HIGHER-ORDER QUASILINEAR PARABOLIC EQUATIONS WITH SINGULAR INITIAL DATA

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Abstract. As a basic model, we study the $2m$th-order quasilinear parabolic equation of diffusion-absorption type

$$(|u|^{q-1}u)_t = \Delta_{m,p} u - |u|^{\lambda-1}u, \quad \text{with } p \geq q \geq 0, \ \lambda > q, \ m \geq 1,$$

where $\Delta_{m,p}$ is the $2m$th-order p-Laplacian

$$\Delta_{m,p} u = (-1)^{m+1} \textstyle\sum_{|\alpha|=m} D^\alpha (|D^m u|^{p-1} D^\alpha u).$$

We consider the Cauchy problem in $\mathbf{R}^N \times \mathbf{R}_+$ with arbitrary *singular* initial data $u_0 \neq 0$ such that $u_0(x) = 0$ for any $x \neq 0$. We prove that, in the most delicate case $p = q$ and $\lambda > \lambda_0 = p + \frac{p(p+1)m}{N}$, this Cauchy problem admits the unique trivial solution $u(\cdot, t) = 0$ for $t > 0$. For $\lambda < \lambda_0$, such nontrivial very singular solutions are known to exist for some semilinear higher-order models. This extends the well-known result by Brezis and Friedman established in 1983 for the semilinear heat equation with $p = q = m = 1$.

1. Introduction: fundamental solutions, semilinear heat equations, Brezis–Friedman results and higher-order PDEs

The *fundamental solutions* naturally occur in the theory of linear operators, where they are associated with Dirac's delta function $\delta(x)$ appearing either on the right-hand side (as for elliptic operators, e.g., $\Delta u = \delta(x)$ in $\mathbf{R}^N$) or as initial data (for parabolic operators, e.g., $u_t = \Delta u$ in $\mathbf{R}^N \times \mathbf{R}_+$ with $u(x, 0) = \delta(x)$ in $\mathbf{R}^N$). For linear differential operators, the fundamental solutions are crucial and create general solutions via convolutions.

Fundamental solutions as *source-type similarity solutions* are also well-known in the theory of quasilinear diffusion equations in divergent form such as the porous medium or the p-Laplacian PDEs,

$$u_t = \Delta |u|^{p-1} u \quad \text{or} \quad u_t = \nabla \cdot (|\nabla u|^{p-1} \nabla u) \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \tag{1.1}$$

where $p > 1$ is a fixed exponent. The source-type solutions having measures as initial data, so $u(x, t) \to \delta(x)$ as $t \to 0$ in the sense of bounded measures in $\mathbf{R}^N$, play a special role in both the general theory and applications of nonlinear parabolic equations. For the

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PME in (1.1), such *Zel'dovich–Kompaneetz–Barenblatt* similarity solutions date back to 1950.

1.1. **Semilinear second-order heat equation with absorption.** A new aspect of the theory of fundamental solutions of nonlinear parabolic equations was discovered more than twenty years ago by H. Brezis and A. Friedman [2]. They studied the semilinear *diffusion-absorption equation* with a measure in the initial condition

$$u_t = \Delta u - |u|^{\lambda-1}u \quad \text{in } \; Q = \mathbf{R}^N \times \mathbf{R}_+, \tag{1.2}$$

$$u(x,0) = \delta(x) \quad \text{in } \; \mathbf{R}^N, \tag{1.3}$$

where $\lambda > 0$ is a fixed exponent and $\delta(x)$ is the Dirac mass at 0. It was established that there exists the critical exponent (the same as *Fujita's exponent* for the reaction-diffusion equation with $+|u|^{\lambda-1}u$ in (1.2))

$$\lambda_0 = 1 + \tfrac{2}{N} \tag{1.4}$$

such that the following results hold.

(I) *Existence*, [2, Thm. 3]: in the subcritical range $\lambda < \lambda_0$, there exists a unique solution $u \in C^{2,1}(Q) \cap L^\lambda(Q)$ satisfying (1.2) in the sense of distributions that takes the initial condition (1.3) in the weak topology of measures,

$$\operatorname{ess}\lim_{t \to 0} \int u(x,t)\varphi(x)\mathrm{d}x = \varphi(0) \quad \text{for any } \; \varphi \in C_{\mathrm{c}}(\mathbf{R}^N), \tag{1.5}$$

where $C_{\mathrm{c}}(\mathbf{R}^N)$ is the space of all continuous compactly supported functions in $\mathbf{R}^N$.

(II) *Nonexistence*, [2, Thm. 1]: for $\lambda \geq \lambda_0$, the Cauchy problem *does not have* a solution $u(x,t) \in L^\lambda_{\mathrm{loc}}(Q)$ satisfying equation (1.2) and (1.5). Moreover, any suitable regular approximation $\{u_j(x,t)\}$ via classical bounded solutions of (1.2) with bounded initial data u_{0j} converging to δ gives the trivial limit, [2, Thm. 4]):

$$u_j(x,t) \to 0 \quad \text{as } \; j \to \infty \; \text{ uniformly on } \mathbf{R}^N \times [\varepsilon, T] \text{ for any small } \varepsilon > 0. \tag{1.6}$$

The paper [2] was the first one that treated remarkable properties of nonlinear parabolic equations with measures as initial functions (earlier references on elliptic problems for $-\Delta u + |u|^{\lambda-1}u = f$, a measure, are presented therein) and established the crucial fact that for a class of "supercritical" nonlinearities, parabolic equations cannot stand measures as initial functions.

Convergence (1.6) for $\lambda \geq \lambda_0$ means that a *boundary layer* occurs at $t = 0$ [2], and that $\bar{u}(x,t) \equiv 0$ for $t > 0$ can be treated as a unique (proper) solution of the Cauchy problem (1.2), (1.3). This was a remarkable example showing that the "extended" semigroup constructed by regular approximations is essentially *discontinues* at $t = 0$ for initial bounded measures from $M(\mathbf{R}^N)$. Later on, it was understood in blow-up and extinction phenomena, that semigroups induced by various nonlinear parabolic equations of reaction-diffusion-absorption kind cannot be continuous in general, and singularities occurring at finite $t = T > 0$ ($T = 0$ implies *initial* singularity as above) can create unavoidable discontinuities of solutions in any weak topology; see [6] and references therein.

It was clear that Brezis–Friedman results were of more general significance and described some general aspects on nonlinear interaction between various operators in nonlinear PDEs of diffusion and lower-order absorption type. Such results were expected to be true for a wider class of semilinear and quasilinear parabolic PDEs. Nevertheless it turned out that their extension even to simple higher-order parabolic equations became a difficult problem and was open for almost twenty years. The original complete proofs in [2] involved various aspects of the Maximum Principle for the second-order PDEs as well as associated properties of order-preserving semigroups, parabolic Kato's inequalities, etc. These classical ideas and techniques do not admit extensions to higher-order PDEs.

It seems that the only known result in this direction is the proof [5] of existence of a solution of the semilinear $2m$th-order equation

$$u_t = -(-\Delta)^m u - |u|^{\lambda-1} u \quad (m \geq 2) \tag{1.7}$$

with initial data (1.3) in the subcritical range

$$1 < \lambda < \lambda_0 = 1 + \tfrac{2m}{N}. \tag{1.8}$$

Some "local" nonexistence results and a formal description of the boundary layer in the nonexistence range $\lambda \geq \lambda_0$ are also available in [5]. In addition, finite subsets of self-similar *very singular solutions* (VSS) of (1.7) in the range (1.8) were detected in [8] by using a bifurcation approach with respect to parameter λ that gave a countable subset of critical (bifurcation) exponents

$$\lambda_l = 1 + \tfrac{2m}{N+l}, \quad l = 0, 1, 2, \ldots, \tag{1.9}$$

so that λ_0 in (1.8) is the first one in this sequence. See Section 4 for extra details on multiplicity of the similarity VSSs.

A general nonexistence conclusion for (1.9) in the range $\lambda \geq 1 + \frac{2m}{N}$ remained open.

1.2. **Higher-order quasilinear parabolic equations and main results.** We now formulate the general class of $2m$th-order parabolic equation with any integer $m \geq 1$ and exponent $q > 0$

$$(|u|^{q-1}u)_t + \textstyle\sum_{|\alpha|=m} (-1)^m D_x^\alpha a_\alpha(x, t, u, \ldots, D_x^m u) + g(x, t, u) = 0 \tag{1.10}$$

posed in $Q = \mathbf{R}^N \times \mathbf{R}_+$. The functions $a_\alpha(x, t, \xi)$ are continuous, and the elliptic operator on the left-hand side of (1.10) is assumed to satisfy the growth and coercivity conditions, so there exist positive constants $p \geq q$ and d_1, d_2 such that

$$\sum_{|\alpha|=m} a_\alpha(x, t, \xi)\xi_\alpha \geq d_1 \Big(\sum_{|\beta|=m} |\xi_\beta| \Big)^{p+1} \text{ and } |a_\alpha(x, t, \xi)| \leq d_2 \Big(\sum_{|\beta|=m} |\xi_\beta| \Big)^p \ \forall\, \alpha, \tag{1.11}$$

for all $(x, t, \xi) \in \bar{Q} \times \mathbf{R}^{n(m)}$, where $n(m)$ denotes the number of distinct multiindices of the length not exceeding m. Here $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = \alpha_1 + \ldots + \alpha_N$ is multiindex, $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}$ and $D_x^m u = \{D_x^\alpha u,\ |\alpha| = m\}$. The function $g(x, t, u)$ in (1.10) satisfies the conditions

$$|g(x, t, u)| \leq d_3\, |u|^\lambda \quad \text{and} \quad g(x, t, u) \cdot u \geq d_4\, |u|^{\lambda+1} \quad (d_4 > 0). \tag{1.12}$$

Typical scaling invariant models are as follows:

$$(|u|^{q-1}u)_t = \Delta_{m,p} u - |u|^{\lambda-1}u, \quad \text{with } \ p \geq q \geq 0, \ \lambda > q, \ m \geq 1, \tag{1.13}$$

where $\Delta_{m,p}$ is the $2m$th-order p-Laplacian

$$\Delta_{m,p} u = (-1)^{m+1} \textstyle\sum_{|\alpha|=m} D^\alpha(|D^m u|^{p-1} D^\alpha u). \tag{1.14}$$

We will consider solutions from the corresponding energy class B_E consisting of functions that satisfy

(i) $u \in C_{\rm loc}(\mathbf{R}_+ \setminus \{0\}; L^{q+1}(\mathbf{R}^N)) \cap L^{p+1}_{\rm loc}(\mathbf{R}_+ \setminus \{0\}; W^m_{p+1}(\mathbf{R}^N))$, and

(ii) $(|u|^{q-1}u)_t \in L^{(p+1)/p}_{\rm loc}(\mathbf{R}_+ \setminus \{0\}; (W^m_{p+1}(\mathbf{R}^N))^*)$.

