SELF-SIMILAR BOUNDARY BLOW-UP FOR HIGHER-ORDER QUASILINEAR PARABOLIC EQUATIONS

V.A. GALAKTIONOV AND A.E. SHISHKOV

Abstract. We study evolution properties of boundary blow-up for $2m$th-order quasilinear parabolic equations in the case where, for homogeneous power nonlinearities, the typical asymptotic behaviour is described by exact or approximate self-similar solutions. Existence and asymptotic stability of such similarity solutions are established by energy estimates and contractivity properties of the rescaled flows.

Further asymptotic results are proved for more general equations by using energy estimates related to Saint-Venant's principle. The established estimates of propagation of singularities generated by boundary blow-up regimes are shown to be sharp by comparing with various self-similar patterns.

1. Introduction: boundary blow-up and main results

Energy estimates are an effective tool in the existence and asymptotic analysis for a wide class of nonlinear parabolic and elliptic equations; see surveys on energy techniques for higher-order PDEs in [15, 3, 4, 20] and an extended list of references in [5]. Energy estimates via Saint-Venant's principle play a special role for higher-order quasilinear equations that do not obey the Maximum Principle and related order-preserving and comparison theorems. These special properties simplify the existence-uniqueness and asymptotic analysis for the second-order equations.

In the present paper we study the blow-up behaviour for $2m$th-order ($m \geq 2$) quasilinear parabolic equations. A typical model is the higher-order *generalized p-Laplacian equation*

$$(|u|^{q-1}u)_t = (-1)^{m+1} \textstyle\sum_{|\beta|=m} D^\beta (|D^m u|^{p-1} D^\beta u), \quad \text{with } p > q > 0. \tag{1.1}$$

Equation (1.1) is considered, for $t \in (0, T)$, in the complement of a smooth bounded domain in $\mathbf{R}^N$ and finite time blow-up singularities as $t \to T^-$ are prescribed on the boundary creating the phenomenon of the *boundary blow-up*. Evolution blow-up singularities play the important role in the general theory of nonlinear PDEs and other infinite-dimensional dynamical systems, and often generate the most difficult and delicate asymptotic patterns admitted by such equations. The blow-up phenomena occur in many applications in physics, mechanics and combustion theory. We refer to the surveys and

Date: June 11, 2004.

1991 *Mathematics Subject Classification.* 35K55, 35K65.

Key words and phrases. $2m$th-order quasilinear parabolic equations, Saint-Venant's principle, energy estimates, global, regional and single-point blow-up, asymptotic behaviour, similarity solutions.

books [2, 8, 9, 12, 17], where further references on the theory and applications to a wide range of other reaction-diffusion problems can be found.

For the second-order parabolic equations, i.e., for $m = 1$ in (1.1), the boundary blow-up is well understood and a classification of *global*, *regional* and *single-point blow-up* is available. The type of blow-up is known to depend on the rate of growth on the boundary; see [17, Chapters 3, 6], [9, Chapters 9, 10] and [20] (energy estimates) for typical results and earlier references going back to the 1960s. The most of these results on blow-up was proved by using various techniques based on the Maximum Principle that cannot be extended to the higher-order case $m \geq 2$.

For the higher-order parabolic PDEs, the analysis of generic blow-up singularities becomes more difficult and has begun recently. In particular, basic properties of boundary blow-up for equations such as (1.1) in the critical "linear" case

$$p = q > 0 \tag{1.2}$$

were studied in [5, 6]. In this case the most interesting blow-up behaviour occurs for boundary data given by the exponential functions

$$u \sim \exp\{(T-t)^{-1/l}\} \quad \text{as} \;\; t \to T^-, \tag{1.3}$$

with an exponent $l > 0$. The principal change of the blow-up type from the single point to global occurs at $l = l_0 = m(p+1) - 1$ corresponding to the regional blow-up. Another important feature of the case (1.2) is that the formation of blow-up singularities is described by similarity solutions of a *quadratic Hamilton-Jacobi equation*, a striking *degeneracy* phenomenon of parabolic equations near the blow-up time. Note that this degeneracy occurs for the classical second-order *heat equation* $u_t = \Delta u$ (see [17, Chapter 3], [9, Chapters 9, 10]) and for the $2m$th-order one [5, Section 3]

$$u_t = (-1)^{m+1} \Delta^m u,$$

for which the blow-up regional localization occurs at $l = 2m - 1$ in (1.3) ($l = 1$ for the heat equation).

In this paper we consider equations like (1.1) in the *quasilinear* case

$$p > q > 0, \tag{1.4}$$

where the asymptotic behaviour is shown to be described by exact or approximate self-similar blow-up solutions of equations like (1.1). This is a principal difference between the parameter ranges (1.2) and (1.4) to be clearly explained later on. Furthermore, this also demands an essential modification and extension of the energy methods for the case (1.4). In particular, we establish that, unlike the exponential blow-up (1.3), for $p > q > 0$, the most interesting blow-up phenomena occur in the class of power functions

$$u \sim (T-t)^{-l} \quad \text{as} \;\; t \to T^-, \tag{1.5}$$

and the critical exponent of the *regional blow-up* is now $l = l_0 = \frac{q+1}{p-q}$. Some other sufficient (but not sharp) conditions for the localized blow-up regimes can be found in [19]. We also consider the exponential blow-up like (1.3) and show that the asymptotics are described by special approximate self-similar patterns.

Our energy approach makes it possible to treat boundary blow-up regimes in a general form to obtain sharp estimates of propagation of the corresponding blow-up waves.

1.1. **Statement of the problem.** Stating our problem, we follow the lines of paper [5], where more details on the statement, mathematical properties, extensions and applications can be found. Given the open complement $\Omega = \{|x| > 1\}$ of the unit ball $B = \{|x| < 1\}$ in $\mathbf{R}^N$, in the cylindrical domain $Q = \Omega \times (0, T)$, $T \in \mathbf{R}_+$, we consider the Cauchy-Dirichlet problem for a general quasilinear $2m$th-order parabolic equation with an integer $m \geq 1$ and exponent $q > 0$

$$(|u|^{q-1}u)_t + \textstyle\sum_{|\alpha|=m} (-1)^m D_x^\alpha a_\alpha(x, t, u, ..., D_x^m u) = 0, \tag{1.6}$$

$$u(x, 0) = u_0(x) \in L_{q+1}(\Omega) \quad \text{in } \Omega, \tag{1.7}$$

$$D_x^\alpha (u - f) = 0 \quad \text{on } \partial\Omega \times (0, T) \text{ for any } |\alpha| \leq m - 1. \tag{1.8}$$

The functions $a_\alpha(x, t, \xi)$ are continuous, and the elliptic operator on the left-hand side of (1.6) satisfies the growth and coercivity conditions so there exist positive constants $p \geq q$ and d_1, d_2 such that

$$\sum_{|\alpha|=m} a_\alpha(x, t, \xi)\xi_\alpha \geq d_1(\sum_{|\beta|=m} |\xi_\beta|)^{p+1} \quad \text{and } |a_\alpha(x, t, \xi)| \leq d_2(\sum_{|\beta|=m} |\xi_\beta|)^p \ \forall \alpha, \tag{1.9}$$

for all $(x, t, \xi) \in \bar{Q} \times \mathbf{R}^{n(m)}$, where $n(m)$ denotes the number of distinct multiindices of the length not exceeding m. Here $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = \alpha_1 + ... + \alpha_N$ is multiindex, $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}$ and $D_x^m u = \{D_x^\alpha u,\ |\alpha| = m\}$. The function $f(x, t)$, determining the boundary values, is assumed to be extended from the lateral boundary $\partial\Omega \times (0, T)$ into the domain Q such that, for any $T_0 < T$, there holds

$$f(\cdot, t) \in C([0, T_0];\ L_{q+1}(\Omega)) \cap L_{p+1}(0, T_0;\ W_{p+1}^m(\Omega)), \tag{1.10}$$

$$f_t(\cdot, t) \in L_1(0, T_0;\ L_{q+1}(\Omega)) \cap L_r(0, T_0; L_r(\Omega)),\ r = \tfrac{p+1}{p+1-q}. \tag{1.11}$$

We assume that the boundary function $f(\cdot, t)$ *blows up* as $t \to T^-$. The rate of blow-up is characterized by the following function defined for $t \in (0, T)$:

$$\begin{aligned} F(t) = &\sup_{0<\tau<t} \left(\int_\Omega |f(x, \tau)|^{q+1} dx \right) + \int_0^t \int_\Omega |D_x^m f(x, \tau)|^{p+1} dx d\tau \\ &+ \left(\int_0^t \left(\int_\Omega |f_\tau(x, \tau)|^{q+1} dx \right)^{\frac{1}{q+1}} d\tau \right)^{q+1} \equiv \tilde{F}^{(1)}(t) + \tilde{F}^{(2)}(t) + \tilde{F}^{(3)}(t) \to \infty \end{aligned} \tag{1.12}$$

as $t \to T^-$. We note that "optimal" extensions of the boundary data to $\Omega \times (0, T)$ will play the key role in obtaining sharp blow-up estimates. Given $S \subset \partial\Omega$, by $W_{p+1}^m(\Omega, S)$ we denote the closure in the norm of the Sobolev space $W_{p+1}^m(\Omega)$ of the subset of functions from $C^\infty(\Omega)$ vanishing in a neighbourhood of S. We set $W_{p+1}^m(\Omega, \emptyset) \equiv W_{p+1}^m(\Omega)$ and denote by $\langle \cdot, \cdot \rangle$ the duality product between $\left(W_{p+1}^m(\Omega, \partial\Omega)\right)^*$ and $W_{p+1}^m(\Omega, \partial\Omega)$.

Local existence of generalized (weak) solutions of the mixed problem for equations (1.6) is a classical important problem of the general theory; see Lions's book [13], Bernis's results [3, 4] and further related references in [5]. We use a standard definition of energy solutions.

Definition 1.1. A function $u(x,t)$ is said to be a *generalized energy solution* of the problem (1.6)–(1.8), if, for any $T_0 < T$, there holds

(i) $u - f \in L_{p+1}(0,T_0; W^m_{p+1}(\Omega, \partial\Omega)) \cap C([0,T_0]; L_{q+1}(\Omega))$,

(ii) $(|u|^{q-1}u)_t \in L_{(p+1)/p}(0,T_0; (W^m_{p+1}(\Omega,\partial\Omega))^*)$,

(iii) u satisfies the initial condition (1.7) and the integral identity

$$\int_0^{T_0} \langle (|u|^{q-1}u)_t, \eta \rangle dt + \int_0^{T_0} \int_\Omega \sum_{|\alpha|=m} a_\alpha(x,t,u(x,t),...,D^m_x u(x,t)) D^\alpha_x \eta(x,t) dxdt = 0, \tag{1.13}$$

where $\eta(x,t) \in L_{p+1}(0,T_0; W^m_{p+1}(\Omega,\partial\Omega))$ is an arbitrary function.

We next introduce typical notations related to blow-up phenomena. We say that $y \in \overline{\Omega}$ is a *singular point* of the energy solution $u(x,t)$ if, for any small $\varepsilon > 0$,

$$\sup_{0<t<T} \left(\int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,t)|^{q+1} dx + \int_0^t \int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,\tau)|^{p+1} dx d\tau \right) = \infty.$$

The subset $\Omega_S \equiv \Omega_S(u)$ of all such singular points is the *singularity* or the *blow-up subset* of solution $u(x,t)$ of problem (1.6)–(1.8). $\Omega_R \equiv \overline{\Omega} \setminus \Omega_S$ is the corresponding *non-singular subset*. For sufficiently regular solutions, it is expected that $u(x_0,t)$ is uniformly bounded as $t \to T^-$ at any interior point $x_0 \in \Omega_R = \Omega \setminus \Omega_S$ and vice-versa $u(x_0,t)$ is not uniformly bounded as $t \to T^-$ for any $x_0 \in \Omega_S$. For such regular solutions, the blow-up set Ω_S is known as the *localization domain* of the solution (or of the boundary blow-up regime). In the one-dimensional case, $L = \operatorname{meas} \Omega_S$ is called the *localization length* of the blow-up regime, see terminology in [17, Chapter 3]. Given bounded initial data, we say that the blow-up regime, prescribed by the function $f(x,t)$ on $\partial\Omega$, is *localized* (respectively, *non-localized*) if it is bounded and $\operatorname{meas}\Omega_S < \infty$ (respectively, $\operatorname{meas}\Omega_S = \infty$). Localized blow-up regime is called the S-regime if $\operatorname{meas}\Omega_S \in (0,\infty)$ (regional blow-up) and the LS-regime if $\operatorname{meas}\Omega_S = 0$ (this includes the case of single point blow-up). A non-localized blow-up regime with $\operatorname{meas}\Omega_S = \infty$ is also called the HS-regime (global blow-up).

1.2. **Main results and plan of the paper.** We study the nonlinear parameter range (1.4) and show that the structure of blow-up singularities is entirely different and, unlike the "linear" case (1.2), can be described by self-similar solutions of special homogeneous parabolic equations like (1.1).

It turns out that regional and global blow-up can occur for special power blow-up boundary data described as $t \to T^-$ in terms of the function (1.12),

$$F(t) \le F_1(t) = w(T-t)^{-l} \quad \text{with } l = \tfrac{q+1}{p-q} + a \text{ and any constant } a \ge 0, \tag{1.14}$$

where $w > 0$ is a fixed constant. We will present examples of higher-order equations (1.1) showing that this is a sharp result and, for $a < 0$ in (1.14), single-point blow-up occurs.

