High Weissenberg number boundary layer structures for UCM fluids

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Abstract

We describe three distinct boundary layer structures that can arise in the high Weissenberg number limit for the upper convected Maxwell (UCM) model. One is a single layer structure previously noted by Renardy [7] and the other two are double layer structures. These latter two structures extend the core flows that can be accommodated by the UCM model in the high Weissenberg regime. The three structures taken together, represent the main dominant balances that occur for the UCM equations near solid boundaries. For each structure, the leading order equations are derived in each region together with particular exact solutions when available. Importantly, the matching conditions between respective regions for each structure are also derived and explained.

Key words and phrases: UCM fluid; high Weissenberg number; boundary layers

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1. Introduction

The high Weissenberg limit for the upper convected Maxwell (UCM) model is a singular limit in which the elastic effects of the model equations are expected to be most pronounced. Such effects in the presence of solid walls have been shown by [7] to manifest themselves through stress boundary layers. As remarked in [7], such elastic layers have been observed for flows between eccentric cylinders and past spheres and can also arise in geometries which give stress singularities, such as the re-entrant corner.

The origin of such boundary layers has been explained in [7] by the incompatibility of the stress behaviour in differing regions. At solid walls, the stresses must be viscometric with the relaxation and rate-of-strain terms in the constitutive equations balancing terms of the upper convected stress derivative. However, at short distances away from the walls, the upper convective stress derivative now dominates with the relaxation and rate-of-strain terms subdominant. Since this latter behaviour is anticipated to occur in the main part of the flow domain, these two solutions for the stresses need to be reconciled through narrow regions or boundary layers. Here we extend the analysis of [7] and propose two additional wall structures that generalize the type of core flows that the UCM model equations can describe.

The dimensionless governing equations that we consider are the momentum and continuity equations

$$\mathrm{Re}\,\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\, p + \nabla\cdot\mathbf{T}, \qquad \nabla\cdot\mathbf{v} = 0, \tag{1.1}$$

together with the constitutive equations

$$\mathbf{T} + \mathrm{We}\,\overset{\triangledown}{\mathbf{T}} = 2\mathbf{D} = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T), \tag{1.2}$$

where $\mathbf{v}$ is the velocity, p is the pressure, $\mathbf{T}$ is the extra-stress tensor and $\mathbf{D}$ is the rate of strain tensor. The Reynolds number Re is assumed order one.

We are interested in describing the high Weissenberg limit $\mathrm{We}\to\infty$. Formally, away from the walls, we expect the upper convected stress derivative to dominate. As the wall is approached, it is the manner in which the relaxation and rate-of-strain terms are recovered in the constitutive equations that gives rise to three possible structures for the boundary layer. They may either be recovered simultaneously giving rise to a single layer structure. Alternatively, they may be recovered separately, in which case an additional region is then required to recover the remaining omitted terms. These latter possibilities, of which there are two, each give rise to double region structures. The three structures are summarised in Figure 1 and can be distinguished by the limiting behaviour of the stream function from the main or core flow, which will be termed the outer region. In section 2, the possible limiting behaviour of the outer solutions are derived as the wall is approached. In section 3 the scalings and leading order equations are derived for the inner regions at the wall. We complete the introduction by considering viscometric behaviour, which guides us in determining the terms we need to recover in the constitutive equations near the wall in any admissible boundary layer structure.

Considering planar flow and introducing the stream function ψ, the component form of (1.2) is

$$T_{11} + \mathrm{We}\left(\frac{\partial\psi}{\partial y}\frac{\partial T_{11}}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T_{11}}{\partial y} - 2\frac{\partial^2\psi}{\partial y^2}T_{12} - 2\frac{\partial^2\psi}{\partial x\partial y}T_{11}\right) = 2\frac{\partial^2\psi}{\partial x\partial y}, \tag{1.3}$$

$$T_{22} + \mathrm{We}\left(\frac{\partial\psi}{\partial y}\frac{\partial T_{22}}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T_{22}}{\partial y} + 2\frac{\partial^2\psi}{\partial x^2}T_{12} + 2\frac{\partial^2\psi}{\partial x\partial y}T_{22}\right) = -2\frac{\partial^2\psi}{\partial x\partial y}, \tag{1.4}$$

$$T_{12} + \mathrm{We}\left(\frac{\partial\psi}{\partial y}\frac{\partial T_{12}}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T_{12}}{\partial y} + \frac{\partial^2\psi}{\partial x^2}T_{11} - \frac{\partial^2\psi}{\partial y^2}T_{22}\right) = \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}. \tag{1.5}$$

We now consider steady shear flow with a Weissenberg dependent shear rate $\dot\gamma \mathrm{We}^{-\frac{(m-2)}{2}}$. Here $\dot\gamma = O(1)$ and the exponent m is at our disposal, with the manner of its occurence being explained below. The stream function then takes the form

$$\psi = \frac{1}{2}\dot\gamma y^2 \mathrm{We}^{-\frac{(m-2)}{2}} \tag{1.6}$$

and (1.3)–(1.5) give

$$T_{11} + \mathrm{We}\left(-2\dot\gamma \mathrm{We}^{-\frac{(m-2)}{2}} T_{12}\right) = 0, \tag{1.7}$$

$$T_{22} = 0, \tag{1.8}$$

$$T_{12} + \mathrm{We}\left(-\dot\gamma \mathrm{We}^{-\frac{(m-2)}{2}} T_{22}\right) = \dot\gamma \mathrm{We}^{-\frac{(m-2)}{2}}. \tag{1.9}$$

Immediately from this we see that viscometric behaviour for the UCM model is given by balances between certain components of the following group of terms in the constitutive equations

$$\left\{\mathbf{T}, \overset{\triangledown}{\mathbf{T}}, 2\mathbf{D}\right\}. \tag{1.10}$$

Consequently any boundary layer equations that are derived generally, need to reflect such a balance near to the wall. The equations (1.7)–(1.9) are readily solved to give

$$T_{22} = 0, \quad T_{12} = \dot\gamma \mathrm{We}^{-\frac{(m-2)}{2}}, \quad T_{11} = 2\dot\gamma^2 \mathrm{We}^{3-m}, \tag{1.11}$$

with the relationship $T_{11} = 2\mathrm{We}T_{12}^2$ holding. We now take the limit $\mathrm{We} \to \infty$ and discuss the two cases $m = 2$ and $m = 3$. The case $m = 2$ is that which gives an order 1 shear rate in the high Weissenberg limit with the inference from (1.11) of the scalings $T_{11} = O(\mathrm{We}), T_{12} = O(1)$. This is the case considered by Renardy in [7], the scalings for these two extra-stress components being those taken within the single boundary layer structure that was considered. The other case of relevance is $m = 3$ so that $T_{11} = O(1), T_{12} = O(\mathrm{We}^{-\frac{1}{2}})$. The shear rate is now Weissenberg dependent and the extra-stress scalings reflect those derived later in section 3.1 for the single layer structure. This is the case we focus upon in this paper, since we will concern ourselves with core flows that give $T_{11} = O(1)$ as the boundary is approached (see (2.3) in section 2). The two cases $m = 2$ and $m = 3$ are related and this is explained in section 3.1. The exponent m has been chosen so that $T_{12} = O(1)$ when $m = 2$ and $T_{11} = O(1)$ when $m = 3$ for the limit $\mathrm{We} \to \infty$. This was done so that the Weissenberg scalings for this simple shear flow correspond to those that are derived later for the single boundary layer structure. As such and with regard to this structure only, we could replace m with n, the main exponent in the paper which is introduced in the next section.

Some final remarks before proceeding:

1. In subsequent sections we perform the analysis generally, with no restriction to a specific geometry or singularity. However, when relevant, for both motivation and to provide an example, we will refer back to the specific case of the re-entrant corner. This is the situation that has been studied successfully in the literature and about which much is now understood after the work of Hinch [4], Renardy [6], Rallinson and Hinch [5] as well as Evans [2] and [3]. Although, in this body of work We $= 1$, the presence of the stress singularity at the re-entrant corner gives rise to stress boundary layers that also occur in the high Weissenberg regime as noted in [7]. Thus comparison is possible in the main, although we note the following subtle difference. For single boundary layer structures the critical exponent occuring in the stream function (this is the index n introduced in (2.2)) is $3-\alpha$ for the re-entrant corner of angle π/α (first derived in [4]), whilst in the high Weissenberg limit we will see that it is 3.

2. As regards notation, it will be convenient in places to use general indices i, j both of which can take the values 1 and 2. Thus the functions $c_{11}(x), c_{12}(x), c_{22}(x)$ can be referred to simply as the $c_{ij}(x)$ for example.

3. For matching purposes between distinct regions in each of the structures we are about consider, it is necessary to ensure that the stream function, all three extra-stresses and pressure simultaneously match. Anything less represents incomplete matching and will be invalid.

2. Core flows - the outer region

At leading order in the outer region, (1.2) gives

$$\overset{\triangledown}{\mathbf{T}} = 0 \qquad \text{as We} \to \infty. \tag{2.1}$$

The general solution to (2.1) with (1.1) (in the creeping flow case Re $= 0$) has been discussed in [8], where it is shown that these equations can be reduced to a form of the compressible Euler equations. For our purposes here, it will be sufficient to only consider the limiting behaviour of these solutions as a solid boundary is approached.

