BLOW-UP AND GLOBAL ASYMPTOTICS OF THE UNSTABLE CAHN-HILLIARD EQUATION WITH A HOMOGENEOUS NONLINEARITY

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ABSTRACT. We study the asymptotic behaviour of classes of global and blow-up solutions to the semilinear parabolic equation of Cahn-Hilliard type

$$u_t = -\Delta(\Delta u + |u|^{p-1}u) \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad p > 1,$$

with bounded integrable initial data. We show that in some $\{p, N\}$-parameter ranges it admits a *countable* subset of blow-up similarity patterns. The most interesting self-similar blow-up patterns are constructed at the first critical exponent $p = p_0 = 1 + 2/N$. Unlike the blow-up case, we show that for $p = p_0$, the subset of global decaying similarity solutions is *continuous* and determine the stable mass-branch. We prove that in general there exists a countable spectrum of critical exponents $\{p = p_l = 1 + 2/(N + l),\ l = 0, 1, 2, ...\}$ generating bifurcation branches which play a key role in general description of solutions globally decaying as $t \to \infty$.

1. Introduction

1.1. The model and discussion. In this paper we study the blow-up and the long-time asymptotic behaviour of solutions of the fourth-order semilinear parabolic equation

$$u_t = -\Delta(\Delta u + u^p) \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad p > 1, \quad \text{where } u^p := |u|^{p-1}u. \tag{1.1}$$

We consider the Cauchy problem for (1.1) with initial data

$$u(x, 0) = u_0(x) \quad \text{in } \mathbf{R}^N, \quad u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N), \tag{1.2}$$

assuming in most cases that $u_0(x)$ has exponential decay as $x \to \infty$.

Equation (1.1) is a model connected with various applications. For instance, it arises as the limit case of the phenomenological, "unstable" Cahn-Hilliard equation for $N = 1$, 2 and $p = 3$, see the references in [31] and [10],

$$u_t = -(\gamma u_{xx} - u^3 + \gamma_1 u)_{xx} - \gamma_2 u.$$

It is also a reduced model from solidification theory with $N = 1$ or 2 and $p = 2$, [29, 2]. Equations of this form arise in the theory of thermo-capillary flows in thin layers of

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viscous fluids with free boundaries, with the anomalous dependence of the surface tension coefficient on temperature, [12, 1].

Another important class of fourth-order models related to (1.1) admitting both blow-up and decaying solutions comes from the theory of thin films and general long-wave unstable equations (see the references in [3]), where a typical *quasilinear* equation takes the form

$$u_t = -(uu_{xxx} + u^3 u_x)_x. \tag{1.3}$$

Equations of this form are known to admit non-negative solutions constructed by special parabolic approximations of the nonlinear coefficients; see [35] and the references therein. Because there is no mechanism to preserve finite support in (1.1) (unlike in (1.3)), there are no compactly supported or steady-state solutions in the subcritical Sobolev range $p < p_S = (N+2)/(N-2)_+$. Thus we will concentrate on solutions defined in $\mathbf{R}^N \times (0,T)$.

From the mathematical point of view, writing (1.1) in the form

$$Pu_t = \Delta u + u^p \tag{1.4}$$

with the positive operator $P = (-\Delta)^{-1}$ on the right-hand side defines a pseudo-parabolic second-order equation. Many aspects of such equations are well understood with both the existence and uniqueness of local and global classical solutions and the blow-up of solutions known from the 1970's; see the first results on blow-up in [26] and the references in the surveys [13, 25].

We are mainly interested in the study of blow-up behaviour of the solutions to (1.1), (1.2). In this sense (1.1) is an exceptional model and (1.4) clearly indicates that, at least formally, we can expect some similarities of blow-up singularity formation phenomena to the classical semilinear heat equation from Combustion Theory (the solid-fuel model)

$$u_t = \Delta u + u^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+ \quad (p > 1). \tag{1.5}$$

There is a huge mathematical literature, developed in the last twenty years, devoted to the study of blow-up solutions of (1.5), where in some ranges of the parameters p and N a complete description of all possible patterns has been achieved; see the references in Chapter 4 in [32] and the survey [13]. On the other hand, the unstable nonlinear operator in (1.1) forms the classical *porous medium equation* but backward in time

$$u_t = -\Delta u^p. \tag{1.6}$$

It is not well-posed and leads to blow-up of u or its derivatives in arbitrarily small times; see the concavity techniques in [27].

In standard form, (1.1) (rather than (1.4)) is a fourth-order semilinear parabolic equation. It is well known that any kind of detailed asymptotic analysis for higher-order equations is much more difficult in comparison with the second-order counterparts in view of the lack of the Maximum Principle, comparison, order-preserving semigroups and potential properties of the operators involved, etc. In this sense, we are going to show that (1.1) can be treated as an "intermediate" model between second and fourth-order parabolic equations, where some questions of blow-up and global asymptotics, at least, partially can be studied by reasonably standard mathematical methods. Note that a completely rigorous analysis of blow-up asymptotics, comparable to that for second-order problems, is not yet available for higher-order semilinear equations such as

$$u_t = -(-\Delta)^m u + u^p, \quad m \geq 2, \tag{1.7}$$

see [5], nor for the generalized Frank-Kamenetskii equation

$$u_t = -\Delta^2 u + e^u$$

with similar evolutionary blow-up properties, nor for the model equation from the Semenov-Rayleigh-Benard problem with principle operators of the form

$$u_t = -u_{xxxx} + \beta[(u_x)^3]_x + e^u, \quad \beta \geq 0,$$

see [14]. The mathematical difficulties of the ODE and PDE patterns increase dramatically with the order of differential operators in the equations.

1.2. **The main conclusions and plan of the paper.** The primary goal of this paper is to present some general principles of formation of stable generic blow-up and global asymptotics for the Cahn-Hilliard equation (1.1). To this end, we construct various subsets of self-similar solutions of (1.1) in different ranges of the parameters p and N. We establish that in the most interesting *conservative* critical case $p = 1 + 2/N$,

(i) (1.1) admits a countable discrete subset of blow-up similarity solutions, and

(ii) there exists an unbounded continuous family of global similarity solutions decaying as $t \to \infty$.

We also study the stability of some crucial branches of similarity solutions and discuss the similarity subsets for arbitrary $p > 1$.

In the remainder of this section we discuss some basic auxiliary properties of (1.1). In Section 2 we introduce the blow-up and global similarity solutions and perform a local asymptotic analysis of the corresponding ODEs. The spectral properties of the key non self-adjoint linearized operators are posed in Section 3. In Section 4 we present an existence analysis of blow-up similarity solutions and show that the minimal profile is an attractor for a wide set of initial data. A countable subset of solutions in the critical case $p = p_0 = 1 + 2/N$ is then constructed via a singular perturbation expansion. In Section 5 we study classes of global solutions of (1.1) decaying as $t \to \infty$ and prove that, unlike the blow-up case, the family of similarity solutions at $p = p_0$ is continuous and we determine the stable branch. We also detect a sequence of critical exponents

$$p_l = 1 + \tfrac{2}{N+l}, \quad l = 0, 1, 2, \dots, \tag{1.8}$$

corresponding to a *transition* phenomenon between the two main classes of asymptotic patterns for the PDE (1.1). Under certain assumptions, we prove that at each $p = p_l$ a bifurcation of similarity solutions occurs, and show that, actually, for $p \neq p_0$, p_1, the subset of exponentially decaying similarity profiles is expected to be countable. Concerning possible *non self-similar* asymptotic behaviour, we show that at the same critical exponents $p = p_l$ unusual centre manifold patterns occur for the complementary *stable* Cahn-Hilliard equation

$$u_t = -\Delta(\Delta u - u^p), \tag{1.9}$$

where the operator is monotone and coercive in $H^{-1}(\mathbf{R}^N)$, so that blow-up cannot occur and the Cauchy problem has a unique classical solution decaying in time. Nevertheless, (1.9) admits an interesting discrete subset of global self-similar *very singular solutions* (VSS) with definite similarities to equations $u_t = -\Delta^2 u - u^p$ studied in [15].

1.3. **A potential operator and gradient system.** Equation (1.1) is uniformly parabolic with all spatial differential operators appearing in divergence form. It admits a unique, classical, local in time solution and the standard parabolic theory applies, [11]. The operator on the right-hand side of (1.1), $-\Delta(\Delta u + u^p)$, is potential in $H^{-1}(\mathbf{R}^N)$, and the dynamical system admits the Lyapunov function (potential)

$$E[u](t) = \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p+1}\|u\|_{p+1}^{p+1}, \quad u \in H^2(\mathbf{R}^N) \cap L^{p+1}(\mathbf{R}^N), \tag{1.10}$$

which is monotone on bounded orbits, $\frac{d}{dt}E[u] = -\|u_t\|_{H^{-1}}^2 \le 0$. For such smooth gradient systems, the ω-limit set of any uniformly bounded orbit,

$$\omega(u_0) = \{f \in C(\mathbf{R}^N) : \exists\ \{t_k\} \to \infty \text{ such that } u(\cdot, t_k) \to\ f \ \text{ uniformly}\}$$

is known to consist of classical stationary solutions: $-\Delta(\Delta f + f^p) = 0$ in $\mathbf{R}^N$ for any $f \in \omega(u_0)$. If the subset of stationary solutions consists of isolated equilibria, the asymptotic behaviour does not essentially differ from the classical second-order theory and any bounded orbit approaches a stationary profile as $t \to \infty$. However, one can see from (1.4) that for this problem with $p < p_S = (N+2)/(N-2)_+$ (the critical Sobolev exponent for the elliptic operator in (1.4)), the only admissible stationary solution is $u \equiv 0$, so that the large-time (and of course blow-up) behaviour is essentially non-stationary.

1.4. **The fundamental solution and local existence.** We first need the fundamental solution of the linear fourth-order parabolic equation

$$u_t = -\Delta^2 u \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+ \tag{1.11}$$

having the similarity form

$$b(x,t) = t^{-N/4} F(y), \quad y = x/t^{1/4}, \tag{1.12}$$

where the rescaled kernel F is the unique radial solution of the elliptic equation

$$\mathbf{B}F \equiv -\Delta^2 F + \tfrac{1}{4} y \cdot \nabla F + \tfrac{N}{4} F = 0 \quad \text{in } \mathbf{R}^N, \quad \text{with } \textstyle\int F = 1. \tag{1.13}$$

The kernel $F = F(|y|)$ has exponential decay, oscillates as $|y| \to \infty$ and satisfies the following estimate [9]: for some positive constants D and d,

$$|F(y)| < D e^{-d|y|^{4/3}} \quad \text{in } \mathbf{R}^N. \tag{1.14}$$

A sharp estimate $d < 3/2^{5/2}$ is obtained from the exponential asymptotics of solutions to the ODE (1.13) as $|y| \to \infty$; see the next section.

Local existence of the unique classical solution of the Cauchy problem (1.1), (1.2) follows from the equivalent integral equation

$$u(t) = \mathbf{M}(u) \equiv b(t) * u_0 - \textstyle\int_0^t \Delta b(t-s) * u^p(s) ds, \quad t > 0, \tag{1.15}$$

where $\mathbf{M}$ is a contraction in the metric space of continuous functions in $\mathbf{R}^N \times [0,\delta]$, $\delta > 0$ small, with the sup-norm. Using (1.12), one can see from (1.15) that any uniformly bounded solution $u(x,t)$ admits a local-in-time classical extension. Therefore, as is usual in the classical parabolic theory, if blow-up of any form occurs it must do so in the L^∞-norm.

1.5. **Finite-time blow-up.** Despite the gradient structure, from known results on blow-up for such pseudo-parabolic equations (1.4) [26] it follows that classical solutions whose initial data satisfy an energy inequality,

$$E(u_0) = \tfrac{1}{2}\|\nabla u_0\|_2^2 - \tfrac{1}{p+1}\|u_0\|_{p+1}^{p+1} < 0 \tag{1.16}$$

cannot be extended beyond a finite blow-up time $T < \infty$. In the particular case of $p = 3$, $N = 1$ this can be interpreted as a condition on the initial mass of positive solutions, [35],

$$m_0 \equiv \textstyle\int u_0 > m_c = \tfrac{4}{3}\sqrt{2}\pi.$$

Finite-time blow-up is a well known phenomenon in long-wave unstable thin-film equations of the form (1.3), [31, 29, 2, 3, 4]. Because not all initial data will satisfy this condition and because of the gradient structure, we will also look for decaying in time solutions. In the case of both decaying and blowing up solutions the role of similarity variables is key.

2. Preliminaries: similarity variables for global and blow-up asymptotics

Equation (1.1) is invariant under the group of scaling transformations

$$t \mapsto \lambda t, \quad x \mapsto \lambda^{1/4} x, \quad u \mapsto \lambda^{-1/2(p-1)} u, \quad \lambda > 0.$$

This symmetry suggests the introduction of the following rescaled variables:

$$u(x,t) = [\sigma(T-t)]^{-1/2(p-1)}\theta(y,\tau), \quad y = x/[\sigma(T-t)]^{1/4}, \quad \tau = -\ln[\sigma(T-t)], \tag{2.1}$$

where $\sigma = 1$ corresponds to blow-up at the unknown blow-up time $t = T$, and $\sigma = -1$ to infinite time decay as $t \to \infty$ with a reference time T (generically, we will take $T = 0$ in this case). Without loss of generality, we suppose that the solution $u(x,t)$ blowing up at the finite time $t = T$ in the L^∞-norm

$$\sup_x |u(x,t)| \to \infty \text{ as } t \to T^-, \tag{2.2}$$

is such that the corresponding blow-up set contains the origin,

$$0 \in B(u_0) = \{x \in \mathbf{R}^N : \ \exists\ \{x_n\} \to x,\ \{t_n\} \to T^- \text{ such that } u(x_n,t_n) \to \infty\}.$$

The rescaled solution $\theta(y,\tau)$ defined as in (2.1) satisfies the semilinear equation

$$\theta_\tau = \mathbf{A}(\theta) \equiv -\Delta(\Delta\theta + \theta^p) - \tfrac{\sigma}{4} y \cdot \nabla\theta - \tfrac{\sigma}{2(p-1)}\theta, \tag{2.3}$$

and we are interested in the possible asymptotic dynamics of solutions for $\tau \gg 1$.

An exact similarity solution of (1.1)

$$u_S(x,t) = [\sigma(T-t)]^{-1/2(p-1)} f(y) \tag{2.4}$$

gives an independent of τ stationary solution $f(y)$ of (2.3) giving a fourth-order elliptic equation for the similarity profile f,

$$\mathbf{A}(f) = 0 \quad \text{in } \mathbf{R}^N. \tag{2.5}$$

It is worth mentioning that classical variational approaches do not apply to equation (2.5) because $\mathbf{A}$ is not potential.