Next, we concentrate on the non-localized energy solutions and establish estimates of the effective length of propagation of such blow-up waves. As a characterization of the

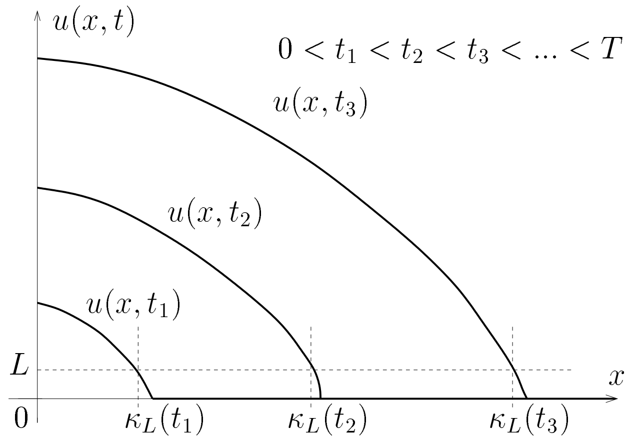


FIGURE 1. Illustration of the definition of $\kappa_L(t)$.

blow-up penetration in the general problem, we introduce the so-called "effective length" of propagation of blow-up waves. Given a sufficiently large constant $L \gg 1$, for any $t \approx T^-$, we define a continuous function $\kappa_L(t)$ as follows:

$$\int_0^t \int_{\{|x|>\kappa_L(t)\}} |u(x,\tau)|^{p+1} dx d\tau = L. \tag{1.15}$$

See Figure 1 for an illustration in one dimension. In the theory of heat transfer, $\kappa_L(t)$ has a natural physical sense as the time-dependent length of the spatial energy distribution. For non-localized HS-regimes, $\kappa_L(t)$ is unbounded as $t \to T^-$.

Theorem 1.1. *Let* (1.4) *and condition* (1.14) *hold. Then, for a fixed* $L \gg 1$, *for any energy solution* $u(x,t)$ *of* (1.6) $-$ (1.8), *as* $t \to T^-$,

$$\kappa_L(t) \le c_1[w(T-t)^{-a}]^{l_1}, \quad \textit{where} \;\; l_1 = \tfrac{p-q}{m(p+1)(q+1)+N(p-q)}, \tag{1.16}$$

with a constant $c_1 = c_1(L) > 0$ *independent of* w.

For convenience, we state separately the critical regional blow-up case.

Corollary 1.1. *If* $l = l_0 = \frac{q+1}{p-q}$, *then* $\kappa_L(t) \le c_1 w^{l_1}$ *for all* $t \in (0,T)$, *i.e., the blow-up regime* (1.14), $l = l_0$ *is, at most, a localized S-regime.*

We next describe the unbounded propagation for exponential blow-up.

Theorem 1.2. *Let* (1.4) *hold and, for* $t \approx T^-$,

$$F(t) \le F_2(t) \equiv \exp\{w(T-t)^{-1/l}\}, \tag{1.17}$$

where $l > 0$. *Then for* $L \gg 1$, *as* $t \to T^-$,

$$\kappa_L(t) \le c_1(T-t)^{\frac{q+1}{a_1}}[w(T-t)^{-1/l}]^{-\frac{q+1}{a_1}} \exp\{\tfrac{p-q}{a_1} w(T-t)^{-1/l}\}, \tag{1.18}$$

where $a_1 = m(p+1)(q+1) + N(p-q)$.

These two theorems are proved in Sections 4–6. In the next Sections 2 and 3 we present an asymptotic blow-up theory for a particular higher-order quasilinear parabolic equation, which shows that the general estimates of blow-up propagation in Theorems 1.1, 1.2 are sharp and cannot be improved.

2. Existence of blow-up similarity solutions for the higher-order p-Laplacian equation

This section is devoted to the existence and asymptotic stability of similarity solutions of a homogeneous equation from the class (1.6), namely, of the $2m$th-order p-Laplacian equation (1.1) with $q = 1$, $p > 1$,

$$u_t = \Delta_{m,p}(u) \equiv (-1)^{m+1} \textstyle\sum_{|\alpha|=m} D^\alpha(|D^m u|^{p-1} D^\alpha u) \quad \text{in } Q = \Omega \times (-1, 0). \tag{2.1}$$

By translation in time, we fix the blow-up time $T = 0$, and then bounded initial data $u_0(x)$ are prescribed at $t = -1$.

2.1. **Blow-up self-similar solutions in 1D.** We begin our similarity analysis for the one-dimensional equation

$$u_t = \Delta_{m,p}(u) \equiv (-1)^{m+1} D^m(|D^m u|^{p-1} D^m u) \quad \text{in } Q = \mathbf{R}_+ \times (-1, 0) \tag{2.2}$$

where $p > 1$, $m \geq 2$ is integer and D denotes d/dx. It is convenient to prescribe the boundary conditions generating blow-up by using the similarity solutions. Equation (2.2) is invariant under a group of scaling transformations $u \mapsto \lambda^n u$, $x \mapsto x/\lambda^\beta$, $t \mapsto t/\lambda$ with $l > 0$, where n is arbitrary fixed exponent and

$$\beta = \tfrac{n(p-1)+1}{m(p+1)}. \tag{2.3}$$

We are interested in self-similar solutions of (2.2) of the form (set $\lambda = -t$)

$$u_*(x, t) = (-t)^n f(y), \quad y = x/(-t)^\beta, \tag{2.4}$$

where f solves the following $2m$th-order ODE:

$$\mathbf{B}_{m,p}(f) \equiv \Delta_{m,p}(f) - \beta f' y + nf = 0 \quad \text{for } y > 0, \quad f(\infty) = 0. \tag{2.5}$$

Profile $f(y)$ has to vanish at infinity that is a natural assumption from combustion theory meaning that the blow-up wave propagates into the unperturbed, “cold” medium.

We assume that $n < 0$ in (2.4) so that if $a_0 = f(0) \neq 0$, then the boundary blow-up occurs in the sense that

$$u_*(0, t) = a_0(-t)^n \to \infty \quad \text{as } t \to 0^-.$$

In general, differentiating (2.4) yields the following Dirichlet boundary conditions:

$$D^i u(0, t) = a_i(-t)^{\gamma_i} \quad \text{for } i = 0, 1, 2, ..., m-1, \quad \text{where } \gamma_i = n - i\beta, \ \textstyle\sum_{(i)} a_i^2 \neq 0, \tag{2.6}$$

and $a_i = f^{(i)}(0)$. One can see that $\gamma_i < 0$ for all i so that $D^i u$ blows up at the origin if $a_i \neq 0$. More general regular (in the classical Lopatinskii sense) boundary blow-up conditions can be considered with proper boundary ($2m$-1)th-order operators B_i, $i =$

$0, 1, 2, ..., m-1$. In particular, the Neumann conditions, where B_i depend on derivatives of order $m, ..., 2m-1$ are always regular.

Assuming that blow-up of u_* or its derivatives occur at $x = 0$, we have from the rescaled variable y in (2.4) that the effective propagation length is $x_0(t) \sim (-t)^\beta$ so, as $t \to 0^-$,

(i) blow-up is *single point* (LS-regime) if $\beta > 0$, i.e., $n > -\frac{1}{p-1}$,

(ii) blow-up is *regional* (S-regime) if $\beta = 0$, i.e., $n = -\frac{1}{p-1}$, and

(iii) blow-up is *global* (HS-regime) if $\beta < 0$, i.e., $n < -\frac{1}{p-1}$.

2.2. **Proof of existence of similarity solutions.** Without loss of generality we restrict our attention to simpler boundary conditions (2.6). It follows from (2.4) that the similarity profile f is a solution of the ODE (2.5) with boundary conditions

$$D^i f(0) = a_i, \quad i = 0, 1, 2, ..., m-1 \qquad (\textstyle\sum_{(i)} a_i^2 \neq 0), \tag{2.7}$$

and we will prove existence and uniqueness of the solution f in a suitable functional setting. Note that, for $m = 1$, (2.5) is a second-order ODE, which is a classical model occurring in the problems of the heat-mass transfer, combustion of solid fuels, dilatant and non-Newtonian liquids as well as plasma physics; see Kalashnikov's survey [10]. It is important that, for $m = 1$, in view of the scaling invariance, (2.5) always reduces to a first-order ODE which can be studied on the phase-plane (see computations below). Such an analysis was first performed by G.I. Barenblatt more than fifty years ago; see [1] and references in [17, Chapter 3].

In the higher-order case $m > 1$, the phase-plane approach does not apply. Of course, (2.5) remains invariant under a scaling group, and if $f(y)$ is a solution, then

$$f_\lambda(y) = \lambda^\mu f(\tfrac{y}{\lambda}) \quad \text{is a solution for any } \lambda > 0 \text{ with } \mu = \tfrac{m(p+1)}{p-1}.$$

Therefore, the corresponding change of variables using the invariants of the group

$$f(y) = e^{\mu\eta}\varphi(\eta), \quad \eta = \ln y, \tag{2.8}$$

reduces (2.5) to an autonomous $2m$th-order ODE. Setting $\varphi' = P(\varphi)$ yields a ($2m$-1)-order ODE for P, which can be resolved relative to the highest-order derivative to obtain

$$P^{(2m-1)} = G(\varphi, P', ..., P^{(2m-2)}). \tag{2.9}$$

As we have mentioned, for $m = 1$, it has a well understood phase plane. For $m \geq 2$, it is difficult to show that (2.9) admits a suitable orbit corresponding to a similarity profile f. Therefore, in the higher-order case, we apply a different technique to establish the existence of a similarity solution.

Theorem 2.1. *Under the condition*

$$2n + \beta < 0, \tag{2.10}$$

problem (2.5), (2.7) *has a unique solution* $f \in W^m_{p+1}(\mathbf{R}_+) \cap L_2(\mathbf{R}_+)$.

Proof. We construct f as the limit as $l \to \infty$ of solutions $\{v_l\}$ of (2.5) on bounded interval $(0, l)$, $l > 1$, with the conditions

$$D^i v_l(0) = a_i, \quad D^i v_l(l) = 0, \quad i = 0, 1, ..., m-1. \tag{2.11}$$

We define an arbitrary function $g \in W^m_{p+1}(\mathbf{R}_+)$ satisfying (2.11) with $l=1$ and such that $g=0$ on $[1,\infty)$. We next introduce the space $W^{m,0}_{p+1}(0,l)$ as the closure of the subset of compactly supported smooth functions on $(0,l)$ in the norm of $W^m_{p+1}(0,l)$, and we will look for a solution v_l for $l>1$ in the form $v_l = w_l + g$, where $w_l \in W^{m,0}_{p+1}(0,l)$.

One can see that the operator in (2.5), $\mathbf{B}_{m,p} : W^{m,0}_{p+1}(0,l) \to W^{-m}_q(0,l)$ with $q = \frac{p+1}{p}$, is continuous, coercive and strong monotone provided that (2.10) holds. The coercivity follows via integration by parts,

$$\langle \mathbf{B}_{m,p}(v), v\rangle = \textstyle\int_0^l |D^m v|^{p+1} dx - \frac{2n+\beta}{2} \int_0^l v^2 dx, \tag{2.12}$$

and the same argument applies to prove the strong (coercive) monotonicity.

Therefore, due to the standard monotonicity theory [13, Chapter 2], there exists the unique solution v_l. Putting the test function $v_l - g$ into the integral identity defining weak solutions of problem (2.5), (2.7), after straightforward calculations by the Hölder and Young inequalities, we obtain that v_l satisfies the uniform bound

$$\textstyle\int_0^l |D^m v_l|^{p+1} dx + |2n+\beta| \int_0^l v_l^2 dx \le K \quad \text{for any } \; l > 1, \tag{2.13}$$

where the constant $K>0$ depends on the norm of g in W^m_{p+1} and is independent of l. The rest of the proof follows from the strong convergence of the sequence $\{v_l\}$ as $l \to \infty$ to be proved below.

Proposition 2.1. *Given any large fixed $l_0 > 0$, $\{v_L, L = 1, 2, ...\}$ is a Cauchy sequence in $W^m_{p+1}(0,l_0)$.*

Proof. We need to estimate the difference $v_L - v_M$ in $W^m_{p+1}(0,l_0)$. To this end, we fix a constant $\tau > l_0$, $\delta > 0$, $\tau + \delta < L, M$, and introduce a cut-off function $\xi(s) \ge 0$ satisfying $\xi(s) = 1$ for $0 \le s < \tau$, $\xi(s) = 0$ for $s > \tau + \delta$ and such that $|\xi^{(i)}(s)| \le c_0/\delta^i$ for all $i = 0, 1, ..., m-1$. Next, we take the test function $(v_L - v_M)\xi$ in the integral identities defining weak solutions v_L and v_M and subtract the resulting equalities to obtain

$$\begin{aligned} &\int_{\mathbf{R}_+} \{(|D^m v_L|^{p-1} D^m v_L - |D^m v_M|^{p-1} D^m v_M) D^m[(v_L - v_M)\xi] \\ &\quad + [\beta y (v_L - v_M)' - n(v_L - v_M)](v_L - v_M)\xi(y)\} dy = 0. \end{aligned} \tag{2.14}$$

Integrating by parts and using the strong monotonicity of the operator $\Delta_{m,p}$ and Lagrange's formula yields

$$\begin{aligned} &\int_{\mathbf{R}_+} [|D^m v_L - D^m v_M|^{p+1} - \tfrac{2n+\beta}{2} |v_L - v_M|^2] \xi dy \le C \int_\tau^{\tau+\delta} (v_L - v_M)^2 |\xi'| y dy \\ &+ C \int_\tau^{\tau+\delta} |D^m v_L - D^m v_M| (|D^m v_L| + |D^m v_M|)^{p-1} \sum_{i \le m-1} |D^i (v_L - v_M)| \, |D^{m-i} \xi(y)| dy \equiv R. \end{aligned} \tag{2.15}$$

Later on, by C we denote different positive constants which depend on parameters of the given problem (2.5), (2.7) and the constant K from the uniform bound (2.13), but are independent of L, M, τ and δ. Using the Hölder and Young inequalities, estimate (2.13) and elementary properties of the cut-off functions ξ, the right-hand side of (2.15) is estimated as follows:

$$
\begin{aligned}
R \le\; & C\left(\int_\tau^{\tau+\delta} |D^m v_L - D^m v_M|^{p+1} dy\right)^{\frac{1}{p+1}} \left(\int_\tau^{\tau+\delta} (|D^m v_L|^{p+1} + |D^m v_M|^{p+1}) dy\right)^{\frac{p-1}{p+1}} \\
& \times \sum_{i\le m-1} \left(\int_\tau^{\tau+\delta} |D^i(v_L - v_M)|^{p+1} dy\right)^{\frac{1}{p+1}} \sup|D^{m-i}\xi(y)| + C\int_\tau^{\tau+\delta} |\xi'|(v_L - v_M)^2 y dy \\
& \le C\int_\tau^{\tau+\delta} |D^m(v_L - v_M)|^{p+1} dy + \delta^{-1} C\int_\tau^{\tau+\delta} (v_L - v_M)^2 y dy \\
& +C\sum_{i\le m-1}\left(\delta^{-(m-i)(p+1)}\int_\tau^{\tau+\gamma} |D^i(v_L - v_M)|^{p+1} dy\right)^{\frac{1}{p}}.
\end{aligned}
\tag{2.16}
$$