We assume a flat boundary (which can be relaxed as discussed in [7]) and adopt Cartesian axes (x, y) orientated with x along the boundary and y along its normal into the flow domain $y > 0$. As the boundary is approached, we make the assumption that the stream function has the leading order limiting behaviour

$$\psi \sim c_0(x) y^n \qquad \text{as } y \to 0, \tag{2.2}$$

for some arbitrary function $c_0(x)$ and power n. The goal of our analysis is to determine the boundary layer structures that can accommodate such behaviour from the core. Although not the most general behaviour, this separable form for the stream function would be anticipated to occur in many flows of relevance and particularly when the core either admits self-similar solutions or has self-similar limiting behaviour. For example, such behaviour has already been recorded for the re-entrant corner of angle π/α where $c_0(x) = C_0 x^{n(\alpha-1)}$ (C_0 a constant) for the core similarity solution of Hinch [4] when expressed in Cartesian coordinates (see also [2]). We also consider extra-stresses with separable behaviour of the form

$$T_{11} \sim c_{11}(x), \quad T_{12} \sim c_{12}(x) y, \quad T_{22} \sim c_{22}(x) y^2 \qquad \text{as } y \to 0, \tag{2.3}$$

where the indices $_1$ and $_2$ are associated with the x and y directions respectively. Explicit expressions for the $c_{ij}(x)$ in terms of $c_0(x)$ are derived in appendix A, the most relevant ones for the analysis below being (A.14)–(A.16). The component form of the momentum equations then give

$$\mathrm{Re}\, n\, c_0(x)c_0'(x)y^{2(n-1)}(1+o(1)) = -\frac{\partial p}{\partial x} + c_{11}'(x) + c_{12}(x) + o(1),$$

$$\mathrm{Re}\, n\, (c_0'^2 - c_0c_0'')y^{(2n-1)})(1+o(1)) = -\frac{\partial p}{\partial y} + O(y),$$

in the limit $y \to 0$. Here $'$ denotes d/dx. Thus for $n > 1$, the inertia terms are uniformly negligible and we have $p = p(x)$ where

$$p'(x) = c_{11}'(x) + c_{12}(x). \tag{2.4}$$

This will be the range for n that we will be concerned with here. It is worth noting that when $n = 1$ we obtain

$$p'(x) = c_{11}'(x) + c_{12}(x) - \mathrm{Re}c_0(x)c_0'(x)$$

in place of (2.4), which is the first point (as n decreases) for which the inertia terms become important. The limiting behaviour of the pressure is still independent of y, although this changes when $n < 1$ with the inertia terms now dominating and the behaviour (2.3) no longer appearing appropriate. The regime $n < 1$ is of less concern to us since the velocity field now has a singular behaviour as the boundary is approached (which is apparent from $\frac{\partial\psi}{\partial y} = O(y^{(n-1)})$).

The outer solution (2.2) and (2.3) does not give viscometric behaviour at the wall appropriate to the UCM equations and we now consider regions in which the terms neglected in (1.2) are recovered at leading order.

3. Boundary layers - the inner regions

The limiting behaviours (2.2) and (2.3), suggest considering inner variables given by the scalings

$$x = X, \quad y = \delta Y, \quad \psi = \delta^n \Psi, \quad p = \bar{p}, \quad T_{11} = \bar{T}_{11}, \quad T_{12} = \delta\bar{T}_{12}, \quad T_{22} = \delta^2\bar{T}_{22}. \tag{3.1}$$

The gauge $\delta = \delta(\mathrm{We})$ is assumed small and is to be determined by balancing terms in the constitutive equations. The component form of (1.1)–(1.2) gives for the inner region $X = O(1), Y = O(1)$ the equations

$$\delta^{2(n-1)}\mathrm{Re}\left(\frac{\partial\Psi}{\partial Y}\frac{\partial^2\Psi}{\partial X\partial Y} - \frac{\partial\Psi}{\partial X}\frac{\partial^2\Psi}{\partial Y^2}\right) = -\frac{\partial\bar{p}}{\partial X} + \frac{\partial\bar{T}_{11}}{\partial X} + \frac{\partial\bar{T}_{12}}{\partial Y}, \tag{3.2}$$

$$\delta^{2n}\mathrm{Re}\left(-\frac{\partial\Psi}{\partial Y}\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial\Psi}{\partial X}\frac{\partial^2\Psi}{\partial Y\partial X}\right) = -\frac{\partial\bar{p}}{\partial Y} + \delta^2\left(\frac{\partial\bar{T}_{12}}{\partial X} + \frac{\partial\bar{T}_{22}}{\partial Y}\right), \tag{3.3}$$

and for the constitutive relations

$$\bar{T}_{11} + \delta^{(n-1)}\mathrm{We}\left(\frac{\partial\Psi}{\partial Y}\frac{\partial\bar{T}_{11}}{\partial X} - \frac{\partial\Psi}{\partial X}\frac{\partial\bar{T}_{11}}{\partial Y} - 2\frac{\partial^2\Psi}{\partial Y^2}\bar{T}_{12} - 2\frac{\partial^2\Psi}{\partial X\partial Y}\bar{T}_{11}\right) = 2\delta^{(n-1)}\frac{\partial^2\Psi}{\partial X\partial Y}, \tag{3.4}$$

$$\bar{T}_{22} + \delta^{(n-1)}\mathrm{We}\left(\frac{\partial\Psi}{\partial Y}\frac{\partial\bar{T}_{22}}{\partial X} - \frac{\partial\Psi}{\partial X}\frac{\partial\bar{T}_{22}}{\partial Y} + 2\frac{\partial^2\Psi}{\partial X^2}\bar{T}_{12} + 2\frac{\partial^2\Psi}{\partial X\partial Y}\bar{T}_{22}\right) = -2\delta^{(n-3)}\frac{\partial^2\Psi}{\partial X\partial Y}, \tag{3.5}$$

$$\bar{T}_{12} + \delta^{(n-1)}\mathrm{We}\left(\frac{\partial\Psi}{\partial Y}\frac{\partial\bar{T}_{12}}{\partial X} - \frac{\partial\Psi}{\partial X}\frac{\partial\bar{T}_{12}}{\partial Y} + \frac{\partial^2\Psi}{\partial X^2}\bar{T}_{11} - \frac{\partial^2\Psi}{\partial Y^2}\bar{T}_{22}\right) = \delta^{(n-3)}\left(\frac{\partial^2\Psi}{\partial Y^2} - \delta^2\frac{\partial^2\Psi}{\partial X^2}\right). \tag{3.6}$$

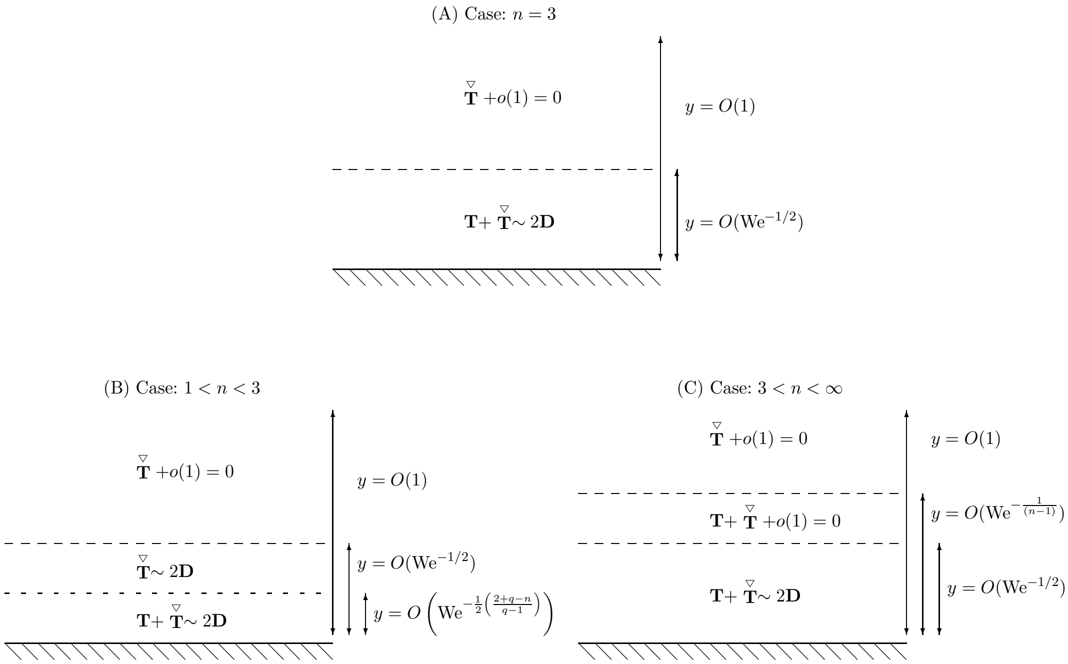

Figure 1: Illustration of the high Weissenberg number boundary layers. We assume that the core stream function has the solid boundary behaviour $\psi \sim c_0(x)y^n$ as $y \to 0$, which introduces a parameter n that may be used to distinguish the structures. (A) shows the case $n = 3$ which gives a single layer inner region. (B) and (C) illustrate the cases $1 < n < 3$ and $3 < n < \infty$ respectively, in which double layer structures are obtained at the boundary. In structures (B) and (C), viscometric behaviour is recovered near the boundary in a layer with the same balance in the constitutive equations as that holding for the single layer structure in (A). It is noted for the structure in (B) that in general the parameter q takes the value $3/2$, so that the lower region is given by $y = O\left(\mathrm{We}^{-\frac{1}{2}(7-2n)}\right)$, although we will also consider the situation when $q = n$ in which the lower region is now $y = O\left(\mathrm{We}^{-\frac{1}{(n-1)}}\right)$. Shown for each region are the balances between the terms of the constitutive equations, conveniently summarised in tensor notation. We emphasize that not all components of the terms are necessarily present and we refer to section 3 for the precise statement of the equations in each region.