We restrict our attention to the one-dimensional or radial geometry, where (2.5) is a fourth-order ODE, and in most cases we impose symmetry conditions at the origin $y = 0$ and a suitable decay condition (possibly exponential) at infinity:

$$f'(0) = 0, \quad f'''(0) = 0, \quad \text{and } f(y) \to 0 \quad \text{as } y \to \infty. \tag{2.6}$$

Existence of non-radial solutions to (2.5) is an interesting open problem.

2.1. **Conservative similarity solutions and the first critical exponent.** Under appropriate decay conditions at infinity (say, exponential), equation (1.1) is *conservative*:

$$\frac{d}{dt}\int_{\mathbf{R}^N} u dx = -\int_{\mathbf{R}^N} \Delta(\Delta u + u^p) dx = 0. \tag{2.7}$$

Given an exact similarity solution (2.4), and assuming that $f \in L^1(\mathbf{R}^N)$, we have that

$$\int_{\mathbf{R}^N} u_S(x,t) dx = [\sigma(T-t)]^{-1/2(p-1)+N/4} \int_{\mathbf{R}^N} f(y) dy, \tag{2.8}$$

which satisfies (2.7) only if $p = p_0$, where

$$p_0 = 1 + \tfrac{2}{N} \tag{2.9}$$

is the *critical exponent* in the problem. It is interesting that p_0 coincides with the Fujita exponent for the semilinear heat equation (1.5). In fact, it is the *first* critical exponent and in Section 5 we show that there exists a countable sequence of further critical exponents (1.8) corresponding to special cases of globally decaying solutions.

It follows from (2.9) that in the one-dimensional case, $N = 1$, we will need to concentrate on the crucial case $p = p_0 = 3$. However, we expect that similarity solutions will govern the dynamics for all p, regardless of the conservation property, see Section 4.6. Moreover, from (2.8) we have

$$\text{for any } p \neq p_0, \quad f \in L^1 \quad \Longrightarrow \quad \textstyle\int f = 0, \tag{2.10}$$

i.e., any L^1-similarity solution must have zero mass, if it is to play a part in the asymptotic blow-up behaviour for the PDE (1.1).

2.2. **Local asymptotic properties of self-similar solutions.** First, we need to describe the possible asymptotics of small solutions to (2.5) satisfying $f(y) \to 0$ as $y \to \infty$, see (2.6). Consider the linearization of (2.5) with operator (2.3) about $f = 0$,

$$-f'''' - \tfrac{2(N-1)}{y} f''' - \tfrac{\mu}{y^2} f'' + \tfrac{\mu}{y^3} f' - \tfrac{\sigma}{4} y f' - \tfrac{\sigma}{2(p-1)} f = 0, \tag{2.11}$$

where $\mu = (N-1)(N-3)$. To determine the balance between the leading terms f'''' and $\frac{\sigma}{4} y f'$, we set $z = y^\alpha$ with $\alpha = 4/3$ reducing the ODE to

$$-f'''' - a_1 f' - a_2 z^{-1} f + \mathbf{D}(z) f = 0. \tag{2.12}$$

Here $a_1 = \sigma/4\alpha^3$, $a_2 = \sigma/2(p-1)\alpha^4$ and $\mathbf{D}(z)f = \sum_{j=1}^{3} \gamma_j z^{j-4} f^{(j)}$ is a linear operator with bounded coefficients as $z \to \infty$, where the first coefficient of the derivative f' is of order $O(z^{-3})$. By the perturbation theory of linear ODEs (see Chapters III - V in [7]), we have that the leading terms of exponentially decaying solutions are described by the operator in (2.12) with constant coefficients,

$$-f'''' - a_1 f' = 0. \tag{2.13}$$

Setting $f = e^{pz}$, $p \neq 0$, gives the characteristic equation $p^4 + a_1 p = 0$, whence

$$p^3 = -a_1 = -\sigma/4\alpha^3 \equiv \rho_0^3(-\sigma), \quad \text{where } \rho_0 > 0. \tag{2.14}$$

There exist three roots of the form

$$p_k = \rho_0 e^{i\pi(4k+\sigma+1)/6}, \quad k = 0, 1, 2, \tag{2.15}$$

where the number of roots with negative real part is two for $\sigma = -1$ and only one for $\sigma = +1$. Thus, as $y \to \infty$, there exists

(2.16) a two-dimensional exponential bundle for the global case $\sigma = -1$, and

(2.17) a one-dimensional exponential bundle for the blow-up case $\sigma = 1$.

On the other hand, equation (2.12) admits solutions with algebraic decay (rather than exponential) as $z \to \infty$ described by the first-order operator

$$-a_1 f' - a_2 z^{-1} f = 0 \quad \Longrightarrow \quad f(z) = A z^{-2/\alpha(p-1)}, \ \ A \neq 0.$$

Existence of solutions with such decay for the perturbed equation (2.12) is established by a standard expansion analysis by calculating solutions via a Kummer-type series converging uniformly for $z \gg 1$. For the linearized equation (2.11), the leading order behaviour is algebraic,

$$f(y) = A|y|^{-2/(p-1)}(1 + o(1)) \quad \text{as } |y| \to \infty, \quad f \in L^1 \ \text{ iff } \ p < p_0, \tag{2.18}$$

with arbitrary constant $A \neq 0$. For blow-up similarity solutions (2.4), the limit-time profile is then bounded for any $x \neq 0$ and has the form

$$u_S(x, T^-) = A|x|^{-2/(p-1)}, \quad x \neq 0.$$

Clearly, for $p \geq p_0$, this is not an admissible solution for the conservative case given finite initial mass, and thus in the critical case $p = p_0$ we must have $A = 0$ and exclusively exponential decay of any similarity profiles.

3. The spectral properties of the rescaled linear operators

The structure of the rescaled equation (2.3) suggests the study of the spectral properties of the linearized operator with $\sigma = -1$ (global solutions), which is conveniently represented in a form associated with the operator in (1.13),

$$\mathbf{A}'(0) = -\Delta^2 + \tfrac{1}{4} y \cdot \nabla + \tfrac{1}{2(p-1)} I \equiv \mathbf{B} + c_* I, \quad c_* = \tfrac{N}{4(p-1)}(p_0 - p). \tag{3.1}$$

The spectral properties of $\mathbf{B}$ and the corresponding adjoint operator $\mathbf{B}^*$ occurring for $\sigma = 1$ will play an important role in the further asymptotic analysis of the nonlinear PDE. The operators are defined in weighted L^2-spaces with the weight functions induced by the exponential estimate of the rescaled kernel (1.14).

3.1. **The point spectrum of the non self-adjoint operator B.** Note that $\mathbf{B}$ is not symmetric and does not admit a self-adjoint extension. We consider $\mathbf{B}$ in the weighted space $L^2_\rho = L^2_\rho(\mathbf{R}^N)$ with the exponentially growing weight function

$$\rho(y) = e^{a|y|^\alpha} > 0 \quad \text{in } \mathbf{R}^N, \quad \text{where } \alpha = 4/3, \tag{3.2}$$

and $a \in (0, 2d)$ is a sufficiently small constant. We ascribe to $\mathbf{B}$ the domain H^4_ρ being a weighted Hilbert space with the norm

$$\|v\|^2 = \textstyle\int \rho(y) \sum_{k=0}^4 |D^k v(y)|^2 \, dy,$$

induced by the corresponding inner product. From [8] we have the following lemma.

Lemma 3.1. $\mathbf{B} : H^4_\rho \to L^2_\rho$ *is a bounded linear operator with the real spectrum*

$$\sigma(\mathbf{B}) = \{\lambda_l = -l/4, \ l = 0, 1, 2, ...\}. \tag{3.3}$$

The eigenvalues λ_l have finite multiplicity with eigenfunctions

$$\psi_\beta(y) = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y), \quad \textit{with any } |\beta| = l, \tag{3.4}$$

and the subset of eigenfunctions $\Phi = \{\psi_\beta, \ |\beta| = 0, 1, 2, ...\}$ is complete in L^2_ρ.

Lemma 3.1 gives the centre and stable subspaces of $\mathbf{B}$, $E^c = \mathrm{Span}\{\psi_0 = F\}$ and $E^s = \mathrm{Span}\{\psi_\beta, \ |\beta| > 0\}$.

3.2. **The polynomial eigenfunctions of the adjoint operator B*.** Consider the operator adjoint to $\mathbf{B}$,

$$\mathbf{B}^* = -\Delta^2 - \tfrac{1}{4} y \cdot \nabla \,, \tag{3.5}$$

which is related to $\mathbf{A}'(0)$ in (2.3) for $\sigma = 1$ (the blow-up case). We consider $\mathbf{B}^*$ in $L^2_{\rho^*}$ with the exponentially decaying weight function $\rho^*(y) = 1/\rho(y) \equiv e^{-a|y|^\alpha} > 0$.

Lemma 3.2. $\mathbf{B}^* : H^4_{\rho^*} \to L^2_{\rho^*}$ *is a bounded linear operator with the same spectrum,* (3.3). *The eigenfunctions $\psi^*_\beta(y)$ with $|\beta| = l$ are l-th order polynomials*

$$\psi^*_\beta(y) = \tfrac{1}{\sqrt{\beta!}} \Big[y^\beta + \textstyle\sum_{j=1}^{\lfloor |\beta|/4 \rfloor} \tfrac{1}{j!} (-\Delta)^{2j} y^\beta \Big] , \tag{3.6}$$

and subset $\{\psi^*_\beta\}$ is complete in $L^2_{\rho^*}$.

With this definition of the adjoint eigenfunctions, the orthonormality condition

$$\langle \psi_\beta, \psi_\gamma^* \rangle = \delta_{\beta,\gamma} \tag{3.7}$$

holds. Here $\langle \cdot, \cdot \rangle$ denotes the standard (dual) L^2 inner product.

4. Blow-up similarity profiles for $p = 3$ in one dimension

The dynamics simplify for the conservative case $p = 1 + 2/N$ as equation (2.5) can be integrated once. The particular critical case $p = 3$ for $N = 1$ is of key importance in our analysis and highlights the typical techniques required to describe a countable subset of self-similar blow-up patterns.

4.1. **Preliminaries.** Thus, the ODE (2.5), (2.3) with $\sigma = 1$ for the similarity profile can be integrated once to give

$$f''' + \tfrac{1}{4} y f + (f^3)' = 0 \quad \text{for } \; y > 0, \quad f'(0) = 0, \tag{4.1}$$

where we have set the constant of integration to zero on the right-hand side. This removes the algebraically decaying mode (2.18), $f(y) = A y^{-1}(1 + o(1))$, by looking for L^1-solutions satisfying the conservation of mass condition (2.7). It follows from (2.17) that we are left with a *one-parameter* family of exponentially decaying functions satisfying

$$f(y) = C y^{-1/3} e^{-\beta y^{4/3}} (1 + o(1)) \quad \text{as } y \to +\infty, \quad C \ge 0, \quad \beta = 3/4^{4/3}. \tag{4.2}$$

This asymptotic bundle is generated by the linear part of the operator in (4.1). In order to construct a solution to (4.1) we will use a shooting type argument from $y = \infty$. Thus, all admissible similarity profiles $f(y)$ have the asymptotic behaviour (4.2) and also satisfy the symmetry condition at the origin.

Let us now study the behaviour of solutions on the manifold (4.2) parameterized by C. Denoting $f(y; C)$ as the function which satisfies equation (4.1) with decay from the bundle (4.2), the goal is to find the set of C such that

$$S(C) \equiv f'(0; C) = 0. \tag{4.3}$$

Because the function $S(C)$ in (4.3) is analytic in C (see below), equation (4.3) has at most a countable subset of roots which can accumulate at $C = \infty$ only and this actually happens. We begin with the global existence for the family $\{f(y; C)\}$.

Lemma 4.1. *For any $C > 0$, the solution $f(y; C)$ to* (4.1), (4.2) *is well defined for all $y \in \mathbf{R}$.*

Proof. This follows from the local properties of the operator in (4.1) which are close to those for the second-order case $f'' + f^3 = 0$. Obviously, it does not admit blow-up of solutions at finite y. Integrating (4.1), $f'' = -f^3 + \frac{1}{4} \int f y dy$, multiplying by f' and integrating again over a sufficiently small interval near any fixed point y yields

$$\tfrac{1}{2} f'^2 = -\tfrac{1}{4} f^4 + \tfrac{1}{4} \textstyle\int f' \int f y + \text{const} \le \tfrac{1}{4} \int |f'| \int |f y| + \text{const}, \tag{4.4}$$

where by Lagrange's formula of finite increments, the right-hand side is not more than quadratic in f' (this recovers the famous Bernstein condition of local solvability). Therefore, $f(y)$ cannot blow-up at a finite point y_* along a sequence since (4.4) guarantees that $f'(y_*)$ is finite. Obviously, this analysis applies for all $p > 1$. $\square$

4.2. **Existence of the first monotone blow-up similarity pattern.** The proof that there exists a first $C_1 > 0$ such that (4.3) holds involves three steps. First, we show that in the limit as $C \to 0^+$ solutions $f(y; C)$ are strictly monotone decreasing on $[0, \infty)$ and, second, that in the other limit $C \to +\infty$ all solutions cannot be monotone. Then, by a standard continuity argument, we have that there must exist an intermediate $C = C_1 > 0$ corresponding to an admissible monotone solution.