In this energy class, we are looking for solutions satisfying the following singular initial condition:

$$u(x,0) = 0 \quad \text{for all } \ x \neq 0. \tag{1.15}$$

In what follows, $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbf{R}^N)$, which is treated as a suitable dual one.

Definition 1.1 A function $u(x,t) \in B_E$ is said to be a *generalized energy solution* of the problem (1.10), (1.15), if the integral identity holds

$$\begin{aligned} &\int_0^T \langle (|u|^{q-1}u)_t, \eta \rangle \, \mathrm{d}t + \int_0^T \int_{\mathbf{R}^N} g(x,t,u)\eta(x,t) \, \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbf{R}^N} \textstyle\sum_{|\alpha|=m} a_\alpha(x,t,u(x,t),...,D^m_x u(x,t)) D^\alpha_x \eta(x,t) \, \mathrm{d}x\mathrm{d}t = 0, \end{aligned} \tag{1.16}$$

with arbitrary functions $\eta(x,t) \in L^{p+1}_{\rm loc}(\mathbf{R}_+; W^m_{p+1}(\mathbf{R}^N))$ vanishing near $(0,0)$, and condition (1.15) is fulfilled in the sense of the following identity:

$$\begin{aligned} &\int_{\mathbf{R}^N} |u(x,T)|^{q-1} u(x,T) \Phi(x,T) \, \mathrm{d}x - \int_0^T \int_{\mathbf{R}^N} |u|^{q-1} u \Phi_t \, \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbf{R}^N} \textstyle\sum_{|\alpha|=m} a_\alpha(x,t,...,D^m_x u) D^\alpha_x \Phi \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbf{R}^N} g(x,t,u)\Phi \, \mathrm{d}x\mathrm{d}t = 0. \end{aligned} \tag{1.17}$$

Here $\Phi(x,t)$ is an arbitrary smooth compactly supported function from $C^{m,1}(\mathbf{R}^N \times \mathbf{R}_+)$ that vanishes in a neighbourhood of $(0,0)$ and $T > 0$ is an arbitrary constant.

Our main result is as follows, where we treat the delicate case $p = q > 0$.

Theorem 1.1. *Let $p = q$ and*

$$\lambda > \lambda_0 = p + \tfrac{p(p+1)m}{N}. \tag{1.18}$$

Then any energy solution $u(x,t)$ of the problem (1.10), (1.15) *satisfies $u(x,0) = 0$ in $\mathbf{R}^N$ in the sense that* (1.17) *holds for arbitrary test function Φ without the assumption on vanishing in a neighbourhood of* $(0,0)$.

This theorem can be treated as a result on removable initial singularity for such class of quasilinear degenerate parabolic PDEs. In particular, in the case $p = 1$, we can consider the semilinear parabolic equation (1.7) with initial data (1.3). Then Theorem 1.1 establishes the nonexistence of a solution in the range

$$\lambda > \lambda_0 = 1 + \tfrac{2m}{N}, \tag{1.19}$$

where this λ_0 is known as the critical Fujita exponent [3, 4].

A stronger and simpler nonexistence result will be proved for fully divergent PDEs and inequalities with a different non-monotone, non-positive absorption term

$$u_t \le L[|u|^{p-1}u] - |u|^\lambda, \quad \lambda > 1, \;\; 0 < p < \lambda, \tag{1.20}$$

where L is a general mth-order differential operator

$$L[v] = \textstyle\sum_{|\alpha|=m} D^\alpha(a_\alpha(x,t,v)v), \tag{1.21}$$

with the coefficient $a_\alpha(x,t,v)$, which are uniformly bounded in $\mathbf{R}^N \times \mathbf{R}_+ \times \mathbf{R}$ for any α.

We consider solutions of the partial differential inequality (1.20) satisfying (1.15) in the following weak sense: if $\eta \in C^{m,1}(\mathbf{R}^N \times \mathbf{R}_+)$, $\eta \ge 0$, vanishes in a neighbourhood of $(0,0)$, then

$$-\textstyle\int \eta_t u \,\mathrm{d}x\mathrm{d}t - \int |u|^{p-1}u L^*_u[\eta]\,\mathrm{d}x\mathrm{d}t \le -\int |u|^\lambda \eta \,\mathrm{d}x\mathrm{d}t. \tag{1.22}$$

The adjoint operator L^* is defined as follows:

$$L^*_u[\eta] = \textstyle\sum_{|\alpha|=m} a_\alpha(x,t,|u|^{p-1}u)(-D)^\alpha \eta.$$

Theorem 1.2. *Let*

$$\lambda \ge \lambda_0 = p + \tfrac{2m}{N}. \tag{1.23}$$

If a function $u(x,t) \in L^\lambda_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}_+ \setminus \{(0,0)\})$ satisfies (1.20) *and* (1.15) *in the sense of* (1.22), *then $u = 0$.*

2. **Proof of Theorem 1.2**

Let us begin with the proof of simpler Theorem 1.2, where we apply a modification of Pohozaev's nonlinear capacity approach; see various techniques and references in the Mitidieri–Pohozaev book [9]. We fix standard C^∞, non-analytic cut-off function $\phi(s) \ge 0$ satisfying

$$\phi(s) = 0, \;\; 0 \le s \le 1; \quad \phi(s) = 1, \;\; s \ge 2, \quad \text{such that} \tag{2.1}$$

$$|\phi'(s)| \le C\phi^{\frac{1}{\lambda}}(s), \quad |\phi^{(m)}(s)| \le C\phi^{\frac{p}{\lambda}}(s). \tag{2.2}$$

For instance, we take the standard $\phi(s) \sim \exp\{-\frac{1}{(1-s)^2}\}$ as $s \to 1^+$. Next, we introduce the test functions with a parameter $R > 0$,

$$\eta = \eta_R(x,t) = \phi\big(\tfrac{t^{2/\sigma}+|x|^2}{R^2}\big), \tag{2.3}$$

where the exponent $\sigma > 0$ will be determined later on.

We now plug the test functions (2.3) into (1.22) and estimate the integral

$$I = \textstyle\int |u|^\lambda \eta \,\mathrm{d}x\mathrm{d}t. \tag{2.4}$$

Using the Young inequality

$$ab \le \varepsilon \, \frac{a^A}{A} + \varepsilon^{1-A'} \, \frac{b^{A'}}{A'}, \quad a,\, b,\, \varepsilon > 0,\ A > 1,\ A' = \frac{A}{A-1}, \tag{2.5}$$

with $A = \lambda$, we obtain

$$\begin{aligned} &|\textstyle\int \eta_t u \, \mathrm{d}x\mathrm{d}t| \le C_1 R^{-\sigma} \int_{\mathrm{supp}\eta_t} \eta^{\frac{1}{\lambda}} |u| \, \mathrm{d}x\mathrm{d}t \\ &\le \tfrac{1}{4} \textstyle\int |u|^\lambda \eta \, \mathrm{d}x\mathrm{d}t + C_2 R^{-\sigma\lambda} \int_{\mathrm{supp}(\eta_t)} \mathrm{d}x\mathrm{d}t \le \frac{1}{4} I + C_3 R^{N+\sigma-\lambda'\sigma}, \end{aligned} \tag{2.6}$$

where $\lambda' = \frac{\lambda}{\lambda-1}$. In the same way, we have

$$|\textstyle\int |u|^{p-1} u L_u^*[\eta] \, \mathrm{d}x\mathrm{d}t| \le \frac{1}{4} I + C_4 R^{N+\sigma-\frac{m\lambda}{\lambda-p}}. \tag{2.7}$$

Using estimates (2.6) and (2.7) in (1.22) imply that

$$I \le C_3 R^{N+\sigma} (R^{-\frac{\lambda\sigma}{\lambda-1}} + R^{-\frac{m\lambda}{\lambda-p}}). \tag{2.8}$$

We now choose σ by "balancing" the two terms on the right-hand side of (2.8), so

$$\frac{\lambda\sigma}{\lambda-1} = \frac{m\lambda}{\lambda-p} \quad \Longrightarrow \quad \sigma = \frac{m(\lambda-1)}{\lambda-p}.$$

Therefore, if

$$N + \sigma - \frac{m\lambda}{\lambda-p} = N - \frac{m}{\lambda-p} > 0, \tag{2.9}$$

then, passing to the limit $R \to 0$ in (2.8) yields $u = 0$. Inequality (2.9) coincides with the strict one in (1.23).

