});\ \Delta\phi \text{ is bounded in } D,\ \partial_n \phi = 0 \text{ on } F_1, \phi = 0 \text{ on } F_2\Big\}.$$

Recall $\varphi \in \mathcal{B}_+(\bar{F}_2)$ is assumed to be a continuous function.

Definition 4.8. *A bounded function $u \in \mathcal{B}_+(\bar{D})$ is called a weak solution of the mixed Dirichlet non-linear Neumann boundary value problem (DNP) given by (24) if $u \in \mathcal{C}(\bar{D})$ and for every test function $\phi \in \mathcal{S}_1$,*

$$\int_D dx\, u(x)\, \Delta\phi(x) = \int_{F_2} \sigma(dy)\, \partial_n \phi(y) \varphi(y) + 2 \int_{F_1} \sigma(dy)\, \phi(y) u^2(y). \tag{25}$$

Remark 4.9. Notice that it follows directly by Greens second identity, that any strong solution is also a weak solution of the (DNP) (24). This indeed motivates Definition 4.8.

Proposition 4.10. *A non-negative function $u \in \mathcal{C}(\bar{D})$ such that $u = \varphi$ on F_2, is a weak solution of (DNP) (24), if the process $M = (M_t, t \geq 0)$ defined on $[0, +\infty)$ by*

$$M_t := u(B_{t\wedge\tau_2}) - u(B_0) - \int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)$$

is a continuous $\mathcal{F}_t$-martingale.

Proof. Assume that $u \in \mathcal{C}(\bar{D})$ is non-negative and $M = (M_t, t \geq 0)$, as defined in the statement of the proposition, is a continuous $\mathcal{F}_t$-martingale. We have,

$$\mathbb{E}_x\big[u(B_{t\wedge\tau_2}) - u(x)\big] = \mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big].$$

Rewriting this equation, we obtain

$$\mathbb{E}_x\big[u(B_t) - u(x)\big] = \mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big] - \mathbb{E}_x\big[\mathbf{1}_{\{\tau_2<t\}}\big(u(B_{\tau_2}) - u(B_t)\big)\big].$$

Multiplying with $\phi \in \mathcal{S}_1$ and integrating over D yields,

$$\begin{aligned}\int_D dx\, \phi(x)\mathbb{E}_x\big[u(B_t) - u(x)\big]\\ = \int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big] - \int_D dx\, \phi(x)\mathbb{E}_x\big[\mathbf{1}_{\{\tau_2<t\}}\big(u(B_{\tau_2}) - u(B_t)\big)\big].\end{aligned} \tag{26}$$

Thanks to the symmetry of the reflecting Brownian motion, we can rewrite the left hand side:

$$\int_D dx\, \phi(x)\mathbb{E}_x\big[u(B_t) - u(x)\big] = \int_D dx\, u(x)\mathbb{E}_x\big[\phi(B_t) - \phi(x)\big].$$

By Lemma 6.1, the process $Y = (Y_t, t \geq 0)$ defined by

$$Y_t := \phi(B_t) - \phi(B_0) - \tfrac{1}{2}\int_0^t ds\, \Delta\phi(B_s) + \tfrac{1}{2}\int_0^t d\ell_s \partial_n\phi(B_s)$$

is a continuous $\mathcal{F}_t$-martingale. Hence, as $0 = \mathbb{E}_x[Y_0] = \mathbb{E}_x[Y_t]$, we have

$$\mathbb{E}_x\big[\phi(B_t) - \phi(x)\big] = \tfrac{1}{2}\mathbb{E}_x\Big[\int_0^t ds\, \Delta\phi(B_s)\Big] - \tfrac{1}{2}\mathbb{E}_x\Big[\int_0^t d\ell_s\, \partial_n\phi(B_s)\Big].$$

Therefore, we can rewrite (26) to

$$\begin{aligned}\frac{1}{t}\int_D dx\, u(x)\mathbb{E}_x\Big[\int_0^t ds\, \Delta\phi(B_s)\Big] - \frac{1}{t}\int_D dx\, u(x)\mathbb{E}_x\Big[\int_0^t d\ell_s\, \partial_n\phi(B_s)\Big]\\ = 2\,\frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big] - 2\,\frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\big[\mathbf{1}_{\{\tau_2<t\}}\big(u(B_{\tau_2}) - u(B_t)\big)\big],\end{aligned}$$

where we also divided by $t > 0$. By Lemma 6.6, 6.7, 6.9 and 6.10, and letting $t \downarrow 0$, we see that

$$\int_D dx\, u(x)\, \Delta\phi(x) - \int_{\partial D} \sigma(dy)\, u(y)\partial_n\phi(y) = 2\int_{F_1} \sigma(dy)\, \phi(y)u^2(y).$$

As $u = \varphi$ on F_2, $\partial_n\phi = 0$ on F_1, we get that u is a weak solution of (DNP) given by (24). $\square$

Corollary 4.11. *The function w given by (23) is a non-negative weak solution of (DNP) (24).*

Proof. That follows directly from Remark 4.7 and Proposition 4.10. $\square$

5. Neumann condition on F_2

In this section, we give a probabilistic representation formula for the boundary value problem (1), where the Dirichlet condition on F_2 is replaced by a Neumann condition. We first consider the approximating problem

$$\begin{cases} \Delta u = 2\theta u \text{ in } D \\ \partial_n u - 2\varphi = 0 \text{ on } F_2 \\ \partial_n u + 2u^2 = 0 \text{ on } F_1 \end{cases}$$

for $\theta > 0$, and then we let θ tend to zero. Similar techniques to those we use can already be found in [2].

5.1. The measure $Z_\theta^{\mathbf{Neu}}$ and its dual.

We use the same notation as in the last sections. For $i = 1, 2$, let ℓ^i denote the local time of B on F_i, i.e.

$$d\ell_r^i = \mathbf{1}_{F_i}(B_r)\, d\ell_r.$$

Let $\mathcal{N}$ be a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx\, dt$, independent of the reflecting Brownian motion B. Denote by (x_i, t_i) the atoms of this measure and set, for $R_0 \in [0, +\infty]$ given,

$$R_t := R_0 \wedge \inf\{x_i \,:\, t_i \leq t\},$$

with the convention $\inf \emptyset = +\infty$. The Markov process $R = (R_t, t \geq 0)$ is a càdlàg decreasing $\mathbb{R}_+ \cup \{\infty\}$ valued process. Moreover, for every $t \geq 0$, $\theta \geq 0$, we have

$$\mathbb{P}(R_t > \theta | R_0) = \mathbf{1}_{\{R_0 > \theta\}} \mathbb{P}(\mathcal{N}([0, \theta] \times [0, t]) = 0) = \mathbf{1}_{\{R_0 > \theta\}}\, \mathrm{e}^{-\theta t}. \tag{27}$$

Let $E' := \mathbb{R}_+ \times F_1 \times [0, \infty]$. In the spirit of Section 3, we define the E'-valued time-homogeneous Markov process $(\zeta_t, t \geq 0)$ by

$$\zeta_t := (\ell_t^{1,-1}, B \circ \ell_t^{1,-1}, R \circ \ell_t^{1,-1})$$

and denote by $\mathbb{P}^\zeta_{t,\hat{x}}$ its law started at $\hat{x} \in E'$ at time $t \geq 0$.

For $\nu \in \mathcal{M}_f(E')$ and $t \geq 0$, let $\mathbb{P}^{X'}_{t,\nu}$ denote the law of the quadratic (non-catalytic) superprocess $X' = (X'_{s'}, s' \geq t)$ with spatial motion ζ, starting at ν at time t. We shall write $\mathbb{P}^{X'}_\nu$ for $\mathbb{P}^{X'}_{0,\nu}$. The total occupation measure Γ^{Neu} of the superprocess X' is defined under $\mathbb{P}^{X'}_{t,\nu}$ by

$$\Gamma^{\mathrm{Neu}}(dr, dx, dk) := \int_t^\infty ds'\, X'_{s'}(dr, dx, dk).$$

Lemma 5.1. *Let $\theta > 0$ and $\tilde{\phi} \in \mathcal{B}_+(E')$ be of the form $\tilde{\phi}(r, x, k) = \mathbf{1}_{\{k > \theta\}} \phi(x)$, where $\phi \in \mathcal{B}_+(F_1)$ is bounded. Then the function $\tilde{v}$ defined on E' by*

$$\mathbb{E}^{X'}_\nu \left[\exp -\langle \Gamma^{\mathrm{Neu}}, \tilde{\phi} \rangle \right] = \exp -\langle \nu, \tilde{v} \rangle,$$

is of the form $\tilde{v}(r, x, k) = \mathbf{1}_{\{k > \theta\}} v(x)$, where $v \in \mathcal{B}(F_1)$ is a non-negative solution of the integral equation on F_1,

$$v(x) + \mathbb{E}_x \Big[\int_0^\infty d\ell_r\, \mathrm{e}^{-\theta r}\, v^2(B_r) \mathbf{1}_{F_1}(B_r) \Big] = \mathbb{E}_x \Big[\int_0^\infty d\ell_r\, \mathrm{e}^{-\theta r}\, \phi(B_r) \mathbf{1}_{F_1}(B_r) \Big]. \tag{28}$$

Remark 5.2. By (39), and as ϕ is bounded, the quantity $\mathbb{E}_x \Big[\int_0^\infty d\ell_r\, \mathrm{e}^{-\theta r}\, \phi(B_r) \mathbf{1}_{F_1}(B_r) \Big]$ is uniformly bounded on F_1. Therefore, v is bounded. Of course, this argument fails for $\theta = 0$.