Proposition 4.2. *For all $0 < C \ll 1$, solutions $f(y; C)$ are strictly monotone decreasing in $y \geq 0$ and $f'(0; C) < 0$.*

Proof. Rescaling $f = Cg$, we have

$$g''' + \tfrac{1}{4} y g + C^2 (g^3)' = 0 \quad \text{in } \mathbf{R}_+. \tag{4.5}$$

In view of the behaviour at infinity (4.2), by standard results on continuous dependence for ODEs [7], it follows that as $C \to 0^+$,

$$f(y; C) = C(\phi_0(y) + o(1)) \quad \text{uniformly in } \mathbf{R}_+, \tag{4.6}$$

where ϕ_0 solves the linear ODE $\phi''' + \frac{1}{4}\phi' y = 0$ and satisfies (4.2) with $C = 1$. All the derivatives of f converge similarly. Let us now show that $\phi_0(y)$ is strictly monotone decreasing. Assume that y_i is the first (from $y = \infty$) local maximum point of ϕ_0, $\phi_0'(y_i) = 0$ and $\phi_0'(y) < 0$ on (y_i, ∞). Integrating over (y_i, ∞), we obtain the contradiction, $\phi_0''(y_i) = \int_{y_i}^{\infty} s\phi_0(s)ds > 0$, thus observing an elementary feature of the Maximum Principle for such third-order equations. $\square$

Proposition 4.3. *There exists a $C_* > 0$ such that $f(y; C_*)$ has a local maximum point at some $y_* \geq 0$ and $f(y; C_*) > 0$ on $[y_*, \infty)$.*

Proof. Assume for contradiction that $f(y; C)$ is strictly monotone decreasing in $\mathbf{R}_+$ for all $C > 0$, and then $f(0; C) > 0$. One can see from the rescaled ODE (4.5) that the solutions $f(y; C)$ cannot be bounded for $y \in \mathbf{R}_+$ uniformly in $C > 0$. Therefore, there exists a sequence $\{C_k\} \to \infty$ such that $a_k = f(0; C_k) \to \infty$. Performing the scaling

$$f(y) = a_k g_k(z), \quad y = z/a_k, \tag{4.7}$$

we arrive at a perturbed ODE for the sequence $\{g_k(z)\}$

$$g_k''' + (g_k^3)' = -\tfrac{1}{4a_k^4} z g_k \quad \text{for } z > 0, \quad g_k(0) = 1, \tag{4.8}$$

where $0 < g_k(z) \leq 1$ and $g_k(z)$ is monotone decreasing in z. Since $\{g_k\}$ is a uniformly bounded sequence of solutions of the asymptotically perturbed ODE (4.8) with regular coefficients, by the Ascoli-Arzelá theorem and standard ODE estimates [7], we have that along a subsequence, $g_k \to \bar{g}$ uniformly on compact subsets in z, where $\bar{g}$, $0 \le \bar{g} \le 1$, must be a monotone decreasing solution of the unperturbed ODE

$$(\bar{g}'' + \bar{g}^3)' = 0, \quad \bar{g}(0) = 1. \tag{4.9}$$

Since all the solutions of (4.9) are oscillatory, which is easily checked by integrating twice, this leads to a contradiction. Therefore, there exists a sufficiently large $C_* > 0$ such that $f(y; C_*)$ is not strictly monotone in $\mathbf{R}_+$ and has a local maximum point. □

Theorem 4.4. *There exists a constant $C_1 \in (0, C_*]$ such that (4.3) holds for $C = C_1$ and $f_1(y) = f(y; C_1)$ is a strictly monotone decreasing symmetric positive similarity profile.*

Proof. Introducing the subset

$$W_1 = \{\mu > 0 : \ f(y; C) \text{ is strictly monotone decreasing in } \mathbf{R}_+ \text{ for all } C \in (0, \mu)\},$$

we have that $W_1 \ne \emptyset$ by Proposition 4.2 and W_1 is bounded above by Proposition 4.3. Hence, there exists

$$C_1 = \sup W_1 \le C_*, \tag{4.10}$$

where by construction $f_1(y) = f(y; C_1)$ is monotone decreasing for $y \ge 0$. One can see that $f_1'(y)$ must vanish at the origin, i.e., (4.3) holds. □

We now briefly describe a countable subset of the similarity profiles satisfying (4.1) and (4.2) with some $C > C_1$. This construction is similar to that for the second-order ODEs from blow-up reaction-diffusion theory; see pp. 190-195 in [32] and related references therein. Similar to Proposition 4.3 we show that as C increases, the function $f(y; C)$ becomes more and more oscillatory for $y > 0$. Indeed, performing the scaling (4.7) and passing to the limit $C = C_k \to \infty$, we obtain a bounded solution $\bar{g}$ satisfying (4.9) admitting oscillatory solutions only. Hence, $f(y; C)$ can have an arbitrarily large number of oscillations for $y > 0$ with $C \gg 1$. As in the proof of Theorem 4.4, for any $k = 2, 3, ...$, we define the subsets

$$W_k = \{\mu > C_{k-1} : \ f(y; C) \text{ has at most } k \text{ local extrema for } y \ge 0 \text{ for all } C \in [C_{k-1}, \mu)\}.$$

Then, once C_{k-1} is known, starting with $k = 2$, we have that $W_k \ne \emptyset$ and by the oscillatory behaviour for $C \gg 1$, W_k is bounded from above. Hence, we define

$$C_k = \sup W_k > C_{k-1}, \quad k = 2, 3, \dots, \tag{4.11}$$

and by construction, $f_k(y) = f(y; C_k)$ satisfies the symmetry condition at the origin (otherwise, it is not the supremum in (4.11)).

Such a construction, while giving an infinite sequence of similarity profiles, does not describe the important properties of the functions $f_k(y)$, such as positivity (or at least "positivity dominance") and the actual distribution of local extrema and inflection points of the profiles. This will be done by a more delicate and partially formal asymptotic matching procedure.

4.3. **Asymptotic construction of a countable subset of similarity profiles.** As is common in matched asymptotics expansions, we will consider three regions as described in Figure 1. The two primary regions are the *outer region* localized near the origin, where the mass of the solution is concentrated with oscillations around a parabolic profile (see (4.19) below), and a *far-field region* as described by the asymptotic bundle (4.2). Joining the two is a narrow *inner* region. The expansion scaling parameters denoted by ε^p, ε^q, etc. are to be determined by matching. Because the characteristics of the family are determined by the far-field behaviour, we will begin by considering refined asymptotics as $y \to \infty$.

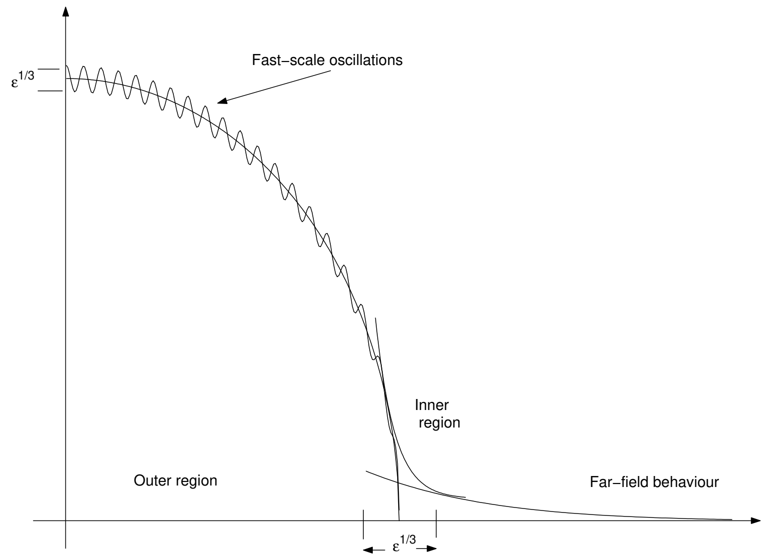

FIGURE 1. Sketch of the regions for construction of asymptotic profiles.

The far-field behaviour. To determine the asymptotic behaviour as $y \to \infty$ in greater detail, we produce an asymptotic description of solutions from the exponential bundle (4.2). Specifically, the solution is given by the formal power series

$$f(y;C) = \textstyle\sum_{n=0}^{\infty} C^{2n+1}\phi_n(y), \tag{4.12}$$

where ϕ_0 is given in (4.6) and the rest of the terms are obtained from the relation

$$\mathbf{B}^*\phi_n \equiv \phi_n''' + \tfrac{1}{4}\, y\phi_n = - \sum_{\substack{1\le i,j,k\le n \\ i+j+k=n+1}} (\phi_i\phi_j\phi_k)', \quad n = 2,3,\dots, \tag{4.13}$$

with the condition $\phi_n(y) = o(\phi_0(y))$ as $y \to \infty$. Using the expansion (4.2), it can be shown by direct calculation that for $y \gg 1$,

$$\phi_n(y) = \left[\gamma_n + o(y^{-1})\right]\, y^{-(4n+1)/3} e^{-(2n+1)\beta y^{4/3}}, \tag{4.14}$$

with suitable constants $|\gamma_n| \le 1$. Moreover, the right-hand side of (4.14) with a sufficiently large constant γ_n gives uniform estimates of ϕ_n in $\mathbf{R}_+$, which establish the uniform convergence of (4.12) on subsets $\{y \ge y_0\}$ with $y_0 \gg 1$. By the Weierstrass theorem, the solution $f(y;C)$ in (4.12) obtained as a uniformly converging series of analytic functions is analytic in C for $y \ge y_0$. Therefore, on extension to $y \in [0, y_0)$, as a solution of an ODE with analytic coefficients and analytic dependence on C in the Dirichlet boundary condition at $y = y_0$, it is analytic in C for any $y \ge 0$ (cf. typical analyticity results in the classical ODE theory, [7], Section 8, Chapter I). Hence, (4.3) is an analytic function having isolated zeros only and we arrive at the following conclusion.

Proposition 4.5. *The problem* (4.1), (4.2) *has at most a countable subset of solutions.*

The discrete nature of the function $S(C)$ will be made evident by close examination of the inner and transition regions.

A singular perturbation problem. We begin by looking for possible solutions localized in a neighbourhood of the origin $y = 0$. We rescale the ODE (4.1) as (cf. (4.7))

$$f(y) = ag(z), \quad y = az, \quad \text{where } a = f(0;C), \tag{4.15}$$

with a an as yet unspecified function of C to be determined for the similarity profiles $f_k(y)$. Under the assumption that $a(C) \to \infty$ as $C \to \infty$, possibly along a subsequence, we define

$$\varepsilon = a^{-4} \ll 1 \quad \text{for } C \gg 1.$$

Under the rescaling (4.15), equation (4.1) leads to a singularly perturbed ODE

$$\varepsilon g''' + \tfrac{1}{4} z g + (g^3)' = 0. \tag{4.16}$$

We need to describe a two-parameter family of solutions of (4.16) defined in a suitable neighbourhood of the origin $z = 0$, which can be asymptotically matched with the exponential bundle (4.2). In order to describe such a family, we use the method of matched asymptotic expansions and supplement the ODE (4.16) with two standard conditions

$$g(0) = 1, \quad g'(0) = 0. \tag{4.17}$$

In addition, we have

$$g''(0) = b\varepsilon^{-m}(1 + o(1)), \tag{4.18}$$

where the constant b and parameter m are as yet unspecified. The third condition (4.18) is introduced for convenience only and we show below that both b and m are determined by the matching procedure. However, their inclusion here is to emphasize that in the outer region, a faster scale is necessary in order to resolve the additional oscillatory structure as described in the proof of Proposition 4.3 by the perturbed problem (4.8).

We will how that the solution to (4.16) has a three layer structure comprising
(i) an *outer region* $0 \le z < \sqrt{12}$ in which $g = O(1)$,
(ii) an *inner region* $z = \sqrt{12} + O(\varepsilon^{1/3})$ in which $g = O(\varepsilon^{1/6})$, and finally

(iii) a *third region* $z > \sqrt{12}$, where g is exponentially small (and belongs to the exponential bundle described in Section 2).

The outer region. Let $g_0(z)$ be the solution of the unperturbed problem (4.16), namely

$$\tag{4.19} \tfrac{1}{4} z g_0 + (g_0^3)' = 0 \quad \text{with} \quad g_0(0) = 1 \implies g_0(z) = \left(1 - \tfrac{1}{12}\, z^2\right)^{1/2}.$$

Close to $z = 0$, we perform the linearization

$$\tag{4.20} g(z) = g_0(z) + G(Z), \quad \text{with the fast variable } Z = z/\varepsilon^{1/2}.$$

Then G satisfies the equation

$$\tag{4.21} G''' + (3g_0^2 G)' + \tfrac{1}{4}\varepsilon Z G + (3 g_0 G^2 + G^3)' + \varepsilon^{3/2} g_0''' = 0, \quad \text{where } g_0^2(Z) = 1 - \tfrac{\varepsilon}{12}\, Z^2.$$

It follows that on compact subsets $0 \le Z \le c$, the principal part of equation (4.21) has constant coefficients,

$$\tag{4.22} G''' + 3G' + \mathcal{N}_1 = 0, \quad G(0) = G'(0) = 0,$$

where $\mathcal{N}_1 = O(\varepsilon G + G^2 + \varepsilon^2)$ on bounded smooth solutions. Therefore, by standard perturbation ODE theory [7], recalling the third condition in (4.17), the solution satisfies the following expansion as $\varepsilon \to 0$ uniformly on subsets $Z \in [0, c]$:

$$\tag{4.23} G(Z) = \theta[1 - \cos(\sqrt{3}\, Z)] + o(\theta), \quad \text{where } \theta = \tfrac{1}{3} b \varepsilon^{1-m}.$$

This rigorously determines the asymptotic behaviour in the region $\{z = O(\varepsilon^{1/2})\}$. For matching reasons, we need to extend expansion (4.23) to larger subsets. This is done by introducing the new independent variable

$$\tag{4.24} t = \int_0^Z \left(1 - \varepsilon \tfrac{u^2}{12}\right)^{1/2} du = \tfrac{1}{\varepsilon^{1/2}} \left[\tfrac{z}{2} \left(1 - \tfrac{z^2}{12}\right)^{1/2} + \sqrt{3} \sin^{-1}\left(\tfrac{z}{2\sqrt{3}}\right)\right],$$

and setting $G(Z) = P(t)/g_0(Z)$. This gives, similar to (4.22), an equation with constant coefficients in the principle part

$$\tag{4.25} P''' + 3P' + \mathcal{N}_2 = 0, \quad P(0) = P'(0) = 0,$$

where the perturbation $\mathcal{N}_2$, in addition to the same small terms as above from (4.22), now contains differential operators with coefficients depending on the derivatives of $g_0 = g_0(z(t))$ satisfying

$$\tfrac{dg_0}{dt} = \tfrac{g_0'}{t'} = O(\varepsilon^{1/2}), \quad \tfrac{d^2 g_0}{dt^2} = O(\varepsilon), \quad \tfrac{d^3 g_0}{dt^3} = O(\varepsilon^{3/2})$$

uniformly on compact subsets in $z \in [0, c]$ for any $c < \sqrt{12}$. Therefore, upon returning to the original variables $\{z,\, g\}$, the outer expansion takes the form

$$\tag{4.26} g(z) = g_0(z) + \theta \tfrac{1}{g_0(z)} (1 - \cos\sqrt{3} t) + ... \equiv g_0 + \theta g_1 + ..., \quad \theta = \tfrac{1}{3} b \varepsilon^{1-m}.$$

Note that we cannot guarantee that it is valid uniformly on subsets $z \in [0, c]$ since though the principle part in (4.25) is exactly the same as in (4.22), equation (4.25) is then considered on expanding subsets in $t \in [0, c\varepsilon^{-1/2}]$ becoming unbounded as $\varepsilon \to 0$. For first-order matching, it suffices to prescribe only the rate of deviation $O(g_0^{-1})$ of $g(z)$ from the unperturbed profile $g_0(z)$ as correctly described by the second term in (4.26).