By Gagliardo-Nirenberg's and Young's inequalities, we have

$$
\begin{aligned}
& \left(\delta^{-(m-i)(p+1)}\int_\tau^{\tau+\delta} |D^i(v_L - v_M)|^{p+1} dy\right)^{\frac{1}{p}} \le C\delta^{-\frac{(m-i)(p+1)}{p}} \left(\int_\tau^{\tau+\delta} |D^m(v_L - v_M)|^{p+1} dy\right)^{\frac{\theta_i}{p}} \\
& \quad \times \left(\int_\tau^{\tau+\delta} |v_L - v_M|^2 dy\right)^{\frac{(1-\theta_i)(p+1)}{2p}} + C\delta^{\nu}\left(\int_\tau^{\tau+\delta} |v_L - v_M|^2 dy\right)^{\frac{p+1}{2p}} \\
& \quad \le C\int_\tau^{\tau+\delta} |D^m(v_L - v_M)|^{p+1} dy + C\delta^{-\mu_i}\left(\int_\tau^{\tau+\delta} |v_L - v_M|^2 dy\right)^{\rho_i} \\
& \quad\quad + C\delta^{-\nu}\left(\int_\tau^{\tau+\delta} |v_L - v_M|^2 dy\right)^{\frac{p+1}{2p}},
\end{aligned}
$$

where $\theta_i = \frac{2(p+1)i+p-1}{2(p+1)m+p-1}$, $\mu_i = \frac{(m-i)(p+1)}{p-\theta_i}$, $\rho_i = \frac{(1-\theta_i)(p+1)}{2(p-\theta_i)}$ and $\nu = \frac{2(p+1)m-1}{p} > 0$. Substituting the last estimate into (2.16) and next into (2.15) yields

$$
\begin{aligned}
I(\tau) &\equiv I_1(\tau) + |\tfrac{2n+\beta}{2}|\, I_2(\tau) \equiv \int_0^\tau |D^m v_L - D^m v_M|^{p+1} dy + |\tfrac{2n+\beta}{2}| \int_0^\tau |v_L - v_M|^2 dy \\
&\le C(I_1(\tau+\delta) - I_1(\tau)) + C\textstyle\sum_{i\le m-1} \delta^{-\mu_i}(I_2(\tau+\delta) - I_2(\tau))^{\rho_i} \\
&+ \delta^{-1} C\tfrac{(\tau+\delta)}{\delta}\,[I_2(\tau+\delta) - I_2(\tau)] + \delta^{-\nu} C[I_2(\tau+\delta) - I_2(\tau)]^{\frac{p+1}{2p}}.
\end{aligned}
\tag{2.17}
$$

We now choose $\delta = \tau$ and then (2.17) takes the form

$$
\begin{aligned}
I(\tau) &\le C[I_1(2\tau) - I_1(\tau)] + C[I_2(2\tau) - I_2(\tau)]^{(p+1)/2p}\tau^{-\nu} \\
&+ C[I_2(2\tau) - I_2(\tau)] + C\textstyle\sum_{i\le m-1}\tau^{-\mu_i}(I_2(2\tau) - I_2(\tau))^{\rho_i}.
\end{aligned}
$$

Using the uniform estimate (2.13), we deduce

$$
I(\tau) \le C(I(2\tau) - I(\tau)) + C\tau^{-\gamma}, \quad \text{where } \gamma = \min_{i\le m-1}\{\nu,\ \tfrac{(m-i)(p+1)}{p-\theta_i}\}. \tag{2.18}
$$

Finally, from (2.18) we derive the following *linear functional inequality*:

$$I(\tau) \le \lambda I(2\tau) + C_1 (2\tau)^{-\gamma} \quad \text{for any } \tau \text{ such that } 2\tau < \min\{L, M\}, \tag{2.19}$$

where $\lambda = \frac{C}{1+C} < 1$ and $C_1 = 2^\gamma \frac{C}{1+C}$ with C from (2.18). Iterating inequality (2.19) j times yields

$$I(\tau) \le \lambda^j I(2^j \tau) + C_2 \tau^{-\gamma} \quad \text{if} \;\; 2^j \tau < \min\{L, M\}. \tag{2.20}$$

Let us fix $\tau = l_0$ and an arbitrarily small $\varepsilon > 0$. Now fix $\tau_1 \ge l_0$ such that

$$C_2 \tau_1^{-\gamma} \le \tfrac{\varepsilon}{2} \implies \tau_1 = \tau_1(\varepsilon, l_0). \tag{2.21}$$

Then we find $j_0 \in \mathbb{N}$ such that

$$2\lambda^{j_0} K \le \tfrac{\varepsilon}{2} \implies j_0 = j_0(\varepsilon, K), \tag{2.22}$$

with K prescribed in (2.13). Let us put $L_0 \ge 2^{j_0}\tau_1$. Then it follows from (2.18) and (2.21), (2.22) that

$$I(l_0) \equiv I_{L,M}(l_0) \le \varepsilon \quad \text{for all } \; L > L_0, \;\; M > L_0. \tag{2.23}$$

It follows from inequality (2.23), due to Gagliardo-Nirenberg's inequality, that $\{v_L\}$ is a Cauchy sequence in $W^m_{p+1}(0, l_0)$ for arbitrary $l_0 > 0$ completing the proof of the proposition. $\square$

By Proposition 2.1, $v_L \to f \in W^m_{p+1,\mathrm{loc}}(\mathbf{R}_+)$ as $L \to \infty$ and, in view of the uniform bound (2.13), we omit the loc completing the existence proof of Theorem 2.1. Uniqueness is established below. $\square$

2.3. **Stability of blow-up similarity solutions.** In order to prove the stability, we use a PDE analysis. We consider equation (2.2) with boundary blow-up conditions (2.6) and sufficiently smooth initial data $u_0 \in L_2(\mathbf{R}_+)$. The operator in (2.2) is coercive and monotone in $L_2(\mathbf{R}_+)$ and this problem falls into the scope of the classical theory, see [13, Chapter 2], so that there exists a unique weak solution $u(x,t)$ in Q. It is convenient to introduce the rescaled solution, where rescaling is performed according to (2.4),

$$\theta(y, \tau) = (-t)^{-n} u, \quad y = x/(-t)^\beta, \quad \tau = -\ln(-t) : (-1, 0) \to \mathbf{R}_+. \tag{2.24}$$

Then θ solves the rescaled equation

$$\theta_\tau = \mathbf{B}_{m,p}(\theta) \quad \text{for } \tau > 0, \quad \theta(0) = u_0 \in L_2(\mathbf{R}_+). \tag{2.25}$$

Note that the operator $\mathbf{B}_{m,p}$ for $n \ne -\frac{1}{p-1}$ (i.e., for $\beta \ne 0$) is not potential and (2.25) is not a gradient system. Actually, this implied the main difficulty in the above proof of existence of similarity profiles. The non-variational structure is a typical feature of the rescaled equations, though note that the original PDE (2.2) contains the potential operator $\mathbf{A}_{m,p}$ which is the derivative in $W_0^{m,p}$ of the functional

$$\Phi(u) = -\tfrac{1}{p+1} \int |D^m u|^{p+1}, \tag{2.26}$$

and (2.2) takes the form $u_t = \Phi'(u)$. This means that $\Phi(u)$ is a Lyapunov function monotone on evolution orbits and $\frac{d}{dt}\Phi(u) = \|u_t\|_2^2 \ge 0$. The extra rescaling (2.24) does not provide us with a suitable Lyapunov function for equation (2.25). Indeed the nonlinear part $\mathbf{A}_{m,p}$ of operator (2.5) is potential, but adding to $\mathbf{A}_{m,p}$ a linear first-order operator as in (2.5) destroys the gradient structure of the equation. As a result, we cannot use a standard dynamical systems theory to ensure existence of non-trivial similarity (stationary) solutions of (2.5) and their stability.

The following L_2-contractivity property of the flow (2.25) implies the stability.

Lemma 2.1. *Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of* (2.2), (2.5) *with initial data $u_{01}, u_{02} \in W^{m,p}(\mathbf{R}_+)$ and let*

$$2n + \beta < 0, \quad i.e., \quad n < n_* = -\tfrac{1}{p-1+2m(p+1)}. \tag{2.27}$$

Then the rescaled solutions $\theta_1(\tau)$ and $\theta_2(\tau)$ satisfy

$$\|\theta_1(\tau) - \theta_2(\tau)\|_2^2 \le C_0 e^{(2n+\beta)\tau} \to 0 \quad as \;\; t \to 0^-, \quad where \;\; C_0 = \|u_{01} - u_{02}\|_2^2. \tag{2.28}$$

Proof. Returning to the original PDE (2.2), in view of the monotonicity of the operator $\Delta_{m,p}$, multiplying the equation for the difference $w = u_1 - u_2$ by w in $L_2(\mathbf{R}_+)$, one obtains that a.e. on $(0,T)$,

$$\|w(t)\|_2^2 \;\; \text{is decreasing,} \tag{2.29}$$

and hence $\|w(t)\|_2^2 \le C_0$. In terms of the rescaled variables (2.24), this is equivalent to (2.28). $\square$

Using $\theta_2(\tau) \equiv f$, which is a self-similar profile from Theorem 2.1, one obtains

Corollary 2.1. *Let* (2.10) *hold. Then, for any initial data $u_0 \in L_2(\mathbf{R}_+)$, the rescaled solution* (2.24) *converges to the self-similar solution* (2.4),

$$\|\theta(\cdot,\tau) - f(\cdot)\|_2^2 = O(e^{(2n+\beta)\tau}) \to 0 \quad as \;\; \tau \to \infty. \tag{2.30}$$

Obviously, this stability result guarantees the uniqueness of the similarity profile that completes the proof of Theorem 2.1.

2.4. **On local behaviour of similarity profiles for even and odd m.** Once we have established existence, uniqueness and exponential stability of similarity solutions, one needs to know more about actual behaviour and asymptotic properties of the profiles f. The essential difference occurring for even and odd m is easy to see in the simpler case of the regional blow-up $n = -\frac{1}{p-1}$, where $\beta = 0$, so (2.4) is a separate variable solution

$$u_*(x,t) = (-t)^{-1/(p-1)} f(x), \tag{2.31}$$

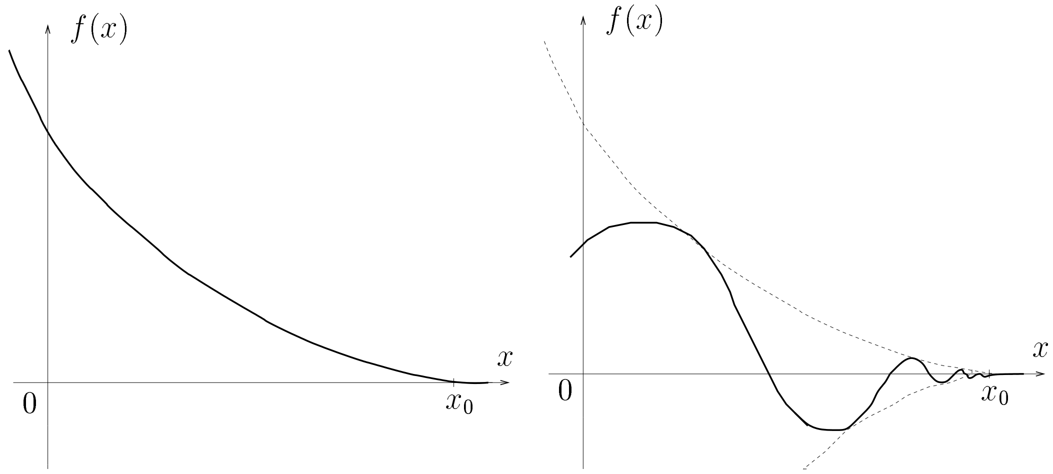
where f solves the autonomous ODE

$$(-1)^{m+1}(|f^{(m)}|^{p-1} f^{(m)})^{(m)} - \tfrac{1}{p-1} f = 0 \quad \text{for} \;\; x > 0. \tag{2.32}$$

Using the same change as in (2.8), we find that (2.32) admits solutions of the form

$$f(x) = x^{\mu} \varphi(\ln x) \quad \text{for} \;\; x > 0, \quad \text{where} \;\; \mu = \tfrac{m(p+1)}{p-1}, \tag{2.33}$$

and $\varphi(\eta)$ satisfies an autonomous $2m$th-order ODE reduced to the lower-order one like (2.9).



(a) Odd m: a nonnegative profile f

(b) An oscillating f

FIGURE 2. Examples of similarity profiles f for various m.

Odd m's. This case includes the classical second-order ODE with $m = 1$, where the boundary blow-up has been studied in detail by using the Maximum Principle; see [17, Chapter 3]. For higher-order equations with $m = 3, 5, ...$, one can choose a constant function $\varphi \equiv \varphi_0 > 0$ in (2.33),

$$\varphi_0 = [(p-1)B(\mu)B((\mu - m)p)]^{-\frac{1}{p-1}}, \tag{2.34}$$

where $B(a)$ denotes the product $a(a-1)...(a-m+1)$. Performing the invariant transformation $x \mapsto x_0 - x$ with a constant $x_0 > 0$, we obtain the similarity profile

$$f(x) = \varphi_0(x_0 - x)_+^{\mu} \quad \text{for} \quad x < x_0, \tag{2.35}$$

which is a weak $C^{[\mu]}$-solution of (2.32) with the corresponding boundary conditions (2.7) generated by the profile. It is a monotone decreasing function vanishing at $x = x_0$ (called a *Kneser*-type solution in the ODE theory) and we put $f(x) \equiv 0$ for $x > x_0$. See Figure 2(a). This gives a clear picture of a localized nonnegative regional blow-up in the domain $x \in [0, x_0]$. On the other hand, for $m = 3, 5, ...$, the equation for φ can admit other changing sign periodic, quasi-periodic or, possibly, chaotic [16, p. 198] solutions describing other blow-up patterns of changing sign (see below), and such behaviour can be generic at least for large m and p close to 1. For $m = 1$, such nonnegative S-regime of localization of boundary blow-up is known for many years; see references in [17, Chapter 3].