In the critical case $\lambda = 1 + \frac{m}{N}$, estimate (2.8) implies that the integral I is uniformly bounded for any R, i.e.,

$$\int_{\mathbf{R}^N \times \mathbf{R}_+} |u|^\lambda \, \mathrm{d}x\mathrm{d}t \quad \text{converges.} \tag{2.10}$$

It follows from (1.22) with $\eta = \eta_R$ that

$$\textstyle\int |u|^\lambda \eta_R \, \mathrm{d}x\mathrm{d}t \le |\int \eta_{Rt} u \, \mathrm{d}x\mathrm{d}t + \int |u|^{p-1} u L^*[\eta_R]|, \tag{2.11}$$

where the terms on the right-hand side correspond to integration over shrinking strips as $R \to 0$,

$$\Omega_R : \quad R^2 \le t^{2/\sigma} + |x|^2 \le 2R^2. \tag{2.12}$$

In the given parameter range, using the Holder inequality as above and integrating over the strips (2.12), one can derive the estimate

$$|\textstyle\int \eta_{Rt} u| + |\int |u|^{p-1} u L^*[\eta_R]| \le C[(\int_{\Omega_R} |u|^\lambda)^{\frac{1}{\lambda}} + (\int_{\Omega_R} |u|^\lambda)^{\frac{p}{\lambda}}]. \tag{2.13}$$

Due to (2.10), the right hand side tends to zero as $R \to 0$, and then using (2.11) completes the proof in the critical case. $\square$

3. **Proof of Theorem 1.1**

The analysis of the diffusion-absorption model with monotone changing sign absorption term $-|u|^{\lambda-1}u$ as in (1.13) turns out to be more difficult. Obviously the nonlinear capacity approach from Section 2 does not apply since the corresponding integral in (2.4) does not have a sign. The present approach is based on a variant of local energy (or Saint-Venant principle) estimates, which was elaborated, in particular, in [10, 11]; see also a survey in [7], where applications to higher-order blow-up parabolic problems are discussed.

Let ϕ be cut-off functions from (2.1). For arbitrary parameters $\tau > \delta > 0$, we introduce a family of main cut-off functions

$$\phi_{\tau,\delta}(x,t) = 1 - [1-\phi(\tfrac{|x|-\tau+\delta}{\delta}+1)][1-\phi(\tfrac{t-(\tau-\delta)^\gamma}{\tau^\gamma-(\tau-\delta)^\gamma}+1)], \quad \gamma = m(p+1). \tag{3.1}$$

We introduce the family of subdomains

$$Q^{(T)}(\tau) := Q^{(T)} \cap \{x: \ |x| > \tau, \, t > \tau^\gamma\}, \quad Q^{(T)} := Q \cap \{t < T\}. \tag{3.2}$$

Obviously

$$\phi_{\tau,\delta}(x,t) = 0 \ \text{ in } \ Q^{(T)} \setminus Q^{(T)}(\tau-\delta), \ \text{ and } \ \phi_{\tau,\delta}(x,t) = 1 \ \text{ in } \ Q^{(T)}(\tau).$$

Due to these properties, we can use the functions $\eta(x,t) := \eta_{\tau,\delta}(x,t) \equiv u(x,t)\phi_{\tau,\delta}(x,t)$ as the test functions in the integral identity (1.16) for arbitrary $\tau > \delta > 0$. On this substitution, we obtain easily, due to (1.11) and (1.12), that

$$\begin{aligned} &\textstyle\int_0^T \langle (|u|^{p-1}u)_t, u\phi_{\tau,\delta}\rangle \, \mathrm{d}t + \int_{Q^{(T)}(\tau-\delta)} (d_1|D_x^m u|^{p+1} + d_4|u|^{\lambda+1})\phi_{\tau,\delta} \, \mathrm{d}x\mathrm{d}t \\ &\quad \le c_1 \textstyle\int_{\{0<t<\tau^\gamma\}\cap\{\tau-\delta<|x|<\tau\}} |D_x^m u|^p \sum_{i=1}^m |D_x^{m-i}u| \, |D_x^i \phi_{\tau,\delta}| \, \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Due to properties (i), (ii) from Definition 1.1 of the energy solution $u(x,t)$, we can use the formulae of integration by parts in the first term, [1]. As a result, we obtain

$$\begin{aligned} &\tfrac{1}{p+1} \textstyle\int_{\mathbf{R}^N} |u(x,\tau)|^{p+1}\phi_{\tau,\delta}(x,\tau) \, \mathrm{d}x + \int_{Q^{(T)}(\tau-\delta)} (d_1|D_x^m u|^{p+1} + d_4|u|^{\lambda+1})\phi_{\tau,\delta} \, \mathrm{d}x\mathrm{d}t \\ &\quad \le C \textstyle\int_{\{0<t<\tau^\gamma\}\cap\{\tau-\delta<|x|<\tau\}} |D_x^m u|^p \sum_{i=1}^m |D_x^{m-i}u||D_x^i\phi_{\tau,\delta}| \, \mathrm{d}x\mathrm{d}t \\ &\qquad + \tfrac{1}{p+1} \textstyle\int_{\{|x|<\tau\}\cap\{(\tau-\delta)^\gamma<t<\tau^\gamma\}} |u|^{p+1} \frac{\partial}{\partial t}\phi_{\tau,\delta} \, \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.3}$$

From (3.3), using (3.1), we obtain the starting relationship

$$\begin{aligned} &I(\tau) := I_1(\tau) + I_2(\tau) = \textstyle\int_{Q^{(T)}(\tau)} |D_x^m u|^{p+1} \, \mathrm{d}x\mathrm{d}t + \int_{Q^{(T)}(\tau)} |u|^{\lambda+1} \, \mathrm{d}x\mathrm{d}t \\ &\quad \le c_1 \textstyle\int_{\Omega(\tau,\delta)\times\{t<\tau^\gamma\}} |D_x^m u|^p \sum_{i=1}^m \delta^{-i}|D_x^{m-i}u| \, \mathrm{d}x\mathrm{d}t \\ &\quad + c_2 \textstyle\int_{\{|x|<\tau\}\times\{(\tau-\delta)^\gamma<t<\tau^\gamma\}} |u|^{p+1}[\tau^\gamma-(\tau-\delta)^\gamma]^{-1} \, \mathrm{d}x\mathrm{d}t \equiv c_1A_1 + c_2A_2, \end{aligned} \tag{3.4}$$

where we denote $\Omega(\tau,\delta)=\{\tau-\delta<|x|<\tau\}$. Let us estimate A_1 and A_2 from above. We begin with A_2,

$$
\begin{aligned}
A_2 &\le \tfrac{c}{\delta\tau^{\gamma-1}} \int_{(\tau-\delta)^\gamma}^{\tau^\gamma} \mathrm{d}t \int_{\{|x|<\tau\}} |u|^{p+1}\,\mathrm{d}x \\
&\le \tfrac{c}{\delta\tau^{\gamma-1}} \int_{(\tau-\delta)^\gamma}^{\tau^\gamma} \mathrm{d}t \big(\int_{\{|x|<\tau\}} |u|^{\lambda+1}\,\mathrm{d}x\big)^{\frac{p+1}{\lambda+1}} \tau^{\frac{N(\lambda-p)}{\lambda+1}} \\
&\le \tfrac{c\,\tau^{\frac{N(\lambda-p)}{\lambda+1}}}{\delta\tau^{\gamma-1}} \left[(\tau^\gamma-(\tau-\delta)^\gamma)\right]^{\frac{\lambda-p}{\lambda+1}} \left(\int_{(\tau-\delta)^\gamma}^{\tau^\gamma}\int_{\{|x|<\tau\}} |u|^{\lambda+1}\mathrm{d}x\mathrm{d}t\right)^{\frac{p+1}{\lambda+1}} \\
&\le c\,\tau^{\frac{N(\lambda-p)}{\lambda+1}-(\gamma-1)\frac{p+1}{\lambda+1}} \left(\tfrac{1}{\delta}\int_{(\tau-\delta)^\gamma}^{\tau^\gamma}\int_{\{|x|<\tau\}} |u|^{\lambda+1}\mathrm{d}x\mathrm{d}t\right)^{\frac{p+1}{\lambda+1}}.
\end{aligned} \tag{3.5}
$$

Consider now A_1,

$$
\begin{aligned}
A_1 &\le \textstyle\sum_{i=1}^m \delta^{-i}\int_0^{\tau^\gamma}\left(\int_{\Omega(\tau,\delta)} |D_x^m u|^p |D_x^{m-i}u|\,\mathrm{d}x\right)\mathrm{d}t \\
&\le \textstyle\sum_{i=1}^m \delta^{-i}\int_0^{\tau^\gamma}\left(\int_{\Omega(\tau,\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\right)^{\frac{p}{p+1}}\left(\int_{\Omega(\tau,\delta)} |D_x^{m-i}u|^{p+1}\,\mathrm{d}x\right)^{\frac{1}{p+1}} \equiv \sum_{i=1}^m A_1^{(i)}.
\end{aligned} \tag{3.6}
$$

Due to the Gagliardo–Nirenberg interpolation inequality, we have

$$
\begin{aligned}
\int_{\Omega(\tau,\delta)} |D_x^{m-i}u|^{p+1}\mathrm{d}x &\le c\left(\int_{\Omega(\tau,\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\right)^{\frac{m-i}{m}}\left(\int_{\Omega(\tau,\delta)} |u|^{p+1}\,\mathrm{d}x\right)^{\frac{i}{m}} \\
&\quad + c_1\delta^{-(m-i)(p+1)}\int_{\Omega(\tau,\delta)} |u|^{p+1}\,\mathrm{d}x.
\end{aligned} \tag{3.7}
$$

Since $\operatorname{meas}\Omega(\tau,\delta)\le c\tau^{N-1}\delta$, by the Hölder inequality, we derive from (3.7) that

$$
\begin{aligned}
\int_{\Omega(\tau,\delta)} |D_x^{m-i}u|^{p+1}\mathrm{d}x &\le c\left(\int_{\Omega(\tau,\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\right)^{\frac{m-i}{m}}\left(\int_{\Omega(\tau,\delta)} |u|^{\lambda+1}\,\mathrm{d}x\right)^{\frac{(p+1)i}{(\lambda+1)m}} \\
&\times \tau^{\frac{(N-1)(\lambda-p)i}{(\lambda+1)m}}\delta^{\frac{(\lambda-p)i}{(\lambda+1)m}} + c_1\tau^{\frac{(N-1)(\lambda-p)}{(\lambda+1)}}\delta^{-(m-i)(p+1)+\frac{\lambda-p}{\lambda+1}}\left(\int_{\Omega(\tau,\delta)} |u|^{\lambda+1}\,\mathrm{d}x\right)^{\frac{p+1}{\lambda+1}}.
\end{aligned} \tag{3.8}
$$