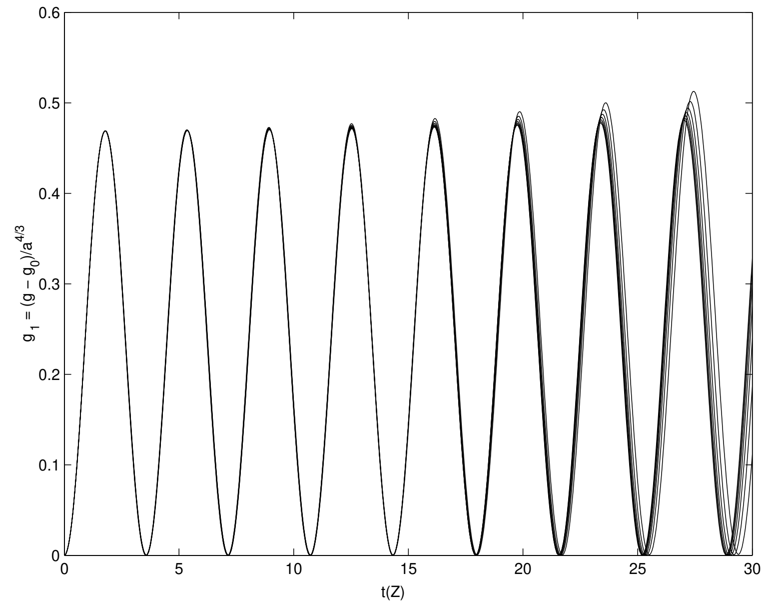

FIGURE 2. Convergence of numerical solutions to $g_1(z,t)$.

The inner region. The outer expansion (4.26) does not reflect the far-field behaviour (4.2). Thus, in order to specify the behaviour for $z \approx \sqrt{12}$, we introduce the inner variables $\bar{z}$ and $R(\bar{z})$ by

$$z = \sqrt{12} + \varepsilon^{1/3}\bar{z}, \qquad g = \varepsilon^{1/6}R, \quad \text{where} \quad R \quad \text{solves} \tag{4.27}$$

$$R''' + \tfrac{1}{4}\left(\sqrt{12} + \varepsilon^{1/3}\bar{z}\right)R + (R^3)' = 0. \tag{4.28}$$

The unperturbed equation

$$R_0''' + \tfrac{1}{2}\sqrt{3}R_0 + (R_0^3)' = 0 \tag{4.29}$$

admits a one-parameter family of suitable exponentially decaying solutions

$$R_0(\bar{z}) \simeq A_0 e^{-3^{1/6}\bar{z}/2^{1/3}} \qquad \text{as} \quad \bar{z} \to +\infty, \tag{4.30}$$

for some constant A_0, and a three-parameter family of slowly growing solutions as $\bar{z} \to -\infty$. Solutions from the bundle (4.30) satisfy

$$R_0(\bar{z}) \simeq R_0 \simeq \left(\frac{-\bar{z}}{3^{1/2}}\right)^{1/2} + \frac{\bar{a}_1}{(-\bar{z})^{1/2}}\left\{1 - \bar{a}_2\cos[c(-\bar{z})^{3/2}] - \bar{a}_3\sin[c(-\bar{z})^{3/2}]\right\}, \tag{4.31}$$

where $c = 2/3^{3/4}$ and $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ are suitable constants. This expansion is determined in the standard way by setting $R_0 = \tilde{R}_0 + \tilde{R}_1$, where $\tilde{R}_0 = (-\bar{z})^{1/2}/3^{1/4}$ under the assumption that $|\tilde{R}_1| \ll |\tilde{R}_0|$. Then the leading linear equation for $\tilde{R}_1$ is

$$\tilde{R}_1''' + \tfrac{\sqrt{3}}{2}\tilde{R}_1 - \tfrac{1}{\sqrt{3}}(\bar{z}\tilde{R}_1)' = 0, \tag{4.32}$$

which has an exact solution in terms of the Airy functions Ai and Bi,

$$\tilde{R}_1 = c_1 Ai^2(\bar{z}) + c_2 Ai(\bar{z}) Bi(\bar{z}) + c_3 Bi^2(\bar{z}), \quad c_i \in \mathbf{R}. \tag{4.33}$$

This fact is important for the reliable numerical integration of (4.29).

Matching. To determine the sequence $\{C_k,\, k \gg 1\}$ and the corresponding values $f(0; C_k) = a(C_k)$, we now match the inner and outer solutions. Expanding the components of the outer solution (4.26) near the transition point $z = \sqrt{12}$, we have

$$g_0 \simeq \varepsilon^{1/6} \left(-\frac{\bar{z}}{3^{1/2}}\right)^{1/2} \left(1 + \varepsilon^{1/3} \frac{\bar{z}}{8 \cdot 3^{1/2}}\right), \tag{4.34}$$

$$g_1 \simeq \varepsilon^{-1/6} \frac{b}{3} \left(-\frac{3^{1/2}}{\bar{z}}\right)^{1/2} \{1 - \cos \frac{\sqrt{3} t_0}{\varepsilon^{1/2}} \cos[c(-\bar{z})^{3/2}] - \sin \frac{\sqrt{3} t_0}{\varepsilon^{1/2}} \sin[c(-\bar{z})^{3/2}]\}, \tag{4.35}$$

where $t_0 = \int_0^{\sqrt{12}} \sqrt{1 - v^2/12}\, dv = \sqrt{3}\pi/2$. Comparing the outer expansion (4.26), using (4.34) and (4.35) with the inner expansion $g = \varepsilon^{1/6} R_0 + ...$, where R_0 is given by (4.31), yields the matching condition $1 - m - 1/6 = 1/6$, if full balance is to be obtained, i.e.,

$$m = \tfrac{2}{3}, \tag{4.36}$$

together with the coefficients

$$\bar{a}_1 = b\, 3^{-3/4}, \qquad \bar{a}_2 = \cos \frac{3^{1/2} t_0}{\varepsilon^{1/2}} \quad \text{and} \quad \bar{a}_3 = \sin \frac{3^{1/2} t_0}{\varepsilon^{1/2}} \quad (t_0 = 3\sqrt{\pi}/2). \tag{4.37}$$

It follows that (4.36) and the first equality in (4.37) are established rigorously since we have used the "envelope" characteristic in expansion (4.26) without specifying the highly oscillatory component. In particular, (4.37) demands

$$\bar{a}_2^2 + \bar{a}_3^2 = 1. \tag{4.38}$$

To determine these values, we observe that equation (4.29) is translation invariant, as is the asymptotic behaviour for $\bar{z} \to \infty$, but not (to leading order) as $\bar{z} \to -\infty$. To see this, we consider the solution to (4.29) as a shooting problem in the single parameter A_0.

However,

$$R_0(\bar{z}; \hat{A}_0) = R_0(\bar{z} - \bar{z}_0; A_0), \quad \text{where} \quad \hat{A}_0 = A_0 e^{3^{1/6} 2^{-1/3} \bar{z}_0}.$$

Clearly this translation does not change the ODE although it will affect the coefficients in the far-field behaviour (4.31). Denoting the new coefficients by $\hat{a}_i$, $i = 1, 2, 3$, we have to leading order that

$$\hat{a}_1 = \bar{a}_1 + \bar{z}_0/2 \cdot 3^{1/4}, \quad \hat{a}_2 = \bar{a}_2, \quad \hat{a}_3 = \bar{a}_3,$$

i.e., the translation only affects the coefficient $\bar{a}_1$ in (4.31). Thus,

$$\hat{A}_0 = A_0 e^{3^{5/12} 2^{2/3} (\hat{a}_1 - \bar{a}_1)}. \tag{4.39}$$

We now seek A_0 such that the form (4.31) satisfies the constraint (4.38). From the translation argument we have that $\bar{a}_1$ is monotone decreasing in A_0 and, it follows from

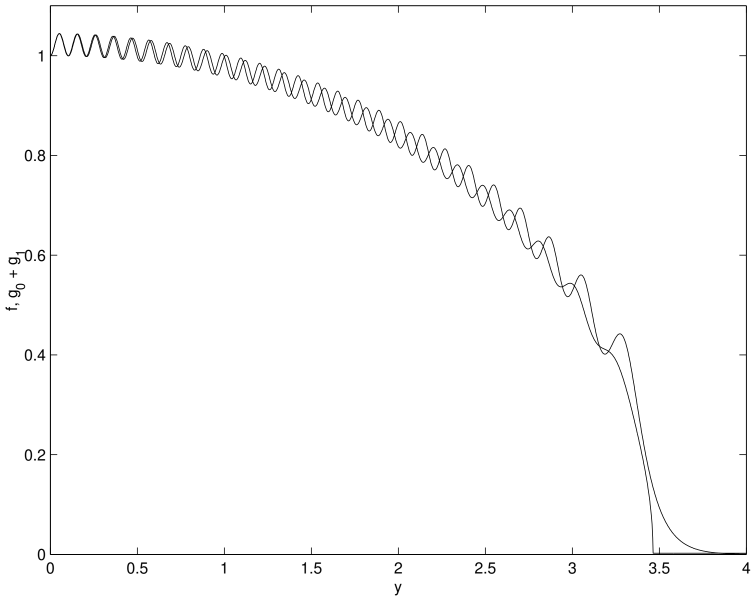

FIGURE 3. Comparison of numerical and asymptotic solutions.

(4.32), (4.31) that there are precisely two possible solutions which are the negatives of each other, i.e., $\bar{a}_1$ and $-\bar{a}_1$. Taking $\hat{a}_1 = -\bar{a}_1$, (4.39) gives

$$\hat{A}_0 = A_0 e^{-2^{5/3}3^{5/12}\bar{a}_1}. \tag{4.40}$$

Numerical computation gives

$$A_0 \approx .448\ldots, \quad \bar{a}_1 = .341\ldots$$

and from (4.40) we deduce the other set of values

$$A_0 \approx .081\ldots, \quad \bar{a}_1 = -.341\ldots.$$

Finally, (4.37) may be used to determine the two values of b. Using the value $\bar{a}_1 \approx .341\ldots$ in the asymptotic expression (4.26) gives very good agreement to full numerical solutions near $z = 0$, as seen in Figure 3. In particular, the amplitude of the fast-scale oscillation is excellent.

The third region. It follows from (4.2) and (4.15) that

$$g(z) \simeq C\varepsilon^{1/3} z^{-1/3} e^{-\beta z^{4/3}/\varepsilon^{1/3}} \tag{4.41}$$

is the appropriate behaviour satisfying (4.16) for $z > \sqrt{12}$. Writing (4.41) in terms of the inner variable $\bar{z}$ from (4.27) gives the leading order expression

$$g(z) \simeq C\varepsilon^{1/3} 12^{-1/6} e^{-\beta 12^{2/3}/\varepsilon^{1/3}}\, e^{-3^{1/6}\bar{z}/2^{1/3}}$$

after using $\varepsilon^{1/3}\bar{z} = O(1)$. Comparing to the inner behaviour (4.30) and using (4.27) gives the matching condition

$$A_0 \simeq C 12^{-1/6} \varepsilon^{1/6} e^{-\beta 12^{2/3}/\varepsilon^{1/3}}, \tag{4.42}$$

Since both values of A_0 are of order $O(1)$, then

$$\varepsilon \sim \frac{3^5}{2^4} \frac{1}{(\ln C)^3} + \frac{3^6}{2^5} \frac{\ln \ln C}{(\ln C)^4} + \frac{3^6}{2^4} \frac{\ln(3^{-2/3} 2 A_0)}{(\ln C)^4} \quad \text{as } C \to \infty.$$

Recall that all the estimates obtained without using the highly oscillatory tails in the singular layer given by (4.26) are rigorous and asymptotically sharp. The fact that there are two values of A_0 and thus two relationships between C and ε corresponds to the two families of solutions, those with $g''(0) > 0$ or $g''(0) < 0$. Most importantly, given that asymptotically $\bar{a}_2 = \cos(3\pi/2\varepsilon^{1/2})$ and $\bar{a}_3 = \sin(3\pi/2\varepsilon^{1/2})$ in (4.37) are *unique* (up to a change in sign in both coefficients), we have that

$$3\pi/2\varepsilon^{1/2} = k\pi \pm \phi\pi(1 + o(1)), \tag{4.43}$$

for an unknown phase shift ϕ (this is a formal conclusion). Using this in the definition of $\varepsilon = a^{-4}$ gives

$$a_k = \sqrt{2k/3}(1 + o(1)) \quad \text{as } k \to \infty.$$

Also, from (4.42) this fixes the corresponding values of the expansion coefficients $C = C_k$ in (4.2) in the outer region

$$C_k \sim k^{1/3} e^{3k^{2/3}/2^{2/3}} \quad \text{for } k \gg 1.$$

Using the computed values of the coefficients and expansions, we present a comparison of the numerical and asymptotic solutions in Figure 3. Notice the phase lag in the t-variable. This can be corrected by further terms in the expansion but, even to this order, captures the correct number of fast scale oscillations. The Z variable over-predicts this number by about one half for the value of $\varepsilon \simeq .00083...$ as shown in Figure (3).

4.4. **Numerical methods. Solution of the full ODEs.** In (4.3) we introduced the shooting function S whose zeroes correspond to admissible blow-up profiles. By numerically integrating the ODE (4.1) with the far-field behaviour (4.2), we have been able to approximate this function, as seen in Figure 4. An asymptotic form of the function $S(C)$ for $C \gg 1$ can be constructed by a similar matching argument by putting an extra term $\tilde{b} \sin(\sqrt{3}t)/g_0(z)$ into (4.26) ($\tilde{b} = 0$ iff $S(C) = 0$). This leads to a harder multi-parameter matching procedure and will not be considered any further. Labeling, as above, the solutions to (4.1) corresponding to the n−th zero of $S(C)$ as f_n, we find that $f_n(y)$ has precisely n maxima on $\mathbf{R}$. In Figure 5 we present the first profile f_1 given in Theorem 4.4, which has the simplest bell-shaped form. It is evolutionary stable for a broad range of initial data (see the next section). For visual clarity we separately present the profiles $\{f_{1+4n}\}$ and $\{f_{2+4n}\}$ for $n = 0, ..., 15$.

The shooting function was approximated numerically for $0 < C \le 10^{12}$ by solving the ODE (4.1) subject to the Dirichlet condition $f(L) = C\phi_0(L)$ with L such that $f(L) = 10^{-10}$. The zeros of $S(C)$ then were used as initial guesses for a boundary value solver

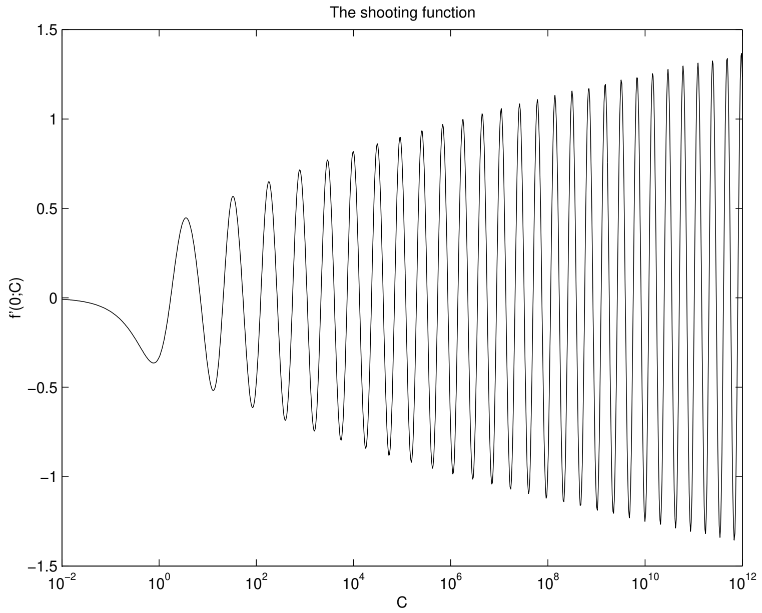

FIGURE 4. Numerical approximation to $S(C)$.