Even m's. Then the ODE for φ does not admit a constant non-zero solution like (2.34). On the other hand, such higher-order autonomous ODEs are known to admit periodic (limit cycle) solutions; see [11, 14, 22] and references therein. In view of (2.33), this means that the corresponding profile $f(x) \mapsto f(x_0 - x)$ is oscillatory as $x \to x_0^-$ with the algebraic envelope as in (2.35). See Figure 2(b). We expect that the oscillatory behaviour is a generic property of the similarity profiles for even m's (and possibly for large odd ones) and this can be observed also in the cases $n \neq -\frac{1}{p-1}$ to be considered next. We also claim that the oscillatory behaviour of solutions at the interfaces is generic for the Cauchy problem for higher-order PDEs like (2.1) and similar degenerate ones.

The oscillatory local behaviour of solutions, at least for p sufficiently close to 1, can be explained by using a natural continuity property of the equation (2.1) relative to the parameter p. Indeed, for $p = 1$, the linear parabolic equation has the oscillating kernel, a property which is supposed to be inherited, at least, for $p \approx 1^+$ (though in the blow-up case, extra positivity features of the operators involved should be taken into account).

2.5. **Preliminary properties of global and single-point blow-up.** We now consider general n's in (2.4).

Global blow-up: $n < -\frac{1}{p-1}$**, i.e.,** $\beta < 0$**.** Looking for a compactly supported f of (2.5) vanishing at finite $y = y_0 > 0$, for odd m's one can get the expansion (cf. (2.35))

$$f(y) = \varphi_0 (y_0 - y)_+^{\nu} + ..., \quad \text{with } \nu = \mu - \tfrac{1}{p-1}, \tag{2.36}$$

where $\varphi_0 = [(p-1)|\beta|\nu y_0 B(\nu) B((\nu - m)p]^{-1/(p-1)}$. Existence of such a solution, which is monotone decreasing in a small neighbourhood of $y = y_0$, can be proved by applying Banach's Contraction Principle to the integral equation for the inverse function $y = y(f)$ obtained from the ODE (2.5) by integration. The monotonicity property makes it possible to establish the contractivity of the integral operator (in this functional setting the trivial solution $f \equiv 0$ is excluded).

For even m's, the monotone expansion like (2.36) does not exist. We expect that, similar to the case $n = -\frac{1}{p-1}$, the similarity profile is oscillating as $y \to y_0^-$ according to (2.36) with the asymptotics $\varphi = \varphi(\ln(y_0 - y)) + \dots$. The existence of such an approximately limit cycle solution is a difficult open problem. Observe that the oscillatory behaviour is expected to be generic also for odd m provided that p is not sufficiently large.

Single point blow-up: $n > -\frac{1}{p-1}$**, i.e.,** $\beta > 0$**.** First, formally, equation (2.5) admits a compactly supported monotone solution (2.36) for even m's and, possibly, oscillatory if m is odd. Second the ODE (2.5) admits non-compactly supported solutions with the following behaviour as $y \to \infty$:

$$f(y) = C y^{\frac{n}{\beta}} + ..., \quad \text{with any constant } C \neq 0. \tag{2.37}$$

The crucial property of the corresponding blow-up solutions is that, passing to the limit $t \to 0^-$ in (2.4), one can conclude that there exists the final-time profile

$$u_*(x, 0^-) = C x^{\frac{n}{\beta}}, \tag{2.38}$$

which is finite everywhere excluding the origin $x = 0$ only (the single point blow-up). This kind of blow-up exists for $m = 1$ and is expected to be generic for any odd and even $m = 2, 3, \dots$.

Remark: on optimality of the existence and stability hypothesis. The asymptotics (2.37), which is expected to be generic, clearly shows that the hypothesis (2.10) of existence and exponential stability of self-similar solutions from L_2 is optimal and cannot be improved. Indeed, $y^{n/\beta} \in L_2(1,\infty)$ if and only if (2.10) holds.

2.6. **On perturbation of similarity solutions.** The theory of approximate self-similar solutions [17, Chapter 6] assumes that generic similarity solutions admit "small" perturbations of boundary data which do not affect the power blow-up functional class. For solutions like (2.4), this means that we can describe the blow-up behaviour with, say, logarithmically perturbed boundary blow-up, where

$$u(0,t) = (-t)^n |\ln(-t)|^\rho \quad \text{with any } \rho \in \mathbf{R} \tag{2.39}$$

(one can take the double logarithm $\ln|\ln(-t)|$ or any function which is slow growing in comparison with the power function $(-t)^n$). See examples in [17, Section 6], where general principles of such perturbation techniques are applied for the case $m = 1$, and in [6, Section 2] for $m > 1$, $p = q$.

2.7. **Blow-up logarithmic travelling wave.** Equation (2.2) admits a special exact solution called the *logarithmic travelling wave*

$$u_*(x,t) = (-t)^{-1/(p-1)} g(z), \quad z = x + \lambda \ln(-t), \tag{2.40}$$

where $\lambda > 0$ is an unknown parameter (a kind of nonlinear eigenvalue to be determined from the solvability of the equation for g with necessary boundary conditions). For the second-order reaction-diffusion-absorption equations, such group-invariant solutions are known to describe some crucial evolution properties of parabolic flows, [17, p. 105].

Let us show that solution (2.40) describes in an explicit form a perturbation of the regional blow-up behaviour. Substituting (2.40) into equation (2.2) yields the ODE

$$\Delta_{m,p}(f) - \tfrac{1}{p-1}\, f + \lambda f' = 0. \tag{2.41}$$

It follows from (2.40) that the corresponding boundary blow-up

$$u_*(0,t) = (-t)^{-1/(p-1)} g(\lambda \ln(-t)) \tag{2.42}$$

is a perturbation of the blow-up S-regime corresponding to standing wave solutions. The rate of perturbation depends on the behaviour of the profile $g(z)$ as $z \to -\infty$. Imposing the condition that $g(z)$ has a typical behaviour like (2.33), i.e.,

$$g(z) = (-z)^\mu \varphi(\ln(-z)) + ... \quad \text{as } z \to -\infty, \tag{2.43}$$

where $\varphi \equiv \varphi_0$ for odd m's and, for instance, φ is a periodic function for m even, we obtain that the interface moves logarithmically, $x_0(t) = \lambda|\ln(-t)| \to \infty$ as $t \to 0^-$, provided that the extra logarithmic factor is available in the S-regime boundary condition (cf. (2.39))

$$u_*(0,t) \sim (-t)^{-1/(p-1)} |\ln(-t)|^\mu.$$

Other boundary conditions are obtained by differentiating (2.42). The problem (2.41), (2.43) for any fixed $\lambda > 0$ is more difficult and can be studied similarly at least for odd $m = 1, 3, 5, ...$, where the asymptotic behaviour (2.43) is purely algebraic and can be derived by standard asymptotic methods. The possible oscillatory behaviour for even and odd m's is more difficult to detect.

2.8. **On blow-up similarity solutions in $\mathbf{R}^N$.** For the case of regional blow-up with $n = -\frac{1}{p-1}$, a similar construction can be performed in the N-dimensional geometry for the PDE (2.1), where the operator is coercive and monotone in $L_2(\Omega)$. Here Ω is the open complement of the unit ball or any sufficiently smooth bounded domain in $\mathbf{R}^N$. The separable self-similar solution has the same form $u_*(x,t) = (-t)^{-1/(p-1)} f(x)$, where f solves the elliptic equation

$$\mathbf{B}_{m,p}(f) \equiv \mathbf{A}_{m,p}(f) - \tfrac{1}{p-1} f = 0 \quad \text{in } \Omega, \quad f(y) \to 0 \quad \text{as} \quad y \to \infty. \tag{2.44}$$

The corresponding boundary conditions are obtained similar to (2.6),

$$\partial^i u / \partial^i \nu = (-t)^{-1/(p-1)} a_i(x) \quad \text{on } \partial\Omega \times (-1,0), \quad i = 0,1,...,m-1, \tag{2.45}$$

where $\partial^i/\partial\nu^i$ are the derivatives with respect to the unit outward normal on $\partial\Omega$ and $a_i(x)$ are sufficiently smooth functions prescribed on $\partial\Omega$. With the given blow-up boundary conditions, existence of the similarity profile f is established via estimates similar to that in Proposition 2.1. The contractivity property of the rescaled parabolic flow (Lemma 2.1) remains valid, and this guarantees the uniqueness and the asymptotic stability of the self-similar solution u_*.

2.9. **Estimates of Theorem 1.1 are sharp.** We now compare the above blow-up similarity analysis of equation (2.2) with the general estimates from Theorem 1.1. In order to apply this theorem, we define the continuation of boundary data (2.6) into the domain $Q = \mathbf{R}_+ \times (-1,0)$. To this end, we fix an arbitrary C^m-smooth function $g(x)$ with the compact support $[0,d]$ and such that $a_i = g^{(i)}(0)$ for $i = 0,1,2,...,m-1$. As the boundary data $f(x,t)$ in Definition 1.1 of the energy solution $u(x,t)$, we take

$$\tilde{f}(x,t) = (-t)^n g(x/(-t)^\beta) \tag{2.46}$$

with the exponent (2.3). Let us compute the corresponding function $F(t)$ given by (1.12) for the boundary data (2.46). It is easy to see that

$$\tilde{F}^{(1)}(t) \sim \tilde{F}^{(2)}(t) \sim \tilde{F}^{(3)}(t) \sim (-t)^{2n+\beta} = (-t)^{2n+\frac{n(p-1)+1}{m(p+1)}}$$

as $t \to 0^-$. This means that the boundary function $F(t)$ satisfies condition (1.14) with the parameter (here $q=1$, $N=1$) $a = -\frac{2}{p-1} - 2n - \frac{n(p-1)+1}{m(p+1)} = -\frac{[n(p-1)+1][2m(p+1)+p-1]}{m(p-1)(p+1)}$. Therefore Theorem 1.1 guarantees the estimate

$$\kappa_L(t) \le c(-t)^{\frac{n(p-1)+1}{m(p+1)}}, \quad t \to 0^-,$$

coinciding with the effective length of propagation of similarity solutions (2.4), (2.3).

3. **Blow-up approximate similarity solutions of exponential growth**

In this section we describe the asymptotic behaviour of boundary blow-up with the exponential growth like

$$u(0,t) = e^{(-t)^{-1}}, \quad t \in (-1,0). \tag{3.1}$$

Obviously, the PDE (2.2) does not admit any exact similarity solutions with such a non-scaling invariant boundary data. Therefore, we construct the *approximate* self-similar solutions following the general classification known for $m = 1$, [17, Chapter 6].

3.1. **Global exponential similarity solutions.** It turns out that, in order to study blow-up for the PDE (2.2) of non-power rate of divergence as in (3.1), we need to describe a class of global similarity solutions of (2.2) of the form

$$u_*(x,t) = e^t f(y), \quad y = x/e^{\beta t}, \quad \beta = \tfrac{p-1}{m(p+1)} > 0, \quad \text{where } f \text{ solves} \tag{3.2}$$

$$\mathbf{B}_{m,p}(f) \equiv \mathbf{A}_{m,p}(f) + \beta f' y - f = 0 \quad \text{for } y > 0, \quad f(\infty) = 0. \tag{3.3}$$

The solution (3.2) determines the corresponding boundary conditions

$$D^i u(0,t) = a_i e^{(1-i\beta)t}, \quad i = 0,1,2,...,m-1; \quad \textstyle\sum a_i^2 \ne 0. \tag{3.4}$$

In terms of the rescaled solution

$$u(x,t) = e^t \theta(y,t), \quad y = x/e^{\beta t}, \tag{3.5}$$

the contractivity property of the rescaled flow (2.25), (3.3) is valid. Indeed, the inequality (2.29) for the difference $w = u_1 - u_2$ implies that (cf. (2.28))

$$\|\theta_1(t) - \theta_2(t)\|_2^2 \le C_0 e^{-(2+\beta)t} \to 0 \quad \text{as } t \to \infty. \tag{3.6}$$

This guarantees the uniqueness and the exponential asymptotic stability of the similarity solution. The existence is proved similar to Theorem 2.1.