Matching to the outer solution (2.2)–(2.3) requires

$$\text{as } Y \to \infty \qquad \Psi \sim c_0(X), \;\; \bar{T}_{11} \sim c_{11}(X), \;\; \bar{T}_{12} \sim c_{12}(X)Y, \;\; \bar{T}_{22} \sim c_{22}(X)Y^2, \tag{3.7}$$

with

$$\bar{p}'(X) = c_{11}'(X) + c_{12}(X), \tag{3.8}$$

where here $' = d/dX$. The solid boundary and no slip conditions at the wall require

$$\text{at} \quad Y = 0, \quad \Psi = \frac{\partial \Psi}{\partial Y} = 0. \tag{3.9}$$

3.1 The single layer structure for $n = 3$

The fullest balance in (3.4)–(3.6) occurs when $n = 3$ and $\delta = \mathrm{We}^{-1/2}$, to give the leading order boundary layer equations in $X = O(1), Y = O(1)$ as

$$0 = -\frac{d\bar{p}}{dX} + \frac{\partial \bar{T}_{11}}{\partial X} + \frac{\partial \bar{T}_{12}}{\partial Y}, \tag{3.10}$$

$$\bar{T}_{11} + \left(\frac{\partial \Psi}{\partial Y}\frac{\partial \bar{T}_{11}}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial \bar{T}_{11}}{\partial Y} - 2\frac{\partial^2 \Psi}{\partial Y^2}\bar{T}_{12} - 2\frac{\partial^2 \Psi}{\partial X \partial Y}\bar{T}_{11} \right) = 0, \tag{3.11}$$

$$\bar{T}_{22} + \left(\frac{\partial \Psi}{\partial Y}\frac{\partial \bar{T}_{22}}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial \bar{T}_{22}}{\partial Y} + 2\frac{\partial^2 \Psi}{\partial X^2}\bar{T}_{12} + 2\frac{\partial^2 \Psi}{\partial X \partial Y}\bar{T}_{22} \right) = -2\frac{\partial^2 \Psi}{\partial X \partial Y}, \tag{3.12}$$

$$\bar{T}_{12} + \left(\frac{\partial \Psi}{\partial Y}\frac{\partial \bar{T}_{12}}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial \bar{T}_{12}}{\partial Y} + \frac{\partial^2 \Psi}{\partial X^2}\bar{T}_{11} - \frac{\partial^2 \Psi}{\partial Y^2}\bar{T}_{22} \right) = \frac{\partial^2 \Psi}{\partial Y^2}. \tag{3.13}$$

These are the boundary layer equations first derived by Renardy [7]. However, we note here that this layer is of width $O(\mathrm{We}^{-1/2})$ with $T_{11} = O(1)$ and $T_{12} = O(\mathrm{We}^{-1/2})$ within the boundary layer. [1]

The equations (3.10)–(3.13) are subject to the matching conditions (3.7) with $n = 3$ and possess the viscometric behaviour

$$\text{as } Y \to 0 \qquad \begin{aligned} \Psi &= \frac{1}{2}a(X)Y^2 + \frac{1}{6}b(X)Y^3 + O(Y^4), \\ \bar{T}_{11} &= 2a(X)^2 + 2a(X)(2b(X) - 3a(X)a'(X))Y + O(Y^2), \\ \bar{T}_{12} &= a(X) + (b(X) - 3a(X)a'(X))Y + O(Y^2), \\ \bar{T}_{22} &= -2a'(X)Y + O(Y^2), \end{aligned} \tag{3.14}$$

with

$$\bar{p}'(X) = a(X)a'(X) + b(X). \tag{3.15}$$

The wall behaviour (3.14) contains two freely specifiable functions $a(X)$ and $b(X)$, which through (3.15) and (3.8) are linked to the outer stress behaviour. We anticipate that these wall and far-field behaviours (3.7) allow for the suitable specification of well-posed boundary value problems for the hyperbolic system (3.10)–(3.13). To comment further upon this, it is important to distinguish cases dependent upon the sign of $a(X)$:

(i) The case $a(X) < 0$. The wall behaviour (3.14) gives the first few terms of a power series expansion in Y with the coefficients of further terms expressible entirely in terms of $a(X)$ and $b(X)$. No further arbitrary functions or constants are introduced, the stream function and extra-stresses being analytic at $Y = 0$ in this case. Consequently, it possible to prescribe $a(X)$ and $b(X)$ and impose (3.14) as initial data for (3.10)–(3.13). Numerically, the stable direction of integration is away from the boundary i.e. Y increasing and it is the pressure gradient, through (3.15) and (3.8), that guides the solution to the far-field behaviour (3.7).

(ii) The case $a(X) > 0$. Now the stream function and extra-stresses are no longer analytic at $Y = 0$ but contain an essential singularity. A power series expansion in Y is no longer sufficient due to the presence of exponentially small terms. Whilst (3.14) is still valid to the order recorded,

[1] In conrast, [7] considered boundary layers with an order one shear stress, $T_{12} = O(1)$, and consequently of width $O(\mathrm{We}^{-1})$. These cases may also be considered here, if we rescale the stresses in (3.1) by $\bar{T}_{ij} = \tilde{T}_{ij}/\delta$ and consider the $\tilde{T}_{ij}$ variables. Full balance in (3.4)–(3.6) now arises when $n = 2$ with $\delta = 1/\mathrm{We}$. Matching to the outer region then requires the limiting behaviour of the outer stresses to be $T_{11} = O(y^{-1}), T_{12} = O(1), T_{22} = O(y)$ as $y \to 0$. The double layer structures that are discussed later then arise for the ranges $1/2 < n < 2$ and $n > 2$.

the full expansion takes the form

$$\begin{aligned}
\Psi &= \frac{1}{2}a(X)Y^2 + \frac{1}{6}b(X)Y^3 + \ldots + D(X)Y^{B+5}e^{-A(X)/Y}(1+O(Y)),\\
\bar{T}_{11} &= 2a(X)^2 + 2a(X)(2b(X) - 3a(X)a'(X))Y + \ldots + D_{11}(X)Y^B e^{-A(X)/Y}(1+O(Y)),\\
\bar{T}_{12} &= a(X) + (b(X) - 3a(X)a'(X))Y + \ldots + \frac{A'(X)D_{11}(X)}{A(X)}Y^{B+1}e^{-A(X)/Y}(1+O(Y)),\\
\bar{T}_{22} &= -2a'(X)Y + \ldots - \frac{A(X)^2D(X)}{a(X)}Y^{B+1}e^{-A(X)/Y}(1+O(Y)) \qquad \text{as } Y \to 0, \qquad (3.16)
\end{aligned}$$

where B is an arbitrary constant and

$$\begin{aligned}
A(X) &= a(X)^{-1/2}\int a(X)^{-1/2}dX,\\
\ln D(X) &= \frac{(B+3)}{2}\ln a(X) + \int\left(\frac{b(X)}{2a(X)^2} - \frac{4}{a(X)A(X)} + \frac{A(X)}{12a(X)^2}(2a(X)b'(X) - 3b(X)a'(X))\right)dX,\\
D_{11}(X) &= D(X)a(X)^{-\frac{1}{2}}e^{\int\frac{6}{a(X)A(X)}dX}\int\left(2a(X)^{\frac{1}{2}}A(X)^2e^{-\int\frac{6}{a(X)A(X)}dX}\right)dX. \qquad (3.17)
\end{aligned}$$

A derivation of the exponentially small terms in (3.16) is given in Appendix B. We note for such terms that although the coefficents of X in the expansions can be obtained in terms of $a(X)$ and $b(X)$, there are arbitrary constants present, namely B together with those associated with the integral expressions. Consequently such terms contain additional information essential to the correct specification for the equations (3.10)–(3.13). As such, it is no longer sufficient to prescribe initial data at $Y = 0$ as in the case $a(X) < 0$, but rather information from the far-field through the imposition of (3.7) will now always be required. Local to $Y = 0$, the presence of the exponentially small terms implies that the stable direction of integration will be towards the boundary i.e. for Y decreasing. We remark that there may be more than one set of such exponentially small terms, in which case the full expansion should include such sets. A specific example is given in Appendix B for the boundary layer similarity solution of the We $= 1$ re-entrant corner problem.