Therefore after straightforward computations, by using (3.8) and Hölder's inequalities, we estimate $A_1^{(i)}$ from (3.6) as follows:

(3.9)

$$
\begin{aligned}
A_1^{(i)} &\le c\delta^{-i+\frac{i(\lambda-p)}{(\lambda+1)m(p+1)}}\tau^{\frac{(N-1+\gamma)(\lambda-p)i}{m(p+1)(\lambda+1)}}\left[\int_0^{\tau^\gamma}\int_{\Omega(\tau,\delta)}(|D_x^m u|^{p+1}+|u|^{\lambda+1})\mathrm{d}x\mathrm{d}t\right]^{1-\frac{i(\lambda-p)}{m(p+1)(\lambda+1)}} \\
&+ c_1\delta^{-m+\frac{(\lambda-p)}{(\lambda+1)(p+1)}}\tau^{\frac{(N-1+\gamma)(\lambda-p)}{(p+1)(\lambda+1)}}\left[\int_0^{\tau^\gamma}\int_{\Omega(\tau,\delta)}(|D_x^m u|^{p+1}+|u|^{\lambda+1})\mathrm{d}x\mathrm{d}t\right]^{1-\frac{(\lambda-p)}{(p+1)(\lambda+1)}}.
\end{aligned}
$$

Inserting estimates (3.9) and (3.5) into (3.6) and (3.4), we obtain

$$
\begin{aligned}
I(\tau) &\le c\,\tau^{\frac{N(\lambda-p)-(\gamma-1)(p+1)}{\lambda+1}}\left[\tfrac{I_2(\tau-\delta)-I_2(\tau)}{\delta}\right]^{\frac{p+1}{\lambda+1}} \\
&+ c_1\textstyle\sum_{i=i}^m \delta^{-(i-1)}\tau^{\frac{i(\lambda-p)(N-1+\gamma)}{m(p+1)(\lambda+1)}}\left[\tfrac{I(\tau-\delta)-I(\tau)}{\delta}\right]^{1-\frac{i(\lambda-p)}{m(p+1)(\lambda+1)}}.
\end{aligned} \tag{3.10}
$$

Our further analysis of inequality (3.10) essentially differs in cases $m=1$ and $m>1$.

3.1. **The second-order equations.** For $m=1$, (3.10) yields

$$
\begin{aligned}
I(\tau) &\le c\,\tau^{\frac{N(\lambda-p)-(\gamma-1)(p+1)}{\lambda+1}}\left[\frac{I(\tau-\delta)-I(\tau)}{\delta}\right]^{\frac{p+1}{\lambda+1}} \\
&+c_1\tau^{\frac{(\lambda-p)(N-1+\gamma)}{(p+1)(\lambda+1)}}\left[\frac{I(\tau-\delta)-I(\tau)}{\delta}\right]^{1-\frac{(\lambda-p)}{(p+1)(\lambda+1)}}.
\end{aligned}
\tag{3.11}
$$

Bearing in mind that now $\gamma=p+1$ and passing to the limit $\delta\to 0$ in (3.11), we obtain the following *differential* inequality:

$$
\begin{aligned}
I(\tau) &\le c\,\tau^{\frac{N(\lambda-p)-p(p+1)}{\lambda+1}}\Big(-I'(\tau)\Big)^{1-\frac{\lambda-p}{\lambda+1}} \\
&+c_1\tau^{\frac{(\lambda-p)(N+p)}{(p+1)(\lambda+1)}}\Big(-I'(\tau)\Big)^{1-\frac{(\lambda-p)}{(p+1)(\lambda+1)}} := cR_1+c_1R_2.
\end{aligned}
\tag{3.12}
$$

Representing R_1 in the form

$$R_1=[(-I'(\tau))^{1-\frac{\lambda-p}{\lambda+1}}\tau^{\frac{(\lambda-p)(N+p)}{A(p+1)(\lambda+1)}}]\tau^{-\frac{(\lambda-p)(N+p)}{A(p+1)(\lambda+1)}+\frac{N(\lambda-p)-p(p+1)}{\lambda+1}},$$

where

$$A=(1-\tfrac{\lambda-p}{(p+1)(\lambda+1)})(1-\tfrac{\lambda-p}{\lambda+1})^{-1}=\tfrac{p(\lambda+1)+p+1}{(p+1)^2},$$

and applying Young's inequality (2.5) with the given A, we easily deduce

$$R_1\le c(\varepsilon)R_2+\sigma\tau^{\frac{N(\lambda-p)-(p+1)^2}{\lambda-p}}\quad\text{for any }\ \varepsilon>0.$$

Due to this inequality, from (3.12) we derive

$$I(\tau)\le c(\varepsilon)\tau^{\frac{(\lambda-p)(N+p)}{(p+1)(\lambda+1)}}(-I'(\tau))^{1-\frac{\lambda-p}{(p+1)(\lambda+1)}}+\varepsilon\tau^{\frac{N(\lambda-p)-(p+1)^2}{\lambda-p}}. \tag{3.13}$$

Let us first consider the case

$$\tfrac{N(\lambda-p)-(p+1)^2}{\lambda-p}\ge 0\quad\Longleftrightarrow\quad \lambda\ge\lambda_1:=p+\tfrac{(p+1)^2}{N}. \tag{3.14}$$

Fix an arbitrary $\tau_0>0$ and suppose that $\varepsilon>0$ is so small that

$$\tfrac12\,\varepsilon\tau_0^{\frac{N(\lambda-p)-(p+1)^2}{\lambda-p}}\le I(\tau)\quad\text{for any }\ \tau\in(0,\tau_0]. \tag{3.15}$$

Then, due to (3.13), we have that $I(\tau)$ satisfies the "homogeneous" inequality

$$I(\tau)\le c_1(\varepsilon)\tau^{\frac{(\lambda-p)(N+p)}{(p+1)(\lambda+1)}}(-I'(\tau))^{1-\frac{\lambda-p}{(p+1)(\lambda+1)}},\quad \tau\in(0,\tau_0],\quad\text{or}$$

$$-I'(\tau)(I(\tau))^{-\frac{(p+1)(\lambda+1)}{p(\lambda+1)+p+1}}\ge c_2\tau^{-\frac{(\lambda-p)(N+p)}{p(\lambda+1)+p+1}},\quad c_2=(c_1(\varepsilon))^{-\frac{(p+1)(\lambda+1)}{p(\lambda+1)+p+1}}. \tag{3.16}$$

Integrating this inequality over the interval (s,τ_0) with $s>0$ yields, for $\lambda>\lambda_1$,

$$I(\tau_0)\le\Big[(I(s))^{-\frac{\lambda-p}{p(\lambda+1)+p+1}}+\frac{c_2(\lambda-p)}{N(\lambda-p)-(p+1)^2}\Big(s^{-\frac{N(\lambda-p)-(p+1)^2}{p(\lambda+1)+p+1}}-\tau_0^{-\frac{N(\lambda-p)-(p+1)^2}{p(\lambda+1)+p+1}}\Big)\Big]^{-\frac{p(\lambda+1)+p+1}{\lambda-p}}. \tag{3.17}$$

If $\lambda=\lambda_1$, integrating yields

$$I(\tau_0)\le\Big[(I(s))^{-\frac{1}{p+N}}+\tfrac{c_2}{p+N}\,(\ln\tfrac1s-\ln\tfrac1{\tau_0})\Big]^{-(p+N)}. \tag{3.18}$$

Passing to the limit $s \to 0$ in (3.17) and (3.18), we obtain $I(\tau_0) = 0$ contradicting the assumption (3.15), so, since $\varepsilon > 0$ is arbitrary, $I(\tau) \equiv 0$. Theorem 1.1 is proved for $m = 1$ and λ satisfying (3.14).

Consider next the case

$$\lambda_2 := p + \tfrac{p(p+1)}{N} < \lambda < \lambda_1 = p + \tfrac{(p+1)^2}{N}. \tag{3.19}$$

Firstly we prove that, for some constant $k > 0$, there exists a sequence $\{\tau_i\} \to 0$ such that $I(\tau)$ solving (3.13) satisfies the following estimate:

$$I(\tau_i) \le k\, \tau_i^{-\frac{(p+1)^2 - N(\lambda-p)}{\lambda-p}}, \quad i = 1, 2, \dots. \tag{3.20}$$

Suppose that it is not true so

$$I(\tau) > k\, \tau^{-\frac{(p+1)^2 - N(\lambda-p)}{\lambda-p}} \quad \text{for all } \tau < \tau_0. \tag{3.21}$$

Then, using (3.21) in estimating the last term in (3.13), we obtain

$$-I'(\tau)(I(\tau))^{-\frac{(p+1)(\lambda+1)}{p(\lambda+1)+p+1}} \ge c_3 \tau^{-\frac{(\lambda-p)(N+p)}{p(\lambda+1)+p+1}}, \quad c_3 = \left(\tfrac{k-\varepsilon}{c(\varepsilon)k}\right)^{\frac{(p+1)(\lambda+1)}{p(\lambda+1)+p+1}}. \tag{3.22}$$

Due to (3.19), $\frac{(\lambda-p)(N+p)}{p(\lambda+1)+p+1} < 1$, so that we integrate (3.22) over $(0, s)$ with $s < \tau_0$ to conclude that

$$I(s) \le c_4 s^{-\frac{(p+1)^2 - N(\lambda-p)}{\lambda-p}}, \quad c_4 = c_4(k,\varepsilon)\left(\frac{(p+1)^2 - N(\lambda-p)}{\lambda-p}\right)^{\frac{p(\lambda+1)+p+1}{\lambda-p}} \left(\tfrac{c(\varepsilon)k}{k-\varepsilon}\right)^{\frac{(p+1)(\lambda+1)}{\lambda-p}}. \tag{3.23}$$

Now we fix an arbitrarily small $\varepsilon > 0$ and choose

$$k_0(\varepsilon) := \inf\{k : \ k > c_4 = c_4(k,\varepsilon)\}. \tag{3.24}$$

It is obvious that, for arbitrary $k > k_0(\varepsilon)$, estimate (3.23) contradicts the assumption (3.21). Therefore, (3.20) holds with arbitrary $k > k_0$.