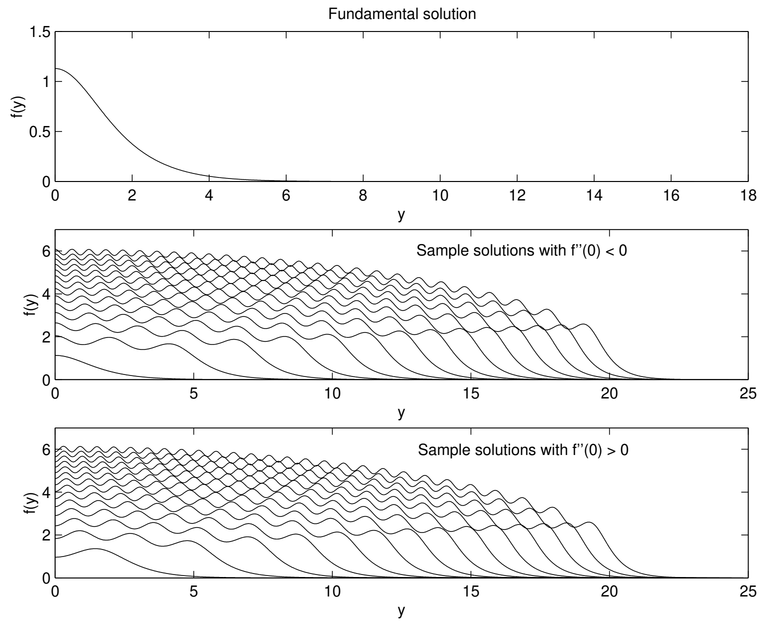

FIGURE 5. Various numerical solutions to (4.1).

where now (4.1) was solved on $y \in (0, 150)$ subject to $f'(0) = 0, f(150) = 0, f'(150) = 0$. The right-hand conditions are satisfied by exponentially decaying solutions within error tolerance.

Numerical solution in the inner region. Because of the fast oscillation in (4.31) and the slow convergence to this profile (the next terms are $O(1/\bar{z})$), the accurate numerical computation of the coefficients (4.42) is not straightforward, in many ways it is the most difficult numerical ODE problem in this paper. In fact, it is known that for second-order equations of the type

$$y''(t) + g(t)y(t) = 0, \quad \text{with } g(t) \to \infty \text{ as } t \to \infty$$

(of which the Airy functions are particular solutions), standard Runge-Kutta or Linear Multi-step Methods are not only poorly suited, but are incapable of producing reliable solutions in double precision arithmetic for large t [21]. As such, the values were computed using a Modified Magnus Method employing Filon quadratures [20, 22].

Numerical solution of the PDE. The possibility of finite-time blow-up suggests that adaptive strategies in both time and space will be required for reliable numerical solution to (1.1). The geometric features of this problem are also key, both the scale invariance and the preservation of mass. As such we use a scale-invariant moving-mesh strategy which dynamically clusters the grid points where a solution indicator, in this case $|u|^3$, is large [18, 6, 19, 34]. This choice does not assume any particular solution structure, but instead follows the scaling structure of the full PDE. This allows the computational grid points to move along level sets of any (emerging) similarity variable. We have also used a *conservative collocation* discretization [18, 34] which preserves the mass of the solution even into blow-up while on a moving grid.

4.5. **Exponential asymptotic stability of the first blow-up pattern with profile f_1: numerical evidence.** Theorem 4.4 together with the above matching analysis for $k \gg 1$ imply that there is a discrete family of solutions to the similarity ODE (4.1), and hence a discrete subset of admissible masses for the final time profiles

$$M_k = \int f_k(y)dy, \quad k = 1, 2, \dots . \tag{4.44}$$

We now justify that the first blow-up pattern with profile f_1 is the only stable one.

Returning to the rescaled PDE (2.3) which for $N = 1$ and $p = 3$ takes the form

$$\theta_\tau = \mathbf{A}(\theta) \equiv -\theta_{yyyy} - \tfrac{1}{4}(\theta y)_y - (\theta^3)_y, \tag{4.45}$$

with initial data θ_0, we study the asymptotic behaviour of the global orbit $\{\theta(\tau)\}$. First we establish the following natural (but not straightforward) property of its ω-limit set $\omega(\theta_0)$.

Proposition 4.6. *With the definition* (2.1) *of the rescaled solution,*

$$0 \notin \omega(\theta_0). \tag{4.46}$$

Proof. We have from (4.45) that $\mathbf{A}'(0) = \mathbf{B}^* - \frac{1}{4}I$ with the known spectral and other properties described in Section 3. By Lemma 3.1, $\sigma(\mathbf{A}'(0)) = \{-(k+1)/4,\ k \geq 0\}$. The principle of linearized stability [28, Chap. 9] implies that any sufficiently small solution $\theta(\tau)$ decays as $\tau \to \infty$ exponentially fast and that for some constant $C_0 > 0$,

$$|\theta(y,\tau)| \le C_0 e^{-\tau/4} \quad \text{uniformly in } \mathbf{R}.$$

Then one can see from scaling (2.1) with $p = 3$ that there holds $|u(x,t)| \le C_0$ for all $t \approx T^-$, i.e., u does not blow-up at $t = T$ contradicting the choice of blow-up time T. □

The linear part of the operator $\mathbf{A}$, $\mathbf{B}^* - \frac{1}{4}I$, is not self-adjoint and we do not expect that $\mathbf{A}$ is a potential operator nor that (4.45) is a gradient system. Therefore, the stability properties of the first similarity profile f_1 are studied via the spectrum of the linearized operator

$$\mathbf{A}'(f_1) = \mathbf{B}^* - \tfrac{1}{4}I - 3\tfrac{d^2}{d^2y}(f_1^2 I), \tag{4.47}$$

which is posed in a functional setting similar to that for $\mathbf{B}^*$ in Section 3.

Proposition 4.7. $\mathbf{A}'(f_1) : H^4_\rho \to L^2_\rho$ *is a bounded linear operator with the discrete spectrum* $\sigma(\mathbf{A}'(f_1)) = \{\mu_l\}$.

Proof. We recall that by Theorem 4.4, $f_1(y)$ has exponential decay as $y \to \infty$, so that (4.47) is a lower-order perturbation with smooth, bounded and exponentially decaying coefficients of the operator (3.1) and hence $\mathbf{A}'(f_1)$ is a bounded operator by Lemma 3.2; see [16]. In view of the known spectrum of $\mathbf{B}^*$ having compact resolvent $(\mathbf{B}^* - \lambda I)^{-1}$ in L^2_ρ [8], and by taking $(\mathbf{A}'(f_1) - cI)^{-1}$ with a large constant c, we have that the additional terms in (4.47) form a compact perturbation of the integral operator. Hence, the spectrum of $\mathbf{A}'(f_1)$ is discrete. □

Note that the two first real positive eigenvalues of $\mathbf{A}'(f_1)$ are easily calculated explicitly:

$$\mu_0 = 1, \;\; \phi_0 = (yf_1(y))' \quad \text{and} \quad \mu_1 = \tfrac{1}{4}, \;\; \phi_1 = f_1'(y), \tag{4.48}$$

corresponding to the invariance of the original PDE (1.1) under the group of translations in t and x respectively. As is usual in blow-up problems, since the blow-up scaling (2.1) does not admit such translations (the time T and the blow-up point $x = 0$ are fixed), these unstable modes are not available for the rescaled PDE (4.45).

A meaningful estimate of the real part of the remainder of the eigenvalues $\{\mu_l,\, l \ge 2\} \subset \mathbb{C}$ of $\mathbf{A}'(f_1)$ is non-trivial. Instead, we present a numerical calculation, from directly simulating the PDE, in Figure 7a. We have chosen symmetric monotone decreasing initial data having a smaller mass than the similarity solution,

$$\textstyle\int u_0 < M_1 = \int f_1.$$

Figure 7b shows the time-evolution of the mass of this solution for $\tau \gg 1$ establishing convergence to M_1 (our numerical scheme was designed to preserve mass for problems of this form). From a variety of numerical experiments, we conjecture that the next eigenvalue satisfies

$$\operatorname{Re}\mu_2 \simeq -.51... \,.$$

We are not aware of a proof of inequality $\operatorname{Re}\mu_2 < 0$ for operator (4.47), and, as is often happens in the operator and stability theory, we rely on rather delicate numerical results.

This calculation is based on carefully measuring the rate of the L^∞-convergence in the rescaled coordinates over compact subsets in the similarity variable y. This is of crucial importance, since measurement over all of x shows another (wrong) estimate $\operatorname{Re}\mu_2 \simeq -1/4$ instead. This is understood analogously to Proposition 4.6. Indeed, solutions with mass different from the final time profile cannot, because of conservation, lose that mass, but, similarly, it cannot contribute to the final-time profile in the rescaled co-ordinates. This extra mass (positive or negative) is damped to zero in the rescaled co-ordinates as described in Proposition 4.6 and hence with exponential rate $e^{-\tau/4}$. However, this is not a property of the linearized operator about the first similarity profile and such an incorrect estimate follows from the scaling (2.1), where $1/2(p-1) = 1/4$ for $p = 3$. This discrepancy is indicated in Figure 6. This type of estimate can of course be made by numerical approximation of the similarity ODE and numerical evaluation of the spectrum of the numerical linearization about such solutions. However, this gives no information about more general convergence behaviour.

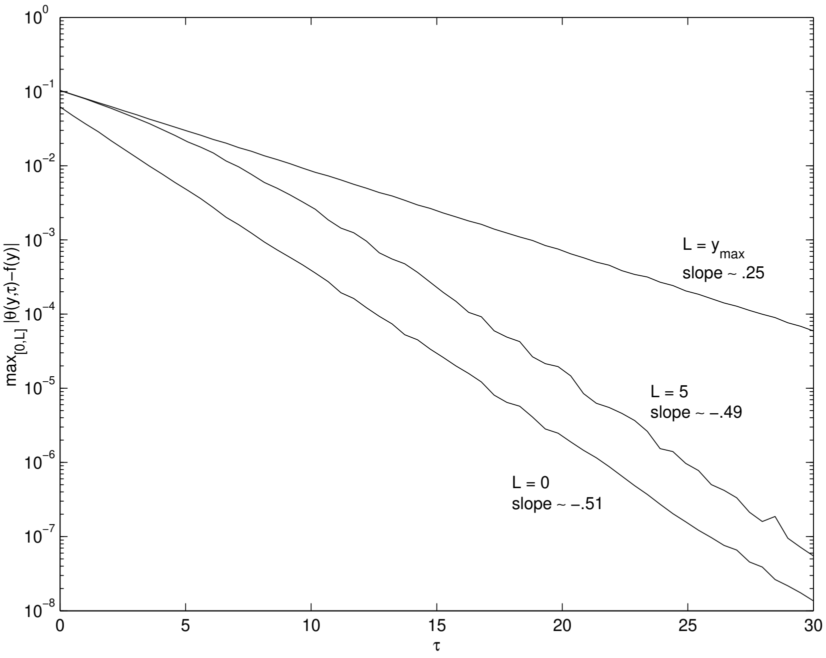

FIGURE 6. Rate of convergence to the similarity profile.

Numerical simulation of the full PDE with stationary profiles other than f_1 shows them to be exponentially unstable and the linearized operator

$$\mathbf{A}'(f_k) = \mathbf{B}^* - \tfrac{1}{4}I - 3\tfrac{d^2}{dy^2}(f_k^2 I) \tag{4.49}$$

has eigenvalues with positive real parts. Ordering the sequence of eigenvalues $\{\mu_l\}$ such that their real parts are non-increasing means that

$$\operatorname{Re}\mu_2 > 0 \quad \text{for all} \quad k \geq 2. \tag{4.50}$$

This has been checked numerically in a number of PDE experiments. For instance, Figures 7c and 7d displays the unstable evolution given by the initial function $\theta_0 = f_5$ with the larger mass $\int \theta_0 = \int f_5 > \int f_1$.

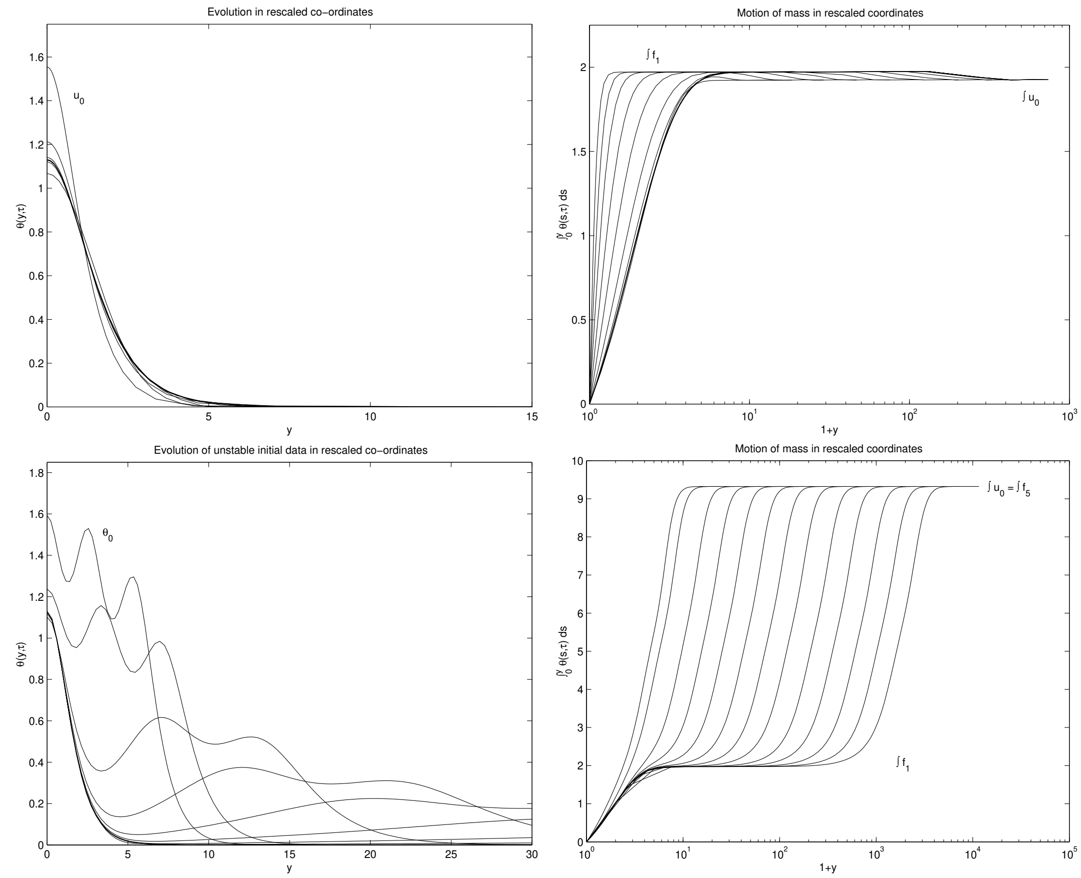

FIGURE 7. Evolution of solutions to (1.1). (a) Convergence to f_1 in rescaled co-ordinates. (b) Motion of the mass for initial mass $\int u_0 < \int f_1$. (c) Instability of f_5. (d) Motion of the mass for initial mass $\int u_0 = \int f_5 > \int f_1$.