Proof. Let $\tilde{\phi} \in \mathcal{B}_+(E')$ be bounded, such that $\tilde{\phi}(r,x,k) = \mathbf{1}_{\{k>\theta\}}\phi(x)$. We proceed as in the proof of Lemma 3.1. As a special case of the weighted occupation time formula (see e.g. [11, II.3]) we have for all functions $h \in \mathcal{B}_+(\mathbb{R}_+)$ with compact support,

$$\mathbb{E}^{X'}_{t,\nu}\Big[-\int_t^\infty ds'\, h(s')\langle X'_{s'}, \tilde{\phi}\rangle\Big] = \exp -\langle \nu, \tilde{v}_t\rangle, \tag{29}$$

where $\tilde{v} \in \mathcal{B}_+(\mathbb{R}_+ \times E')$ is the unique non-negative solution of the integral equation,

$$\tilde{v}_t(\hat{x}) + \mathbb{E}^{\zeta}_{t,\hat{x}}\Big[\int_t^\infty ds'\, \tilde{v}^2_{s'}(\zeta_{s'})\Big] = \mathbb{E}^{\zeta}_{t,\hat{x}}\Big[\int_t^\infty ds'\, h(s')\tilde{\phi}(\zeta_{s'})\Big].$$

Using the definition of ζ and the substitution $\ell^1_r = s'$ we obtain with $\hat{x} = (s,x,k) \in E'$,

$$\begin{aligned}\tilde{v}_t(s,x,k) + \mathbb{E}^{\zeta}_{t,(s,x,k)}\Big[\int_{\ell^{1,-1}_t = s}^\infty d\ell^1_r\, \tilde{v}^2_{\ell^1_r}(r, B_r, R_r)\Big]&\\ = \mathbb{E}^{\zeta}_{t,(s,x,k)}\Big[\int_{\ell^{1,-1}_t = s}^\infty d\ell^1_r\, h(\ell^1_r)\mathbf{1}_{\{R_r>\theta\}}\phi(B_r)\Big]&.\end{aligned} \tag{30}$$

Using time homogeneity for ζ and B, independence between B and R, and (27), we have

$$\mathbb{E}^{\zeta}_{t,(s,x,k)}\Big[\int_{\ell^{1,-1}_t = s}^\infty d\ell^1_r\, h(\ell^1_r)\mathbf{1}_{\{R_r>\theta\}}\phi(B_r)\Big] = \mathbf{1}_{\{k>\theta\}}\mathbb{E}_x\Big[\int_0^\infty d\ell^1_r\, h(\ell^1_r + t)\,\mathrm{e}^{-\theta r}\,\phi(B_r)\Big].$$

In particular, this quantity vanishes for $k \le \theta$. Since $\tilde{v}_t$ is non-negative, we deduce from (30) that $\tilde{v}(r,x,k) = 0$ if $k \le \theta$. Also notice, that for $k > \theta$, the left hand side of (30) does not depend on k. In particular, $\tilde{v}^{k_0}_t$ defined by $\tilde{v}^{k_0}_t(s,x,k) = \tilde{v}_t(s,x,k\wedge k_0)$ also solves (30) for any $k_0 > \theta$. By uniqueness, we get that $\tilde{v}$ does not depend on k on $\{k > \theta\}$. Hence, we deduce that $\tilde{v}_t(r,x,k) = \mathbf{1}_{\{k>\theta\}}\bar{v}_t(r,x)$, where $\bar{v}_t$ is the unique non-negative solution on F_1 of the integral equation,

$$\bar{v}_t(s,x) + \mathbb{E}_x\Big[\int_0^\infty d\ell^1_r\ \mathrm{e}^{-\theta r}\,\bar{v}^2_{\ell^1_r+t}(r+s, B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell^1_r\, h(\ell^1_r + t)\,\mathrm{e}^{-\theta r}\,\phi(B_r)\Big].$$

We complete the proof using similar arguments as those following equation (13) in the proof of Lemma 3.1. □

Remark 5.3. It is not clear that (28) has a unique solution. However, if $\|\phi\|_\infty$ is small enough (depending on $\theta > 0$), then arguing as in the end of the proof of Proposition 3.3, one can show that (28) has a unique solution. Moreover, Lemma 5.1 allows us to compute the first moment of Γ^{Neu}: for all $\phi \in \mathcal{B}_+(F_1)$,

$$\mathbb{E}^{X'}_{\nu}\big[\langle\Gamma^{\mathrm{Neu}}, \tilde{\phi}\rangle\big] = \int_{E'} \nu(ds,dx,dk)\,\mathbf{1}_{\{k>\theta\}}\mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\phi(B_r)\mathbf{1}_{F_1}(B_r)\Big], \tag{31}$$

where $\tilde{\phi}(r,x,k) = \mathbf{1}_{\{k>\theta\}}\phi(x)$.

Let $\eta \in \mathcal{M}_f(\bar{D})$ and define $\nu_{\eta,\theta}$ to be the law of (τ_1, B_{τ_1}), killed at an independent exponential time of rate θ, with B_0 distributed according to η:

$$\langle \nu_{\eta,\theta}, \psi\rangle = \int \eta(dx)\,\mathbb{E}_x\big[\mathrm{e}^{-\theta\tau_1}\,\psi(\tau_1, B_{\tau_1})\big]. \tag{32}$$

Moreover, we write $\nu_\eta = \nu_{\eta,0}$.

We write $\nu \ge \nu'$ for $\nu,\nu' \in \mathcal{M}_f(E)$ if $\langle \nu, g\rangle \ge \langle \nu', g\rangle$ for any $g \in \mathcal{B}_+(E)$. Notice that $(\nu_{\eta,\theta}, \theta \ge 0)$ is a decreasing sequence of measures.

Remark 5.4. Let us write $\Gamma_\theta^{\mathrm{Neu}}$ for the random measure Γ^{Neu} defined under $\mathbb{P}^{X'}_{\nu_{\eta,\theta}\otimes\delta_\infty}$. Thanks to the Poissonian representation of superprocesses, due to the branching property (see e.g. Theorem 4.2.1 [7]), one can construct all the family $(\Gamma_\theta^{\mathrm{Neu}}, \theta \geq 0)$ on the same probability space in such a way, that this family is a decreasing sequence of measures. We shall use this remark later.

Definition 5.5. *Let $\theta \geq 0$. We define the random measure Z_θ^{Neu} on $\bar F_2$ by: for all $\varphi \in \mathcal{B}_+(\bar F_2)$,*

$$\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle = \langle \eta, Q_\theta\varphi\rangle + \int_{E'} \Gamma_\theta^{\mathrm{Neu}}(ds\, dx\, dk)\, \mathbf{1}_{\{k>\theta\}}\rho(x) H^x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(e_r)\Big],$$

where

$$Q_\theta\varphi(x) := \mathbb{E}_x\Big[\int_0^{\tau_1} d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(B_r)\Big].$$

We call Z_θ^{Neu} the Neumann boundary measure and denote by $\mathbb{P}^Z_{\eta,\theta}$ its law.

From now on, we assume that $\theta > 0$.

Remark 5.6. To see that the random measure Z_θ^{Neu} is finite for $\theta > 0$, we can perform a first moment calculation. Using (31), (32), Lemma 2.1, the exit formula (6), the strong Markov property of B and the definition of Q_θ, we get

$$\begin{aligned}
&\mathbb{E}^Z_{\eta,\theta}\left[\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle\right]\\
&\quad = \langle \eta, Q_\theta\varphi\rangle + \int_{\mathbb{R}_+\times F_1} \nu_{\eta,\theta}(ds, dx)\\
&\qquad\qquad \times \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ \rho(B_r)\mathbf{1}_{F_1}(B_r) H^{B_r}\Big[\int_0^\infty d\ell_{r'}\ \mathrm{e}^{-\theta r'}\ \varphi(e_{r'}))\Big]\Big]\\
&\quad = \langle \eta, Q_\theta\varphi\rangle\\
&\qquad + \int \eta(dx)\, \mathbb{E}_x\Big[\mathrm{e}^{-\theta\tau_1}\, \mathbb{E}_{B_{\tau_1}}\Big[\int_0^\infty dL_r\ \mathrm{e}^{-\theta r}\ H^{B_r}\Big[\int_0^\infty d\ell_{r'}\ \mathrm{e}^{-\theta r'}\ \varphi(e_{r'})\Big]\Big]\Big]\\
&\quad = \langle \eta, Q_\theta\varphi\rangle + \int \eta(dx)\, \mathbb{E}_x\Big[\mathrm{e}^{-\theta\tau_1}\, \mathbb{E}_{B_{\tau_1}}\Big[\int_{\tau_1}^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big]\Big]\\
&\quad = \langle \eta, Q_\theta\varphi\rangle + \int \eta(dx)\, \mathbb{E}_x\Big[\int_{\tau_1}^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big]\\
&\quad = \int \eta(dx)\, \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big],
\end{aligned}$$

which is finite, thanks to (39). This argument fails if $\theta = 0$, as the first moment is infinite if $\int_{F_2} \sigma(dy)\ \varphi(y) > 0$.

Recall the notation of the constants $(c_\theta, \theta > 0)$ from (39).