Next, let for some subsequence $\{\tilde\tau_j\}$ of the sequence $\{\tau_i\}$ from (3.20), there are points $\{\tilde\tau_{j+1} \le \tilde\tau'_{j+1} < \tilde\tau'_j < \tilde\tau_j\}$ such that

$$I(\tilde\tau'_{j+1}) = k(\tilde\tau'_{j+1})^{-\frac{(p+1)^2 - N(\lambda-p)}{\lambda-p}}, \tag{3.25}$$

$$I(\tau) > k\tau^{-\frac{(p+1)^2 - N(\lambda-p)}{\lambda-p}} \quad \text{for } \tau \in (\tilde\tau'_{j+1}, \tilde\tau'_j). \tag{3.26}$$

By (3.26), it follows from (3.13) that (3.22) holds on the interval $(\tilde\tau'_{j+1}, \tilde\tau'_j)$. Integrating it over $(\tilde\tau'_{j+1}, s)$ with $s < \tilde\tau'_j$ yields

$$\begin{aligned} I(s) \le \Big[&(I(\tilde\tau'_{j+1}))^{-\frac{\lambda-p}{p(\lambda+1)+p+1}} + \tfrac{c_3(\lambda-p)}{(p+1)^2 - N(\lambda-p)} \\ &\times \Big(s^{\frac{(p+1)^2 - N(\lambda-p)}{p(\lambda+1)+p+1}} - (\tilde\tau'_{j+1})^{\frac{(p+1)^2 - N(\lambda-p)}{p(\lambda+1)+p+1}} \Big)\Big]^{-\frac{p(\lambda+1)+p+1}{\lambda-p}} \\ &\le \langle \text{due to (3.24), (3.22) and (3.25)} \rangle \le k\, s^{-\frac{(p+1)^2 - N(\lambda-p)}{\lambda-p}}. \end{aligned} \tag{3.27}$$

The last inequality contradicts the assumption (3.26). Hence

$$I(\tau) \le k\,\tau^{-\frac{(p+1)^2-N(\lambda-p)}{\lambda-p}} \quad \text{for all } \tau<\tau_0. \tag{3.28}$$

Our next step is to find a sharp estimate of the "entropy" functions connected with our solution $u(x,t)$,

$$J_1(\tau)+J_2(\tau) := \int_{Q^{(T)}(\tau)} |D_x u|^p \,\mathrm{d}x\mathrm{d}t + \int_{Q^{(T)}(\tau)} |u|^\lambda \,\mathrm{d}x\mathrm{d}t. \tag{3.29}$$

It follows from (3.28) that, for any $\tau>0$,

$$I_1(\tfrac{\tau}{2}) - I_1(\tau) + I_2(\tfrac{\tau}{2}) - I_2(\tau) \le c\,\tau^{-\frac{(p+1)^2-N(\lambda-p)}{\lambda-p}}. \tag{3.30}$$

Therefore, by Hölder's inequality and (3.30) there holds

$$\begin{aligned} J_2(\tfrac{\tau}{2}) - J_2(\tau) &= \int_{Q^{(T)}(\frac{\tau}{2})\setminus Q^{(T)}(\tau)} |u|^\lambda \,\mathrm{d}x\mathrm{d}t \\ &\le (\mathrm{meas}\,\{Q^{(T)}(\tfrac{\tau}{2})\setminus Q^{(T)}(\tau)\})^{\frac{1}{\lambda+1}} \Big(\int_{Q^{(T)}(\frac{\tau}{2})\setminus Q^{(T)}(\tau)} |u|^{\lambda+1}\,\mathrm{d}x\mathrm{d}t\Big)^{\frac{\lambda}{\lambda+1}} \\ &\le c[I_2(\tfrac{\tau}{2}) - I_2(\tau)]^{\frac{\lambda}{\lambda+1}} (\tau^{N+p+1})^{\frac{1}{\lambda+1}} \\ &\le c\,\tau^{-\frac{[(p+1)^2-N(\lambda-p)]\lambda}{(\lambda-p)(\lambda+1)}+\frac{N+p+1}{\lambda+1}} = c\,\tau^{\frac{N(\lambda-p)-p(p+1)}{\lambda-p}}. \end{aligned} \tag{3.31}$$

Now, since

$$N(\lambda-p)-p(p+1)>0 \iff \lambda>\lambda_2 = p+\tfrac{p(p+1)}{N}, \tag{3.32}$$

(3.31) implies

$$J_2(\tfrac{\tau}{2}) - J_2(\tau) \le c\,\tau^\alpha, \quad \alpha = \tfrac{N(\lambda-p)-p(p+1)}{\lambda-p} > 0. \tag{3.33}$$

Consequently

$$J_2(\tfrac{\tau_0}{2^{j+1}}) - J_2(\tfrac{\tau_0}{2^j}) \le \tfrac{c}{2^{\alpha j}}\,\tau_0^\alpha, \quad j=1,2,\dots. \tag{3.34}$$

Summing up these inequalities for all $j\le i$, we obtain

$$J_2(\tfrac{\tau_0}{2^{i+1}}) \le J_2(\tau_0) + \tfrac{c\,\tau_0^\alpha}{1-2^{-\alpha}} \quad \text{for any } i=1,2,\dots. \tag{3.35}$$

Hence the full integral converges,

$$\int_{Q^T} |u(x,t)|^\lambda \,\mathrm{d}x\mathrm{d}t < \infty. \tag{3.36}$$

In a similar way, we deduce using (3.28) that

$$\begin{aligned} J_1(\tfrac{\tau}{2}) - J_1(\tau) &= \int_{Q^{(T)}(\frac{\tau}{2})\setminus Q^{(T)}(\tau)} |D_x u|^p \,\mathrm{d}x\mathrm{d}t \\ &\le \tau^{\frac{N+p+1}{p+1}} \Big(\int_{Q^{(T)}(\frac{\tau}{2})\setminus Q^{(T)}(\tau)} |D_x u|^{p+1}\,\mathrm{d}x\mathrm{d}t\Big)^{\frac{p}{p+1}} \\ &\le c\,\tau^{-\frac{[(p+1)^2-N(\lambda-p)]p}{(\lambda-p)(p+1)}+\frac{N+p+1}{p+1}} = c\,\tau^{\alpha+1}, \end{aligned} \tag{3.37}$$

where α is as in (3.33).

Now we will prove immediately that $u(0,x) \equiv 0$ in $\mathbf{R}^N$. We introduce a sequence of smooth cut-off functions

$$\varphi_i(x,t) := \varphi_{\tau_0/2^i,\tau_0/2^{i+1}}(x,t), \quad i = 1,2,\dots, \tag{3.38}$$

with $\varphi_{\tau,\delta}$ given in (3.1). Let $\Phi(x,t) \in C^{1,1}(\mathbf{R}^N \times \mathbf{R}_+)$ be arbitrary compactly supported function. Then, by Definition 1.1, the following identity holds:

$$\begin{aligned}
&\textstyle\int_{\mathbf{R}^N} |u(x,T)|^{p-1}u(x,T)\Phi(x,T)\,\mathrm{d}x - \int_0^T \int_{\mathbf{R}^N} |u|^{p-1}u(\Phi\varphi_i)_t\,\mathrm{d}x\mathrm{d}t \\
&\textstyle+ \int_0^T \int_{\mathbf{R}^N} \sum_{j=1}^N a_j(x,t,u,Du)D_{x_j}(\Phi\varphi_i)\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbf{R}^N} g(x,t,u)\Phi\varphi_i\,\mathrm{d}x\mathrm{d}t \\
&\qquad := R_1 + R_2(i) + R_3(i) + R_4(i) = 0
\end{aligned} \tag{3.39}$$

for any $i = 1,2,\dots$. We now pass to the limit $i \to \infty$. First we rewrite

$$R_4(i) = \textstyle\int_{Q_i^{(T)}} g(x,t,u)\Phi\phi_i\,\mathrm{d}x\mathrm{d}t + \int_{Q_{i+1}^{(T)}\setminus Q_i^{(T)}} g(x,t,u)\Phi\phi_i\,\mathrm{d}x\mathrm{d}t := R_4^{(1)}(i) + R_4^{(2)}(i),$$

where $Q_i^{(T)} = Q^{(T)}(\frac{\tau_0}{2^i})$. By assumption (1.12) and the estimate (3.34) we have

$$|R_4^{(2)}(i)| \le c\,\tfrac{\tau_0^\alpha}{2^{i\alpha}} \to 0 \quad \text{as } i \to \infty. \tag{3.40}$$

Therefore, due to (3.36), as $i \to \infty$,

$$R_4(i) \to \textstyle\int_0^T \int_{\mathbf{R}^N} g(x,t,u)\Phi\,\mathrm{d}x\mathrm{d}t. \tag{3.41}$$

Next we have

$$\begin{aligned}
R_3(i) &= \textstyle\int_0^T \int_{Q_i^{(T)}} \sum_{j=1}^N a_j(x,t,u,Du)\phi_i D_{x_j}\Phi\,\mathrm{d}x\mathrm{d}t \\
&\textstyle+ \int_0^T \int_{Q_{i+1}^{(T)}\setminus Q_i^{(T)}} \sum_{j=1}^N a_j(x,t,u,Du)\Phi D_{x_j}\varphi_i\,\mathrm{d}x\mathrm{d}t := R_3^{(1)}(i) + R_3^{(2)}(i).
\end{aligned} \tag{3.42}$$