4.6. **General** p. The case of general p is more complicated than for the conservative case as without conservation the ODE (2.5) remains truly fourth-order. One seeming exception to this is the second critical exponent $p_1 = 2$ for which one can also derive a third-order ODE. These are the solutions which conserve momentum. For simplicity, we consider only $N = 1$. Multiplying by x in the PDE (1.1) and integrating by parts, we have

$$0 = -\int_{\mathbf{R}} (u_{xx} + u^p)_{xx} dx = \frac{d}{dt} \int_{\mathbf{R}} ux\, dx = \frac{d}{dt} \left[(T-t)^{-1/2(p-1)+1/2} \int_{\mathbf{R}} \theta(y,t) y\, dy \right].$$

Thus, $p = 2$ defines the exponent for which momentum is conserved. Generically, in $\mathbf{R}^N$ this leads to the second critical exponent $p_1 = 1 + 2/(N+1)$ in the sequence (1.8) and similar to (2.10), we conclude that

$$\text{for any } p \neq p_1, \quad yf \in L^1 \quad \Longrightarrow \quad \int yf = 0. \tag{4.51}$$

Multiplying the ODE (2.5) in $\mathbf{R}$ with y and integrating once, we have

$$f'''y - f'' + \tfrac{1}{4}y^2 f + (f^2)'y - f^2 = -f''(0) - f^2(0) \equiv \tfrac{1}{4}A, \tag{4.52}$$

where the constant $A \in \mathbf{R}$ determines the rate of algebraic decay at infinity. Recall that from (2.18) with $p = 2$ the asymptotic algebraic behaviour is

$$f(y) = Ay^{-2}(1 + o(1)) \quad \text{as } y \to \infty. \tag{4.53}$$

For the case $A = 0$, the far-field behaviour is purely exponential with a 1D bundle there, see (2.17). In this case, equation (4.52) may be re-written as

$$f'' + (f^2)' + \tfrac{1}{4}\, yF = 0 \qquad F = \textstyle\int_0^y f(s)ds.$$

We have got some numerical and analytical evidence that there is no exponentially decaying solutions for $p = 2$.

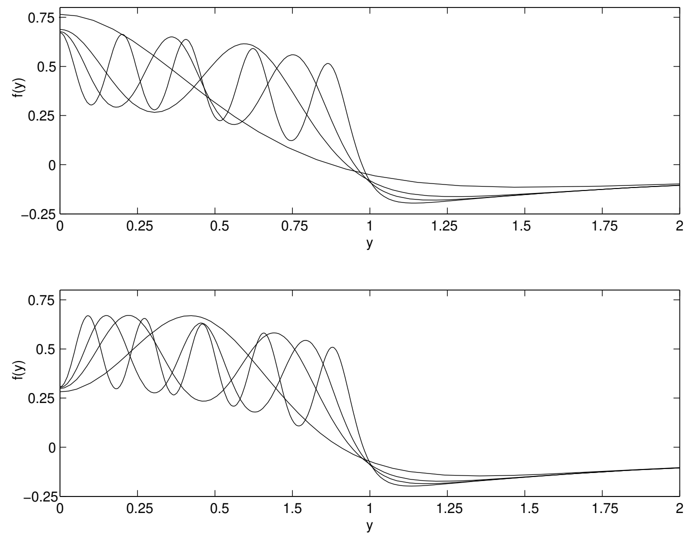

FIGURE 8. (a) Examples from the family of algebraically decaying solutions with $f''(0) < 0$. (b) Examples from the family of algebraically decaying solutions with $f''(0) > 0$. (All solutions are shown with f rescaled with $(-A^{-1/2})$.)

For $A \neq 0$, solutions can again be constructed by continuation in A, but now the condition of zero mass must be enforced creating, once again, a fourth-order system. For all $A \neq 0$, we solve the ODE (4.52), (4.53) imposing the condition $\int_{\mathbf{R}} f = 0$ by setting $f(y) = F'(y)$, where F is odd, $F(0) = 0$ and $F(y) \to 0$ as $y \to \infty$, finding A such that $f'(0; A) = 0$. In Figure 8 we present example solutions with algebraic decay as well as the exponentially decaying profile. These solutions are presented with f rescaled with $(-A)^{-1/2}$. The main difference from the case $p = 3$ is that now the shooting from

$y = \infty$ is two-dimensional including the parameter A in (2.18) and an additional $C \in \mathbf{R}$ from the corresponding exponential bundle (2.17). This problem exhibits typical "multi-dimensional" features of higher-order ODEs.

The details of a matched asymptotic construction are given in Appendix, which are now more involved than the $p = 3$ case. Figure 8 illustrates that the inner solutions have oscillations whose amplitudes are not asymptotically small (cf. (4.26)). However, we expect that the existence result similar to that in Theorem 4.4 as well as the multiplicity analysis below can still be performed but become essentially more delicate.

Finally, by direct simulation of the full PDE with various values of $p \neq 3$, we conjecture that stable solutions to (2.5) govern the blow-up dynamics for all p, not just in the conservative case. For instance, for $p = 2$ solutions converge exponentially fast in τ on compact subsets of y to the first fundamental algebraically decaying solution in Figure 8a.

5. Global similarity and approximate similarity patterns

We now turn our attention to the more mathematically tractable case of global solutions. Taking $\sigma = -1$ and $T = 0$ in (2.1) yields the rescaling of global in time solutions. We study the asymptotic behaviour as $\tau \to \infty$ of solutions satisfying the parabolic PDE

$$\theta_\tau = \mathbf{A}(\theta) \equiv -\Delta(\Delta\theta + \theta^p) + \tfrac{1}{4} y \cdot \nabla\theta + \tfrac{1}{2(p-1)}\theta. \tag{5.1}$$

5.1. **Similarity patterns for the one-dimensional equation with $p = 3$.** As in the case of blow-up in Section 4, we begin with the analysis of global self-similar solutions

$$u_S(x,t) = t^{-1/4} f(y), \quad y = x/t^{1/4},$$

where f satisfies the ODE obtained from (2.5), (2.3) by integration

$$f''' - \tfrac{1}{4} y f + (f^3)' = 0, \quad f'(0) = 0. \tag{5.2}$$

We are looking for profiles f with exponential decay at infinity, so that these are L^1-solutions satisfying the conservation law (2.7). Recall that unlike the blow-up case (4.1), the ODE (5.2) admits a two-dimensional exponential bundle as $y \to \infty$, see (2.16). This essentially simplifies the existence analysis and implies a continuous subset of similarity profiles.

Theorem 5.1. *The ODE* (5.2) *has an unbounded continuous family of exponentially decaying solutions.*

Proof. Step 1: monotone solutions as $y \to \infty$. By a standard local analysis, one can check that, in addition to the two-dimensional exponential bundle of solutions mentioned in (2.16) and those with the algebraic decay (2.18), (5.2) has a three-dimensional bundle of asymptotically monotone growing solutions as $y \to \infty$

$$f(y) = \frac{y}{\sqrt{12}} + \frac{1}{y}\left[C_1 \cos\left(\frac{y^{3/2}}{3}\right) + C_2 \sin\left(\frac{y^{3/2}}{3}\right) + C_3\right] + ..., \tag{5.3}$$

where C_1, C_2, C_3 are arbitrary parameters.

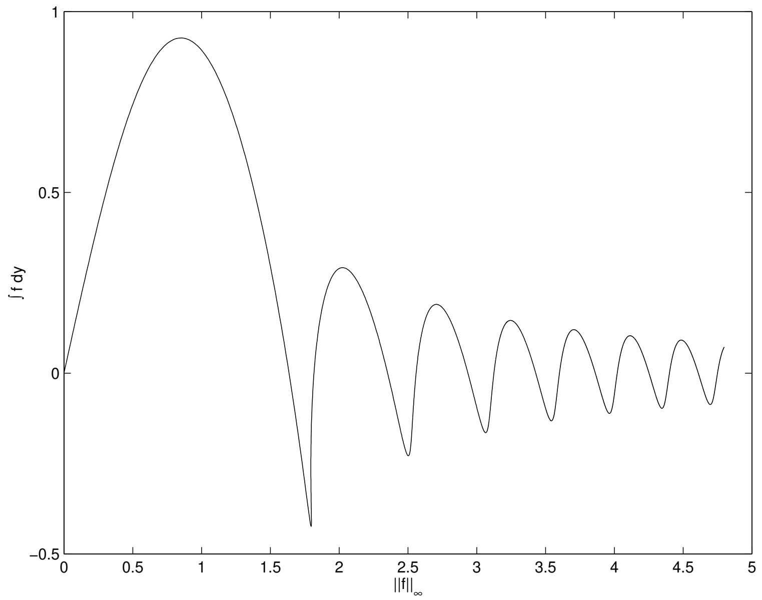

FIGURE 9. Mass bifurcation diagram.

Step 2: shooting argument. For a fixed a constant $a \ge 0$, denote by $f(y;C)$ the solution of (5.2) with conditions

$$f(0) = a, \quad f''(0) = C.$$

As in Section 4, one can show that $f(y;C)$ is globally defined. Using the stability of the three-dimensional bundle (5.3) concentrated around the stable explicit solution

$$f_*(y) = y/\sqrt{12} \to \infty, \quad y \to \infty, \tag{5.4}$$

we have that there exists a sufficiently large $C_1 > 0$ such that $f(y;C)$ belongs to the bundle in (5.3) for all $C \ge C_1$. On the other hand, via symmetry by reflection, for all $-C \gg 1$, $f(y)$ approaches as $y \to \infty$ the bundle around the explicit profile $-f_*(y)$. Introducing the subset

$$W = \{\mu < C_1 : \ f(y;C) \text{ belongs to (5.3) for all } C \in (\mu, C_1)\},$$

we have that there exists a finite $C = \inf W$, and by construction, $C = C(a)$ provides us with a profile $f(y;C(a))$ which belongs to the exponential bundle (2.16) as $y \to \infty$. $\square$

Note that this implies that there exists a solution with all $f(0) = a \in \mathbf{R}$. On Figure 9 we present the bifurcation diagram with respect to mass of this continuous family of similarity profiles. As we already know, there exists another case $p = 2$, $N = 1$, where the fourth-order ODE (2.5) reduces to a simpler third-order one. The existence results here are quite similar and the family of solutions is continuous. However, continuous families for $p = p_0$ and $p = p_1$ (quite special conservative cases associated with the equation (1.1) in divergence form) are exceptional and these are not the generic situation for arbitrary p; see Section 5.3, where discrete subsets will be detected (cf. the VSSs in [15]).

5.2. **The minimal mass-branch is evolutionary stable.** It follows from Figure 9 that for any fixed initial mass $m_0 = \int u_0 \ne 0$, there exists a finite (or empty if $|\int u_0| > m^* \approx .9271...$) subset of similarity solution with the given mass. Moreover, for $m_0 = 0$ besides $f(y) \equiv 0$, a countable family of such solutions is expected to exist. The crucial problem for such a case of multiple solutions is the stability of those solutions in the PDE sense. Let us show that the *minimal branch* in Figure 9 corresponding to the limit

$$f \to 0 \quad \text{as} \ m_0 \to 0 \tag{5.5}$$

is stable at least for all small masses $m_0 > 0$ (or $m_0 < 0$ replacing $f \mapsto -f$). As a typical example, we perform such computations for the N-dimensional case bearing in mind that the first critical exponent is now $p = p_0 = 1 + 2/N$. Consider the corresponding elliptic equation

$$\mathbf{A}(f) \equiv -\Delta^2 f + \tfrac{1}{4} y \cdot \nabla f + \tfrac{N}{4} f - \Delta f^p = 0, \quad \int f = m_0 > 0. \tag{5.6}$$

First, as an example, we establish a local result on the existence of such similarity patterns. For $N = 2$ local existence was established in [1].

Proposition 5.2. *Let $p = p_0$. For any sufficiently small $m_0 > 0$,* (5.6) *admits a solution satisfying*

$$f = m_0 F + O(m_0^p) \qquad (\textit{with } F \textit{ is as in } (1.13)). \tag{5.7}$$

Proof. Setting $f = m_0 v$ and introducing operator $\mathbf{C} = \mathbf{B} - I$ with the strictly negative point spectrum, $\sigma(\mathbf{C}) = \{\lambda_l = -1 - l/4\}$, we consider the equivalent integral equation

$$v = \mathbf{D}(v) \equiv -\mathbf{C}^{-1} v + |m_0|^{p-1} \mathbf{C}^{-1} \Delta v^p, \quad \text{with condition } \int v = 1. \tag{5.8}$$

Since $\mathbf{B}$ has compact resolvent in $L^2_\rho(\mathbf{R}^N)$ [8], we have that after a suitable smooth truncation of the nonlinearity v^p for $|v| \gg 1$ (see further comments below), $\mathbf{D}$ is a compact Hammerstein operator [23]. Since $\mathbf{D}'(0) = \mathbf{C}^{-1}$ has the simple eigenvalue $\lambda_0 = 1$, bifurcation occurs at $m_0 = 0$ and by the Lyapunov-Schmidt method [23], the solution can be represented in the form $v = F + w$, where $w \in \mathcal{L}^\perp\{F\}$. On substitution into (5.8) we obtain that $w = O(|m_0|^{p-1}) \in L^2_\rho$ is small for $|m_0| \ll 1$. Hence, w is small in H^{2m}_ρ and by the known asymptotic properties of the ODE under consideration, w is small uniformly. This means that the truncation of the unbounded nonlinearity v^p does not affect the solution bifurcating at $m_0 = 0$. Finally, we obtain the representation (5.7) in H^{2m}_ρ and uniformly in $\mathbf{R}^N$. See more details in [15], Section 6. $\square$

To study the stability of this branch, we consider the linearized operator

$$\mathbf{A}'(f)Y = \mathbf{B}Y - p\Delta(|f|^{p-1}Y) \equiv \mathbf{B}Y - |m_0|^{2/N} p \Delta(|F|^{p-1}Y) + \dots . \tag{5.9}$$

For small m_0, $\mathbf{A}'(f)$ is a perturbation of $\mathbf{B}$ with the known spectrum as given in (3.3). In 1D and in the radial geometry all the eigenvalues are simple and we will denote the eigenfunctions by ψ_l instead of ψ_β with $l = |\beta|$. Similarly to Proposition 4.7, we have that

$\mathbf{A}'(f)$ has a discrete spectrum which is a perturbation of that of $\mathbf{B}$, [16]. Recall from the orthogonality condition (3.7) that $\psi_0 = F$ is the only eigenfunction of $\mathbf{B}$ with non-zero mass, see (1.13)). Thus, requiring that the mass of profiles be preserved, we have to take into account perturbations of eigenvalues corresponding to perturbed eigenfunctions from $\mathcal{L}^{\perp}\{F\}$, i.e., of eigenvalues of $\mathbf{B}$ for $l = 1, 2, ...$ ($l = 0$ not included). By direct calculation from (5.9), one can estimate this point spectrum given, to leading order, by

$$\sigma(\mathbf{A}'(f)) = \left\{-l/4 + |m_0|^{2/N} p \left\langle \Delta(|F|^{p-1}\psi_l, \psi_l^*\right\rangle + ..., l = 1, 2, ...\right\}.$$

This guarantees that, for all small $|m_0|$, the real parts of the eigenvalues are bounded away from zero from above and that this branch is exponentially stable. Note that this expansion is not valid near the subsequent zeros of mass (other than 0) seen in Figure 9, as the L^∞ norm of such solutions is not zero for $m_0 = 0$ and hence expansion (5.7) does not hold. Therefore, other continuous monotone increasing or decreasing sub-branches are not stable.