(iii) The case $a(X) = 0$. This is the borderline case and now we drop down the order in the analytic part of the wall expansion with the previous comments made for $a(X)$ now pertaining to $b(X)$. The stream function and extra-stresses will be analytic at the boundary in the case $b(X) < 0$ and non-analytic when $b(X) > 0$ due to the presence of exponentially small terms. In this case, the relationship (3.15) may impose a sign restriction on $b(X)$ if the pressure gradient can only take one sign. The case $b(X) = 0$ (remembering that $a(X) = 0$) is degenerate and gives the trivial zero solution.

The presence of exponentially small terms and non-analyticity (through the presence of an essential singularity) of the stream function and extra-stresses at a solid boundary was first discussed by Rallinson and Hinch [5] in the context of the downstream boundary layer at the re-entrant corner. There, a clear discussion is given on the significance of the sign of the leading order stream function for the solution behaviour at the wall. Many of the above remarks, simply reiterate their results.

3.2 A double layer structure for $1 < n < 3$

A second distinguished limit occurs in (3.4)–(3.6) for $1 < n < 3$ and $\delta = \mathrm{We}^{-1/2}$. The leading order boundary layer equations in $X = O(1), Y = O(1)$ are now

$$0 = -\frac{d\bar{p}}{dX} + \frac{\partial \bar{T}_{11}}{\partial X} + \frac{\partial \bar{T}_{12}}{\partial Y}, \tag{3.18}$$

$$\left(\frac{\partial \Psi}{\partial Y}\frac{\partial \bar{T}_{11}}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial \bar{T}_{11}}{\partial Y} - 2\frac{\partial^2 \Psi}{\partial Y^2}\bar{T}_{12} - 2\frac{\partial^2 \Psi}{\partial X \partial Y}\bar{T}_{11}\right) = 0, \tag{3.19}$$

$$\left(\frac{\partial \Psi}{\partial Y}\frac{\partial \bar{T}_{22}}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial \bar{T}_{22}}{\partial Y} + 2\frac{\partial^2 \Psi}{\partial X^2}\bar{T}_{12} + 2\frac{\partial^2 \Psi}{\partial X \partial Y}\bar{T}_{22}\right) = -2\frac{\partial^2 \Psi}{\partial X \partial Y}, \tag{3.20}$$

$$\left(\frac{\partial \Psi}{\partial Y}\frac{\partial \bar{T}_{12}}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial \bar{T}_{12}}{\partial Y} + \frac{\partial^2 \Psi}{\partial X^2}\bar{T}_{11} - \frac{\partial^2 \Psi}{\partial Y^2}\bar{T}_{22}\right) = \frac{\partial^2 \Psi}{\partial Y^2}. \tag{3.21}$$

The omission of the relaxation terms in this system, suggests that we require another region in which they are recovered. This inner inner region will be considered later, but in order for the solution here to match to it, we are interested in limiting stress behaviour of the form

$$\text{as } Y \to 0 \qquad \bar{T}_{11} = O(Y^{-2}), \quad \bar{T}_{12} = O(Y^{-1}), \quad \bar{T}_{22} = O(1). \tag{3.22}$$

As such we note that (3.18)–(3.21) admits the limiting behaviour

$$\text{as } Y \to 0 \qquad \Psi \sim \hat{c}_0(X) Y^q, \quad \bar{T}_{11} \sim \frac{\hat{c}_{11}(X)}{Y^2}, \quad \bar{T}_{12} \sim \frac{\hat{c}_{12}(X)}{Y}, \quad \bar{T}_{22} \sim -1 + \hat{c}_{22}(X), \tag{3.23}$$

where admissible $\hat{c}_0(X)$ and appropriate expressions for the $\hat{c}_{ij}(X)$ are given in appendix B for $q \geq 1$. However, it is noted that a general expression for the extra-stresses, which can be expressed in a form relevant to the natural stress basis, is only obtainable for $q = 3/2$. We anticipate that in general this is the necessary value of q, which is independent of n and for which we have

$$\hat{c}_0(X) = \hat{C}_0, \quad \hat{c}_{11}(X) = \hat{C}_1 + \hat{C}_2 X + \hat{C}_3 X^2, \quad \hat{c}_{12}(X) = \hat{C}_2 + 2\hat{C}_3 X, \quad \hat{c}_{22}(X) = 4\hat{C}_3, \tag{3.24}$$

for arbitrary constants $\hat{C}_0, \hat{C}_1, \hat{C}_2, \hat{C}_3$. Thus at leading order, the limiting form of the stream function is independent of X with the implication of the flow being parallel to the boundary. [2]

We now consider another region in which the relaxation terms are retained at leading order, in order to recover viscometric behaviour. This inner inner region is given by the scalings

$$Y = \theta_1 \hat{Y}, \quad \Psi = \theta_1^q \hat{\Psi}, \quad \bar{T}_{11} = \frac{\hat{T}_{11}}{\theta_1^2}, \quad \bar{T}_{12} = \frac{\hat{T}_{12}}{\theta_1}, \quad \bar{T}_{22} = \hat{T}_{22}, \tag{3.25}$$

[2] It is worth mentioning, that (3.18)–(3.21) also possess the exact solution

$$\Psi = c_0(X) Y^n, \quad \bar{T}_{11} = c_{11}(X) + \frac{\hat{c}_{11}(X)}{Y^2}, \quad \bar{T}_{12} = c_{12}(X) Y + \frac{\hat{c}_{12}(X)}{Y},$$

$$\bar{T}_{22} = -1 + c_{22}(X) Y^2 + \hat{c}_{22}(X),$$

where $\hat{c}_0(X) = c_0(X)$ and the $\hat{c}_{ij}(X)$ satisfy (C.5)–(C.7) in appendix B for $q = n \neq 3/2$ and (C.9) if $q = n = 3/2$. This exact solution satisfies the matching conditions (3.7), although we have the requirement that $\hat{c}_0(X) = c_0(X)$ must satisfy (C.8) for $q = n \neq 3/2$. As such, we do not expect this exact solution to be generally applicable, although it may find relevance in certain situations. It also noteworthy that when $n = 2$ and $\hat{c}_{ij} = 0$, this solution gives viscometric behaviour appropriate to the elastic form of the UCM equations, namely $\overset{\triangledown}{\mathbf{T}} = 2\mathbf{D}$. However, this truncated form of the exact solution does not appear to be relevant to the full constitutive equations (1.2).

where $\theta_1 = \mathrm{We}^{\frac{(n-3)}{2(q-1)}}$. Since $1 < n < 3$ and $q > 1$ (whether $q = 3/2$ or $q = n$), then θ_1 is small. In the specific case $q = 3/2$ then $\theta_1 = \mathrm{We}^{-(3-n)}$. The leading order equations in $X = O(1), \hat{Y} = O(1)$ are now

$$0 = \frac{\partial \hat{T}_{11}}{\partial X} + \frac{\partial \hat{T}_{12}}{\partial \hat{Y}}, \tag{3.26}$$

$$\hat{T}_{11} + \left(\frac{\partial \hat{\Psi}}{\partial \hat{Y}} \frac{\partial \hat{T}_{11}}{\partial X} - \frac{\partial \hat{\Psi}}{\partial X} \frac{\partial \hat{T}_{11}}{\partial \hat{Y}} - 2\frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2} \hat{T}_{12} - 2\frac{\partial^2 \hat{\Psi}}{\partial X \partial \hat{Y}} \hat{T}_{11} \right) = 0, \tag{3.27}$$

$$\hat{T}_{22} + \left(\frac{\partial \hat{\Psi}}{\partial \hat{Y}} \frac{\partial \hat{T}_{22}}{\partial X} - \frac{\partial \hat{\Psi}}{\partial X} \frac{\partial \hat{T}_{22}}{\partial \hat{Y}} + 2\frac{\partial^2 \hat{\Psi}}{\partial X^2} \hat{T}_{12} + 2\frac{\partial^2 \hat{\Psi}}{\partial X \partial \hat{Y}} \hat{T}_{22} \right) = -2\frac{\partial^2 \hat{\Psi}}{\partial X \partial \hat{Y}}, \tag{3.28}$$

$$\hat{T}_{12} + \left(\frac{\partial \hat{\Psi}}{\partial \hat{Y}} \frac{\partial \hat{T}_{12}}{\partial X} - \frac{\partial \hat{\Psi}}{\partial X} \frac{\partial \hat{T}_{12}}{\partial \hat{Y}} + \frac{\partial^2 \hat{\Psi}}{\partial X^2} \hat{T}_{11} - \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2} \hat{T}_{22} \right) = \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2}. \tag{3.29}$$

At the wall, we have the viscometric behaviour

$$\begin{aligned}
\hat{\Psi} &= \frac{1}{2}a(X)\hat{Y}^2 - \frac{1}{6}a(X)a'(X)\hat{Y}^3 + O(\hat{Y}^4), \\
\hat{T}_{11} &= 2a(X)^2 - 10a(X)^2a'(X)\hat{Y} + O(\hat{Y}^2), \\
\hat{T}_{12} &= a(X) - 4a(X)a'(X)\hat{Y} + O(\hat{Y}^2), \\
\hat{T}_{22} &= -2a'(X)\hat{Y} + O(\hat{Y}^2), \qquad \text{as } \hat{Y} \to 0.
\end{aligned} \tag{3.30}$$

Similar remarks to those made following (3.14) apply here. The cases $a(X) < 0$ and $a(X) > 0$ need to be distinguished with analytic behaviour being obtained in the former and non-analytic behaviour in the latter. We note that (3.30) follows from (3.14) on setting $b(X) = -a(X)a'(X)$. Matching to the limiting behaviour (3.23) of the inner solution gives the conditions

$$\text{as } \hat{Y} \to \infty \quad \hat{\Psi} \sim \hat{c}_0(X)\hat{Y}^q, \;\; \hat{T}_{11} \sim \frac{\hat{c}_{11}(X)}{\hat{Y}^2}, \;\; \hat{T}_{12} \sim \frac{\hat{c}_{12}(X)}{\hat{Y}}, \;\; \hat{T}_{22} \sim -1 + \hat{c}_{22}(X), \tag{3.31}$$

where $q = 3/2$ and $\hat{c}_0(X), \hat{c}_{ij}(X)$ are given in (3.24) (or as described in footnote 2 if $q = n$). The pressure gradient no longer enters the momentum equation at leading order and we have to go higher order terms within this inner inner region in order to recover it.