Lemma 5.7. *Let $\theta > 0$. We have for all $\varphi \in \mathcal{B}_+(\bar F_2)$,*

$$\mathbb{E}^Z_{\eta,\theta}\left[\exp -\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle\right] = \exp -\langle \eta, w_\theta\rangle,$$

where $(w_\theta(x), x \in \bar D)$ is a non-negative solution of the integral equation on $\bar D$,

$$w_\theta(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big]. \tag{33}$$

If additionally we assume that φ is bounded with $2c_\theta^2\|\varphi\|_\infty < 1$, then the integral equation (33) has a unique solution.

Remark 5.8. If φ is bounded, (33) implies that w_θ is bounded by $c_\theta\|\varphi\|_\infty$. In general, for $\theta = 0$, i.e. without killing, the right hand side of the integral equation (33) is infinite. See also Proposition 5.16.

Proof. Let $\tilde{\phi} \in \mathcal{B}_+(E')$ defined by $\tilde{\phi}(s,x,k) = \mathbf{1}_{\{k>\theta\}}\phi(x)$, where

$$\phi(x) = \rho(x) H^x\Big[\int_0^\infty d\ell_r \ \mathrm{e}^{-\theta r}\ \varphi(e_r)\Big].$$

We have

$$\begin{aligned}\mathbb{E}^Z_{\eta,\theta}\left[\exp -\langle Z^{\mathrm{Neu}}_\theta, \varphi\rangle\right] &= \mathbb{E}^{X'}_{\nu_{\eta,\theta}}\left[\exp -\big(\langle \eta, Q_\theta\varphi\rangle + \langle \Gamma^{\mathrm{Neu}}_\theta, \phi\rangle\big)\right]\\ &= \exp -\big(\langle \eta, Q_\theta\varphi\rangle + \langle \nu_{\eta,\theta}, \tilde{v}_\theta\rangle\big),\end{aligned}$$

where thanks to Lemma 5.1 $\tilde{v}_\theta(s,x) = v_\theta(x)$ is a non-negative solution on F_1 of

$$\begin{aligned}v_\theta(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r \ \mathrm{e}^{-\theta r}\ v^2_\theta(B_r)\mathbf{1}_{F_1}(B_r)\Big]&\\
&= \mathbb{E}_x\Big[\int_0^\infty d\ell_r \ \mathrm{e}^{-\theta r}\ \rho(B_r)\mathbf{1}_{F_1}(B_r) H^{B_r}\Big[\int_0^\infty d\ell_{r'} \ \mathrm{e}^{-\theta r'}\ \varphi(e_{r'})\Big]\Big].\\
&= \mathbb{E}_x\Big[\int_0^\infty dL_r \ \mathrm{e}^{-\theta r}\ H^{B_r}\Big[\int_0^\infty d\ell_{r'} \ \mathrm{e}^{-\theta r'}\ \varphi(e_{r'})\Big]\Big].\\
&= \mathbb{E}_x\Big[\int_{\tau_1}^\infty d\ell_r \ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big],\end{aligned}\tag{34}$$

where we used Lemma 2.1 for the second equality and the exit-formula (6) for the last. Define for $x \in \bar{D}$,

$$w_\theta(x) := Q_\theta\varphi(x) + \mathbb{E}_x\left[\mathrm{e}^{-\theta\tau_1}\ v_\theta(B_{\tau_1})\right],$$

and notice that $w_\theta = v_\theta$ on F_1. Moreover, we have by construction that

$$\langle \eta, w_\theta\rangle = \langle \eta, Q_\theta\varphi\rangle + \langle \nu_{\eta,\theta}, \tilde{v}_\theta\rangle.$$

Using the strong Markov property of B and (34) one checks that w_θ solves (33). If $2c^2_\theta\|\varphi\|_\infty < 1$, we get the uniqueness as in the end of the proof of Proposition 3.3. $\square$

The following Lemma play the same rôle in this section as Lemma 4.4 in Section 4.2 and can be proved using the same techniques.

Lemma 5.9. *Let $\theta > 0$ and φ bounded. Let T be a finite $\mathcal{F}_t$-stopping time, then*

$$\begin{aligned}w_\theta(x) + \mathbb{E}_x\Big[\int_0^T d\ell_r \ \mathrm{e}^{-\theta r}\ w^2_\theta(B_r)\mathbf{1}_{F_1}(B_r)\Big]&\\
&= \mathbb{E}_x\Big[\mathrm{e}^{-\theta T}\ w_\theta(B_T)\Big] + \mathbb{E}_x\Big[\int_0^T d\ell_r \ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big].\end{aligned}$$

5.2. **Weak solution of the θ-approximation.** Fix a continuous non-negative function $\varphi \in \mathcal{C}(\bar{F}_2)$. And define a function w_θ on $\bar{D}$ by

$$w_\theta(x) := -\log \mathbb{E}^Z_{\delta_x,\theta}\left[\exp -\langle Z^{\mathrm{Neu}}_\theta, \varphi\rangle\right].$$

We assume throughout this section that $\theta > 0$. By Remark 5.8, we have that w_θ is bounded.

Proposition 5.10. *Let $\theta > 0$. The function w_θ belongs to $\mathcal{C}^2(D)$ and solves $\Delta w_\theta = 2\theta w_\theta$.*

Proof. This can be proved from Lemma 5.9, using standard results on killed Brownian motion, in the same way as Lemma 4.5 is deduced from Lemma 4.4. $\square$

Lemma 5.11. *The function w_θ is continuous on $\bar{D}$.*

Proof. Lemma 5.9 applied to the deterministic time $T = t > 0$, yields

$$w_\theta(x) = \mathbb{E}_x\left[\mathrm{e}^{-\theta t}\, w_\theta(B_t)\right] + \mathbb{E}_x\left[\int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\right]\right].$$

As φ and w_θ are bounded, we have thanks to (40), that the last term of this equality decreases to 0 as $t \downarrow 0$ uniformly in x. As the second term is continuous in x the proof is complete. □

Remark 5.12. In particular, the process $M^{\mathrm{Neu}} = (M_t^{\mathrm{Neu}}, t \geq 0)$ defined by

$$M_t^{\mathrm{Neu}} := \mathrm{e}^{-\theta t}\, w_\theta(B_t) - w_\theta(B_0) + \int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\right],$$

is a continuous $\mathcal{F}_t$-martingale. Thus the process $N^{\mathrm{Neu}} = (N_t^{\mathrm{Neu}}, t \geq 0)$ defined by $N_0^{\mathrm{Neu}} = 0$ and $dN_t^{\mathrm{Neu}} = \mathrm{e}^{\theta t}\, dM_t^{\mathrm{Neu}}$, that is

$$N_t^{\mathrm{Neu}} = w_\theta(B_t) - w_\theta(B_0) - \theta\int_0^t dr\, w_\theta(B_r) + \int_0^t d\ell_r\ \left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\right],$$

is also a continuous $\mathcal{F}_t$-martingale.

Let us define a space of test functions $\mathcal{S}_2$ by

$$\mathcal{S}_2 := \left\{\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D});\ \Delta\phi \text{ bounded};\ \partial_n\phi = 0 \text{ on } \partial D\right\}.$$

Definition 5.13. *Let $\theta \geq 0$. A function $u \in \mathcal{B}_+(\bar{D})$ is said to be a weak solution of the boundary value problem*

$$\begin{cases} \Delta u = 2\theta u \ \textit{in } D, \\ \partial_n u - 2\varphi = 0 \ \textit{on } F_2, \\ \partial_n u + 2u^2 = 0 \ \textit{on } F_1, \end{cases} \tag{35}$$

if $u \in \mathcal{C}(\bar{D})$ and for all $\phi \in \mathcal{S}_2$,

$$\int_D dx\, u(x)\Delta\phi(x) = 2\theta\int_D dx\, u(x)\phi(x) - 2\int_{F_2}\sigma(dy)\,\phi(y)\varphi(y) + 2\int_{F_1}\sigma(dy)\,u^2(y)\phi(y).$$

Notice any non-negative strong solution of (35) is a weak solution.

Proposition 5.14. *A non-negative function $u \in \mathcal{C}(\bar{D})$ is a weak solution of the boundary value problem (35) if and only if the process $N = (N_t, t \geq 0)$ defined by*

$$N_t = u(B_t) - u(B_0) - \theta\int_0^t dr\, u(B_r) + \int_0^t d\ell_r\ \left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - u^2(B_r)\mathbf{1}_{F_1}(B_r)\right],$$

is a continuous $\mathcal{F}_t$-martingale.

Proof. First assume that $u \in \mathcal{C}(\bar{D})$ is a weak solution of (35) and let $x \in \bar{D}$. Thanks to the Markov property of B, we have for $0 < s < t$,

$$\mathbb{E}_x[N_t|\mathcal{F}_s] = N_s + \mathbb{E}_{B_s}[N_{t-s}].$$

Thus, to prove the process N is a $\mathcal{F}_t$-martingale, it is enough to check that $\mathbb{E}_x[N_t] = 0$ for all $t > 0$. Let $s > 0$. As $p_s(x, \cdot) \in \mathcal{S}_2$ (see appendix, section 6.1), we compute, using the integral equation for u and $\phi(y) = p_s(x, y)$,

$$\begin{aligned}\frac{d}{ds}\mathbb{E}_x[u(B_s)] &= \int_D dy\, u(y)\partial_s p_s(x,y)\\ &= \int_D dy\, u(y)\tfrac{1}{2}\Delta_y p_s(x,y)\\ &= \theta\int_D dy\, u(y)p_s(x,y) - \int_{F_2}\sigma(dy)\, p_s(x,y)\varphi(y) + \int_{F_1}\sigma(dy)\, p_s(x,y)u^2(y).\end{aligned}$$

For $\varepsilon > 0$, integrating from ε to t gives,

$$\mathbb{E}_x[u(B_t)] - \mathbb{E}_x[u(B_\varepsilon)] = \theta\int_\varepsilon^t dr\, \mathbb{E}_x[u(B_r)] - \mathbb{E}_x\Big[\int_\varepsilon^t d\ell_r^2\, \varphi(B_r)\Big] + \mathbb{E}_x\Big[\int_\varepsilon^t d\ell^1\, u^2(B_r)\Big].$$

Hence, by continuity of u, we see that $\mathbb{E}_x[N_t] = 0$ as $\varepsilon \downarrow 0$.