Hence, for almost all initial data with $|\int u_0| \le m^*$ there exists the stable branch of similarity solutions which can attract the solution $t \to \infty$. If $|\int u_0| > m^*$, then, as is seen from Figure 9, self-similar asymptotic behaviour is not available and the solution is expected to blow-up in finite time.

5.3. **The p-bifurcation diagram for similarity profiles and asymptotic behaviour.** In order to understand the global bifurcation diagram of similarity profiles, we first determine the spectrum of critical exponents as bifurcation points. Writing the elliptic equation (2.5) in the form

$$\mathbf{B}f + c_* f = \Delta f^p \quad \text{in } \mathbf{R}^N, \quad f(y) \to 0 \text{ exponentially fast as } y \to \infty, \tag{5.10}$$

with $c_* = N(p_0 - p)/4(p-1)$, we obtain the following result.

Proposition 5.3. *Let, for an integer $l \ge 0$, the eigenvalue $\lambda_l = -l/4$ of the operator* (3.1) *be of odd multiplicity. Then the critical exponent p_l given in* (1.8) *is a bifurcation point for the problem* (5.10).

Proof. After performing, as above, a smooth truncation of the nonlinearity f^p, the proof follows analogously to that of Proposition 5.2 (cf. Proposition 6.1 in [15]). By using the explicit representation of the resolvent of $\mathbf{B}$ [8], this differential equation reduces to an integral one with compact Hammerstein operators to which the classical bifurcation techniques apply [23]. □

Using standard bifurcation theory, we briefly describe the results calculated directly from the differential equation. By Lemma 3.1, the linear operator in (5.10) has the spectrum $\sigma(\mathbf{B} + c_* I) = \{c_* - l/4, \, l \ge 0\}$. Therefore, any $p = p_l$ for which $c_* - l/4 = 0$ giving the critical exponents (1.8), determines a bifurcation point for problem (5.10) provided that $\lambda_l = -l/4$ is of odd multiplicity (e.g., this is always true for $N = 1$ or in the radial geometry where the eigenvalues are always simple). In order to describe the local behaviour of the bifurcation branches at $p \approx p_l$, we fix an l and set

$$p = p_l + \varepsilon, \quad \text{with } p_l = 1 + \tfrac{2}{N+l}.$$

Since $c_* = \frac{l}{4} - \mu_l \varepsilon + O(\varepsilon^2)$, with $\mu_l = \frac{1}{8}(N+l)^2$, equation (5.10) takes the form

$$\big(\mathbf{B} + \tfrac{l}{4} I\big) f - \mu_l \varepsilon f + O(\varepsilon^2) = \Delta f^p. \tag{5.11}$$

Using the Lyapunov-Schmidt method ([23], Chapt. 8), setting $f = C\psi_l + w$, where $w \in \mathcal{L}^\perp\{\psi_l\}$, one can find the algebraic equations on C and hence near the bifurcation point the solution takes the form

$$f = [\nu_l(p - p_l)]^{1/(p-1)} \psi_l + \ldots \quad \text{as } p \to p_l^+, \ \text{ where } \nu_l = -(N+l)^2/8\kappa_l \ \text{ and} \tag{5.12}$$

$$\kappa_l = \langle \Delta \psi_l^{p_l}, \psi_l^* \rangle = \langle \psi_l^{p_l}, \Delta \psi_l^* \rangle < 0. \tag{5.13}$$

The expansion (5.12) is written down under the crucial assumption that (5.13) holds. The strict inequality (5.13) has been checked numerically for various $l \ge 2$. It is worth mentioning that an analytical proof of (5.13) is not straightforward even in the simplest case $l = 2$, $N = 1$, where $\psi_2 = F''(y)/\sqrt{2}$, $(\psi_2^*)'' \equiv \sqrt{2}$ and $p_2 = 5/3$, though the positivity dominance the rescaled kernel F in the sense that $\int F = 1$ directly suggests $\kappa_2 < 0$.

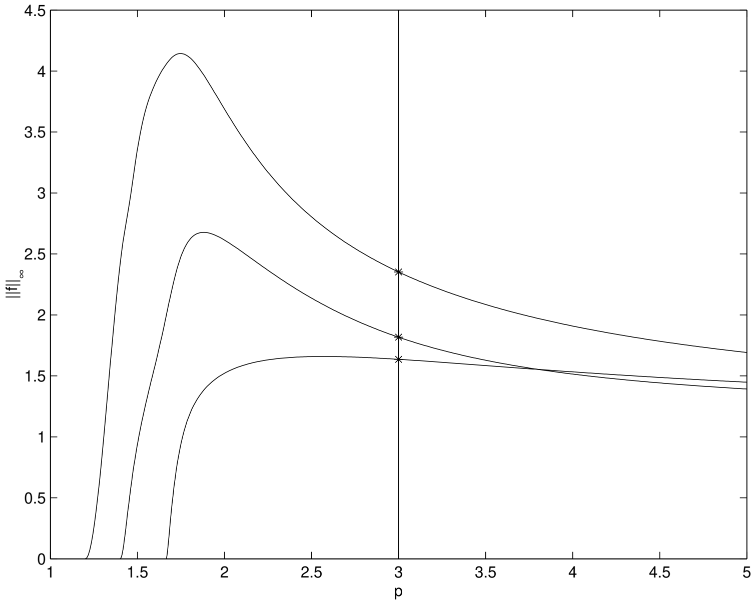

FIGURE 10. Bifurcation diagram of (5.10) for $N = 1$ with respect to p with even symmetry conditions at $y = 0$. There is a continuous branch of solutions from $p = 3$ and 2 but only a discrete set of solutions for all $p \neq 3, 2$. Another vertical branch of solutions bifurcates from $p = 2$.

Notice however, that

$$\kappa_0 = \kappa_1 = 0 \quad \text{for all} \quad N \ge 1,$$

as $\psi_0^* = 1$ and $\psi_1^*(y)$ is linear in y. Thus, the corresponding branches leave these bifurcation points vertically, suggesting a continuous family of solutions for these critical exponents described above. In Figure 10 we see that the remaining branches bifurcate with increasing p which is guaranteed by (5.13). Then, bearing in mind the countability of bifurcation points (1.8), Figure 10 clearly suggests that for any $p > 1$, $p \neq p_0$, p_1, there exists a countable, unbounded subset of solutions to (5.10). For non-coercive smooth operators, this is a typical result from Lusternik-Schnirel'man theory (see the classical formulation in [23], p. 381, and [24, 33] on variational methods in weighted Sobolev spaces like L^2_ρ for problems in $\mathbf{R}^N$), though problem (5.10) is not variational. The countability of solutions is not necessarily associated with the existence of a variational formulation.

It follows from (2.4) that for $p \neq p_0$, any solution of (5.10) must have zero mass, see (2.10). The similarity profiles on all the bifurcation branches in Figure 10 satisfy (2.10). Lastly, recall from Figure 9 that there is a discrete subset of μ_k such that $f(0) = \mu_k$ and $\int f_{\mu_k} = 0$. Clearly, on Figure 10 this countable subset of points corresponds to intersections of the bifurcation branches with the vertical one $\{p = 3\}$. To investigate problem (5.10) numerically, we have solved the fourth-order system

$$f''' = \tfrac{1}{4} f y + c_* F - |f|^{p-1} f', \quad F' = f,$$

coupled with the boundary conditions

$$f'(0) = 0,\ F(0) = 0,\ \lim_{y \to \infty} f(y) = 0,\ \lim_{y \to \infty} F(y) = 0,$$

as a continuation problem in $p < 3$ and $p > 3$ starting from those already known profiles at the intersection points with $\{p = 3\}$.

Figure 10 shows branching of solutions of (5.10), i.e., similarity profiles with exponential decay at infinity. Concerning possible solutions with the algebraic decay (2.18), one can see that such $f \in L^1(\mathbf{R}^N)$ for $p < p_0$, so that the blow-up rate (2.4) implies the zero mass condition (2.10). We expect that there exists a continuous family of such solutions with certain domain of attraction which do not play any role for exponentially decaying initial data.

On stability of the p_2-branch for $p \in (p_2, p_0)$. Let us show that the first non-vertical branch, from $p = p_2$, is stable for $p \approx p_2$. This is done by estimating the real parts of the eigenvalues of the linearized operator given in (5.10), which for $l = 2$ has the form

$$\mathbf{A}'(f) = \mathbf{B} + \tfrac{1}{2} I - \varepsilon \mathbf{C} + o(\varepsilon), \quad \mathbf{C} = \mu_2 I + p_2 \nu_2 \Delta(|\psi_2|^{p_2 - 1} I), \tag{5.14}$$

where we substitute the expansion of f given by (5.12) with $l = 2$. By $\{\bar{\lambda}_l = \bar{\lambda}_l(\varepsilon)\}$, $\varepsilon = p_2 - p$, we denote the discrete spectrum of $\mathbf{A}'(f)$.

For $N = 1$, due to the conservation of mass (for any $p \leq p_0 = 3$) and of momentum (for any $p \leq p_1 = 2$), the first modes with positive unperturbed eigenvalues of $\mathbf{B} + \frac{1}{2} I$ ($\varepsilon = 0$), $\bar{\lambda}_0(0) = 1/2$ and $\bar{\lambda}_1(0) = 1/4$ are not taken into account and we need to check the perturbed third eigenvalue $\bar{\lambda}_2(\varepsilon)$. The same holds for $N > 1$ in the radial setting deleting all eigenvalues corresponding to non-symmetric eigenfunctions and hence $\bar{\lambda}_1$. Since this unperturbed eigenvalue vanishes, $\bar{\lambda}_2(0) = 0$, by the perturbation theory [16], the eigenvalue expansion takes the form $\bar{\lambda}_2(\varepsilon) = \varepsilon \rho + o(\varepsilon)$ with the eigenfunction given by $\bar{\psi}_2(\varepsilon) = \psi_2 + \varepsilon \varphi + o(\varepsilon)$. Substituting these approximations into the eigenvalue equation, we find

$$\mathbf{A}'(f) \bar{\psi}_2(\varepsilon) = \bar{\lambda}_2(\varepsilon) \bar{\psi}_2(\varepsilon),$$

and using (5.14), we obtain from the equation at $O(\varepsilon)$ that the expansion coefficients ρ and φ satisfy

$$(\mathbf{B} + \tfrac{1}{2} I)\varphi = \rho\psi_2 + \mathbf{C}\psi_2.$$

Hence, by the orthogonality condition,

$$\rho = -\langle \mathbf{C}\psi_2, \psi_2^* \rangle = -\mu_2 - \tfrac{1}{8}\, p_2(N+2) = -\tfrac{1}{8}\,(N+2)(N+3),$$

and, for small $\varepsilon = p_2 - p > 0$,

$$\bar{\lambda}_2(\varepsilon) = -\tfrac{1}{4}\,(p_2 - p)(N+2)(N+3) + o(p_2 - p), \quad \text{i.e., } \operatorname{Re}\lambda_2(\varepsilon) < 0. \tag{5.15}$$

This means the exponential asymptotic stability of the similarity patterns on the p_2-bifurcation branch for all $p_2 - p > 0$ sufficiently small. We expect that the whole branch remains stable for all $p \in (p_2, p_0)$. Note that for $p = p_0$ it is not stable as Figure 9 suggests since this solution has $\|f\|_\infty \approx 1.6$ and the stable solutions have smaller L^∞-norm.

It follows from (5.10) that $c_* < 0$ for $p > p_0$, so that 0 becomes stable in equation (5.1). Therefore, we apply for (1.1) the scaling corresponding to the fundamental solution (1.12),

$$u(x,t) = t^{-N/4} g(y,\tau), \quad y = x/t^{1/4}, \tag{5.16}$$

to get an exponentially perturbed equation

$$g_\tau = \mathbf{B}g - e^{\gamma\tau}\Delta g^p, \quad \text{where } \gamma = N(p_0 - p)/4 < 0 \text{ if } p > p_0. \tag{5.17}$$

Recall that by Lemma 3.1, $\mathbf{B}$ has a one-dimensional centre subspace with the eigenfunction F, see (1.13). Therefore, we arrive at a typical result in the supercritical range: for all sufficiently small initial data with $m_0 = \int u_0 \neq 0$, the solution of (1.1) is global and has the asymptotic behaviour of the fundamental solution (1.12) with the same mass,

$$u(x,t) = m_0 t^{-N/4}[F(x/t^{1/4}) + o(1)) \quad \text{as } t \to \infty.$$

See a rigorous proof in [8] for equation (1.7) in the range $p > p_0 = 1 + \frac{2m}{N}$.

5.4. **Transitional non-self-similar patterns at critical exponents for the stable Cahn-Hilliard equation.** Our analysis shows that at the bifurcation points (1.8) there occurs a transition from the linearized patterns corresponding to the stable manifold of $\mathbf{A}$ with the linearization (3.1) to the essentially nonlinear similarity patterns described in Figure 10. This is quite a general phenomenon in nonlinear evolutionary parabolic equations observed in various higher-order models, [5, 8, 14, 15].