The possibility of a double layer structure of this form was noted in [2] for re-entrant corner flows.

3.3 A double layer structure for $3 < n < \infty$

In this case we have $\delta = \mathrm{We}^{-\frac{1}{(n-1)}}$ which balances the relaxation terms and the upper convected stress derivative in (3.4)–(3.6), with the rate-of-strain terms subdominant. The leading order boundary layer equations in $X = O(1), Y = O(1)$ are now

$$0 = -\frac{d\bar{p}}{dX} + \frac{\partial \bar{T}_{11}}{\partial X} + \frac{\partial \bar{T}_{12}}{\partial Y}, \tag{3.32}$$

$$\bar{T}_{11} + \left(\frac{\partial \Psi}{\partial Y} \frac{\partial \bar{T}_{11}}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \bar{T}_{11}}{\partial Y} - 2\frac{\partial^2 \Psi}{\partial Y^2} \bar{T}_{12} - 2\frac{\partial^2 \Psi}{\partial X \partial Y} \bar{T}_{11} \right) = 0, \tag{3.33}$$

$$\bar{T}_{22} + \left(\frac{\partial \Psi}{\partial Y} \frac{\partial \bar{T}_{22}}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \bar{T}_{22}}{\partial Y} + 2\frac{\partial^2 \Psi}{\partial X^2} \bar{T}_{12} + 2\frac{\partial^2 \Psi}{\partial X \partial Y} \bar{T}_{22} \right) = 0, \tag{3.34}$$

$$\bar{T}_{12} + \left(\frac{\partial \Psi}{\partial Y} \frac{\partial \bar{T}_{12}}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \bar{T}_{12}}{\partial Y} + \frac{\partial^2 \Psi}{\partial X^2} \bar{T}_{11} - \frac{\partial^2 \Psi}{\partial Y^2} \bar{T}_{22} \right) = 0. \tag{3.35}$$

For matching with the inner inner region described below, we are interested in solutions to (3.32)–(3.35) with limiting behaviour

$$\text{as } Y \to 0 \qquad \Psi \sim \bar{c}_0(X)Y, \ \ \bar{T}_{11} \sim \bar{c}_{11}(X), \ \ \bar{T}_{12} \sim \bar{c}_{12}(X)Y, \ \ \bar{T}_{22} \sim \bar{c}_{22}(X)Y^2. \tag{3.36}$$

In appendix C, it is shown that the $\bar{c}_{ij}(X)$ may be expressed as follows

$$\bar{c}_{11}(X) = \bar{C}_1\bar{c}_0(X)^2 e^{-\bar{I}(X)}, \tag{3.37}$$

$$\bar{c}_{12}(X) = \left(-\bar{C}_1\bar{c}_0'(X) + \bar{C}_2\right)\bar{c}_0(X)e^{-\bar{I}(X)}, \tag{3.38}$$

$$\bar{c}_{22}(X) = \left(\bar{C}_1\bar{c}_0'(X)^2 - 2\bar{C}_2\bar{c}_0'(X) + \bar{C}_3\right)e^{-\bar{I}(x)}, \tag{3.39}$$

in terms of $\bar{c}_0(X)$ where

$$\bar{I}(X) = \int \bar{c}_0(X)^{-1} dX \tag{3.40}$$

and $\bar{C}_1, \bar{C}_2, \bar{C}_3$ are arbitrary constants. The momentum equation (3.32) requires that

$$0 = -\frac{d\bar{p}}{dX} + \bar{c}_0(X)e^{-\bar{I}(x)}\left(\bar{C}_1(\bar{c}_0'(X) - 1) + \bar{C}_2\right).$$

However, the pressure gradient is also given by (3.8) and thus we must have

$$\bar{c}_0(X)e^{-\bar{I}(x)}\left(\bar{C}_1(\bar{c}_0'(X) - 1) + \bar{C}_2\right) = \frac{c_0(X)^{\frac{2}{n}-1}}{n}c_0(X)'(c_1 + c_2 I(X) + c_3 I(X)^2) + \frac{(2n-1)}{2(n-1)}c_0(X)^{\frac{1}{n}}(c_2 + 2c_3 I(X)), \tag{3.41}$$

where

$$I(X) = \int c_0(X)^{-\frac{1}{n}} dX. \tag{3.42}$$

This expression may be thought of as a constraint on the admissible $\bar{c}_0(X)$ and constants $\bar{C}_1, \bar{C}_2$ for specified $c_0(X)$ and constants c_1, c_2, c_3 from the outer solution. It may be simplified by introducing

$$J(X) = e^{\bar{I}(X)},$$

so that $\bar{c}_0(X) = J/J'$ and then becomes

$$-\bar{C}_1\frac{J''J}{J'^3} + \frac{\bar{C}_2}{J'} = -\frac{I''}{I'^3}(c_1 + c_2 I(X) + c_3 I(X)^2) + \frac{(2n-1)}{2(n-1)I'}(c_2 + 2c_3 I(X)).$$

We note the following explicit solutions:

(i) In the case $c_2 = c_3 = 0$ we have

$$J = I^2, \bar{C}_1 = 4c_1, \bar{C}_2 = 2c_1, \bar{c}_0(X) = \frac{J}{J'} = \frac{1}{2}\frac{I}{I'} = \frac{1}{2}c_0(X)^{\frac{1}{n}}\int c_0(X)^{-\frac{1}{n}} dX.$$

(ii) In the case $c_1 = c_3 = 0$ we have

$$J = I, \bar{C}_1 = -c_2, \bar{C}_2 = \frac{(2n-1)}{2(n-1)}c_2, \bar{c}_0(X) = \frac{I}{I'} = c_0(X)^{\frac{1}{n}}\int c_0(X)^{-\frac{1}{n}} dX.$$

Further, we note the following important exact solution to (3.32)–(3.35)

$$\Psi = c_0(X)Y^n + \bar{c}_0(X)Y, \ \ \bar{T}_{11} = c_{11}(X), \ \ \bar{T}_{12} = c_{12}(X)Y, \ \ \bar{T}_{22} = c_{22}(X)Y^2, \tag{3.43}$$

where $\bar{p}'$ satisfies (3.8) and now

$$\bar{c}_0(X) = \frac{1}{2} c_0(X)^{\frac{1}{n}} \int c_0(X)^{-\frac{1}{n}} \, dX. \tag{3.44}$$

The constant contained within the indefinite integral can be determined either by considering higher order terms in the outer region or by specifying the behaviour of the stream function within this inner region for a suitable limit of X. This solution satisfies the matching condition (3.7) and the limiting behaviour (3.36) with $\bar{c}_{ij}(X) = c_{ij}(X)$ and $\bar{c}_0(X)$ as given in (3.44). The extra-stresses thus maintain their far-field behaviour throughout this inner region. The interesting observation for this case is that the stream function in (3.43) suggests that the tangential velocity component $\partial\Psi/\partial Y = nc_0(X)Y^{n-1} + \bar{c}_0(X)$ will change sign if $c_0(X)\bar{c}_0(X) < 0$ (for given X), with the consequence of reverse flow occuring between the top and bottom of this region. Such solutions have been used in [3] to describe the local behaviour at the re-entrant in the presence of a lip vortex confined to the near upstream wall region.

In order to recover viscometric behaviour at the wall, we consider another region in which the rate-of-strain terms are retained at leading order. This inner inner region is given by the scalings

$$Y = \theta_2 \hat{Y}, \quad \Psi = \theta_2 \hat{\Psi}, \quad \bar{T}_{11} = \hat{T}_{11}, \quad \bar{T}_{12} = \theta_2 \hat{T}_{12}, \quad \bar{T}_{22} = \theta_2^2 \hat{T}_{22}, \tag{3.45}$$

where $\theta_2 = \delta^{\frac{(n-3)}{2}} = \mathrm{We}^{-\frac{(n-3)}{2(n-1)}}$. It is noted that $\theta_2 \ll 1$ since $n > 3$ and the thickness of this inner inner region is $O(\delta\theta_2) = O(\mathrm{We}^{-\frac{1}{2}})$, the same as the thickness of the single layer $n = 3$ structure.