Let $u \in \mathcal{C}(\bar{D})$ and assume now that for any $x \in \bar{D}$, the process N is a continuous $\mathcal{F}_t$-martingale. As $\mathbb{E}_x[N_t] = 0$, we have

$$\mathbb{E}_x[u(B_t)] - u(x) = \theta\int_0^t dr\, \mathbb{E}_x[u(B_r)] - \mathbb{E}_x\Big[\int_0^t d\ell_r^2\, \varphi(B_r)\Big] + \mathbb{E}_x\Big[\int_0^t d\ell^1\, u^2(B_r)\Big].$$

Let $\phi \in \mathcal{S}_2$. Multiplying the last equation by ϕ and integrating over D yields

$$\begin{aligned}\int_D dx\, u(x)\mathbb{E}_x[\phi(B_t) - \phi(x)] &= \theta\int_D dx\, \phi(x)\int_0^t dr\, \mathbb{E}_x[u(B_r)]\\ &\quad - \int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_r\, \big[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - u^2(B_r)\mathbf{1}_{F_1}(B_r)\big]\Big],\end{aligned} \tag{36}$$

where we used for the first term the symmetry of the reflecting Brownian motion. Since $\phi \in \mathcal{S}_2$, by Lemma 6.1 the process $Y = (Y_t, t \geq 0)$ defined by

$$Y_t := \phi(B_t) - \phi(B_0) - \tfrac{1}{2}\int_0^t ds\, \Delta\phi(B_s)$$

is also an $\mathcal{F}_t$-martingale. Hence, we have $\mathbb{E}_x[\phi(B_t) - \phi(x)] = \frac{1}{2}\mathbb{E}_x\Big[\int_0^t ds\, \Delta\phi(B_s)\Big]$. Thus, dividing (36) by $t > 0$, gives

$$\begin{aligned}&\frac{1}{2t}\int_D dx\, u(x)\mathbb{E}_x\Big[\int_0^t ds\, \Delta\phi(B_s)\Big]\\ &\quad= \frac{1}{t}\theta\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t dr\, u(B_r)\Big] - \frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_r\, \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big]\\ &\qquad + \frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_r\, u^2(B_r)\mathbf{1}_{F_1}(B_r)\Big].\end{aligned}$$

Hence, we complete the proof applying Lemmas 6.6 and 6.7. □

Proposition 5.15. *The function w_θ is a non-negative weak solution of the boundary value problem (35). If additionally $2c_\theta^2\|\varphi\|_\infty < 1$, then the solution w_θ is also unique.*

Proof. It follows immediately from Remark 5.12 and Proposition 5.14 that w_θ is a weak solution of (35). To prove uniqueness, let $u \in \mathcal{B}_+(\bar{D})$ be a weak solution of (35) and assume

$2c_\theta^2\|\varphi\|_\infty < 1$. Thanks to Proposition 5.14 and Remark 5.12, the process $M = (M_t, t \geq 0)$ defined by

$$M_t := \mathrm{e}^{-\theta t}\, u(B_t) - u(B_0) + \int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r) - \int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\ u^2(B_r)\mathbf{1}_{F_1}(B_r),$$

is a continuous $\mathcal{F}_t$-martingale, as well as $dM_t = \mathrm{e}^{-\theta t}\, dN_t$. As u and φ are bounded and thanks to (39), we have that M is a uniformly integrable martingale. Hence $(M_t, t \geq 0)$ converges almost surely and in L^1 to a limit, say M_∞, with $\mathbb{E}_x[M_\infty] = \mathbb{E}_x[M_0] = 0$. Therefore, u is a non-negative solution of the integral equation,

$$u(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ u^2(B_r)\mathbf{1}_{F_1}(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\ \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big].$$

As $2c_\theta^2\|\varphi\|_\infty < 1$, by Lemma 5.7 w_θ is the only non-negative solution of the last displayed equation. Hence, we have $u = w_\theta$. $\square$

5.3. **The case $\theta \downarrow 0$.** Let $\varphi \in \mathcal{B}_+(F_2)$ be bounded.

Observe that thanks to Remark 5.4, one can assume that $(\Gamma_\theta^{\mathrm{Neu}}, \theta > 0)$ is an increasing sequence of measures as $\theta \downarrow 0$. Notice also that $(Q_\theta\varphi, \theta > 0)$ is also an increasing sequence of functions as $\theta \downarrow 0$. From the definition of Z_θ^{Neu}, we deduce that the sequence $(Z_\theta^{\mathrm{Neu}}, \theta > 0)$ is also an increasing sequence of measures as $\theta \downarrow 0$. Let Z^{Neu} be its limit as $\theta \downarrow 0$. (One could check that Z^{Neu} as the same law as Z_0^{Neu}.) By dominated convergence, we get that $(w_\theta, \theta > 0)$ increases to a limit, say w, as $\theta \downarrow 0$, defined on $\bar{D}$ by

$$w(x) = -\log \mathbb{E}^Z_{\delta_x}\left[\exp -\langle Z^{\mathrm{Neu}}, \varphi\rangle\right].$$

From now on, we assume that $\partial F = \emptyset$, that is $\bar{F}_1 \cap \bar{F}_2 = \emptyset$.

Proposition 5.16. *The function w is bounded on $\bar{D}$. More precisely, there exists a finite constant c independent of φ, such that for any $x \in \bar{D}$,*

$$w(x) \leq c\left(\|\varphi\|_\infty + \sqrt{\|\varphi\|_\infty}\right).$$

Proof. For $\varepsilon > 0$, we set $F_1^\varepsilon = \{x \in \bar{D} : d(x, F_1) \leq \varepsilon\}$, the ε-neighborhood of F_1 in $\bar{D}$. Since $\partial F = \emptyset$, there exists $\varepsilon > 0$, such that $F_1^\varepsilon \cap F_2 = \emptyset$. Let τ_1^ε the first exit time of F_1^ε:

$$\tau_1^\varepsilon(e) = \inf\{s > 0 : e(s) \notin F_1^\varepsilon\},$$

for $e \in \mathbb{D}$ (recall notations from Section 2.2). In particular, using the strong Markov property of the exit measure H^x, we have that for any $x \in F_1$,

$$H^x\Big[\int_0^\infty d\ell_r\Big] = H^x\Big[\mathbf{1}_{\{\tau_1^\varepsilon < \infty\}}\int_{\tau_1^\varepsilon}^{\tau_1} d\ell_r\Big] = H^x\Big[\mathbf{1}_{\{\tau_1^\varepsilon < \infty\}}\mathbb{E}_{e(\tau_1^\varepsilon)}[\ell_{\tau_1}]\Big] \leq cH^x\Big[\tau_1^\varepsilon < \infty\Big],$$

where we used Lemma 6.3 for the last inequality. Arguing as in the proof of Lemma 8.3 of [6], we have that

$$\sup_{x \in F_1} H^x\Big[\tau_1^\varepsilon < \infty\Big] < \infty.$$

This implies that $H^x\Big[\int_0^\infty d\ell_r\Big]$ is bounded on F_1 say by C_0.

Since, thanks to Corollary 6.11, ρ is bounded by a constant, say C_1, we get from Definition 5.5, that for $\varphi \geq 0$,

$$0 \leq \langle Z_\theta^{\mathrm{Neu}}, \varphi \rangle \leq \| \varphi \|_\infty \left[\int \eta(dx) \mathbb{E}_x[\ell_{\tau_1}] + C_0 C_1 \langle \Gamma_\theta^{\mathrm{Neu}}, \mathbf{1} \rangle \right].$$

From Remark 5.4, and Lemma 6.3, we get there exists a finite constant c, such that

$$0 \leq \langle Z^{\mathrm{Neu}}, \varphi \rangle \leq c \| \varphi \|_\infty \left[\langle \eta, \mathbf{1} \rangle + \langle \Gamma_0^{\mathrm{Neu}}, \mathbf{1} \rangle \right].$$