Due to (1.11) we obtain

$$\begin{aligned}
|R_3^{(2)}(i)| &\le c\,\tfrac{2^{i+1}}{\tau_0} \textstyle\int_{Q_{i+1}^{(T)}\setminus Q_i^{(T)}} |D_x u|^p\,\mathrm{d}x\mathrm{d}t \le \langle \text{by (3.39) with } \tau = \tau_0 2^{-i} \rangle \\
&\le c(\tau_0 2^{-i})^{\alpha+1}\tfrac{2^{i+1}}{\tau_0} \le 2c\,\tau_0^\alpha 2^{-i\alpha} \to 0, \quad i \to \infty.
\end{aligned}$$

Therefore in view of the boundedness, $I_1(0) < \infty$, which follows from (3.37), we obtain

$$R_3(i) \to \textstyle\int_0^T \int_{\mathbf{R}^N} \sum_{j=1}^N a_j(x,t,u,Du)D_{x_j}\Phi\,\mathrm{d}x\mathrm{d}t, \quad i \to \infty. \tag{3.43}$$

Finally consider

$$\begin{aligned}
-R_2(i) &= \textstyle\iint_{Q_i^{(T)}} |u|^{p-1}u\Phi_t\,\mathrm{d}x\mathrm{d}t + \iint_{Q_{i+1}^{(T)}\setminus Q_i^{(T)}} |u|^{p-1}u\Phi\varphi_{it}\,\mathrm{d}x\mathrm{d}t \\
&\textstyle+ \iint_{Q_{i+1}^{(T)}\setminus Q_i^{(T)}} |u|^{p-1}u\Phi_t\varphi_i\,\mathrm{d}x\mathrm{d}t := R_2^{(1)}(i) + R_2^{(2)}(i) + R_2^{(3)}(i).
\end{aligned} \tag{3.44}$$

It is easy to see that

$$\begin{aligned}
|R_2^{(2)}(i)| &\le c(\tfrac{\tau_0}{2^i})^{-(p+1)} \textstyle\iint_{Q_{i+1}^{(T)}\setminus Q_i^{(T)}} |u|^p\,\mathrm{d}x\mathrm{d}t \\
&\le c(\tfrac{\tau_0}{2^i})^{-(p+1)}(\operatorname{meas}\{Q_{i+1}^{(T)} \setminus Q_i^{(T)}\})^{\frac{\lambda+1-p}{\lambda+1}} \times \left(\textstyle\iint_{Q_{i+1}^{(T)}\setminus Q_i^{(T)}} |u|^{\lambda+1}\,\mathrm{d}x\mathrm{d}t\right)^{\frac{p}{\lambda+1}} \\
&\le \langle \text{by (3.28)} \rangle \le c(\tfrac{\tau_0}{2^i})^{-(p+1)}(\tfrac{\tau_0}{2^i})^{\frac{(N+p+1)(\lambda+1-p)}{\lambda+1}}(\tfrac{\tau_0}{2^i})^{-\frac{[(p+1)^2 - N(\lambda-p)]p}{(\lambda-p)(\lambda+1)}} = c(\tfrac{\tau_0}{2^i})^\alpha,
\end{aligned}$$

where α is given in (3.10). Therefore $R_2^{(2)}(i) \to 0$ as $i \to \infty$. It is easier to check that $R_2^{(3)}(i) \to 0$ as $i \to \infty$.

Consequently, due to (3.36), we have

$$R_2(i) \to -\int_0^T \int_{\mathbf{R}^N} |u|^{p-1} u \Phi_t \, dx dt \quad \text{as } i \to \infty. \tag{3.45}$$

Combining (3.41), (3.43) and (3.45), we deduce that (1.17) is valid for any smooth function $\phi(x,t)$ *without* the assumption on vanishing near $(0,0)$. This means that, in the weak sense, $u(x,0) = 0$ in $\mathbf{R}^N$ completing the proof of Theorem 1.1 for $m = 1$.

3.2. **Higher-order parabolic PDEs.** We return to the inequality (3.10) and will analyze it for arbitrary $m > 1$. We rewrite the general term in the sum in (3.10) in the following form:

$$\begin{aligned} B_i &:= \delta^{-(i-1)} \tau^{\frac{i(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)}} \left(-\frac{\Delta I}{\delta}\right)^{1-\frac{i(\lambda-p)}{\gamma(\lambda+1)}} \\ &= \left[\tau^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)A_i}} \left(-\frac{\Delta I}{\delta}\right)^{(1-\frac{(\lambda-p)}{\gamma(\lambda+1)})\frac{1}{A_i}}\right] \tau^{\frac{i(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)} - \frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)A_i}} \delta^{-(i-1)}, \end{aligned} \tag{3.46}$$

where $\gamma = m(p+1)$ and

$$A_i = [1 - \tfrac{\lambda-p}{\gamma(\lambda+1)}][1 - \tfrac{i(\lambda-p)}{\gamma(\lambda+1)}]^{-1} = \tfrac{\gamma(\lambda+1)-(\lambda-p)}{\gamma(\lambda+1)-i(\lambda-p)} > 1.$$

Applying Young's inequality with exponents A_i and $A_i' = \frac{A_i}{A_i-1}$, one easily obtains

$$B_i \le c(\varepsilon) \tau^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)}} \left(-\frac{\Delta I}{\delta}\right)^{1-\frac{\lambda-p}{\gamma(\lambda+1)}} + \varepsilon \delta^{-\frac{\gamma(\lambda+1)-(\lambda-p)}{\lambda-p}} \tau^{N-1+\gamma} \tag{3.47}$$

for any $\varepsilon > 0$. Similarly we estimate the first term on the right-hand side of (3.10),

$$\begin{aligned} B_0 &:= c\,\tau^{\frac{N(\lambda-p)-(\gamma-1)(p+1)}{\lambda+1}} \left(-\frac{\Delta I_2}{\delta}\right)^{\frac{p+1}{\lambda+1}} \\ &= c\left[\tau^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)A_0}} \left(-\frac{\Delta I_2}{\delta}\right)^{1-\frac{\lambda-p}{\lambda+1}}\right] \tau^{-\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)A_0} + \frac{N(\lambda-p)-(\gamma-1)(p+1)}{\lambda+1}}, \end{aligned} \tag{3.48}$$

where

$$A_0 = [1 - \tfrac{\lambda-p}{\gamma(\lambda+1)}](1 - \tfrac{\lambda-p}{\lambda+1})^{-1} = \tfrac{\gamma(\lambda+1)-(\lambda-p)}{\gamma(p+1)}.$$

Using Young's inequality with exponents A_0 and $A_0' = \frac{A_0}{A_0-1}$ yields

$$B_0 \le c(\varepsilon) \tau^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)}} \left(-\frac{\Delta I_2}{\delta}\right)^{1-\frac{\lambda-p}{\gamma(\lambda+1)}} + \varepsilon \tau^{\frac{N(\lambda-p)-\gamma(p+1)}{\lambda-p}}. \tag{3.49}$$

Combining (3.47) and (3.49), we obtain from (3.10) that

$$\begin{aligned} I(\tau) &\le c(\varepsilon) \tau^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)}} \left[\frac{I(\tau-\delta)-I(\tau)}{\delta}\right]^{1-\frac{\lambda-p}{\gamma(\lambda+1)}} \\ &+ \varepsilon \tau^{\frac{N(\lambda-p)-\gamma(p+1)}{\lambda-p}} + \varepsilon \delta^{-\frac{\gamma(\lambda+1)-(\lambda-p)}{\lambda-p}} \tau^{N-1+\gamma}, \end{aligned} \tag{3.50}$$

where $\varepsilon > 0$ can be arbitrarily small. We define $I(\tau) = \infty$ for all $\tau \le 0$, and then inequality (3.50) is valid for arbitrary $\tau > 0$ and $\delta > 0$ without the restriction $\delta < \tau$. Therefore, we can choose $\delta = \delta_1(\tau)$ from the equality

$$\tau^{N-1+\gamma}\delta_1^{-\frac{\gamma(\lambda+1)-(\lambda-p)}{\lambda-p}} = I(\tau), \tag{3.51}$$

where $I(\tau)$ is still the unknown function to be estimated. It follows from (3.51) that

$$\delta_1 = \delta_1(\tau) = \tau^{\frac{(N-1+\gamma)(\lambda-p)}{\gamma(\lambda+1)-(\lambda-p)}}\big(I(\tau)\big)^{-\frac{\lambda-p}{\gamma(\lambda+1)-(\lambda-p)}}. \tag{3.52}$$

Substituting this into (3.50) yields

$$I(\tau) \le c_1(\varepsilon)\tau^{\frac{(N-1+\gamma)(\lambda-p)}{\gamma(\lambda+1)}}\left[\frac{I(\tau-\delta_1(\tau))-I(\tau)}{\delta_1(\tau)}\right]^{1-\frac{\lambda-p}{\gamma(\lambda+1)}} + \varepsilon\tau^{\frac{N(\lambda-p)-\gamma(p+1)}{\lambda-p}}. \tag{3.53}$$

Firstly consider the case

$$\tfrac{N(\lambda-p)-\gamma(p+1)}{\lambda-p} \ge 0 \quad \Longrightarrow \quad \lambda \ge \lambda_1 := p + \tfrac{\gamma(p+1)}{N} = p + \tfrac{m(p+1)^2}{N}. \tag{3.54}$$

Due to the monotonicity of $I(\tau)$, it follows from (3.53) that the following "homogeneous" inequality holds:

$$I(\tau) \le c_1\tau^{\frac{(N-1+\gamma)(\lambda-p)}{\gamma(\lambda+1)}}\left[\frac{I(\tau-\delta_1)-I(\tau)}{\delta_1}\right]^{1-\frac{\lambda-p}{\gamma(\lambda+1)}}. \tag{3.55}$$