It remains to determine what happens at the critical values $p = p_l$ for the original PDE (1.1). Here one can expect some non-trivial centre manifold patterns in the rescaled equation (5.1). It turns out that this centre manifold behaviour is unstable (possibly, with blow-up) for (1.1) but leads to non-similarity patterns for the stable equation (1.9). The construction of global self-similar solutions (2.4), $\sigma = -1$, is quite the same as that already considered in Section 5.3 but instead of (5.10) we arrive at the equation

$$\mathbf{B}f + c_* f = -\Delta f^p. \tag{5.18}$$

Therefore, in the bifurcation analysis similar to Proposition 5.3 we obtain expansions (5.12), where $\nu_l = (N+l)^2/8\kappa_l < 0$, i.e., the branches bifurcate at $p = p_l^-$ concentrating at $p = 1^+$, and hence p is expected to decrease along them (no rigorous proof of nonlocal properties of branches is still available). A similar p-diagram occurs for very singular similarity solutions for the semilinear parabolic equations with absorption, [15].

It is important to understand how such bifurcations can affect the orbits of the original PDE (1.9). Consider the corresponding rescaled equation (5.1), which at $p = p_l$ takes the form

$$\theta_\tau = (\mathbf{B} + \tfrac{l}{4} I)\theta + \Delta \theta^p \quad \text{for } \tau \gg 1. \tag{5.19}$$

We fix $l \geq 1$ so that, by orthogonality, ψ_l has zero mass. By (3.3), $\mathbf{B} + \frac{l}{4}I$ has a non-trivial kernel, and therefore, by the general invariant manifold theory (see e.g. [28]) one can expect that a centre manifold behaviour exists. This is expressed by looking for solutions in the form

$$\theta(\tau) = a_l(\tau)\psi_l + w(\tau), \quad \text{with } w(\tau) \in \mathcal{L}^\perp\{\psi_l\}, \tag{5.20}$$

where the centre manifold dominance in the asymptotic behaviour means that $w(\tau) = o(a_l(\tau))$ as $\tau \to \infty$. Substituting this expansion into (5.19) and multiplying by the adjoint eigenfunction ψ_l^* in $L^2(\mathbf{R}^N)$ yields $\dot{a}_l = \kappa_l a_l^p + o(a_l^p)$, where κ_l is as given in (5.13). Since $\kappa_l < 0$, integrating this asymptotic ODE gives

$$a_l(\tau) = [(p-1)|\kappa_l|\tau]^{-1/(p-1)}(1 + o(1)). \tag{5.21}$$

In the original $\{x, t, u\}$-variables, this defines the following asymptotic patterns for the PDE (1.1) at $p = p_l$, $l \geq 2$ as $t \to \infty$:

$$u(x,t) = C_l (t \ln t)^{-(N+l)/2} [\psi_l(y/t^{1/4}) + o(1)], \quad \text{where } C_l = [2|\kappa_l|/(N+l)]^{-(N+l)/2}.$$

Obviously, using (3.1) as the linearization in the rescaled equation (5.1), one can see that in the supercritical range $p > p_l$ there exist stable manifold patterns with the behaviour

$$\theta(y,\tau) = C e^{-(N+l)(p-p_l)\tau/4(p-1)} [\psi_l(y) + o(1)] \quad \text{as } \tau \to \infty \quad (C \neq 0) \tag{5.22}$$

(cf. (5.20), (5.21) for $p = p_l$). Therefore, the centre manifold patterns (5.20), (5.21) are exactly the transitional ones at $p = p_l$ describing the exchange between the linearized stable manifold ones (5.22) and the nonlinear similarity patterns studied above.

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Appendix

The limit $A \to -\infty$ may be considered by recasting (4.52) with (4.53) as a singular perturbation problem in a manner similar to that developed above for $p = 3$. Introducing the small parameter $\epsilon > 0$ by

$$A = -\epsilon^{-1/2},$$

the rescaling $y = \epsilon^{-1/4}\bar{y}$ gives the following problem for $f(\bar{y})$,

$$\epsilon\left(\frac{f''}{\bar{y}}\right)' + \epsilon^{1/2}\left(\frac{f^2}{\bar{y}}\right)' + \frac{1}{4}f = -\frac{1}{4\bar{y}^2}, \quad \text{with} \tag{A.1}$$

$$f'(0) = 0, \qquad f''(0) + \frac{f(0)^2}{\epsilon^{1/2}} = \frac{1}{4\epsilon}, \quad \text{and} \tag{A.2}$$

$$f = -\frac{1}{\bar{y}^2}(1 + o(1)) \quad \text{as } \bar{y} \to +\infty. \tag{A.3}$$

Here $'$ denotes $d/d\bar{y}$. In the limit $\epsilon \to 0^+$, we obtain a four layer structure illustrated schematically in Figure 11. We have an outer region $\bar{y} = O(1)$ in which f takes the far-field behaviour (A.3) and an inner region $\bar{y} = O(\epsilon^{1/8})$ in which $f = O(\epsilon^{-1/4})$. This inner region is partitioned into two distinct regions, an inner 1 where $\bar{y} > \epsilon^{1/8}\hat{a}$ and an inner 2 where $\bar{y} < \epsilon^{1/8}\hat{a}$. The partition is through an inner region located at $\bar{y} = \epsilon^{1/8}\hat{a}$, which is of width $O(\epsilon^{3/8})$. The constant $\hat{a}$ will be shown below to be close to the location of the first singularity of a Painleve transcendent type equation.

The outer region $\bar{y} = O(1)$. At leading order (A.1) gives

$$f = -\frac{1}{\bar{y}^2}, \tag{A.4}$$

which satisfies (A.3) but not (A.2). This gives rise to the consideration of the following inner region in which terms involving the first derivatives of f are retrieved.

The inner 1 region $\bar{y} > \epsilon^{1/8}\hat{a}$. Introducing the scalings

$$\bar{y} = \epsilon^{1/8}\hat{y}, \qquad f = \epsilon^{-1/4}\hat{f}, \tag{A.5}$$

(A.1) becomes

$$\epsilon^{1/2}\left(\frac{\hat{f}''}{\hat{y}}\right)' + \left(\frac{\hat{f}^2}{\hat{y}}\right)' + \frac{1}{4}\hat{f} = -\frac{1}{4\hat{y}^2}, \tag{A.6}$$

where $'$ now denotes $d/d\hat{y}$. Posing $\hat{f} \sim \hat{f}_0$, gives the leading order problem,

$$\left(\frac{\hat{f}_0^2}{\hat{y}}\right)' + \frac{1}{4}\hat{f}_0 = -\frac{1}{4\hat{y}^2}, \tag{A.7}$$

with the matching condition,

$$\hat{f}_0 \sim -\frac{1}{\hat{y}^2} \quad \text{as } \hat{y} \to +\infty. \tag{A.8}$$

We next consider a region located at $\hat{y} = \hat{a}$, so that

$$\hat{f}_0 \sim \hat{f}_0(\hat{a}) \quad \text{as } \hat{y} \to \hat{a}^+. \tag{A.9}$$

The inner region $\bar{y} = \epsilon^{1/8}\hat{a} + O(\epsilon^{3/8})$. Introducing the new variables,

$$\hat{y} = \hat{a} + \epsilon^{1/4}\hat{Y}, \quad \hat{f} = \hat{F}, \tag{A.10}$$

we have from (A.6) that

$$\left(\frac{\hat{F}''}{(\hat{a} + \epsilon^{1/4}\hat{Y})}\right)' + \left(\frac{\hat{F}^2}{\hat{a} + \epsilon^{1/4}\hat{Y}}\right)' + \frac{\epsilon^{1/4}}{4}\left(\hat{F} + \frac{1}{(\hat{a} + \epsilon^{1/4}\hat{Y})^2}\right) = 0, \tag{A.11}$$

where $'$ is now $d/d\hat{Y}$. Posing $\hat{F} \sim \hat{F}_0$ gives at leading order,

$$\hat{F}_0'' + \hat{F}_0^2 = (\hat{f}_0(\hat{a}))^2, \tag{A.12}$$

after matching to (A.9). This gives the constant solution $\hat{F}_0 = \hat{f}_0(\hat{a})$ at leading order within this region.

The inner 2 region $\bar{y} < \epsilon^{1/8}\hat{a}$. We now consider the scalings,

$$\bar{y} = \epsilon^{1/8}\hat{y}, \quad f = \epsilon^{-1/4}F, \tag{A.13}$$

where $F = F(\hat{y}, Y)$ is also taken to be a function of the fast scale $\hat{y} = \epsilon^{1/4}Y$. Defining the multiple scales operator $\mathcal{L}$ by

$$\mathcal{L} = \frac{\partial}{\partial Y} + \epsilon^{1/4}\frac{\partial}{\partial \hat{y}},$$

(A.1) becomes,

$$\mathcal{L}\left(\frac{1}{\hat{y}}\mathcal{L}^2 F\right) + \mathcal{L}\left(\frac{F^2}{\hat{y}}\right) + \frac{1}{4}\left(F + \frac{1}{\hat{y}^2}\right) = 0. \tag{A.14}$$

Posing $F \sim F_0 + \epsilon^{1/4}F_1$, we have at leading order,

$$\frac{\partial^2 F_0}{\partial Y^2} + F_0^2 = A_0(\hat{y}), \tag{A.15}$$

for some function $A_0(\hat{y})$ and at the next order,

$$\frac{\partial}{\partial Y}\left(\frac{\partial^2 F_1}{\partial Y^2} + 2F_0F_1\right) - 2\frac{\partial F_0^2}{\partial \hat{y}} + \frac{\hat{y}}{4}F_0 + 3\hat{y}^{1/3}\frac{\partial}{\partial \hat{y}}\left(\frac{A_0}{\hat{y}^{1/3}}\right) + \frac{1}{4\hat{y}} = 0. \tag{A.16}$$

The secular terms involving Y in (A.16) are removed if

$$3\hat{y}^{1/3}\frac{d}{d\hat{y}}\left(\frac{A_0}{\hat{y}^{1/3}}\right) = -\frac{1}{4\hat{y}},$$

which gives

$$A_0(\hat{y}) = \frac{1}{4} - B_0\hat{y}^{1/3},$$

for some constant B_0 (assumed positive). Thus F_0 satisfies,

$$F_0'' + F_0^2 = \tfrac{1}{4} - B_0\epsilon^{1/12}Y^{1/3}, \tag{A.17}$$

where $' \equiv d/dY$ and is subject to (A.2), which at leading order in the current variables becomes,

$$F_0'(0) = 0, \qquad F_0''(0) + F_0(0)^2 = \tfrac{1}{4}. \tag{A.18}$$

The second condition in (A.18) is satisfied by F_0 as seen from (A.17). The equation (A.17) is noted to have movable singularities that are second order poles. Denoting the first (i.e. smallest) positive pole location by $\hat{a}_0$, we have the local behaviour,

$$F_0 \sim \tfrac{-6\epsilon^{1/2}}{(\hat{a}_0-\hat{y})^2} \qquad \text{as } \hat{y} \to \hat{a}_0^-. \tag{A.19}$$

Matching F_0 with $\hat{F}_0$ from the inner region requires $F_0(\hat{a}) = \hat{f}_0(\hat{a})$, where $\hat{a} < \hat{a}_0$.

It is worth remarking that the leading order solution $\hat{f}_s(\hat{y})$ in this inner 2 region on the slow scale $\hat{y}$ satisfies (A.7). As $\hat{y} \to 0^+$, the only bounded positive solution satisfying $\hat{f}_s'(0) = 0$ has the behaviour,

$$\hat{f}_s \sim \tfrac{1}{2} - \tfrac{1}{8}\,\hat{y}^2 \quad \text{as } \hat{y} \to 0^+.$$

For the solution satisfying this behaviour, we have that $\hat{f}_s = 0$ at $\hat{y} \approx 1.6$ with $\hat{f}_s'$ unbounded which gives a first approximation for $\hat{a}$. We now have from the numerical solution of (A.7) and (A.8) the estimate $\hat{f}_0(\hat{a}) \approx -0.2$ for $\hat{a} = 1.6$. Further, imposing this condition on the two-scale approximation (A.17) suggests taking $B_0 \approx 1/4\hat{a}^{1/3}$.

Full numerical solution of (4.52) gives the values $y \approx 1.5\epsilon^{-1/8}, f \approx -0.21\epsilon^{-1/4}$ for the location and value of the minimum of f when $\epsilon = 6.6 \times 10^{-9}$. This is in very good agreement with the values $\hat{a} = 1.6, \hat{f}_0(\hat{a}) = -0.2$ from the asymptotics.

There are two sets of solutions to (A.17) satisfying (A.18), which are distinguished by the their values for $F(0)$. Full numerical solution of (4.52), gives the estimates $F_0(0) \approx 0.5 \pm 0.167$.

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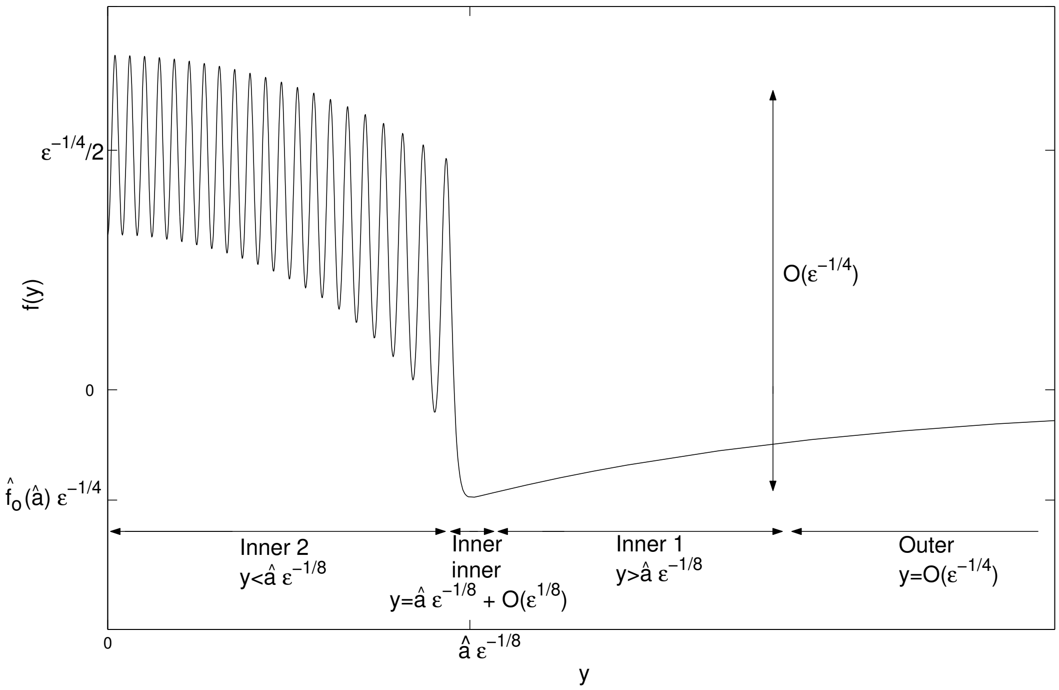

FIGURE 11. Schematic illustration of the asymptotic regions for the singular perturbation problem (A.1)–(A.3) expressed on the y variable scale.