It is well known that the total mass of the superprocess X', $\langle \Gamma_0^{\mathrm{Neu}}, \mathbf{1} \rangle$, started at ν_η is distributed according the law of a stable subordinator of index 1/2 at time $\langle \nu_\eta, \mathbf{1} \rangle$. (The solution of the integral equation (30), with $\theta = 0$, $\phi = \lambda \mathbf{1}$ and $h(t) = \mathbf{1}_{[0,T]}(t)$ is given by $\tilde{v}_t = \sqrt{\lambda} \dfrac{\sinh((T-t)/4\sqrt{\lambda})}{\cosh((T-t)/4\sqrt{\lambda})}$ for $t \in [0, T]$. Then, letting $T \to \infty$, we deduce from (29) that the log-Laplace transform of $\langle \Gamma_0^{\mathrm{Neu}}, \mathbf{1} \rangle$ is exactly $\sqrt{\lambda} \langle \nu_\eta, \mathbf{1} \rangle$.) In particular, we deduce that

$$\mathbb{E}_\eta^Z \left[\mathrm{e}^{-\langle Z^{\mathrm{Neu}}, \varphi \rangle} \right] \leq \mathrm{e}^{-c \langle \eta, \mathbf{1} \rangle \left(\|\varphi\|_\infty + \sqrt{\|\varphi\|_\infty} \right)},$$

for a finite constant c independent of φ and η. Since this holds for any finite measure η, this implies the proposition. □

Lemma 5.17. *The function w is continuous on $\bar{D}$.*

Proof. As w is bounded, we obtain from Lemma 5.9 applied to the deterministic time $T = t > 0$ and dominated convergence,

$$w(x) = \mathbb{E}_x[w(B_t)] + \mathbb{E}_x \Big[\int_0^t d\ell_r^2 \, \varphi(B_r) \Big] - \mathbb{E}_x \Big[\int_0^t d\ell_r^1 \, w^2(B_r) \Big].$$

Then, we can deduce the continuity of w, following the proof of Lemma 5.11. □

The following Proposition is now obvious from Proposition 5.15 and dominated convergence:

Proposition 5.18. *The function w is a weak solution to the nonlinear Neumann boundary value problem (35) with $\theta = 0$.*

6. Appendix

6.1. **Reflecting Brownian motion in D.** The reflecting Brownian motion $B = (B_t, t \geq 0)$ is a strong Markov process on $\bar{D}$, with transition density $p_t(x, y)$ defined on $(0, \infty) \times \bar{D} \times \bar{D}$. The density has the following properties (see [9] or [16]):

(i) $p_t(x, y)$ is continuously differentiable in $t > 0$ for fixed $(x, y) \in \bar{D} \times \bar{D}$, and for $\varepsilon > 0$, its derivative is uniformly bounded for $t \geq \varepsilon$, $(x, y) \in \bar{D} \times \bar{D}$. As a function of x, $p_t(x, y)$ belongs to $\mathcal{C}^1(\bar{D}) \cap \mathcal{C}^2(D)$ for fixed $t \in \mathbb{R}_+, y \in \bar{D}$.

(ii) $p_t(x, y)$ solves the heat equation inside D

$$\partial_t p_t(x, y) = \tfrac{1}{2} \Delta_x p_t(x, y) \quad \text{for } (t, x, y) \in \mathbb{R}_+ \times D \times \bar{D},$$

with the boundary condition

$$\partial_{n_x} p_t(x, y) = 0 \quad \text{for } (t, x, y) \in \mathbb{R}_+ \times \partial D \times \bar{D}.$$

(iii) For any $x \in \bar{D}$ and $f \in \mathcal{B}(\bar{D})$, bounded and continuous at x, we have

$$\lim_{t \downarrow 0} \int_{\bar{D}} dy \, f(y) p_t(x, y) = f(x).$$

The function $p_t(x,y)$ is symmetric in x and y, positive and satisfies $\displaystyle\int_D dy\, p_t(x,y) = 1$. Moreover, for any bounded $f \in \mathcal{B}(\bar{D})$, $t>0$, the function $x \mapsto \int dy\, p_t(x,y) f(y)$ is in $\mathcal{C}(\bar{D})$.

We denote by $\mathbb{P}_x$ the law of B starting in $B_0 = x \in \bar{D}$. Let $(\mathcal{F}_t, t \geq 0)$ be the filtration generated by B completed the usual way. We have the following martingale problem characterization of the reflecting Brownian motion:

Lemma 6.1. [5] *For every $\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$, with $\Delta\phi$ bounded on D,*

$$\phi(B_t) - \phi(B_0) - \frac{1}{2}\int_0^t ds\, \Delta\phi(B_s) + \frac{1}{2}\int_0^t d\ell_s\, \partial_n \phi(B_s),$$

is a continuous $\mathcal{F}_t$-martingale.

6.2. **Estimates for reflecting Brownian motion.** Following [9], we have the following estimates: there exists a constant c such that for all $x \in \bar{D}$ and all $t \in (0,1]$,

$$\int_{\partial D} \sigma(dy)\, p_t(x,y) \leq c/\sqrt{t}, \tag{37}$$

where σ is the surface measure on ∂D. Moreover, there exist two positive constants c' and β such that for all $x,y \in \bar{D}$, $t \geq 1$, we have

$$|p_t(x,y) - a_D| \leq c' \,\mathrm{e}^{-\beta t}, \tag{38}$$

where $a_D^{-1} := \int_D dy$ is the d-dimensional Lebesgue measure of D. We deduce from those inequalities that for any $\theta > 0$, there is a constant $c_\theta > 0$ such that, for all $x \in \bar{D}$

$$\int_0^\infty dr \int_{\partial D} \sigma(dy)\, \mathrm{e}^{-\theta r}\, p_r(x,y) \leq c_\theta. \tag{39}$$

From (2), (37) and (38) we get there exists a constant K such that for all $t \geq 0$, we have

$$\sup_{x \in \bar{D}} \mathbb{E}_x[\ell_t] \leq K\,(\sqrt{t} + t).$$

By induction, we deduce that for $n \in \mathbb{N}$, there exists $K_n > 0$ such that for all $t \geq 0$,

$$\sup_{x \in \bar{D}} \mathbb{E}_x[(\ell_t)^n] \leq K_n\,(t^{n/2} + t^n). \tag{40}$$

Thanks to [9, Theorem 2.5], the reflecting Brownian motion in D has the same modulus of continuity as a standard Brownian motion in $\mathbb{R}^d$. In particular, for $T > 0$, there exists a constant K, such that for all $t \in [0,T]$, $x \in \bar{D}$, $a \geq 0$,

$$\mathbb{P}_x\Big(\sup_{0 \leq s \leq t} |B_s - x| \geq a\Big) \leq P_x\Big(\sup_{0 \leq s \leq t} |W_s - x| \geq a/K\Big), \tag{41}$$

where $W = (W_t, t \geq 0)$ is under P_x a standard Brownian motion in $\mathbb{R}^d$ started at x.

For $i = 1,2$, let $\tau_i := \inf\{t > 0 : B_t \in F_i\}$ be the first hitting time of F_i, with the convention that $\inf \emptyset = +\infty$.

Lemma 6.2. *For any $t > 0$, the function $x \mapsto \mathbb{P}_x(\tau_i > t)$ is upper semi continuous in $\bar{D}$. In particular, for all $y \in \bar{F}_i$, we have*

$$\lim_{x \to y; x \in \bar{D}} \mathbb{P}_x(\tau_i > t) = 0.$$

Proof. Notice that $\mathbb{P}_x(\tau_i > t)$ is the non-increasing limit as $\varepsilon \downarrow 0$ of

$$\mathbb{E}_x[\mathbb{P}_{B_\varepsilon}(\tau_i > t - \varepsilon)],$$

which are continuous functions of $x \in \bar{D}$. Thus the function $x \mapsto \mathbb{P}_x(\tau_i > t)$ is upper semi continuous for $t > 0$. To conclude, notice that, since ∂D and ∂F are smooth, any point of $\bar{F}_i$ is regular for F_i, and thus $\mathbb{P}_y(\tau_i > t) = 0$ for all $y \in \bar{F}_i$. □

Lemma 6.3. *The functions $x \mapsto \mathbb{E}_x[\tau_i]$ and $x \mapsto \mathbb{E}_x[\ell_{\tau_i}]$ are bounded on $\bar{D}$. Moreover, we have for all $y \in \bar{F}_i$,*

$$\lim_{x \to y; x \in \bar{D}} \mathbb{E}_x[\tau_i] = 0 \quad \text{and} \quad \lim_{x \to y; x \in \bar{D}} \mathbb{E}_x\left[\ell_{\tau_i}\right] = 0.$$

Proof. Since $\mathbb{P}_x(\tau_i > 1) < 1$ for all $x \in \bar{D}$, we deduce from Lemma 6.2, that $\delta := \sup_{x \in \bar{D}} \mathbb{P}_x(\tau_i > 1) < 1$. By the strong Markov property of the reflecting Brownian motion, we have for any $n \in \mathbb{N}^*$,

$$\mathbb{P}_x(\tau_i > n) = \mathbb{E}_x\left[\mathbf{1}_{\{\tau_i > n-1\}} \mathbb{P}_{B_{n-1}}(\tau_i > 1)\right] \leq \delta \mathbb{P}_x(\tau_i > n-1),$$

and hence, by induction $\sup_{x \in \bar{D}} \mathbb{P}_x(\tau_i > n) \leq \delta^n$. Therefore,

$$\mathbb{E}_x[\tau_i] = \int_0^\infty dt\, \mathbb{P}_x(\tau_i > t) \leq \sum_{n=0}^\infty \mathbb{P}_x(\tau_i > n) \leq \frac{1}{1-\delta} < \infty. \tag{42}$$

Hence, $x \mapsto \mathbb{E}_x[\tau_i]$ is bounded on $\bar{D}$. Moreover, for $y \in \bar{F}_i$, the estimate in (42) allows us to use dominated convergence in

$$\lim_{x \to y; x \in \bar{D}} \mathbb{E}_x[\tau_i] = \lim_{x \to y; x \in \bar{D}} \int_0^\infty dt\, \mathbb{P}_x(\tau_i > t) = \int_0^\infty dt \lim_{x \to y; x \in \bar{D}} \mathbb{P}_x(\tau_i > t),$$

and the last expression is equal to zero by Lemma 6.2.