The leading order equations in $X = O(1), \hat{Y} = O(1)$ are now

$$0 = -\frac{d\bar{p}}{dX} + \frac{\partial \hat{T}_{11}}{\partial X} + \frac{\partial \hat{T}_{12}}{\partial \hat{Y}}, \tag{3.46}$$

$$\hat{T}_{11} + \left(\frac{\partial \hat{\Psi}}{\partial \hat{Y}} \frac{\partial \hat{T}_{11}}{\partial X} - \frac{\partial \hat{\Psi}}{\partial X} \frac{\partial \hat{T}_{11}}{\partial \hat{Y}} - 2\frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2} \hat{T}_{12} - 2\frac{\partial^2 \hat{\Psi}}{\partial X \partial \hat{Y}} \hat{T}_{11} \right) = 0, \tag{3.47}$$

$$\hat{T}_{22} + \left(\frac{\partial \hat{\Psi}}{\partial \hat{Y}} \frac{\partial \hat{T}_{22}}{\partial X} - \frac{\partial \hat{\Psi}}{\partial X} \frac{\partial \hat{T}_{22}}{\partial \hat{Y}} + 2\frac{\partial^2 \hat{\Psi}}{\partial X^2} \hat{T}_{12} + 2\frac{\partial^2 \hat{\Psi}}{\partial X \partial \hat{Y}} \hat{T}_{22} \right) = -2\frac{\partial^2 \hat{\Psi}}{\partial X \partial \hat{Y}}, \tag{3.48}$$

$$\hat{T}_{12} + \left(\frac{\partial \hat{\Psi}}{\partial \hat{Y}} \frac{\partial \hat{T}_{12}}{\partial X} - \frac{\partial \hat{\Psi}}{\partial X} \frac{\partial \hat{T}_{12}}{\partial \hat{Y}} + \frac{\partial^2 \hat{\Psi}}{\partial X^2} \hat{T}_{11} - \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2} \hat{T}_{22} \right) = \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2}, \tag{3.49}$$

with $\bar{p}$ as given in (3.8). At the solid boundary, (3.14) gives the viscometric behaviour (adjusted for current hat variables as appropriate) with the same remarks made for the presence (or not) of exponentialy small terms applying here. Matching to the inner solution (3.36) gives the conditions

$$\text{as } \hat{Y} \to \infty \quad \hat{\Psi} \sim \bar{c}_0(X)\hat{Y}, \quad \hat{T}_{11} \sim \bar{c}_{11}(X), \quad \hat{T}_{12} \sim \bar{c}_{12}(X)\hat{Y}, \quad \hat{T}_{22} \sim \bar{c}_{22}(X)\hat{Y}^2, \tag{3.50}$$

which completes the problem specification for this region. Since this far-field behaviour generally needs to be imposed on the equations (3.46)–(3.49), it is likely that we are restricted to the non-analytic case $a(X) > 0$ for the retrieval of viscometric behaviour (3.14) at the boundary.

4. Discussion

The purpose of this paper was to systematically describe the dominant balances that can occur near solid boundaries for the UCM constitutive equations in the high Weissenberg limit. Three structures have been identified, one a single layer region and the other two, double layer regions. The two double layer regions each have the balance of the single layer holding as an inner inner region at the solid boundary. This is not unexpected since it is the balance of the single layer that recovers viscometric behaviour and remarked upon in the introduction. These are anticipated to be the most likely boundary layer structures that the UCM model forms at solid boundaries. In both of the double layer structures, the inner region (i.e. the upper part of the boundary layer) may be thought of as suitable extensions of the outer region, accommodating more general outer solutions and hence core flows. However, it is noted that in the inner inner region for $1 < n < 3$, the pressure gradient is not present at leading order which distinguishes this structure from the other. In fact, it may be this feature that is contributory in the inner inner region in this case being able to accommodate a greater variety of limiting behaviours from the inner region. Consequently the two double layer structures should be distinguished and the structure occuring for $3 < n < \infty$ referred to as a type I double layer structure and that for $1 < n < 3$ as a type II structure.

These boundary layer structures have been distinguished here by the limiting behaviour of the outer (core) stream function (as mentioned in the caption of Figure 1). This, together with the influence of any local singularities would be expected to aid in determining flow patterns within the structures. Moreover, we have proceeded relatively generally and have not assumed explicit forms for the function $c_0(x)$ arising within the limiting stream function behaviour (2.2). In subsequent work, specific forms for $c_0(x)$ will be chosen and a systematic classification given for associated similarity solutions of the boundary layer equations.

A final question for these structures regards the determination of the exponent n in the outer (core) stream function behaviour (2.2), which is important as its value influences the sizes of the regions within the boundary layer structures. In the single layer structure, $n = 3$ is the unique value forced by the balance of terms in the constitutive equations. Thus n is determined for this structure by local considerations. In the two double layer structures n can now take a range of values. The balance of terms in the equations no longer fixing its value. It is thus required to be specified by the outer core flow and hence requires global information. This situation is analogous to the first and second kind similarity solutions (see, for example, [1]), where the exponents for the similarity variables are determined by local considerations for first kind and global considerations (usually through the construction of an appropriate eigenvalue problem) for second kind. We note that the structures are continuous in the limits $n \to 3^{\pm}$, with the two double layer structures collapsing to the single layer structure.

The distinction between the structures regarding the determination of the exponent n is critical. One possible reason the outer core solution may carry sufficient information to fix n is due to the dominance of the Oldroyd (i.e. the upper convected) stress derivative in this region. This is the fast time scale response for the fluid in which its memory is strongest, where it advects and deforms affinely. It thus may not have the capacity to respond to a solid boundary in encounters with sufficient freedom to adjust its velocity behaviour, as a fluid with weaker memory may be able to do. In other words, the fluid in the core may be able to carry n with it (i.e. is able to speficy its own velocity behaviour) and the consequent structure picked out at the boundary to accommodate it. This should be compared to a Newtonian fluid, where the fluid is able to respond instantaneously to the boundary it encounters by adjusting its velocity profile appropriately.

It remains finally to comment on the importance and possible occurence of these structures. Arguably, the single layer structure will play the main and central role for high Weissenberg structures. Physically this is plausible on the grounds of it being the simplest structure (mathematically). The role of the two double layer structures is less clear, particularly the structure for $1 < n < 3$. However, a key observation for the double layer structure for $3 < n < \infty$ is that its thickness is $O(\mathrm{We}^{-\frac{1}{(n-1)}})$. This width retrieves the single layer structure width $O(\mathrm{We}^{-\frac{1}{2}})$ as $n \to 3^+$ and increases, reaching an $O(1)$ width as $n \to \infty$ (where this structure breaks down). Let us now consider the situation of the re-entrant corner with a lip vortex at the upstream wall. The lip vortex is represented locally at the corner by the presence of reverse flow at the upstream wall. Such solutions are possible for (at least) the single layer structure and the double layer structure $3 < n < \infty$ (see [3]), with the separating streamline occuring at the edge of the boundary layers. As a consequence these two structures can describe a continuous spectrum of situations distinguished by the thickness of the boundary layer in which the lip vortex is confined to a narrow region at the upstream wall. As n increases, the region of reverse flow widens until the situation of a fully developed lip vortex appears. As such the double layer structure acts as bridging structure between the lip vortex confined to a narrow region at the upstream wall associated with the single layer structure and that of the fully developed lip vortex. Local asymptotic solutions for geometries involving singularities other than the re-entrant corner have yet to be addressed fully in the literature, and it will interesting to see what relevance the structures presented here will have to them.

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Appendix A. Limiting core behaviour

In component form, (2.1) is

$$\frac{\partial\psi}{\partial y}\frac{\partial T_{11}}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T_{11}}{\partial y}-2\frac{\partial^2\psi}{\partial y^2}T_{12}-2\frac{\partial^2\psi}{\partial x\partial y}T_{11}=0, \tag{A.1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T_{22}}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T_{22}}{\partial y}+2\frac{\partial^2\psi}{\partial x^2}T_{12}+2\frac{\partial^2\psi}{\partial x\partial y}T_{22}=0, \tag{A.2}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T_{12}}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T_{12}}{\partial y}+\frac{\partial^2\psi}{\partial x^2}T_{11}-\frac{\partial^2\psi}{\partial y^2}T_{22}=0, \tag{A.3}$$

For the behaviour (2.2), we consider solutions of the form

$$T_{11}=c_{11}(x)y^r,\quad T_{12}=c_{12}(x)y^{r+1},\quad T_{22}=c_{22}(x)y^{r+2}, \tag{A.4}$$

with r as yet unspecified. Then (A.1)–(A.3) give

$$nc_0(x)c_{11}'-(2n+r)c_0'(x)c_{11}-2n(n-1)c_0(x)c_{12}=0, \tag{A.5}$$

$$nc_0(x)c_{22}'+(2(n-1)-r)c_0'(x)c_{22}+2c_0''(x)c_{12}=0, \tag{A.6}$$

$$nc_0(x)c_{12}'-(1+r)c_0'(x)c_{12}+c_0''(x)c_{11}-n(n-1)c_0(x)c_{22}=0. \tag{A.7}$$