Plugging δ_1 in (3.52) into (3.55), we obtain after simple manipulations that

$$I(\tau) \le c_2[I(\tau-\delta_1(\tau)) - I(\tau)], \quad \text{with } c_2 = c_1^{\frac{\gamma(\lambda+1)}{\gamma(\lambda+1)-(\lambda-p)}}, \quad \text{so} \tag{3.56}$$

$$I(\tau) \le \theta I\Big(\tau - \tau^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)-(\lambda-p)}}\big(I(\tau)\big)^{-\frac{\lambda-p}{\gamma(\lambda+1)-(\lambda-p)}}\Big) \tag{3.57}$$

for any $\tau \in \mathbf{R}$, where $\theta = \frac{c_2}{1+c_2} < 1$. Introducing the new energy function

$$J(\tau) := \tau^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)-(\lambda-p)}}\big(I(\tau)\big)^{-\frac{\lambda-p}{\gamma(\lambda+1)-(\lambda-p)}}, \tag{3.58}$$

by a simple transformation, we derive from (3.57) the following functional inequality:

$$J(\tau - J(\tau)) \le \theta^{\frac{\lambda-p}{\gamma(\lambda+1)-(\lambda-p)}}\left[\frac{\tau-J(\tau)}{\tau}\right]^{\frac{(\lambda-p)(N-1+\gamma)}{\gamma(\lambda+1)-(\lambda-p)}} J(\tau). \tag{3.59}$$

In particular, (3.59) implies that

$$J(\tau - J(\tau)) \le \theta_1 J(\tau) \quad \text{for any } \tau \in \mathbf{R} \quad (\theta_1 < 1). \tag{3.60}$$

Lemma 3.1. *Let a nonnegative continuous nondecreasing function $J(\tau)$ satisfy inequality (3.60) for any $\tau \le \tau_0$. Then $J(\tau) \equiv 0$ for all $\tau \le \tau_0 - (1-\theta_1)^{-1}J(\tau_0)$.*

Proof. Define a decreasing sequence $\{\tau_i\}$ by $\tau_{i+1} = \tau_i - J(\tau_i)$ for $i = 0, 1, 2, \ldots$. Due to (3.60), we have

$$J(\tau_{i+1}) \le \theta_1 J(\tau_i) \le \ldots \le \theta_1^{i+1} J(\tau_0) \to 0 \quad \text{as } i \to \infty. \tag{3.61}$$

Furthermore we also have

$$\begin{aligned}\tau_{i+1} &= \tau_i - J(\tau_i) = \tau_{i-1} - J(\tau_{i-1}) - J(\tau_i) = ... = \tau_0 - \textstyle\sum_{j=0}^{i} J(\tau_j) \\ &\geq \langle \text{due to (3.61)} \rangle \geq \tau_0 - J(\tau_0) \textstyle\sum_{j=0}^{i} \theta_1^j \geq \tau_0 - \frac{1}{1-\theta_1} J(\tau_0).\end{aligned} \tag{3.62}$$

The Lemma is an immediate consequence of (3.61) and (3.62). $\square$

We return to the energy function $J(\tau)$ defined in (3.58) and satisfying

$$J(\tau) > 0 \quad \text{for all } \tau > 0.$$

Therefore by Lemma 3.1 we have

$$\tau_0 - \tfrac{1}{1-\theta_1} J(\tau_0) \leq 0 \quad \Longleftrightarrow \quad J(\tau_0) \geq (1-\theta_1)\tau_0.$$

Since τ_0 is arbitrary, the last inequality implies that

$$J(\tau) \geq (1-\theta_1)\tau \quad \text{for all } \tau > 0, \tag{3.63}$$

which, by (3.58), is equivalent to

$$I(\tau) \leq (1-\theta_1)^{-\frac{\gamma(\lambda+1)-(\lambda-p)}{\lambda-p}} \tau^{\frac{N(\lambda-p)-\gamma(p+1)}{\lambda-p}} \quad \text{for } \tau \geq 0. \tag{3.64}$$

By assumption (3.54) and bearing in mind that the energy function $I(\tau)$ is nonincreasing, (3.64) implies that $I(\tau) \equiv 0$. This completes the proof of Theorem 1.1 for any $m \geq 1$ and $\lambda \geq \lambda_1$.

Consider now the parameter range

$$\lambda_0 = p + \tfrac{mp(p+1)}{N} < \lambda < \lambda_1 = p + \tfrac{m(p+1)^2}{N}. \tag{3.65}$$

We are starting from the "inhomogeneous" inequality (3.53). Substituting $\delta_1(\tau)$ from (3.52), after simple manipulations, we derive

$$I(\tau) \leq c_2[I(\tau - \delta_1(\tau)) - I(\tau)] + \varepsilon \tau^{-\frac{\gamma(\lambda+1)G}{(\lambda-p)L}} (I(\tau))^{-\frac{\lambda-p}{L}}, \tag{3.66}$$

where $c_2 = c_2(\varepsilon)$ is from (3.56) and

$$G = -N(\lambda - p) + \gamma(p+1) > 0, \quad L = \gamma(\lambda+1) - (\lambda-p) > 0. \tag{3.67}$$

It follows from (3.66) that

$$I(\tau) \leq \theta I(\tau - \delta_1(\tau)) + \varepsilon \tau^{-\frac{\gamma(\lambda+1)G}{(\lambda-p)L}} (I(\tau))^{-\frac{\lambda-p}{L}}, \quad \theta = \tfrac{c_2}{1+c_2} < 1. \tag{3.68}$$

Firstly we prove that there exists a constant $M_0 > 0$ such that, for some sequence $\{\tau_i\} \to 0$,

$$I(\tau_i) \leq M_0 \tau_i^{-\frac{G}{\lambda-p}}, \quad i = 1, 2, \ldots. \tag{3.69}$$

Suppose on the contrary that (3.69) is not true and so, for arbitrary $M_0 > 0$,

$$I(\tau) > M_0 \tau^{-\frac{G}{\lambda-p}} \quad \text{for any } \tau < \tau_0(M_0). \tag{3.70}$$

Then, due to (3.70), it follows from (3.68) that

$$I(\tau) \leq \theta I(\tau - \delta_1(\tau)) + \varepsilon M_0^{-\frac{\lambda-p}{L}} \tau^{-\frac{G}{\lambda-p}} \quad \text{for any } \tau < \tau_0, \tag{3.71}$$

or, using (3.70) once more, $I(\tau) \leq \theta I(\tau - \delta_1(\tau)) + \varepsilon M_0^{-\frac{\gamma(\lambda+1)}{L}} I(\tau)$, so finally

$$(3.72) \qquad I(\tau) \leq \theta_2 I(\tau - \delta_1(\tau)), \quad \text{where } \theta_2 = \theta(1 - \varepsilon M_0^{-\frac{\gamma(\lambda+1)}{L}})^{-1} < 1.$$

We have obtained for $I(\tau)$ an estimate similar to (3.57). Therefore, similar to Lemma 3.1, we derive the following estimate:

$$(3.73) \qquad I(\tau) \leq (1 - \theta_2^{\frac{\lambda-p}{L}})^{-\frac{L}{\lambda-p}} \tau^{-\frac{G}{\lambda-p}}, \quad \tau < \tau_0,$$

which contradicts the assumption (3.70) for sufficiently large M_0.

Secondly our aim is to prove an estimate like (3.69) for all $\tau > 0$, namely that, for some constant $M_0 > 0$,

$$(3.74) \qquad I(\tau) \leq M_0 \tau^{-\frac{G}{\lambda-p}} \quad \text{for all } \tau > 0.$$

Suppose again that (3.74) does not holds, so, by (3.69), there exists a sequence of intervals $\Delta_i = (\tau_i, \tau_i')$ such that $\Delta_i \cap \Delta_j = \emptyset$ for all $i \neq j$, $\tau_i, \tau_i' \to 0$ as $i \to \infty$, and

$$(3.75) \qquad I(\tau_i) = M_0 \tau_i^{-\frac{G}{\lambda-p}}, \quad I(\tau_i') = M_0 (\tau_i')^{-\frac{G}{\lambda-p}}, \quad I(\tau) > M_0 \tau^{-\frac{G}{\lambda-p}} \text{ on } \Delta_i.$$

Take an arbitrary point $\tau \in \Delta_i$. In a way similar to that for obtaining (3.72), we deduce

$$(3.76) \qquad I(\tau) \leq \theta_2 I(\tau - \delta_1(\tau)) \quad \text{on } \Delta_i,$$

where θ_2 is given in (3.72). Due to (3.75), it follows from the definition (3.52) of $\delta_1(\tau)$ that

$$(3.77) \qquad \delta_1(\tau) \leq M_0^{-\frac{\lambda-p}{L}} \tau \quad \text{on } \Delta_i.$$

Then, by (3.77), we have that (3.76) implies

$$(3.78) \qquad I(\tau) \leq \theta_2 I((1 - M_0^{-\frac{\lambda-p}{L}})\tau) \quad \text{on } \Delta_i.$$

Iterating this inequality j times yields on Δ_i

$$(3.79) \qquad I(\tau) \leq \theta_2^j I((1 - M_0^{-\frac{\lambda-p}{L}})^j \tau).$$

We pick j from the condition

$$(3.80) \qquad (1 - M_0^{-\frac{\lambda-p}{L}})^{j+1} \tau < \tau_i \leq (1 - M_0^{-\frac{\lambda-p}{L}})^j \tau.$$

Then, by (3.79) and (3.75),

$$(3.81) \qquad I(\tau) \leq \theta_2^{-1} (\tfrac{\tau}{\tau_i})^{-R} I(\tau_i) = \frac{M_0 \tau^{-R}}{\theta_2 \tau_i^{-R} \tau_i^{\frac{G}{\lambda-p}}},$$

where $R = R(M_0) = (\ln \theta_2^{-1})[\ln(1 - M_0^{-\frac{\lambda-p}{L}})^{-1}]^{-1} \to \infty$ as $M_0 \to \infty$.