Let us now treat the function $x \mapsto \mathbb{E}_x[\ell_{\tau_i}]$. It follows from the Cauchy-Schwarz inequality and (40), that

$$\begin{aligned}
\mathbb{E}_x[\ell_{\tau_i}] &= \sum_{n=0}^\infty \mathbb{E}_x\left[\mathbf{1}_{\{n < \tau_i \leq n+1\}} \ell_{\tau_i}\right] \\
&\leq \sum_{n=0}^\infty \mathbb{E}_x\left[\mathbf{1}_{\{\tau_i > n\}} \ell_{n+1}\right] \\
&\leq \sum_{n=0}^\infty \mathbb{P}_x\left(\tau_i > n\right)^{1/2} \mathbb{E}_x\left[(\ell_{n+1})^2\right]^{1/2} \\
&\leq c \sum_{n=0}^\infty \delta^{n/2}(n+1),
\end{aligned}$$

where c is a finite constant independent of $x \in \bar{D}$. Hence, the function $x \mapsto \mathbb{E}_x[\ell_{\tau_i}]$ is bounded on $\bar{D}$.

The same arguments as in the previous part of the proof, show that the function $x \mapsto \mathbb{E}_x[(\ell_{\tau_i})^2]$ is bounded. Let $\varepsilon \in (0, 1]$. Using the Cauchy-Schwarz inequality for the third line and (40), with $n = 2$, for the fourth, we obtain for all $x \in \bar{D}$,

$$\begin{aligned}\mathbb{E}_x[\ell_{\tau_i}] &= \mathbb{E}_x[1_{\{\tau_i>\varepsilon\}}\ell_{\tau_i}] + \mathbb{E}_x[1_{\{\tau_i\leq\varepsilon\}}\ell_{\tau_i}] \\ &\leq \mathbb{E}_x[1_{\{\tau_i>\varepsilon\}}\ell_{\tau_i}] + \mathbb{E}_x[1_{\{\tau_i\leq\varepsilon\}}\ell_{\varepsilon}] \\ &\leq \mathbb{P}_x(\tau_i>\varepsilon)^{1/2}\mathbb{E}_x[(\ell_{\tau_i})^2]^{1/2} + \mathbb{P}_x(\tau_i\leq\varepsilon)^{1/2}\mathbb{E}_x[(\ell_{\varepsilon})^2]^{1/2} \\ &\leq c(\mathbb{P}_x(\tau_i>\varepsilon)^{1/2} + \sqrt{\varepsilon}),\end{aligned}$$

where the constant c is independent of x. We conclude using Lemma 6.2. □

Lemma 6.4. *For all $\eta > 0$ and all $y \in \bar{F}_2$ we have*

$$\lim_{x\to y;\, x\in\bar{D}} \mathbb{P}_x\big(|B_{\tau_2} - x| \geq \eta\big) = 0.$$

Proof. First notice, that by Markov's inequality,

$$\mathbb{P}_x\big(|B_{\tau_2} - x| \geq \eta\big) \leq \eta^{-2}\mathbb{E}_x\left[|B_{\tau_2} - x|^2\right].$$

Applying Lemma 6.1 to the function $\gamma(z) := |z - x|^2$ yields that

$$M_t := |B_t - x|^2 - d\,t + \int_0^t d\ell_r\, \partial_n\gamma(B_r),$$

is a $\mathcal{F}_t$-martingale under $\mathbb{P}_x$. Notice that $|\partial_n\gamma|$ is bounded from above by a constant independent of x. Hence, the optional stopping theorem applied to the stopping time $t \wedge \tau_2$ and the martingale convergence theorem imply that

$$\mathbb{E}_x\left[|B_{\tau_2} - x|^2\right] \leq C\big(\mathbb{E}_x[\tau_2] + \mathbb{E}_x\left[\ell_{\tau_2}\right]\big).$$

Hence, the assertion follows by Lemma 6.3. □

Lemma 6.5. *Let $y \in \bar{F}_2$ and $\varphi \in \mathcal{C}(\bar{F}_2)$, then*

$$\lim_{x\to y;\, x\in\bar{D}} \mathbb{E}_x[\varphi(B_{\tau_2})] = \varphi(y).$$

Proof. Let $\varepsilon > 0$ and $y \in \bar{F}_2$. As φ is continuous on $\bar{F}_2$, there exists $\delta > 0$ such that $|\varphi(y) - \varphi(z)| < \varepsilon$ for all $z \in O_\delta(y) \cap \bar{F}_2$, where $O_\delta(y)$ is the ball of radius δ centered at y. Hence, we have for all $x \in O_{\delta/2}(y) \cap \bar{D}$

$$\begin{aligned}\mathbb{E}_x\left[|\varphi(B_{\tau_2}) - \varphi(y)|\right] &= \mathbb{E}_x\left[|\varphi(B_{\tau_2}) - \varphi(y)|\mathbf{1}_{\{|B_{\tau_2}-y|<\delta\}}\right] + \mathbb{E}_x\left[|\varphi(B_{\tau_2}) - \varphi(y)|\mathbf{1}_{\{|B_{\tau_2}-y|\geq\delta\}}\right] \\ &\leq \varepsilon + 2\,\|\varphi\|_\infty\, \mathbb{P}_x\big(|B_{\tau_2} - y| \geq \delta\big) \\ &\leq \varepsilon + 2\,\|\varphi\|_\infty\, \mathbb{P}_x\big(|B_{\tau_2} - x| \geq \delta/2\big).\end{aligned}$$

We conclude using Lemma 6.4. □

6.3. **Convergence Lemmas.** In this section we give a series of technical Lemmas on convergence.

Lemma 6.6. *For every bounded function $\phi \in \mathcal{B}(D)$ and every bounded function $\psi \in \mathcal{C}(D)$,*

$$\lim_{t\downarrow 0} \frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t ds\, \psi(B_s)\Big] = \int_D dx\, \phi(x)\psi(x).$$

Proof. Since ψ is continuous and bounded, we have that $\lim_{t\downarrow 0}\int p_t(x,y)\psi(y)\,dy = \psi(x)$ for all $x \in D$. This implies,

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t ds\int p_s(x,y)\psi(y)\,dy = \psi(x).$$

As ϕ and ψ are bounded, we can use dominated convergence to complete the proof. □

Lemma 6.7. *For every $\phi \in \mathcal{C}(\bar{D})$ and every bounded $\psi \in \mathcal{B}(\partial D)$,*

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D dx\,\phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_s\,\psi(B_s)\Big] = \int_{\partial D}\sigma(dy)\,\phi(y)\psi(y).$$

Proof. From (2), and the symmetry of the density kernel p, we have

$$\begin{aligned}\frac{1}{t}\int_D dx\,\phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_s\psi(B_s)\Big] &= \frac{1}{t}\int_D dx\,\phi(x)\int_0^t ds\int_{\partial D}\sigma(dy)\psi(y)p_s(x,y)\\ &= \int_{\partial D}\sigma(dy)\psi(y)\,\frac{1}{t}\int_0^t ds\int_D dx\,\phi(x)p_s(y,x).\end{aligned}$$

Then, we get the result using arguments similar to the proof of Lemma 6.6. □

Denote by $d(x) := d(x, F_2)$ the distance between x and F_2.

Lemma 6.8. *For all $T > 0$, there exist constants $c > 0, K > 0$ (depending on T) such that for all $t \in [0,T]$, $x \in \bar{D}$ with $d(x) > 0$,*

$$\mathbb{P}_x(\tau_2 \le t) \le c\frac{\sqrt{t}}{d(x)}\exp -\Big(\frac{d(x)^2}{Kt}\Big).$$

Proof. We have $\mathbb{P}_x(\tau_2 \le t) \le \mathbb{P}_x\Big(\sup_{0\le s\le t}|B_s - x| \ge d(x)\Big)$. Then the Lemma follows from (41) and standard result on Brownian motion. □

Recall from Section 4 that

$$\mathcal{S}_1 = \Big\{\phi \in \mathcal{C}^2(D)\cap\mathcal{C}^1(\bar{D});\,\Delta\phi \text{ is bounded in } D,\ \partial_n\phi = 0 \text{ on } F_1, \phi = 0 \text{ on } F_2\Big\}.$$

Lemma 6.9. *For any $\phi \in \mathcal{S}_1$ and every bounded $\psi \in \mathcal{B}(\partial D)$,*

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D dx\,\phi(x)\mathbb{E}_x\Big[\int_0^{\tau_2\wedge t} d\ell_s\,\psi(B_s)\Big] = \int_{F_1}\sigma(dy)\,\phi(y)\psi(y).$$