The similarity profiles $f(y)$ are expected to be compactly supported with the asymptotics of the form (2.36) or oscillatory (changing sign) at least for $p \approx 1^+$. In this case, the similarity solution exhibits infinite propagation with the effective length

$$x_0(t) = y_0 e^{\beta t} \to \infty \quad \text{as } t \to \infty. \tag{3.7}$$

3.2. **Stability of approximate similarity solutions.** Bearing in mind the typical boundary blow-up growth driven by smooth monotone increasing function $\psi(t) \to \infty$ as $t \to 0^-$ in (3.1), we introduce the rescaled variables

$$u(x,t) = \psi(t)\theta(y,\tau), \quad y = x/\phi(t), \quad \tau = \tfrac{1}{\beta}\,\phi(t), \quad \text{where} \tag{3.8}$$

$$\phi^{m(p+1)}(t) = (p-1)\textstyle\int \psi^{p-1}(t)dt > 0, \tag{3.9}$$

describes the actual propagation of the boundary blow-up into the space $\{x > 0\}$. With such a definition of ϕ, the rescaled solution θ solves the perturbed PDE

$$\theta_\tau = \mathbf{A}_{m,p}(\theta) + \beta\theta_y y + r_1(t)\theta, \quad \text{where } r_1(t) = -\phi^{m(p+1)}\psi'/\psi^p \equiv -\beta(\ln\psi)'/(\ln\phi)'. \tag{3.10}$$

It follows that, under the hypothesis on the boundary blow-up regime,

$$(\psi/\psi')'(t) \to 0 \quad \text{as } t \to 0^-, \tag{3.11}$$

so the right-hand side of (3.10) is an asymptotically small perturbation of the rescaled equation (2.25). Indeed, (3.10) can be written in the form

$$\theta_\tau = \mathbf{B}_{m,p}(\theta) + [r_1(t) + 1]\theta, \tag{3.12}$$

where the last perturbation term is vanishing since, by using (3.11) via L'Hospital's rule, we have (note that $\beta m(p+1) = p-1$)

$$\lim r_1 = -\lim\,(p-1)\tfrac{\psi' \int \psi^{p-1}}{\psi^p} = -\lim \tfrac{p-1}{p-1+(\psi/\psi')'} = -1. \tag{3.13}$$

We now apply to the asymptotically perturbed equation (3.12) the stability theorem from [7, Section 3]; see other applications in [9]. Namely, consider the approximate similarity solution

$$u_*(x,t) = \psi(t) f(y), \quad y = x/\phi(t), \tag{3.14}$$

where (3.9) and (3.11) hold. Let $u(x,t)$ be a solution of the PDE with bounded compactly supported initial data that satisfy the boundary conditions generated by (3.14), i.e.,

$$D^i u(x,t) = a_i \psi(t)/\phi^{i\beta}(t), \quad i = 0, 1, ..., m-1. \tag{3.15}$$

The rescaled solution given by (3.8) solves the perturbed equation (3.12) and the forward orbit is uniformly bounded in $W^{m,p}$. This is established similar to the unperturbed case. Then the main hypothesis (H3) of Theorem 3 in [7] is the Lyapunov uniform stability of the Ω-limit set of the unperturbed equation (2.25). One can see that the contractivity property (3.6) is stronger than this uniform stability. Therefore we obtain the necessary stability result (2.30) showing that the exponential global similarity solutions (3.2) describe the asymptotics of a wide class of boundary blow-up (3.8) with $\psi(t)$ satisfying (3.11). Note that, in view of (3.15), (2.30) guarantees that the solution $u(x,t)$ blows up globally

$$\sup_x |u(x,t)| \to \infty \quad \text{on subsets } \{0 \le x \le c\phi(t)\} \quad \text{as} \quad t \to 0^- \tag{3.16}$$

with any constant $c \in (0, y_0)$, where y_0 is the interface point of the similarity profile $f(y)$.

It is easy to evaluate that, for the exponential blow-up function (3.1), the actual propagation length $\phi(t)$ in (3.9) is

$$\phi^{m(p+1)}(t) = (-t)^2 e^{(p-1)(-t)^{-1}}(1+o(1)) \quad \text{as } t \to 0^-. \tag{3.17}$$

Actually, the general estimate (1.18) in Theorem 1.2 applied to the present problem describes the correct rate of propagation including the lower order power factor.

Similar estimates are easily obtained for the doubly exponential blow-up

$$\psi(t) = \exp\{e^{(-t)^{-1}}\} \implies \phi^{m(p+1)}(t) \sim (-t)^2 e^{-(-t)^{-1}} \exp\{(p-1)e^{(-t)^{-1}}\},$$

and for other extensions with an arbitrary large number of exponents on the right-hand side. These exotic functions show that the approximate similarity solutions can be constructed for boundary blow-up data $\psi(t)$ with arbitrarily steep monotone growth as $t \to 0^-$. This means that there exists an unbounded class of blow-up solutions of the PDE that is attracted, in the rescaled (structural) sense as $t \to 0^-$, to the approximate similarity solutions (3.14). This completes a part of the general classification of solutions to (2.2) on a "plane of the boundary-value problems"; see [17, Section 6].

3.3. **Estimates of Theorem 1.2 are sharp.** Let us show that the general estimate of Theorem 1.2 are sharp. We consider the case $q = 1$, $l = 1$ and $N = 1$. It is convenient (see below) to introduce the following functions

$$\phi(t) = w^{-\frac{2}{a_1}}(-t)^{\frac{4}{a_1}} \exp\{\tfrac{w(p-1)}{a_1}(-t)^{-1}\}, \ \ \psi(t) = w^{\frac{1}{a_1}}(-t)^{-\frac{2}{a_1}} \exp\{\tfrac{wm(p+1)}{a_1}(-t)^{-1}\},$$

where $a_1 = 2m(p+1) + p - 1$. We choose the extended boundary data $f(x,t)$ for the equation (2.2) as follows:

$$\tilde{f}(x,t) = \psi(t) g(x/\phi(t)) \quad \text{in} \quad Q,$$

where a C^{2m}-smooth function $g(s)$ is as in (2.46). It is easy to see that the above functions ϕ, ψ asymptotically satisfy (3.9), i.e.,

$$\phi^{m(p+1)}(t) \sim \int^t \psi^{p-1}(\tau)d\tau \quad \text{as} \quad t \to 0^-. \tag{3.18}$$

Using the asymptotics (3.18), by routine calculations, we obtain that the functions given in (1.12) satisfy

$$\begin{gathered}\tilde{F}^{(1)}(t) \sim \phi(t)\psi^2(t) \sim \tilde{F}^{(2)}(t) \sim \int^t \psi^{p+1}(\tau)\phi^{m(p+1)}(\tau)d\tau \sim \tilde{F}^{(3)}(t) \\ \sim (\int^t \psi'(\tau)\phi^{\frac{1}{2}}(\tau)d\tau)^2 \sim (\int^t \psi(\tau)\phi'(\tau)\phi^{-1}(\tau)d\tau)^2 \sim \exp\{w(-t)^{-1}\} \quad \text{as} \quad t \to 0^-.\end{gathered}$$

Therefore Theorem 1.2 guarantees that $\kappa_l(t) \le c\,\phi(t)$ for any $t \in (0,T)$ that, due to (3.18) and (3.9), corresponds to the above analytic results; cf. formula (3.16).

4. **Energy a priori estimates: derivation of the main system of functional inequalities**

4.1. **Energy functions.** We return to the general problem (1.6)–(1.8) and begin with introducing the families of the subdomains of Q. For any fixed $s > 1$ and $\delta > 0$, we define $\Omega(s) = \{|x| > s\}$, $\Omega(s,\delta) = \Omega(s) \setminus \Omega(s+\delta)$ ($\Omega = \Omega(1)$), $Q_a^b(s,\delta) = \Omega(s,\delta) \times (a,b)$ for $0 \le a \le b \le T$, $Q_a^b(s) \equiv Q_a^b(s,\infty)$.

We next introduce the energy functions connected with the fixed solution $u(x,t)$

$$h(s,t) = \int_{\Omega(s)} |u|^{q+1}dx, \ \ I(s,t) = \int_0^t \int_{\Omega(s)} |D_x^m u|^{p+1}dxdt, \ \ J(s,t) = \int_0^t \int_{\Omega(s)} |u|^{p+1}dxdt. \tag{4.1}$$

Lemma 4.1. ([5, Section 4]) *For any energy solution $u(x,t)$ of problem* (1.6)–(1.8)*, the following global a priori estimate holds for any $t \in (0,T)$:*

$$h(1,t) + I(1,t) \le d_3[\|u_0\|^{q+1}_{L_{q+1}(\Omega(1))} + F(t)] \le d_4 F(t), \tag{4.2}$$

where $F(t)$ is as defined in (1.12)*, $h(s,t)$, $I(s,t)$ are from* (4.1) *and constants d_3, d_4 depend on known parameters of the problem only.*

Next we derive a priori estimates of solutions depending on families of subdomains introduced above.

Lemma 4.2. *Let $u(x,t)$ be an arbitrary energy solution of problem* (1.6)–(1.8). *Then, for any $0 \le a < b \le T$, $s > 1$, $\delta > 0$, the following estimate holds:*

$$\int_{\Omega(s+\delta)} |u(x,b)|^{q+1} dx + d_1 \int_a^b \int_{\Omega(s+\delta)} |D_x^m u|^{p+1} dx dt \le \int_{\Omega(s)} |u(x,a)|^{q+1} dx$$

$$+ c_1 \delta^{-\theta} \left(\int_{Q_a^b(s,\delta)} |D_x^m u|^{p+1} dx dt \right)^{\frac{p}{p+1}} \left(\int_a^b \left(\int_{\Omega(s,\delta)} |u|^{q+1} dx \right)^{\frac{p+1}{q+1}} dt \right)^{\frac{1}{p+1}} \tag{4.3}$$

$$+ c_2 \sum_{i=1}^m \delta^{-i} \left(\int_{Q_a^b(s,\delta)} |D_x^m u|^{p+1} dx dt \right)^{\frac{p+\theta_i}{p+1}} \left(\int_a^b \left(\int_{\Omega(s,\delta)} |u|^{q+1} dx \right)^{\frac{p+1}{q+1}} dt \right)^{\frac{1-\theta_i}{p+1}},$$

where $\theta_i = \frac{(m-i)(p+1)(q+1)+N(p-q)}{a_1}$, $\theta = \frac{a_1}{(p+1)(q+1)}$, $a_1 = m(p+1)(q+1) + N(p-q)$ and the constant $d_1 > 0$ is as prescribed in (1.9).

Proof. Let $\xi(y)$ be the cut-off function as defined in the proof of Proposition 2.1. Substituting the test function $\eta = u(1 - \xi(|x|))$ into the integral identity (1.13) and using the corresponding formula of integration by parts, by condition (1.9) and the Hölder inequality we obtain

$$\int_{\Omega(s+\delta)} |u(x,b)|^{q+1} dx + d_1 \int_{Q_a^b(s+\delta)} |D_x^m u|^{p+1} dx dt \le \int_{\Omega(s)} |u(x,a)|^{q+1} dx$$

$$+ c \sum_{i=1}^m \delta^{-i} \left(\int_{Q_a^b(s,\delta)} |D_x^m u|^{p+1} dx dt \right)^{\frac{p}{p+1}} \left(\int_{Q_a^b(s,\delta)} |D_x^{m-i} u|^{p+1} dx dt \right)^{\frac{1}{p+1}}. \tag{4.4}$$

Due to the Gagliardo–Nirenberg interpolation inequality, there holds

$$\int_{\Omega(s,\delta)} |D_x^{m-i} u|^{p+1} dx \le k_1 \left(\int_{\Omega(s,\delta)} |D_x^m u|^{p+1} dx \right)^{\theta_i} \left(\int_{\Omega(s,\delta)} |u(x,t)|^{q+1} dx \right)^{\frac{(1-\theta_i)(p+1)}{q+1}}$$

$$+ k_2 \left(\int_{\Omega(s,\delta)} |u(x,t)|^{q+1} dx \right)^{\frac{p+1}{q+1}} \delta^{-\frac{(m-i)(p+1)(q+1)+N(p-q)}{q+1}}, \tag{4.5}$$

where constants k_1 and k_2 do not depend on s and δ. Integrating in t the last inequality and substituting into (4.4) yields (4.3) after routine calculations. □

4.2. **System of functional inequalities.** We introduce a strictly monotone sequence

$$\{t_j\} \to T^-, \quad \Delta_j = t_j - t_{j-1} > 0, \quad j = 0, 1, 2, ...; \quad t_{-1} = 0, \ t_0 > 0, \tag{4.6}$$

and define, for $s > 1$, $j \ge 1$, the following three families of energy functions associated with the energy solution $u(x,t)$:

(4.7)

$$I_j(s) = \int_{\Delta_j} \int_{\Omega(s)} |D_x^m u|^{p+1} dx dt, \ J_j(s) = \int_{\Delta_j} \int_{\Omega(s)} |u|^{p+1} dx dt, \ h_j(s) = \sup_{t \in \Delta_j} \int_{\Omega(s)} |u|^{q+1} dx.$$

Lemma 4.3. *Given an arbitrary energy solution of* (1.6)–(1.8), *for any* $s > 1$ *and* $\delta > 0$, *functions* (4.7) *satisfy the following system of functional inequalities:*

$$h_j(s+\delta) \le (1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\Delta I_j(s)P_j(s,\delta), \tag{4.8}$$

$$I_j(s+\delta) \le c_1 h_{j-1}(s) + c_2\Delta I_j(s)P_j(s,\delta), \quad j = 1, 2, \dots, \tag{4.9}$$

where θ, θ_i *are as in* (4.3), $\Delta I_j \equiv I_j(s) - I_j(s+\delta)$, *and*

$$P_j(s,\delta) \equiv R_j^{\frac{1}{q(p+1)}} + \sum_{i=1}^{m} R_j^{\frac{1-\theta_i}{(p+1)(q+\theta_i)}}, \quad R_j(s,\delta) \equiv I_j^{p-q}(s)\Delta_j^{q+1}\delta^{-a_1}, \quad a_1 \text{ from } (4.3).$$

Proof. We put $a = t_{j-1}$ in the inequality (4.3) and choose $b \in \bar{\Delta}_j$ such that $h_j(s+\delta) = \int_{\Omega(s+\delta)} |u(x,b)|^{q+1}dx$. Using Young's inequality yields

$$h_j(s+\delta) \le h_{j-1}(s) + c\delta^{-\theta}(\Delta I_j(s))^{\frac{p}{p+1}} h_j^{\frac{1}{q+1}}(s)\Delta_j^{\frac{1}{p+1}} + c_1\sum_{i=1}^{m}\delta^{-i}(\Delta I_j(s))^{\frac{p+\theta_i}{p+1}} h_j^{\frac{1-\theta_i}{q+1}}(s)\Delta_j^{\frac{1-\theta_i}{p+1}}$$

$$\le h_{j-1}(s) + \varepsilon h_j(s) + c(\varepsilon)\left[\delta^{-\theta}(\Delta I_j(s))^{\frac{p}{p+1}}\Delta_j^{\frac{1}{p+1}}\right]^{\frac{q+1}{q}} + c(\varepsilon)\sum_{i=1}^{m}\left[\delta^{-i}(\Delta I_j(s))^{\frac{p+\theta_i}{p+1}}\Delta_j^{\frac{1-\theta_i}{p+1}}\right]^{\frac{q+1}{q+\theta_i}}.$$

It is easy to check that

$$\tfrac{(p+\theta_i)(q+1)}{(q+\theta_i)(p+1)} - 1 = \tfrac{(p-q)(1-\theta_i)}{(q+\theta_i)(p+1)}, \quad \tfrac{p(q+1)}{q(p+1)} - 1 = \tfrac{p-q}{q(p+1)}, \quad \tfrac{\theta(q+1)}{q} = \tfrac{a_1}{q(p+1)}, \quad \tfrac{i(q+1)}{q+\theta_i} = \tfrac{a_1(1-\theta_i)}{(p+1)(q+\theta_i)}.$$