Here and in the rest of this appendix, $'$ denotes d/dx. For later reference, the components of the momentum equation are

$$\mathrm{Re}nc_0c_0'y^{2(n-1)}=-\frac{\partial p}{\partial x}+y^r\left(c_{11}'+(r+1)c_{12}\right), \tag{A.8}$$

$$\mathrm{Re}n(c_0'^2-c_0c_0'')y^{2n-1}=-\frac{\partial p}{\partial y}+y^{r+1}\left(c_{12}'+(r+2)c_{22}\right). \tag{A.9}$$

It is worth remarking that (A.4) can be related directly to the form of the extra-stress represented in its natural basis. This natural basis has been described in [8]. Explicitly, as the boundary is approached this formulation has the limiting form

$$T_{11}=\lambda u^2,\quad T_{12}=\lambda uv+\mu,\quad T_{22}=\lambda v^2+\frac{2\mu v}{u}+\frac{\nu}{u^2},$$

where $u=\dfrac{\partial\psi}{\partial y}, v=-\dfrac{\partial\psi}{\partial x}$ are the usual velocity components. Here then we may write

$$\lambda(x,y)=\lambda_0(x)y^{2-2n+r},\quad \mu(x,y)=\mu_0(x)y^{1+r},\quad \nu(x,y)=\nu_0(x)y^{2n+r}, \tag{A.10}$$

where

$$c_{11}(x)=n^2c_0(x)^2\lambda_0(x), \tag{A.11}$$

$$c_{12}(x)=-\frac{c_0'(x)}{nc_0(x)}c_{11}(x)+\mu_0(x), \tag{A.12}$$

$$c_{22}(x)=\left(\frac{c_0'(x)}{nc_0(x)}\right)^2c_{11}(x)-\frac{2c_0'(x)}{nc_0(x)}\mu_0(x)+\frac{\nu_0(x)}{n^2c_0(x)^2}. \tag{A.13}$$

It is noted that the constraint $\lambda\nu=\mu^2$ for $\mathbf{T}$ to be a rank one tensor in the outer region is satisfied if $c_{11}c_{22}=c_{12}^2$.

The equations (A.5)–(A.7) are a third-order linear system of ODEs for the unknowns c_{11},c_{12},c_{22}. We now record explicit solutions for these expressions in the two specific cases $r=0$ with $n\neq 1$ and $n=1$ with r left general. Unfortunately, for general $c_0(x)$, the case $r\neq 0$ only admits an explicit quadrature solution in the case $n=1$. However, for certain $c_0(x)$ the general r case admits explicit quadrature solutions, and we record these below for the case in which $c_0(x)$ is algebraic power of x.

A.1 The case $r = 0$ and $n \neq 1$

In this case we have the exact solution

$$c_{11}(x) = c_0(x)^{\frac{2}{n}} \left(c_1 + c_2 I(x) + c_3 I(x)^2\right), \tag{A.14}$$

$$c_{12}(x) = -\frac{c_0(x)^{\frac{2}{n}-1}}{n} c_0'(x) \left(c_1 + c_2 I(x) + c_3 I(x)^2\right) + \frac{c_0(x)^{\frac{1}{n}}}{2(n-1)}(c_2 + 2c_3 I(x)), \tag{A.15}$$

$$c_{22}(x) = \frac{c_0(x)^{\frac{2}{n}-2}}{n^2} c_0'(x)^2 \left(c_1 + c_2 I(x) + c_3 I(x)^2\right) - \frac{c_0(x)^{\frac{1}{n}-1}}{n(n-1)} c_0'(x)(c_2 + 2c_3 I(x)) + \frac{c_3}{(n-1)^2}, \tag{A.16}$$

where

$$I(x) = \int c_0(x)^{-\frac{1}{n}} dx. \tag{A.17}$$

The constant of integration in $I(x)$ can be absorbed into the arbitrary constants c_1, c_2, c_3. Relevant to the pressure gradient, we record the quantity

$$c_{11}'(X) + c_{12}(X) = \frac{c_0^{\frac{2}{n}-1}}{n} c_0'(c_1 + c_2 I(x) + c_3 I(x)^2) + \frac{(2n-1)}{2(n-1)} c_0^{\frac{1}{n}} (c_2 + 2c_3 I(x)).$$

In terms of the natural stress variables we have the identification

$$\lambda_0(x) = \frac{c_{11}(x)}{n^2 c_0(x)^2} = \frac{c_0(x)^{\frac{2}{n}-2}}{n^2} \left(c_1 + c_2 I(x) + c_3 I(x)^2\right), \tag{A.18}$$

$$\mu_0(x) = \frac{c_0(x)^{\frac{1}{n}}}{2(n-1)}(c_2 + 2c_3 I(x)), \tag{A.19}$$

$$\nu_0(x) = \frac{n^2 c_0(x)^2 c_3}{(n-1)^2}. \tag{A.20}$$

Using these in (A.10), the constraint $\lambda\nu = \mu^2$ is now satisfied if the constants satisfy $c_2^2 = 4c_1c_3$. We note that if $I(x) = o(1)$ then these expressions reduce to

$$\lambda \sim \frac{c_1}{n^2}\psi^{\frac{2}{n}-2}, \quad \mu \sim \frac{c_2}{2(n-1)}\psi^{\frac{1}{n}}, \quad \nu \sim \frac{c_3 n^2}{(n-1)^2}\psi^2, \qquad \text{as } \psi \to 0,$$

when expressed in terms of the stream function using (2.2), implying that the limiting forms of these variables from the outer region are constant along streamlines. This observation has already been made and used in [5] for re-entrant corner geometries. In addition, when $c_2 = c_3 = 0$ we have

$$T_{11} \sim \frac{c_1}{n^2}\psi^{\frac{2}{n}-2}u^2, \quad T_{12} \sim \frac{c_1}{n^2}\psi^{\frac{2}{n}-2}uv, \quad T_{22} \sim \frac{c_1}{n^2}\psi^{\frac{2}{n}-2}v^2 \quad \text{as } y \to 0.$$

This is the limiting behaviour of the well known stretching solution $\mathbf{T} = g(\psi)\,\mathbf{v}\mathbf{v}^T$ in the outer region, noted in [4] for the re-entrant corner.

A.2 The case $n = 1$ and general r

In this case we have the exact solution

$$c_{11}(x) = c_1 c_0(x)^{r+2}, \tag{A.21}$$

$$c_{12}(x) = \left(-c_1 c_0'(x) + c_2\right) c_0(x)^{r+1}, \tag{A.22}$$

$$c_{22}(x) = \left(c_1 c_0'(x)^2 - 2c_2 c_0'(x) + c_3\right) c_0(x)^r, \tag{A.23}$$

with c_1, c_2, c_3 arbitrary constants. In terms of the natural stress variables we have the identification

$$\lambda_0(x) = \frac{c_{11}(x)}{c_0(x)^2} = c_1 c_0(x)^r, \qquad \mu_0(x) = c_2 c_0(x)^{r+1}, \qquad \nu_0(x) = c_3 c_0(x)^{r+2}. \tag{A.24}$$

The constraint $\lambda\nu = \mu^2$ is now satisfied if $c_2^2 = c_1 c_3$. We note that the expressions for λ, μ, ν can be written in terms of the stream function only, namely

$$\lambda = c_1 \psi^r, \quad \mu = c_2 \psi^{r+1}, \quad \nu = c_3 \psi^{2+r},$$

again implying that the limiting forms of these variables from the outer region are constant along streamlines.

Relevant to the pressure gradient, we note the quantity

$$c_{11}'(x) + c_{12}(x) = \left(c_1 c_0'(x) + (r+1)c_2\right) c_0(x)^{r+1}.$$

A.3 The specific case $c_0(x) = C_0 x^{mn}$ for general r

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Due to its relevance to the re-entrant corner (and may be to other situations as well), we record here solutions for the specific case

$$c_0(x) = C_0 x^{mn}, \tag{A.25}$$

where C_0 and m are arbitrary constants. We now note that (A.5)–(A.7) have the general solution

$$c_{11}(x) = C_1 x^{m(2+r)} + C_2 x^{1+m(1+r)} + C_3 x^{2+mr}, \tag{A.26}$$

$$c_{12}(x) = -mC_1 x^{-1+m(2+r)} - \frac{(2mn-m-1)}{2(n-1)} C_2 x^{m(1+r)} - \frac{(mn-1)}{(n-1)} C_3 x^{1+mr}, \tag{A.27}$$

$$c_{22}(x) = m^2 C_1 x^{-2+m(2+r)} + \frac{m(mn-1)}{(n-1)} C_2 x^{-1+m(1+r)} + \frac{(mn-1)^2}{(n-1)^2} C_3 x^{mr}, \tag{A.28}$$

for arbitrary constants C_1, C_2, C_3 and $m \neq 1, n \neq 1$. We note that the case $n = 1$ is covered in the previous subsection A.2. Here we have

$$\lambda_0(x) = \frac{c_{11}(x)}{n^2 c_0(x)^2} = x^{mr} \frac{c_0(x)^{\frac{2}{n}-2}}{n^2 C_0^{\frac{2}{n}}} \left(C_1 + C_2 x^{1-m} + C_3 x^{2(1-m)}\right), \tag{A.29}$$

$$\mu_0(x) = x^{mr} \frac{c_0(x)^{\frac{1}{n}}}{2(n-1)} \frac{(1-m)}{C_0^{\frac{1}{n}}} \left(C_2 + 2C_3 x^{(1-m)}\right), \tag{A.30}$$

$$\nu_0(x) = x^{mr} \frac{n^2 c_0(x)^2 (1-m)^2 C_3}{(n-1)^2}, \tag{A.31}$$

and we note that if $m < 1$ and $x = o(1)$ then these expressions reduce to

$$\lambda \sim \frac{C_1}{n^2 C_0^{\frac{2+r}{n}}} \psi^{\frac{2-2n+r}{n}}, \quad \mu \sim \frac{(1-m)C_2}{2(n-1)C_0^{\frac{1+r}{n}}} \psi^{\frac{(1+r)}{n}}, \quad \nu \sim \frac{n^2(1-m)^2 C_3}{(n-1)^2 C_0^{\frac{r}{n}}} \psi^{2+\frac{r}{n}}, \qquad \text{as } \psi \to 0.$$