Proof. As $\phi \in \mathcal{S}_1$, we have in particular that $\phi \in \mathcal{C}^1(\bar{D})$ and $\phi = 0$ on F_2. Hence, there is a constant $K > 0$ such that $\phi(x) \leq K d(x)$. Let $T > 0$. We have for $t \in [0, T]$,

$$\begin{aligned}
\Big| \frac{1}{t} \int_D dx\, \phi(x) \mathbb{E}_x \Big[\int_0^t d\ell_s \psi(B_s) \Big] - \frac{1}{t} \int_D dx\, \phi(x) \mathbb{E}_x \Big[\int_0^{\tau_2 \wedge t} d\ell_s \psi(B_s) \Big] \Big| & \\
\leq \int_D dx\, |\phi(x)| \frac{1}{t} \mathbb{E}_x \Big[\mathbf{1}_{\{\tau_2 \leq t\}} \int_{t \wedge \tau_2}^t d\ell_r\, \psi(B_r) \Big] & \\
\leq K \, \| \psi \|_\infty \int_D dx\, \frac{d(x)}{t} \mathbb{E}_x \big[\mathbf{1}_{\{\tau_2 \leq t\}} \ell_t \big] & \\
\leq K \, \| \psi \|_\infty \int_D dx\, \frac{d(x)}{t} \mathbb{P}_x(\tau_2 \leq t)^{1/2}\, \mathbb{E}_x [(\ell_t)^2]^{1/2} & \\
\leq c \int_D dx\, \frac{d(x)}{\sqrt{t}} \mathbb{P}_x(\tau_2 \leq t)^{1/2}, & \qquad (43)
\end{aligned}$$

where c is a constant independent of $t \in (0, T]$, and where we used the Cauchy-Schwarz inequality and (40), for the third inequality and the fourth. By Lemma 6.8, we have for all $x \in D$,

$$\lim_{t \downarrow 0} \frac{1}{\sqrt{t}} \mathbb{P}_x(\tau_2 \leq t)^{1/2} = 0 \quad \text{and} \quad \frac{d(x)}{\sqrt{t}} \mathbb{P}_x(\tau_2 \leq t)^{1/2} \leq c,$$

where c is a constant independent of $t \in (0, T]$ and $x \in D$. Therefore we can apply dominated convergence in (43) to get the result. □

Lemma 6.10. *For every $\phi \in \mathcal{S}_1$ and every $\psi \in \mathcal{C}(\bar{D})$, we have*

$$\lim_{t \downarrow 0} \tfrac{1}{t} \int_D dx\, |\phi(x)|\, \mathbb{E}_x \Big[|\psi(B_{\tau_2}) - \psi(B_t)| \mathbf{1}_{\{\tau_2 < t\}} \Big] \;=\; 0.$$

Proof. Let $T > 0$. Let c denote a constant independent of $t \in (0, T]$, which may vary. From Lemma 6.8, we have for all $t \in [0, T]$,

$$\int_D dx\; d(x) \mathbb{P}_x(\tau_2 < t) \leq c \int_D dx\; d(x) \frac{\sqrt{t}}{d(x)} \exp -\Big(\frac{d(x)^2}{Kt} \Big) \leq c\sqrt{t} \int_0^\infty dr\;\, \mathrm{e}^{-r^2/Kt} \leq ct.$$

As $\phi \in \mathcal{S}_1$, there is a constant $K' > 0$ such that $|\phi(x)| \leq K' d(x)$. Hence, we have for all $t \in [0, T]$,

$$\begin{aligned}
\frac{1}{t} \int_D dx\;\, |\phi(x)|\, \mathbb{E}_x \big[\, |\psi(B_{\tau_2}) - \psi(B_t)|\, \mathbf{1}_{\{\tau_2 < t\}} \big] & \\
\leq \frac{K'}{t} \int_D dx\;\, d(x) \mathbb{E}_x \Big[\mathbf{1}_{\{\tau_2 < t\}} \mathbb{E}_{B_{\tau_2}} \Big[\sup_{0 \leq s \leq t} |\psi(B_s) - \psi(B_0)|\, \Big] \Big] & \\
\leq c \sup_{x \in \partial D} \mathbb{E}_x \Big[\sup_{0 \leq s \leq t} |\psi(B_s) - \psi(x)|\, \Big]. &
\end{aligned}$$

Let $\varepsilon > 0$. As $\psi \in \mathcal{C}(\bar{D})$ and $\bar{D}$ is compact, ψ is uniformly continuous on $\bar{D}$ and hence there exists $\delta > 0$, such that, $|\psi(y) - \psi(x)| < \varepsilon$ for all $x, y \in \bar{D}$ with $|x - y| < \delta$. Then, we have

$$\begin{aligned}
\sup_{x \in \partial D} \mathbb{E}_x \Big[\sup_{0 \leq s \leq t} |\psi(B_s) - \psi(x)|\, \Big] &\leq \varepsilon + \sup_{x \in \partial D} \mathbb{E}_x \Big[\sup_{0 \leq s \leq t} |\psi(B_s) - \psi(x)|\, \mathbf{1}_{\{\sup_{0 \leq s \leq t} |B_s - x| > \delta\}} \Big] \\
&\leq \varepsilon + 2 \, \| \psi \|_\infty \sup_{x \in \partial D} \mathbb{P}_x \Big(\sup_{0 \leq s \leq t} |B_s - x| > \delta \Big).
\end{aligned}$$

And therefore it follows by (41), that

$$\lim_{t \to 0} \sup_{x \in \partial D} \mathbb{E}_x \Big[\sup_{0 \leq s \leq t} |\psi(B_s) - \psi(x)|\, \Big] = 0.$$

This completes the proof. □

6.4. **Proof of Lemma** (2.1)**.** In a first step, we give a representation formula for μ. For $x \in D$, define the measure $h(x, dy)$ on F_1, for any Borel subset $A \subset \mathbb{R}^d$, by

$$h(x, A) = \mathbb{E}_x[\mathrm{e}^{-\tau_1}\, \mathbf{1}_A(B_{\tau_1})].$$

We set $\tilde{\mu}(dy) = \int_D dz\; h(z, dy)$, and we want to prove that $\mu = \tilde{\mu}$.

From potential theory (see [4], Proposition VI.1.15), it is enough to check that $G^1\mu = G^1\tilde{\mu}$ almost everywhere on $\bar{D}$, where the function $G^1\nu$ is the 1-potential of the bounded measure ν on $\bar{D}$, defined by

$$G^1\nu(x) = \int G^1(x, y)\; \nu(dy),$$

where $G^1(x, y) = \int_0^\infty \mathrm{e}^{-t}\, p_t(x, y)\; dt$. Let ψ be a non-negative bounded measurable function defined on $\bar{D}$. We have,

$$\begin{aligned}
\int_D G^1\tilde{\mu}(x)\psi(x)\; dx &= \int_D \psi(x)\; dx \int_0^\infty \mathrm{e}^{-t}\; dt \int_D dz \int p_t(x, y) h(z, dy) \\
&= \int_D \psi(x)\; dx \int_0^\infty \mathrm{e}^{-t}\; dt \int_D dz \int p_t(y, x) h(z, dy) \\
&= \int_D dz\; \mathbb{E}_z\Big[\mathrm{e}^{-\tau_1}\, \mathbb{E}_{B_{\tau_1}}\Big[\int_0^\infty \mathrm{e}^{-t}\, \psi(B_t)\; dt\Big]\Big] \\
&= \int_D dz\; \mathbb{E}_z\Big[\int_{\tau_1}^\infty \mathrm{e}^{-t}\, \psi(B_t)\; dt\Big] \\
&= \int_D dz\; \mathbb{E}_z\Big[\int_0^\infty \mathrm{e}^{-t}\, \psi(B_t)\; dt\Big] - \int_D dz\; \mathbb{E}_z\Big[\int_0^{\tau_1} \mathrm{e}^{-t}\, \psi(B_t)\; dt\Big],
\end{aligned}$$

where we used the symmetry of p for the second and the strong Markov property for the fourth equality. Using again the symmetry of p for the first term of the last equation, we get

$$\begin{aligned}
\int_D dz\; \mathbb{E}_z\Big[\int_0^\infty \mathrm{e}^{-t}\, \psi(B_t)\; dt\Big] &= \int_D dz \int_0^\infty \mathrm{e}^{-t}\; dt \int_D dy\; p_t(z, y)\psi(y) \\
&= \int_D dz \int_0^\infty \mathrm{e}^{-t}\; dt \int_D dy\; p_t(y, z)\psi(y) \\
&= \int_D dy\; \psi(y).
\end{aligned}$$

Let $p_t^{F_1}$ be the density of the transition kernel of B killed on F_1. For $t > 0$, the function $p_t^{F_1}(x, y)$ is symmetric (the proof of this fact is similar to the case where B is a Brownian motion, see for example the proof of Theorem 2.4.3 in [14]). For the second term, we have

$$\begin{aligned}
\int_D dz\; \mathbb{E}_z\Big[\int_0^{\tau_1} \mathrm{e}^{-t}\, \psi(B_t)\; dt\Big] &= \int_D dz \int_0^\infty \mathrm{e}^{-t}\; dt \int_D dy\; p_t^{F_1}(z, y)\psi(y) \\
&= \int_D dz \int_0^\infty \mathrm{e}^{-t}\; dt \int_D dy\; p_t^{F_1}(y, z)\psi(y) \\
&= \int_D dy\; \psi(y)\mathbb{E}_y\Big[\int_0^{\tau_1} \mathrm{e}^{-t}\; dt\Big].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_D G^1\tilde\mu(x)\psi(x)\ dx &= \int_D dy\ \psi(y) - \int_D dy\ \psi(y)\mathbb{E}_y\Big[\int_0^{\tau_1} \mathrm{e}^{-t}\ dt\Big]\\ &= \int_D dy\ \psi(y)\mathbb{E}_y[\mathrm{e}^{-\tau_1}]\\ &= \int_D G^1\mu(y)\psi(y)\ dy.\end{aligned}$$

And we get $G^1\mu = G^1\tilde\mu$ a.e. in D. Thus we have

$$\mu(dy) = \int_D dz\ h(z,dy). \tag{44}$$

In a second step, we prove that for any $z\in D$, the measure $h(z,dy)$ is absolutely continuous with respect to the surface measure on F_1 (recall that $h(z,F_1^c)=0$ for all $z\in D$).