Using these relations, we rewrite the last inequality as

$$h_j(s+\delta) \le h_{j-1}(s) + \varepsilon h_j(s) + c(\varepsilon)\Delta I_j(s)\left[R_j^{\frac{1}{q(p+1)}}(s,\delta) + \sum_{i=1}^{m} R_j^{\frac{1-\theta_i}{(p+1)(q+\theta_i)}}(s,\delta)\right]. \tag{4.10}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, (4.10) implies (4.8); see more details in [18, p. 1337]. In order to derive (4.9), we again begin with (4.3) where $a = t_{j-1}$ and $b = t_j$. Using again Young's inequality then yields, for any $\varepsilon > 0$,

$$I_j(s+\delta) \le h_{j-1}(s) + \varepsilon h_j(s) + c(\varepsilon)\Delta I_j(s)\left[R_j^{\frac{1}{q(p+1)}}(s,\delta) + \sum_{i=1}^{m} R_j^{\frac{1-\theta_i}{(p+1)(q+\theta_i)}}(s,\delta)\right]. \tag{4.11}$$

Summing up inequalities (4.10) and (4.11) and using Lemma 1 in [18], we obtain (4.9). □

Lemma 4.4. ([5, p. 1095]) *Energy functions* $h_j(s)$, $J_j(s)$ *defined by* (4.7) *satisfy the following system of inequalities for all* $s > 1$ *and* $\delta > 0$*:*

$$J_i(s+\delta) \le c_1\Delta_j^{1-\theta_m}\left[(1+\varepsilon)h_{j-1}(s) + c(\varepsilon)\tfrac{\Delta J_j(s)}{\delta^{m(p+1)}}\right]^{1+\nu}, \tag{4.12}$$

$$h_j(s+\delta) \le (1+\varepsilon)h_{j-1}(s) + c(\varepsilon)\tfrac{\Delta J_j(s)}{\delta^{m(p+1)}}, \quad j = 1, 2, \dots, \tag{4.13}$$

where $\varepsilon > 0$ *is arbitrary,* $c(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$, θ_m *is as defined in* (4.3), $\nu = \frac{(p-q)(1-\theta_m)}{q+1}$ *and* $\Delta J_j(s) = J_j(s) - J_j(s+\delta)$.

Introduce now an additional family of energy functions

$$E_j(s) = J_j^{\frac{1}{1+\nu}}(s), \quad j = 0, 1, 2, \dots . \tag{4.14}$$

It is not difficult to check by iterating (4.12), (4.13) that these functions satisfy the following system (see details in [19]):

$$E_j(s + \textstyle\sum_{i=1}^j \delta_j) \le \tilde{c}_1(\varepsilon) \sum_{k=1}^j \frac{\Delta_k^{\frac{1-\theta_m}{1+\nu}} E_k^{1+\nu}(s)(1+\varepsilon)^{j-k}}{\delta_k^{m(p+1)}} \left(\frac{\Delta_j}{\Delta_k}\right)^{\frac{1-\theta_m}{1+\nu}}$$

$$+ c_2(1+\varepsilon)^j \Delta_j^{\frac{1-\theta_m}{1+\nu}} h_0(s) \quad \text{for any } \varepsilon > 0,\ s > 1,\ \delta_i > 0,\ j = 1, 2, \dots . \tag{4.15}$$

The main application of the functional system (4.12), (4.13) and (4.15) in our study of propagation of the blow-up waves is as follows.

Lemma 4.5. *Let an energy solution $u(x,t)$ of problem* (1.6)–(1.8) *satisfy, for some $s_0 \ge 1$, the estimate*

$$h(s_0, t) + I(s_0, t) \le cF^{(\gamma)}(t) \equiv c(T-t)^{-\gamma - \frac{q+1}{p-q}}, \quad t \in (0, T), \tag{4.16}$$

for any exponent $\gamma \in (-\frac{q+1}{p-q}, 0)$. Then, for any $s > s_0$,

$$J(s,t) \le C = C(s) < \infty, \quad t \in (0, T), \tag{4.17}$$

where I, J and h are energy functions (4.7).

Proof. Fix an arbitrary $\tilde{s} > s_0$ such that

$$J(\tilde{s}, t) \to \infty \quad \text{as } t \to T^- \tag{4.18}$$

(if such an $\tilde{s} > s_0$ would not exist, then the lemma would have been proved). Now we define displacements $\{\Delta_j = \Delta_j(\tilde{s})\}$. Namely, we introduce the free parameter $\nu_1 \in (0,1)$ that will be defined later on, and introduce the sequence $\{t_j\}$ by induction

$$(t_j - t_{j-1})^{1-\theta_m} (J(\tilde{s}, t_j) - J(\tilde{s}, t_{j-1}))^{\nu+\nu_1} = g^{1+\nu}, \quad j \ge 0, \tag{4.19}$$

where ν is as given in (4.12). Here $g > 0$ is another free parameter to be specified. It follows from (4.18) that (4.19) correctly defines this sequence $\{t_j\} \to T$ as $j \to \infty$.

Let us derive an upper estimate of $J_j(\tilde{s}) \equiv J(\tilde{s}, t_j) - J(\tilde{s}, t_{j-1})$. By the interpolation inequality (4.5), setting $s = \tilde{s}$ and letting $\delta \to \infty$, we obtain

$$\int_{\Omega(\tilde{s})} |u|^{p+1} dx \le k_1 \left(\int_{\Omega(\tilde{s})} |D_x^m u|^{p+1} dx \right)^{\theta_m} h^{\frac{(1-\theta_m)(p+1)}{q+1}}(\tilde{s}, t).$$

Integrating this inequality over $t \in \Delta_j$ yields

$$J_j(\tilde{s}) \le k_1 (I_j(\tilde{s}))^{\theta_m} h_j^{\frac{(1-\theta_m)(p+1)}{q+1}}(\tilde{s}) \Delta_j^{1-\theta_m}. \tag{4.20}$$

Due to assumption (4.16), it follows from (4.20) that

$$J_j(\tilde{s}) \le c_0 (F^{(\gamma)}(t_j))^{1+\nu} \Delta_j^{1-\theta_m}, \quad j = 0, 1, 2, \dots . \tag{4.21}$$

Therefore definition (4.19) yields $c_0^{\nu+\nu_1}\Delta_j^{1-\theta_m}(F^{(\gamma)}(t_j))^{(1+\nu)(\nu+\nu_1)}\Delta_j^{(1-\theta_m)(\nu+\nu_1)} \geq g^{1+\nu}$, or, due to (4.16),

$$c_1\Delta_j \geq g^{\frac{1+\nu}{1-\theta_m)(1+\nu+\nu_1)}}(T-t_j)^B, \tag{4.22}$$

where $c_1 = c_0^{\frac{\nu+\nu_1}{(1-\theta_m)(1+\nu+\nu_1)}}$ and $B = \frac{(\frac{q+1}{p-q}+\gamma)(1+\nu)(\nu+\nu_1)}{(1-\theta_m)(1+\nu+\nu_1)}$. It is easy to check that any of two conditions

$$-\tfrac{q+1}{p-q} < \gamma \leq -\tfrac{q+1}{(p-q)(1+\nu)}, \quad \nu_1 > 0, \tag{4.23}$$

$$-\tfrac{q+1}{(p-q)(1+\nu)} < \gamma < 0, \quad 0 < \nu_1 \leq -\frac{\gamma\nu}{\frac{q+1}{(p-q)(1+\nu)}+\gamma}, \tag{4.24}$$

are sufficient to guarantee inequalities $0 < B \equiv B(\gamma,\nu_1) \equiv 1 + \frac{(q+1)\nu_1+\gamma(1+\nu)(\nu+\nu_1)(p-q)}{\nu(1+\nu+\nu_1)} \leq 1$. Therefore, if parameters γ, ν_1 satisfy (4.23) or (4.24), then it follows from (4.22) that $c_1\Delta_j \geq g^{\frac{1+\nu}{1-\theta_m)(1+\nu+\nu_1)}}(T-t_j)$ for $j = 0,1,2,\ldots$, and hence

$$\Delta_{j+1} \leq r_1\Delta_j, \quad j \geq 0; \quad r_1 = c_1 g^{-\frac{1+\nu}{(1-\theta_m)(1+\nu+\nu_1)}}. \tag{4.25}$$

Using property (4.25) and definitions (4.19), (4.14), we derive

$$E_i(\tilde{s}) = g^{\frac{1}{\nu+\nu_1}}\Delta_i^{-\frac{1-\theta_m}{(1+\nu)(\nu+\nu_1)}} \leq r_1^{\frac{(j-i)(1-\theta_m)}{(1+\nu)(\nu+\nu_1)}}E_j(\tilde{s}) \quad \text{for any } i<j, \ \ j \geq 0. \tag{4.26}$$

We now define the displacements $\{\delta_i,\ i \leq j\}$ in system (4.15). Namely, we fix a $j \geq 1$, $\delta > 0$, $b \in (0,1)$ and set

$$\delta_i = \delta\, b^{j-i} \quad \text{for} \quad i \leq j. \tag{4.27}$$

Taking into account (4.19), (4.25), (4.26), we derive from (4.15) that

$$\begin{aligned} E_j(\tilde{s}+\tfrac{\delta}{1-b}) &\leq \frac{g\tilde{c}_1(\varepsilon)E_j^{1-\nu_1}(\tilde{s})}{\delta^{m(p+1)}}\ \sum_{k=1}^{j}\left(\frac{(1+\varepsilon)\tilde{r}_1}{b^{m(p+1)}}\right)^{j-k} \\ &+c_2\Delta_0^{\frac{1-\theta_m}{1+\nu}}(1+\varepsilon)^j\tilde{r}_1^j\, h_0(\tilde{s}), \quad j=1,2,\ldots; \quad \tilde{r}_1 = r_1^{\frac{1-\theta_m}{1+\nu}}. \end{aligned} \tag{4.28}$$

Fix $\varepsilon = \varepsilon_0 > 0$ and next fix the free parameter g from (4.19) so large that

$$(1+\varepsilon_0)b^{-m(p+1)}\tilde{r}_1 = \tfrac{1}{2}, \tag{4.29}$$

where $\tilde{r}_1 = \tilde{r}_1(g)$ is given in (4.28), (4.25). In this case (4.28) yields

$$E_j(\tilde{s}+\tfrac{\delta}{1-b}) \leq \frac{2g\tilde{c}_1(\varepsilon_0)E_j^{1-\nu_1}(\tilde{s})}{\delta^{m(p+1)}} + c_3\Delta_0^{\frac{1-\theta_m}{1+\nu}}b^{m(p+1)j}h_0(\tilde{s}), \quad j=1,2,\ldots.$$

Raising these inequalities to the power $1+\nu$, summing up over j from $j=1$ to k and taking into account inequalities (4.26), we deduce

$$\begin{aligned} J(\tilde{s}+\delta,t_k) &\leq A_1\delta^{-m(p+1)(1+\nu)}J_k^{1-\nu_1}(\tilde{s}) + B_1 \\ &\leq A_1\delta^{-m(p+1)(1+\nu)}J^{1-\nu_1}(\tilde{s},t_k) + B_1 \quad \text{for any} \quad k \geq 1 \text{ and } \delta > 0, \end{aligned} \tag{4.30}$$

where constants $A_1 = A_1(t_0)$ and $B_1 = B_1(t_0)$ do not depend on k, $\tilde{s}$, δ. By the Stampacchia Lemma [21, Lemma 4.1], inequality (4.30) yields $J(s_0+\delta,t_k) \leq c\,\delta^{-\frac{m(p+1)(1+\nu)}{(1-\nu_1)\nu_1}} + c_1$ for any $\delta > 0$ and $k \geq 1$, and this completes the proof of Lemma 4.5. $\square$

5. General analysis of the functional system (4.8), (4.9)

Our upper estimate of the propagation of blow-up waves is based on Lemma 4.5 and on the detailed analysis of the decrease rate of solutions of system (4.8), (4.9). The optimal choice of the sequence of displacements $\{\Delta_j\}$ will play an important role. This optimal sequence has to depend on the current spatial point s, at which the energy functions $I(s,t)$ and $h(s,t)$ are considered. Roughly speaking, the optimal decay estimates of these energy functions with respect to s are connected with displacements (the partitions) $\{\Delta_i = \Delta_i(s)\}$ that provide us with the following uniform bounds from below and above:

$$0 < c_1 \le \tfrac{I_j(s)}{I_{j-1}(s)} \le c_2 < \infty \quad \text{for any} \quad j \ge 1. \tag{5.1}$$

In particular, for $s=1$ according to (5.1) and Lemma 4.1, we will define sequence $\{t_j\}$ such that

$$r^{-1} \le F(t_j)(F(t_{j-1}))^{-1} \le r_1^{-1}, \quad j \ge 1, \quad \Delta_i \equiv t_i - t_{i-1}, \tag{5.2}$$

where $r_1 < r < 1$ are free positive parameters. We now define the sequence of displacements by

$$\delta_j = (F^{p-q}(t_j)\Delta_j^{q+1})^{\frac{1}{a_1}}, \quad j \ge 1; \quad \delta_0 = 0, \quad a_1 = m(p+1)(q+1) + N(p-q). \tag{5.3}$$

Lemma 5.1. *Let functions $I_j(s)$, $h_j(s)$ from* (4.7) *satisfy the functional system* (4.8), (4.9) *and let the global a priori estimate* (4.2) *hold. Let displacements $\{\Delta_i\}$, $\{\delta_i\}$ be given by* (5.2), (5.3) *and suppose that sequence $\{\delta_i\}$ is increasing. Then, for any $b \in (0,1)$, there exist $r = r(b) > 0$ in the definition* (5.2) *of displacements $\{\Delta_i\}$ ($r(b) \to 0$ as $b \to 1$) and constants $c_0 > 0$, $C_0 > 0$ such that*

$$I_j(1 + c_0\delta_j) \le C_0 r^{jb} F(t_j), \tag{5.4}$$

$$h_j(1 + c_0\delta_j) \le C_0 r^{jb} F(t_j), \quad j \ge 0. \tag{5.5}$$

Proof. Lemma 4.1 implies the following starting a priori estimates:

$$I_j(1) \le d_5 F(t_j), \quad h_j(1) \le d_5 F(t_j), \quad j \ge 0. \tag{5.6}$$

In particular, these estimates prove (5.4), (5.5) for $j=0$. For functions R_j from system (4.8), (4.9), we now have

$$R_j(\delta, 1) \le \delta^{-a_1}(d_5 F(t_j))^{p-q}\Delta_j^{q+1}, \quad j \ge 1. \tag{5.7}$$