We continue the estimate (3.81),

$$(3.82) \qquad I(\tau) \leq \theta_2^{-1} M_0 \tau^{-\frac{G}{\lambda-p}} (\tfrac{\tau_i}{\tau})^{R - \frac{G}{\lambda-p}}.$$

If we fix M_0 so large that

$$(3.83) \qquad R(M_0) - \tfrac{G}{\lambda-p} > 0,$$

then (3.82) contradicts the third inequality in (3.75). Therefore, our assumption (3.75) is false and (3.74) follows.

The rest of proof of Theorem 1.1 repeats, with minor changes, the proof for $m = 1$. So, using (3.74) and the fact that

$$\text{meas}\,\{Q^{(T)}(\tfrac{\tau}{2}) \setminus Q^{(T)}(\tau)\} = c\,\tau^{N+m(p+1)}, \tag{3.84}$$

we obtain, similar to (3.32), that

$$J_2(\tfrac{\tau}{2}) - J_2(\tau) \le c\tau^{\alpha}, \quad \text{where } \alpha = \tfrac{N(\lambda-p)-mp(p+1)}{\lambda-p} > 0, \tag{3.85}$$

for $\lambda > \lambda_2 = p + \frac{mp(p+1)}{N}$. Analogously to (3.37), using (3.74) and (3.84), we derive

$$J_1(\tfrac{\tau}{2}) - J_1(\tau) = \int_{Q^{(T)}(\frac{\tau}{2})\setminus Q^{(T)}(\tau)} |D_x^m u|^p \mathrm{d}x\mathrm{d}t \le c\,\tau^{\alpha+1}. \tag{3.86}$$

Applying estimates (3.86) and (3.85), similar to the case $m = 1$, we establish the identity (1.17) for arbitrary test functions $\Phi(x,t) \in C^{m,1}(\mathbf{R}^N \times \mathbf{R}_+)$ without the assumption on vanishing in a neighbourhood of the origin $(0,0)$. Theorem 1.1 is proved. $\square$

4. **On existence of VSSs in the subcritical range**

In this final section we intend to justify that, for the above class of higher-order parabolic equations including (1.13), Theorems 1.1 and 1.2 provide us with the optimal nonexistence range of VSSs.

4.1. **A simple characterization of the VSS existence range.** Consider the Cauchy problem for the PDE (1.13) with sufficiently smooth compactly supported nonnegative initial data $\tilde{u}_0(x)$, and now fix λ in the subcritical range

$$q < \lambda < \lambda_0 = p + \tfrac{q(p+1)m}{N}. \tag{4.1}$$

Assume that there exists a global in time solution $\tilde{u}(x,t)$ satisfying certain suitable estimates. In other words, we assume that the parabolic problem under consideration is well-posed for classes of regular bounded initial data.

Then we apply the standard scaling technique and introduce a family of global solutions of (1.13),

$$u_s(x,t) = s^{\beta}\tilde{u}(\tfrac{x}{s^{\alpha}}, \tfrac{t}{s}), \quad \text{with parameter } s > 0, \quad \text{where} \tag{4.2}$$

$$\beta = -\tfrac{1}{\lambda-q} < 0, \quad \alpha = \tfrac{\lambda-p}{m(p+1)(\lambda-q)}. \tag{4.3}$$

Obviously, (4.2) is a global solution of (1.13) with scaled initial data

$$u_{0s}(x) = u_0(\tfrac{x}{s^{\alpha}}) \quad \text{in } \mathbf{R}^N. \tag{4.4}$$

Let us check the "singularity" properties of such solutions as $s \to 0^+$. Integrating equation (1.13) over $\mathbf{R}^N$ yields the unbounded singular limit

$$\int_{\mathbf{R}^N} |u|^{q-1}u\,\mathrm{d}x = s^{\mu}\int_{\mathbf{R}^N} |u_0|^{q-1}u_0\,\mathrm{d}x \to \infty \quad \text{as } s \to 0, \tag{4.5}$$

iff the exponent μ satisfies

$$\mu = -\tfrac{q}{\lambda - q} + \alpha N < 0. \tag{4.6}$$

One can see that (4.6) is equivalent to (4.1). Proving the existence of a VSS assumes sharp estimates as $s \to \infty$ of the family $\{u_s(x,t)\}$, or, which is the same, sharp characterization of the asymptotic behaviour of the fixed global "characteristic" solution $\tilde u(x,t)$ as $t \to \infty$. If such estimates (from above and below) was available, passing to the limit $s \to 0$ in (4.2) would guarantee existence of a VSS. This is still an open problem.

4.2. **On the λ-bifurcation of the VSSs.** We next discuss some refined properties of VSSs concerning the total number of such solutions. We describe existence and multiplicity of the similarity VSSs for the semilinear model (1.7); see [8].

In the subcritical range $\lambda \in (1, \lambda_0)$, where λ_0 is as in (1.8), we look for VSSs in the self-similar form

$$u_*(x,t) = t^{-1/(\lambda-1)} V(y), \quad y = x/t^{1/2m}, \tag{4.7}$$

where V is a non-trivial radial solution of the elliptic equation

$$-(-\Delta)^m V + \tfrac{1}{2m}\, \nabla V \cdot y + \tfrac{1}{\lambda - 1}\, V - |V|^{\lambda-1} V = 0 \quad \text{in } \mathbf{R}^N, \tag{4.8}$$

$$V(y) \;\; \text{decays exponentially fast as} \;\; |y| \to \infty. \tag{4.9}$$

Condition (4.9) on exponential decay at infinity is naturally enforced by introducing weighted L^2 and Sobolev spaces; see [8, Sect. 3]. On the basis of the classical bifurcation theory, it was shown [8] that the critical exponents (1.9) are bifurcation points for the elliptic problem (4.8), at which bifurcation branches of VSS profiles V appear. The global behaviour of such bifurcation branches was studied numerically and a typical bifurcation diagram is shown in Figure 1. If all the branches can be extended to $\lambda = 1^+$, for which there exists a strong numerical and formal analytical evidence (such an extension remains an open mathematical problem), we arrive at the following estimate of the total number of VSS in the radial ODE setting: there exists at least

$$M(m, \lambda, N) = \sharp_{\rm even}\big(\tfrac{N(\lambda_0 - \lambda)}{\lambda - 1}\big) \tag{4.10}$$

different non-trivial radial solution $V = V(|y|)$, where $\sharp_{\rm even}(z)$ for $z \ge 0$ denotes the number of non-negative even numbers $0, 2, 4, ...$ not exceeding the integer part $\lfloor z \rfloor$.

This shows that in the semilinear model (1.7), for any $\lambda \in (1, \lambda_0)$, there exists a *finite* number of VSSs bifurcating at critical values (1.9), and M increases to infinity as $\lambda \to 1^+$. It was shown in [8, Sect. 6.1] that a countable subset of λ-bifurcation branches are concentrated at the point $\lambda = 1^+$ (as shown in Figure 1). Along these branches, as $\lambda \to 1^+$, the rescaled similarity profiles $\{V_l(y)\}$ become close to eigenfunctions $\{\psi_l(y)\}$ of the linear non-self-adjoint operator

$$\mathbf{B} = -(-\Delta)^m + \tfrac{1}{2m}\, y \cdot \nabla + \tfrac{N}{2m}\, I.$$

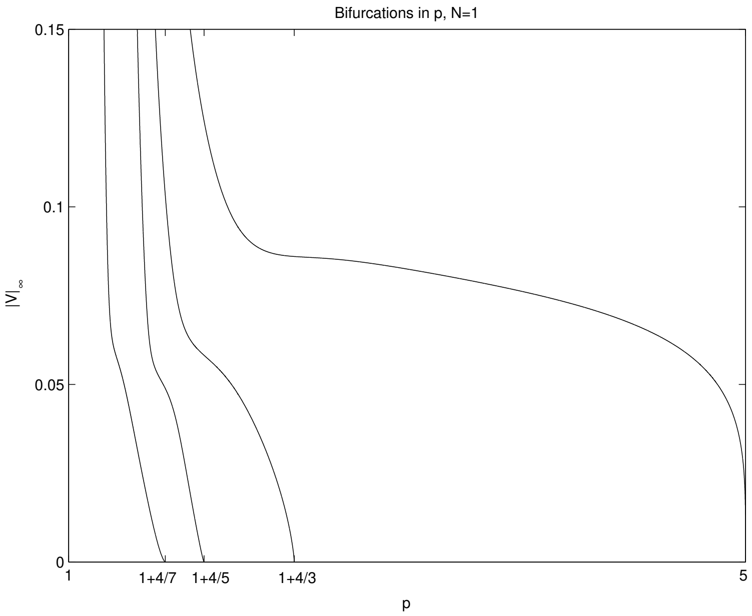

FIGURE 1. Four first branches of the λ-bifurcation diagram for problem (4.8), (4.9) with $m = 2$ and $N = 1$. The branches are originated at the bifurcation points (1.9), $\lambda = \lambda_l$ for $l = 0, 1, 2, 3$.

Its spectral properties have been studied in [4]. These eigenfunctions are associated with the linear polyharmonic equation

$$u_t = -(-\Delta)^m u - u$$

obtained from (1.7) by setting $\lambda = 1$.

For the quasilinear PDEs like (1.13), we also expect a finite number of VSSs to exist, though a rigorous analysis of "bifurcation" phenomena is associated with unknown "nonlinear eigenfunctions" of the corresponding nonlinear purely diffusion operator, and remain an open problem. On the other hand, some local results on existence of VSSs in a neighbourhood of bifurcation points (1.9) can be obtained by the branching analysis of solutions of (1.13) for $\lambda \approx 1^+$. It can be shown that the orthogonality condition for such an application of the Lyapunov–Schmidt method is valid. This guarantees the local existence of a countable subset of bifurcation branches. The question of their global continuation in λ is a challenging open problem for nonlinear non-potential degenerate operators involved.

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