Let ψ be a non-negative continuous function defined on ∂D, with closed support in F_1. We have, for $z\in D$,

$$\begin{aligned}h(z,\psi) &= \mathbb{E}_z[\mathrm{e}^{-\tau_1}\ \psi(B_{\tau_1})]\\ &= \mathbb{E}_z[\mathrm{e}^{-\tau_1}\ \psi(B_{\tau_1})\mathbf{1}_{\{\tau_1<\tau_2\}}] + \mathbb{E}_z[\mathrm{e}^{-\tau_1}\ \psi(B_{\tau_1})\mathbf{1}_{\{\tau_1>\tau_2\}}].\end{aligned} \tag{45}$$

Let $\tau=\tau_1\wedge\tau_2$ be the first hitting time of ∂D. Since $\psi=0$ on F_2,

$$\mathbb{E}_z[\mathrm{e}^{-\tau_1}\ \psi(B_{\tau_1})\mathbf{1}_{\{\tau_1<\tau_2\}}] = \mathbb{E}_z[\mathrm{e}^{-\tau}\ \psi(B_\tau)].$$

From similar arguments to those used in the proof of Proposition 3.11 in [3], there is a (negative) constant c_d (dependent only on d), such that

$$\mathbb{E}_z[\mathrm{e}^{-\tau}\ \psi(B_\tau)] = c_d\int_{\partial D}\psi(y)\frac{\partial g^1(z,y)}{\partial n(y)}\ \sigma(dy), \tag{46}$$

where $g^1(x,y)=\int_0^\infty \mathrm{e}^{-t}\,p_t^{\partial D}(x,y)\ dt$, and $p_t^{\partial D}$ is the density of the transition kernel of the Brownian motion killed on ∂D.

From [12], there exists a continuous additive functional of B, such that

$$\mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-t}\ d\tilde L_t\Big] = \mathbb{E}_x[\mathrm{e}^{-\tau_2}].$$

Let $\tilde G$ be defined as G in Section 2.1 but for F_1 replaced by F_2. Using Theorem 2.2, with F_1 replaced by F_2, we get the existence of a family of universally measurable σ-finite measures $(\tilde H^x, x\in F_2)$, on $(\Omega,\mathcal{F}_\infty)$, such that for any non-negative predictable process $(Z_s, s\geq 0)$ and for any non-negative function $f\in\mathcal{F}_\infty$, such that $f(\delta)=0$, we have

$$\mathbb{E}_z\Big[\sum_{s\in\tilde G} Z_s f\circ i_s\Big] = \mathbb{E}_z\Big[\int_0^\infty Z_s\tilde H^{B_s}(f)d\tilde L_s\Big].$$

From (4), with obvious changes, we deduce that

$$\mathbb{E}_z[\mathrm{e}^{-\tau_1}\ \psi(B_{\tau_1})\mathbf{1}_{\{\tau_1>\tau_2\}}] = \mathbb{E}_z\Big[\int_0^{\tau_1}\mathrm{e}^{-s}\ \tilde H^{B_s}[\mathrm{e}^{-\tau_1}\ \psi(e_{\tau_1})\mathbf{1}_{\{\tau_1<\infty\}}]\ d\tilde L_s\Big].$$

Let $\varepsilon>0$ and consider the compact set

$$K = \{x\in\bar D; d(x,F_1)\leq\varepsilon, d(x,F_1)\leq d(x,F_2)\}, \tag{47}$$

and $\tau_K = \inf\{t > 0, B_t \in K\}$ the hitting time of K. For $x \in F_2$, we have, using the strong Markov property of $\tilde{H}^x$ with respect to Q_t^2, the kernel of the reflected Brownian motion killed on F_2, (see [12], Theorem 5.1),

$$\begin{aligned}
\tilde{H}^x[\mathrm{e}^{-\tau_1}\,\psi(e_{\tau_1})\mathbf{1}_{\{\tau_1<\infty\}}] &= \tilde{H}^x[\mathrm{e}^{-\tau_K}\,\mathbb{E}_{e_{\tau_K}}[\mathrm{e}^{-\tau_1}\,\psi(B_{\tau_1})\mathbf{1}_{\{\tau_1<\tau_2\}}]] \\
&= \tilde{H}^x[\mathrm{e}^{-\tau_K}\,\mathbb{E}_{e_{\tau_K}}[\mathrm{e}^{-\tau}\,\psi(B_\tau)]] \\
&= \tilde{H}^x\Big[\,\mathrm{e}^{-\tau_K}\,c_d\int_{\partial D}\psi(y)\frac{\partial g^1(e_{\tau_K},y)}{\partial n(y)}\,\sigma(dy)\Big] \\
&= c_d\int_{\partial D}\psi(y)\tilde{H}^x\Big[\,\mathrm{e}^{-\tau_K}\,\frac{\partial g^1(e_{\tau_K},y)}{\partial n(y)}\Big]\,\sigma(dy),
\end{aligned}$$

where we used (46) for the second equality. From this last expression, (46) and (45), we deduce that there exists a measurable non-negative function $\tilde{f}$ defined on $D \times F_1$ such that for $z \in D$,

$$h(z,\psi) = \int_{F_1}\tilde{f}(z,y)\psi(y)\sigma(dy).$$

From (44), we deduce that μ is absolutely continuous with respect to σ and the density is given by $\rho(y) = \int_D \tilde{f}(z,y)\,dz$, that is

$$\rho(y) = c_d\int_D dz\,\left[\frac{\partial g^1(z,y)}{\partial n(y)} + \mathbb{E}_z\Big[\int_0^{\tau_1} d\tilde{L}_s\;\mathrm{e}^{-s}\,\tilde{H}^{B_s}[\mathrm{e}^{-\tau_K}\,\frac{\partial g^1(e_{\tau_K},y)}{\partial n(y)}]\Big]\right].$$

Corollary 6.11. *If $\partial F = \emptyset$, then the function ρ is bounded.*

Proof. We keep the notations of this section. Since $\bar{F}_1 \cap \bar{F}_2 = \emptyset$, we can choose $\varepsilon > 0$ small enough so that for any $(x,y) \in F_1 \times F_2$, $|x-y| \geq 3\varepsilon$. In particular K defined by (47) is in fact equal to $\{x \in \bar{D}; d(x,F_1) \leq \varepsilon\}$.

Let P_D be the Poisson kernel of the Brownian motion in D. There exists a positive constant C_D, such that for any $(z,y) \in D \times \partial D$,

$$P_D(z,y) \leq C_D d(z,\partial D)|z-y|^{-d}. \tag{48}$$

As $\displaystyle\int_{\partial D}\sigma(dy)\;P_D(z,y)\psi(y) = \mathbb{E}_z[\psi(B_\tau)]$, for any $\psi \in \mathcal{B}_+(\partial D)$, we deduce from (46) that

$$0 \leq c_d\frac{\partial g^1(z,y)}{\partial n(y)} \leq P_D(z,y). \tag{49}$$

From this inequality and (48), we deduce easily that $c_d\displaystyle\int_D dz\;\frac{\partial g^1(z,y)}{\partial n(y)}$ is bounded from above by a finite constant, say C_0, independent of $y \in F_1$. Since by construction $d(e_{\tau_K},\partial D) > \varepsilon$ (on $\{\tau_K < \infty\}$ under $\tilde{H}^x$), we get that for any $x \in F_2$, $y \in F_1$,

$$\begin{aligned}
c_d\tilde{H}^x[\mathrm{e}^{-\tau_K}\,\frac{\partial g^1(e_{\tau_K},y)}{\partial n(y)}] &\leq \tilde{H}_x[\tau_K<\infty]\sup_{\{(z,y');d(z,\partial D)\geq\varepsilon\;,y'\in F_1\}} c_d\frac{\partial g^1(z,y')}{\partial n(y')} \\
&= c\tilde{H}^x[\tau_K<\infty],
\end{aligned}$$

for a finite constant c independent of $x \in F_2$ and $y \in F_1$, thanks to (49) and (48). Arguing as in the proof of Lemma 8.3 of [6], we have that

$$\sup_{x\in F_2}\tilde{H}^x\Big[\tau_K<\infty\Big] < \infty.$$

This implies that $c_d\tilde{H}^x[\mathrm{e}^{-\tau_K}\dfrac{\partial g^1(e_{\tau_K},y)}{\partial n(y)}]$ is bounded from above for $x \in F_2$ and $y \in F_1$ say by C_1. In particular we have

$$\rho(y) \leq C_0 + C_1 \int_D dz\ \mathbb{E}_z\left[\int_0^\infty \mathrm{e}^{-s}\, d\tilde{L}_s\right] = C_0 + C_1 \int_D dz\ \mathbb{E}_z\left[\mathrm{e}^{-\tau_2}\right],$$

using the definition of $\tilde{L}$. This last inequality implies that ρ is bounded. $\square$

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