Define the new displacements

$$\delta_j^{(1)} = (\omega^{-1} d_5^{p-q})^{\frac{1}{a_1}} \delta_j, \quad j \ge 0, \tag{5.8}$$

where δ_j and a_1 have been defined in (5.3) and $\omega > 0$ is a small parameter to be defined later. It is easy to see that, for $j \ge 1$,

$$R_j(\delta_j^{(1)}, 1) \le \omega \quad \Longrightarrow \quad P_j(\delta_j^{(1)}, 1) \le \omega_1(\omega) = \omega^{\frac{1}{q(p+1)}} + \textstyle\sum_{i=1}^m \omega^{\frac{1-\theta_m}{(p+1)(q+\theta_i)}}. \tag{5.9}$$

Due to (5.6)–(5.9), it follows from (4.8), (4.9) that, for $j \ge 1$,

$$I_j(1 + \delta_j^{(1)}) \le c_1 d_5 F(t_{j-1}) + c_2\omega_1 d_5 F(t_j) \le ((5.2)) \le d_5 F(t_j)(c_1 r + c_2\omega_1), \quad j \ge 1. \tag{5.10}$$

Similarly, for any $j \geq 1$,

$$h_j(1+\delta_j^{(1)}) \leq (1+\varepsilon)d_5F(t_{j-1}) + c(\varepsilon)\omega_1 d_5F(t_j) \leq d_5F(t_j)[(1+\varepsilon)r + c(\varepsilon)\omega_1]. \tag{5.11}$$

Fix now a $b \in (0,1)$, then an $\varepsilon > 0$ and next the parameter r from definition (5.2) such that $r^{1-b} < \max\{(1+\varepsilon)^{-1}, c_1^{-1}\}$ with c_1 as in (5.10). Then we fix $\omega = \omega(\varepsilon, c_1, c_2, r) > 0$ such that the following inequalities:

$$c_1r + c_2\omega < r^b, \quad (1+\varepsilon)r + c(\varepsilon)\omega < r^b, \tag{5.12}$$

are satisfied. Then (5.10), (5.11) imply

$$h_j(1+\delta_j^{(1)}) \leq r^b d_5F(t_j), \quad I_j(1+\delta_j^{(1)}) \leq r^b d_5F(t_j), \quad j \geq 1 \tag{5.13}$$

We introduce the next sequence of displacements

$$\delta_j^{(2)} = r^{\frac{b(p-q)}{a_1}}\delta_j^{(1)}, \quad j \geq 1, \quad \text{where } \delta_j^{(1)} \text{ from (5.8).} \tag{5.14}$$

Due to (5.13), it follows that

$$R_j(1+\delta_j^{(1)}, \delta_j^{(2)}) \leq \omega \quad \Longrightarrow \quad P_j(1+\delta_j^{(1)}, \delta_j^{(2)}) \leq \omega_1(\omega), \quad j \geq 1. \tag{5.15}$$

As a result, we deduce from (4.9) with $s = 1+\delta_j^{(1)}$ $\delta = \delta_j^{(2)}$ that, for $j \geq 2$,

$$\begin{aligned} I_j(1+\delta_j^{(1)}+\delta_j^{(2)}) &\leq c_1h_{j-1}(1+\delta_j^{(1)}) + c_2I_j(1+\delta_j^{(1)})P_j(1+\delta_j^{(1)}, \delta_j^{(2)}) \\ &\leq (\text{by (5.15) and the monotonicity of sequence } \{\delta_j^{(1)}\}) \\ &\leq c_1h_{j-1}(1+\delta_{j-1}^{(1)}) + c_2\omega_1I_j(1+\delta_j^{(1)}) \leq ((5.13)) \leq c_1d_5r^bF(t_{j-1}) \\ &+c_2\omega_1d_5r^bF(t_j) \leq ((5.2)) \leq r^b(c_1r+c_2\omega_1)d_5F(t_j) \leq ((5.12)) \leq r^{2b}d_5F(t_j). \end{aligned} \tag{5.16}$$

Similarly, from (4.8) with $b = 1+\delta_j^{(1)}$, $\delta = \delta_j^{(2)}$, we obtain $h_j(1+\delta_j^{(1)}+\delta_j^{(2)}) \leq r^{2b}d_5F(t_j)$ for $j \geq 2$. This procedure can be repeated j times for functions $h_j(s)$ and $I_j(s)$ for each $j \geq 1$. As a result, we obtain

$$I_j(1+\textstyle\sum_{i=1}^{j}\delta_j^{(i)}) + h_j(1+\sum_{i=1}^{j}\delta_j^{(i)}) \leq 2d_5r^{jb}F(t_j), \ \ j \geq 0, \tag{5.17}$$

where $\delta_j^{(k+1)} = \tilde{r}\delta_j^{(k)}$ for $k \geq 1$ with $\tilde{r} = r^{b(p-q)/a_1}$. From inequalities (5.17), we obtain estimates (5.4), (5.5) with $c_0 = (1-\tilde{r})^{-1}(\omega^{-1}d_5^{p-q})^{1/a_1}$, $C_0 = 2d_5$.

6. **Propagation of blow-up waves: proofs of Theorems**

6.1. **Proof of Theorem 1.1.** According to Lemma 5.1, firstly we find the optimal starting partition $\{\Delta_i\}$ and the decay estimates of the corresponding energy functions. To this end, we set

$$\Delta_i = \Delta_0 r^{\frac{i}{l}}, \quad i \geq 1; \quad r \in (0,1), \tag{6.1}$$

where $l > 0$ is as given in (1.14) and then $\Delta_0 = t_0 = T(1 - r^{\frac{1}{l}})$. It is easy to see that $T - t_j = \sum_{i=j+1}^{\infty} \Delta_i = \Delta_0 r^{\frac{j+1}{l}}/(1 - r^{\frac{1}{l}}) = \Delta_j r^{\frac{1}{l}}/(1 - r^{\frac{1}{l}})$. Therefore $F_1(t_j) = w(T - t_j)^{-l} = w r^{-1}(1 - r^{\frac{1}{l}})^l \Delta_j^{-l}$ and hence

$$F_1(t_j)/F_1(t_{j-1}) = 1/r, \quad j \geq 1, \tag{6.2}$$

that corresponds to the "optimality condition" (5.2). Due to (6.1), we also have

$$r^j = T^{-l} r^{-1} (T - t_j)^l, \quad j \geq 1. \tag{6.3}$$

It is easy to check that definition (5.3) now gives that $\delta_j = c\, r^{-\frac{ja(p-q)}{a_1}}$. Therefore, for any $a \geq 0$, this sequence is increasing so Lemma 5.1 is applicable. We fix a $\gamma \in (0, \frac{q+1}{p-q})$ and define the parameter b from Lemma 5.1 such that

$$(1 - b)l \equiv (1 - b)(\tfrac{q+1}{p-q} + a) = \tfrac{q+1}{p-q} - \gamma \implies b = \tfrac{a+\gamma}{a + \frac{q+1}{p-q}}. \tag{6.4}$$

By (6.4) and (6.3), inequalities (5.4), (5.5) imply that

$$I_j(1 + c_0\delta_j) \leq C_0 r^{jb} F_1(t_j) = \tfrac{C_0 w}{(T^l r)^b} (T - t_j)^{-(\frac{q+1}{p-q} - \gamma)}, \tag{6.5}$$

$$h_j(1 + c_0\delta_j) \leq \tfrac{C_0 w}{(T^l r)^b} (T - t_j)^{-(\frac{q+1}{p-q} - \gamma)}, \quad j \geq 0. \tag{6.6}$$

Summing up the inequalities (6.5) for $j = 0, 1, 2, ..., i$ with any $i \geq 1$, we deduce

$$\begin{aligned} I(s_i, t_i) &\equiv \textstyle\int_0^{t_i} \int_{\Omega(s_i)} |D_x^m u|^{p+1} dx dt \leq \frac{C_0 w}{(T^l r)^b} \sum_{j=0}^{i} (T - t_j)^{-(1-b)l} \\ &= \tfrac{C_0 w}{(T^l r)^b} \left(\tfrac{r^{\frac{1}{l}}}{1 - r^{\frac{1}{l}}} \right)^{-(1-b)l} \textstyle\sum_{j=0}^{i} \Delta_j^{-(1-b)l} \leq ((6.1)) \leq \frac{C_0 w r^{(1-b)i}}{r T^l (1 - r^{1-b})} \\ &= ((6.3)) = \tfrac{C_0 w (T - t_i)^{-l(1-b)}}{r^b T^{lb} (1 - r^{1-b})}, \quad i \geq 0, \quad \text{where} \end{aligned} \tag{6.7}$$

$$s_i = c_0 (F_1^{p-q}(t_i) \Delta_i^{q+1})^{\frac{1}{a_1}} = c_0 \left[\frac{w^{p-q}(1 - r^{\frac{1}{l}})^{q+1}}{r^{\frac{q+1}{l}} (T - t_i)^{a(p-q)}} \right]^{\frac{1}{a_1}},$$

with c_0 defined after (5.17). From (6.7), by using the monotonicity of function $I(s, t)$ relative to both arguments and (6.4), we obtain the following estimate:

$$I(\chi(t), t) \leq d_6 w (T - t)^{-\frac{q+1}{p-q} + \gamma}, \quad t \in (0, T), \ \gamma > 0, \tag{6.8}$$

where $\chi(t) = d_7 (T - t)^{-\frac{a(p-q)}{a_1}}$ and d_6, d_7 are some easily computed constants. Similarly, from the family of inequalities (6.6) we deduce the estimate

$$h(\chi(t), t) \equiv \textstyle\int_{\Omega(\chi(t))} |u(x, t)|^{q+1} dx \leq d_6 w (T - t)^{-\frac{q+1}{p-q} + \gamma}, \quad t \in (0, T). \tag{6.9}$$

It follows from (6.8), (6.9) that, by using Lemma 4.5, there holds

$$J(\chi(t) + \delta, t) \leq \tilde{c}(\delta) < \infty \quad \text{for any} \ \ t \in (0, T) \text{ and } \delta > 0, \tag{6.10}$$

so $I(\chi(t) + \delta, t) + h(\chi(t) + \delta, t) \leq \tilde{c}_0(\delta) < \infty$ for any $t \in (0, T)$ and $\delta > 0$. The function $\chi(t)$ is directly related to the effective length $\kappa_L(t)$ introduced in (1.15). Indeed, (6.10) means that, for a fixed δ_0 and arbitrary $L \geq \tilde{c}_0(\delta_0)$, there holds $\kappa_l(t) \leq \chi(t) + \delta_0$ for all $t \in (0, T)$. This completes the proof of Theorem 1.1. □

6.2. **Proof of Theorem 1.2.** Keeping the usual notations and condition (5.2), we set

$$\Delta_i = (\tfrac{w}{1+\mu})^l (i+K)^{-1-l}, \quad i \geq 1, \tag{6.11}$$

where $l > 0$ is as given in (1.17) and $\mu > 0$, $K > 0$ are free parameters to be specified. Since $T - t_j = \sum_{i=j+1}^{\infty} \Delta_j$, it is easy to check that

$$l^{\frac{1}{l}}(1+\mu)(j+K) \leq w(T-t_j)^{-\frac{1}{l}} \leq l^{\frac{1}{l}}(1+\mu)(j+K+1), \quad j \geq 0. \tag{6.12}$$

We need to guarantee that $T - \sum_{i=1}^{\infty} \Delta_i = \Delta_0 = t_0 > 0$. Due to the left-hand inequality (6.12), this is true provided that

$$l^{\frac{1}{l}}(1+\mu)K > wT^{-\frac{1}{l}} \quad \Longrightarrow \quad K > l^{-\frac{1}{l}}(1+\mu)^{-1}wT^{-\frac{1}{l}} \equiv K_0(\mu). \tag{6.13}$$

We now check condition (5.2). Since $\varphi(t) = (T-t)^{-\frac{1}{l}}$ is convex, we have

$$(T-t_j)^{-\frac{1}{l}} - (T-t_{j-1})^{-\frac{1}{l}} \equiv \varphi(t_j) - \varphi(t_{j-1}) \geq \varphi'(t_{j-1})\Delta_j$$

$$= \frac{\Delta_j}{l(T-t_{j-1})^{\frac{l+1}{l}}} \geq ((6.11), (6.12)) \geq \frac{w^l l^{\frac{l+1}{l}}(1+\mu)^{l+1}(j+K-1)^{l+1}}{(1+\mu)^l (j+K)^{l+1} l w^{l+1}}$$

$$\geq l_1(1+\mu)w^{-1}, \quad l_1 = l^{\frac{1}{l}}(1-K^{-1})^{l+1} > 0, \quad \text{if} \tag{6.14}$$

$$K > \max\{1, K_0\} \quad (K_0 \text{ is given in (6.13)}). \tag{6.15}$$

Thus, under conditions (6.15), it follows from (6.14) that, for any $j \geq 1$,

$$\frac{F_2(t_j)}{F_2(t_{j-1})} \geq \exp\{l_1(1+\mu)\} \equiv \frac{1}{r(\mu)}, \ \ r < 1. \tag{6.16}$$

In a similar manner, we obtain, for $j \geq 1$,

$$\frac{F_2(t_j)}{F_2(t_{j-1})} \leq \exp\{l_2(1+\mu)\} \equiv \frac{1}{r_1(\mu)}, \ \ r_1 < 1, \tag{6.17}$$

where $l_2 = l^{\frac{1}{l}}(1+\frac{1}{K})^{l+1}$. In addition, due to (6.12), there holds

$$\frac{F_2(t_0)}{F_2(0)} \leq \exp\{l_1(1+\mu)(K+1)\} \equiv \frac{1}{r_2(\mu)}, \ \ r_2 < 1. \tag{6.18}$$

Since $r(\mu) \to \infty$ as $\mu \to \infty$ for any $b \in (0,1)$, we can fix a μ such that conditions (5.12), (5.2) hold. It easily follows from (6.11) and (6.16) that the sequence

$$\delta_j = (F_2^{p-q}(t_j)\Delta_j^{q+1})^{\frac{1}{a_1}}, \quad j \geq 1, \quad \delta_0 = 0, \tag{6.19}$$

is increasing. Therefore, for $F_2(t)$ and $\{\Delta_j\}$ from (6.11), Lemma 5.1 yields

$$I_j(1+c_0\delta_j) \leq C_0 r^{jb} F_2(t_j) \leq ((6.12), (6.16)) \leq$$

$$C_0 \exp\{l^{\frac{1}{l}}(1+\mu)(j+K+1-(1-\tfrac{1}{K})^{l+1}bj\} \leq C_0 K_1 F_2^g(t_j), \tag{6.20}$$

$$h_j(1+c_0\delta_j) \leq C_0 K_1 F_2^g(t_j), \quad j \geq 0, \tag{6.21}$$

where $K_1 = \exp\{1 + (1+\frac{1}{K})^{l+1}bK\}$, $g = 1 - b(1-\frac{1}{K})^{l+1} < 1$. Then (6.20), (6.16) imply

$$I(1+c_0\delta_j, t_j) \leq C_0 K_2 F_2^g(t_j), \quad j \geq 0; \quad K_2 = K_1(1-r^g)^{-1}. \tag{6.22}$$

It follows from (6.12) that

$$T - t_{j-1} \leq K_3(T-t_j), \quad j \geq 1; \quad K_3 = (\tfrac{K+1}{K-1})^l. \tag{6.23}$$

By (6.23), (6.12), from (6.11) it follows that

$$\Delta_j \le l^{\frac{l+1}{l}}(1+\mu)K_3^{\frac{l+1}{l}} w^{-1}(T-t_j)^{\frac{l+1}{l}}, \quad j \ge 1. \tag{6.24}$$

So (6.19) gives

$$\delta_j \le K_4 \psi_w^{-\frac{q+1}{a_1}}(t_j)(T-t_j)^{\frac{q+1}{a_1}} \exp\{(p-q)a_1^{-1}\psi_w(t_j)\}, \quad j \ge 1, \tag{6.25}$$

where $\psi_w(t) \equiv w(T-t)^{-1/l}$ and K_4 is a positive constant.

We now use the constant K in (6.11) as a variable parameter. We fix $K = \bar{K}$ and consider the partitions $\{\Delta_i(K)\}$ for $K \in [\bar{K}, \bar{K}+1]$ defined by (6.11) for all $i \ge 1$ with the corresponding function $\Delta_0(K)$. One can see that $\Delta_i(\bar{K}+1) = \Delta_{i+1}(\bar{K})$ for all $i \ge 1$ and $\Delta_0(\bar{K}+1) = \Delta_0(\bar{K}) + \Delta_1(\bar{K})$. It follows from these properties that, for arbitrary $\bar{t} \in (t_0(\bar{K}), T)$, there exist j and $\tilde{K} \in [\bar{K}, \bar{K}+1]$ such that $\bar{t} = \sum_{i=0}^{j} \Delta_j(\tilde{K})$. Hence (6.12), (6.22), (6.25) imply

$$I(1+c_0\chi_w(t), t) \le C_0 \tilde{K}_2 F_2^g(t), \ \ t \in (0,T); \ \ \tilde{K}_2 = \tilde{K}_2(\tilde{K}), \ \text{ where} \tag{6.26}$$

$$\chi_w(t) = \begin{cases} 0, & \text{if } t \le \tilde{t}_0 = T - \frac{1}{l}\left(\frac{w}{(1+\mu)\tilde{K}}\right)^l \equiv \tilde{t}_0(w), \\ \tilde{K}_4 (T-t)^{\frac{q+1}{a_1}} \psi_w^{-\frac{q+1}{a_1}}(t) \exp\{\frac{p-q}{a_1}\psi_w(t)\}, & t > \tilde{t}_0, \end{cases}$$

and $\tilde{K}_4 = \tilde{K}_4(\tilde{K})$. In a similar way, we deduce that $h(1+c_0\chi_w(t), t) \le C_0\tilde{K}_2 F_2^g(t)$ for any $t \in (0,T)$. Fix now an arbitrary $\bar{t} \in (0,T)$. Then, in view of (6.26), there holds

$$I(1+c_0\chi_w(\bar{t}), t) \le C_0 \tilde{K}_2 \exp\{\psi_{gw}(t)\}, \tag{6.27}$$

$$h(1+c_0\chi_w(\bar{t}), t) \le C_0 \tilde{K}_2 \exp\{\psi_{gw}(t)\}, \quad t \in (0, \bar{t}). \tag{6.28}$$

Thus, starting from a priori estimate (4.2), we have obtained (6.27), (6.28). We now take these estimates as the starting ones instead of (4.2). As a result, we deduce in a similar manner by using Lemma 5.1 that $I(1 + c_0 s_1(t,\bar{t}), t) + h(1 + c_0 s_1(t,\bar{t}), t) \le C_0\tilde{K}_2^2 \exp\{\psi_{g^2 w}(t)\}$ for any $t \in (0,\bar{t})$, where, for $i = 1$, we use the following notation:

$$s_i(t,\bar{t}) = \sum_{j=0}^{i-1} \tilde{K}_2^{\frac{j(p-q)}{a_1}} \chi_{g^j w}(\bar{t}) + \tilde{K}_2^{\frac{i(p-q)}{a_1}} \chi_{g^i w}(t), \quad i \ge 1; \quad s_i(t,t) \equiv s_i(t). \tag{6.29}$$

In particular, this means that

$$I(1+c_0 s_1(\bar{t}), t) + h(1+c_0 s_1(\bar{t}), t) \le C_0 \tilde{K}_2^2 \exp\{\psi_{g^2 w}(t)\}, \quad t \in (0, \bar{t}]. \tag{6.30}$$

As the next step, we use (6.30) as the starting estimate and perform the same round of computations. After j of such iterations, we deduce the estimate

$$I(1+c_0 s_j(t,\bar{t}), t) + h(1+c_0 s_j(t,\bar{t}), t) \le C_0 \tilde{K}_2^{j+1} \exp\{\psi_{g^{j+1} w}(t)\}, \quad t \in (0, \bar{t}]. \tag{6.31}$$

The required number of iterations $j_0 = j_0(t)$ is chosen so that $\chi_{g^{j_0} w}(t) > 0$. Due to (6.26), this means that

$$t < T - \frac{1}{l}\left[\frac{w g^{j_0}}{(1+\mu)\tilde{K}}\right]^l \quad \Longrightarrow \quad j_0(t) \le \frac{\ln(w(T-t)^{-\frac{1}{l}}) - \ln(l^{\frac{1}{l}}(1+\mu)\tilde{K})}{\ln g^{-1}}. \tag{6.32}$$

Thus, fixing a sufficiently large constant $K_5 > \ln(l^{\frac{1}{l}}(1+\mu)\tilde{K})/\ln g^{-1}$, we define the number of iterations required

$$
(6.33) \qquad j_0(t) = \begin{cases} 0, & \text{if } \ln \psi_w(t) \le K_5 \ln g^{-1} \;\Rightarrow\; t \le T - w^l g^{-K_5 l} \equiv t', \\ & \langle (\ln g^{-1})^{-1} \ln \psi_w(t) - K_5 \rangle, \;\; \text{otherwise}, \end{cases}
$$

where $\langle \cdot \rangle$ denotes the integral part. Using (6.33), it follows from (6.30) that

$$
(6.34) \qquad I(1 + c_0 s_{j_0}(\bar{t}), t) + h(1 + c_0 s_{j_0}(\bar{t}), t) \le C_0 K_6 (w(T-t)^{-\frac{1}{l}})^{\ln \tilde{K}_2 / \ln g^{-1}}
$$

for $t \in (0, \bar{t})$, where some finite $K_6 = K_6(\tilde{K}_2, K_5)$. It now remains to find a suitable upper estimate of $s_{j_0}(t)$. In the case $j_0(\bar{t}) > 0$, we have, for any $i \le j_0(\bar{t})$, that

$$
\alpha_i \equiv \tilde{K}_2^{\frac{i(p-q)}{a_1}} \chi_{g^i w}(\bar{t}) \left[\tilde{K}_2^{\frac{(i-1)(p-q)}{a_1}} \chi_{g^{i-1} w}(\bar{t}) \right]^{-1} \le \tilde{K}_2^{\frac{p-q}{a_1}} g^{-\frac{q+1}{a_1}} \exp\{ -\frac{(p-q) w g^{j_0} (\frac{1}{g} - 1)}{a_1 (T-t)^{\frac{1}{l}}} \}
$$

$$
(6.35) \qquad \le (\text{by } (6.33)) \le g^{-\frac{q+1}{a_1}} \tilde{K}_2^{\frac{p-q}{a_1}} \exp\{ \frac{(p-q)(\frac{1}{g}-1)}{a_1} g^{-(K_5+1)} \} \equiv \tilde{\alpha}.
$$

Finally fix K_5 large enough that $\tilde{\alpha} = \frac{1}{2}$ so $\alpha_i \le \frac{1}{2}$ for any $i \le j_0(\bar{t})$. Then, due to (6.35), we have from (6.29) that $s_{j_0}(\bar{t}) \le \chi_w(\bar{t}) \left(1 + \sum_{i=1}^{j_0} \alpha_i \right) \le 2\chi_w(\bar{t})$. As a consequence, it follows from (6.34) that $I(1 + 2c_0 \chi_w(\bar{t})) + h(1 + 2c_0 \chi_w(\bar{t})) \le C_0 K_6 (w(T-t)^{-\frac{1}{l}})^{\ln \tilde{K}_2 / \ln g^{-1}}$ for any $t \in (0, \bar{t})$. Since $\bar{t} \in (0, T)$ is arbitrary here, by using Theorem 1.1 we complete the proof of Theorem 1.2. $\square$

Acknowledgement. The second author thanks Department of Mathematical Sciences of the University of Bath for its hospitality during his visits. The work has been partially supported by the INTAS Projects CERN-INTAS00-0136 and INTAS 03-51-5007.

References

[1] G.I. Barenblatt, *On some unsteady motions of a liquid and a gas in a porous medium*, Prikl. Mat. Mekh., **16** (1952), 67-78.

[2] J. Bebernes and D. Eberly, Mathematical Problems in Combustion Theory, Appl. Math. Sci., Vol. **83**, Springer-Verlag, Berlin, 1989.

[3] F. Bernis, *Qualitative properties for some nonlinear higher-order degenerate parabolic equations*, Houston J. Math., **14** (1988), 319-352.

[4] F. Bernis, *Existence results for doubly nonlinear higher order parabolic equations on unbounded domains*, Math. Ann., **279** (1988), 373-394.

[5] V.A. Galaktionov and A.E. Shishkov, *Saint-Venant's principle in blow-up for higher-order quasilinear parabolic equations*, Proc. Roy. Soc. Edinburgh, **133A** (2003), 1075-1119.

[6] V.A. Galaktionov and A.E. Shishkov, *Structure of boundary blow-up for higher-order quasilinear parabolic equations*, Proc. Royal Soc. London, to appear (http://www.maths.bath.ac.uk/ MATHEMATICS/ preprints. html).

[7] V.A. Galaktionov and J.L. Vazquez, *Asymptotic behaviour of nonlinear parabolic equations with critical exponents. A dynamical systems approach,* J. Funct. Anal., **100** (1991), 435-462.

[8] V.A. Galaktionov and J.L. Vazquez, *The problem of blow-up in nonlinear parabolic equations*, Discr. Cont. Dyn. Syst., **8** (2002), 399-433.

[9] V.A. Galaktionov and J.L. Vazquez, A Stability Technique for Evolution Partial Differential Equations. A Dynamical Systems Approach, Birkhäuser, Boston/Berlin, 2004.

[10] A.S. Kalashnikov, *Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations,* Russian Math. Surveys, **42** (1987), 169-222.
[11] I.T. Kiguradze and T. Kusano, *Periodic solutions of nonautonomous ordinary differential equations of higher order*, Differ. Equat., **35** (1999), 71-77.
[12] H.A. Levine, *The role of critical exponents in blow-up problems*, SIAM Rev., **32** (1990), 262-288.
[13] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier–Villars, Paris, 1969.
[14] Z. Liu and Y. Mao, *Existence theorems for periodic solutions of higher order nonlinear differential equations*, J. Math. Anal. Appl., **216** (1997), 481-490.
[15] O.A. Oleinik and E.V. Radkevich, *Method of introducing of a parameter for evolution equations*, Russian Math. Surveys, **33** (1978), 7-84.
[16] L.A. Peletier and W.C. Troy, Spatial Patterns. Higher Order Models in Physics and Mechanics, Birkhäuser, Boston/Berlin, 2001.
[17] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin/New York, 1995.
[18] A.E. Shishkov, *Estimates of rate of propagation of perturbations in quasilinear divergent degenerate parabolic equations of high order*, Ukrain Math. J., **44** (1992), 1335-1340.
[19] A.E. Shishkov, *Localized boundary blow-up regimes for general quasilinear parabolic equations of arbitrary order in divergence form*, Proc. Steklov Inst. Math., **1 (236)** (2002), 341-356.
[20] A.E. Shishkov and A.G. Shchelkov, *Blow-up boundary regimes for general quasilinear parabolic equations in multidimensional domains*, Sbornik: Mathematics, **190** (1999), 447-479.
[21] G. Stampacchia, Equations elliptiques du second ordre a coefficients discontinue, Les presses de l'Universite de Montreal, 1966.
[22] J.R. Ward, *Asymptotic conditions for periodic solutions of ordinary differential equations*, Proc. Amer. Math. Soc., **81** (1981), 415-420.

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK and Keldysh Institute of Applied Mathematics, Miusskaya Sq. 4, 125047 Moscow, RUSSIA
E-mail address: `vag@maths.bath.ac.uk`

Institute of Applied Mathematics and Mechanics of NAS of Ukraine, R. Luxemburg str. 74, 83114 Donetsk, UKRAINE
E-mail address: `shishkov@iamm.ac.donetsk.ua`