The quantity occuring in the x momentum equation is now

$$c_{11}'(x) + (r+1)c_{12}(x) = mC_1 x^{-1+m(2+r)} - \frac{(1+m-2n+r(m-1))}{2(n-1)} C_2 x^{m(1+r)} - \frac{(1+(m-2)n+r(m-1))}{(n-1)} C_3 x^{1+mr}. \tag{A.32}$$

In the particular case $r = 0$, (A.26)–(A.28) recover the solution (A.14)-(A.16) for $c_0(x)$ given by (A.25) and where the constants are related by

$$C_1 = c_1 C_0^{\frac{2}{n}}, \quad C_2 = \frac{c_2 C_0^{\frac{1}{n}}}{(1-m)}, \quad C_3 = \frac{c_3}{(1-m)^2}.$$

The case $m = 1$ when $n \neq 1$ needs separate discussion, in which we have

$$c_{11}(x) = x^{2+r}\left(C_1 + C_2 \ln x + C_3 (\ln x)^2\right), \tag{A.33}$$

$$c_{12}(x) = -x^{1+r}\left(\left(C_1 + C_2 \ln x + C_3(\ln x)^2\right) - \frac{1}{2(n-1)}\left(C_2 + 2C_3 \ln x\right)\right), \tag{A.34}$$

$$c_{22}(x) = x^{r}\left(\left(C_1 + C_2 \ln x + C_3(\ln x)^2\right) - \frac{1}{(n-1)}\left(C_2 + 2C_3 \ln x\right) + \frac{C_3}{(n-1)^2}\right), \tag{A.35}$$

in place of (A.26)–(A.28). The condition $c_{11}(x)c_{22}(x) = c_{12}(x)^2$ is satisfied if $C_2^2 = 4C_1C_3$. For the natural stress basis, we now have

$$\lambda_0(x) = \frac{x^{2-2n+r}}{n^2 C_0^2}\left(C_1 + C_2 \ln x + C_3(\ln x)^2\right), \tag{A.36}$$

$$\mu_0(x) = \frac{x^{1+r}}{2(n-1)}\left(C_2 + 2C_3 \ln x\right), \tag{A.37}$$

$$\nu_0(x) = \frac{x^{2n+r} n^2 C_0^2 C_3}{(n-1)^2}. \tag{A.38}$$

The quantity occuring in the x momentum equation is now

$$c_{11}'(x) + (r+1)c_{12}(x) = x^{1+r}\left(C_1 + \frac{(r-1+2n)C_2}{2(n-1)} + \left(C_2 + \frac{(2n-1+r)C_3}{(n-1)}\right)\ln x + C_3(\ln x)^2\right). \tag{A.39}$$

In the particular case $r = 0$, we again recover the solution (A.14)-(A.16) for $c_0(x)$ given by (A.25) and where the constants are related by

$$C_1 = c_1 C_0^{\frac{2}{n}}, \quad C_2 = c_2 C_0^{\frac{1}{n}}, \quad C_3 = c_3.$$

Appendix B. Exponentially small terms

Here we derive the exponentially small terms present in the wall behaviour (3.16) for the boundary layer equations (3.10)–(3.13). We write

$$\Psi \sim \Psi^0 + \Psi^1, \qquad \bar{T}_{ij} \sim T_{ij}^0 + T_{ij}^1 \qquad \text{as } Y \to 0, \tag{B.1}$$

where Ψ^0, T^0_{ij} represent the assumed known analytic power series expansion in Y (the first few terms of which are recorded in (3.14)) and Ψ^1, T^1_{ij} are the exponentially small terms to be derived that satisfy $|\Psi^1, T^1_{ij}| \ll Y^N$ for any integer N. Substituting into (3.10)–(3.13) and linearizing about Ψ^0, T^0_{ij} gives the system

$$0 = \frac{\partial T^1_{11}}{\partial X} + \frac{\partial T^1_{12}}{\partial Y}, \tag{B.2}$$

$$\begin{aligned} T^1_{11} &+ \left(\frac{\partial \Psi^0}{\partial Y}\frac{\partial T^1_{11}}{\partial X} - \frac{\partial \Psi^0}{\partial X}\frac{\partial T^1_{11}}{\partial Y} - 2\frac{\partial^2 \Psi^0}{\partial Y^2}T^1_{12} - 2\frac{\partial^2 \Psi^0}{\partial X \partial Y}T^1_{11}\right) \\ &+ \left(\frac{\partial \Psi^1}{\partial Y}\frac{\partial T^0_{11}}{\partial X} - \frac{\partial \Psi^1}{\partial X}\frac{\partial T^0_{11}}{\partial Y} - 2\frac{\partial^2 \Psi^1}{\partial Y^2}T^0_{12} - 2\frac{\partial^2 \Psi^1}{\partial X \partial Y}T^0_{11}\right) = 0, \end{aligned} \tag{B.3}$$

$$\begin{aligned} T^1_{22} &+ \left(\frac{\partial \Psi^0}{\partial Y}\frac{\partial T^1_{22}}{\partial X} - \frac{\partial \Psi^0}{\partial X}\frac{\partial T^1_{22}}{\partial Y} + 2\frac{\partial^2 \Psi^0}{\partial X^2}T^1_{12} + 2\frac{\partial^2 \Psi^0}{\partial X \partial Y}T^1_{22}\right) \\ &+ \left(\frac{\partial \Psi^1}{\partial Y}\frac{\partial T^0_{22}}{\partial X} - \frac{\partial \Psi^1}{\partial X}\frac{\partial T^0_{22}}{\partial Y} + 2\frac{\partial^2 \Psi^1}{\partial X^2}T^0_{12} + 2\frac{\partial^2 \Psi^1}{\partial X \partial Y}T^0_{22}\right) = -2\frac{\partial^2 \Psi^1}{\partial X \partial Y}, \end{aligned} \tag{B.4}$$

$$\begin{aligned} T^1_{12} &+ \left(\frac{\partial \Psi^0}{\partial Y}\frac{\partial T^1_{12}}{\partial X} - \frac{\partial \Psi^0}{\partial X}\frac{\partial T^1_{12}}{\partial Y} + \frac{\partial^2 \Psi^0}{\partial X^2}T^1_{11} - \frac{\partial^2 \Psi^0}{\partial Y^2}T^1_{22}\right) \\ &+ \left(\frac{\partial \Psi^1}{\partial Y}\frac{\partial T^0_{12}}{\partial X} - \frac{\partial \Psi^1}{\partial X}\frac{\partial T^0_{12}}{\partial Y} + \frac{\partial^2 \Psi^1}{\partial X^2}T^0_{11} - \frac{\partial^2 \Psi^1}{\partial Y^2}T^0_{22}\right) = \frac{\partial^2 \Psi^1}{\partial Y^2}. \end{aligned} \tag{B.5}$$

We now assume the WKBJ ansatz

$$\begin{aligned} \Psi^1 &= \exp\left(-\frac{A(X)}{Y} + (B+5)\ln Y + \ln D(X) + O(Y)\right), \\ T^1_{ij} &= \exp\left(-\frac{A(X)}{Y} + B_{ij}\ln Y + \ln D_{ij}(X) + O(Y)\right), \end{aligned}$$

where the constants B_{ij} and functions $A(X), D(X), D_{ij}(X)$ are to be determined. Substituting into (B.2)–(B.5) (taking at least the first four terms in each of Ψ^0, T^0_{ij}) and equating powers of Y we find

$$B_{11} = B, \quad B_{12} = B_{22} = 1 + B$$

for the fullest consistent balance between terms. The leading and first order terms then give

$$a(X)A'(X) + \frac{1}{2}A(X)a'(X) - 1 = 0, \tag{B.6}$$

$$D_{12}(X) = \frac{D_{11}(X)A'(X)}{A(X)}, \qquad D_{22} = -\frac{A(X)^2 D(X)}{a(X)},$$

together with

$$(\ln D(X))' = \frac{(3+B)}{2}\frac{a'(X)}{a(X)} + \frac{b(X)}{2a(X)^2} - \frac{4}{a(X)A(X)} + \frac{A(X)}{12a(X)^2}\left(2a(X)b'(X) - 3b(X)a'(X)\right), \tag{B.7}$$

$$\begin{aligned} D'_{11}(X) - D_{11}(X)\left(\frac{(2+B)}{2}\frac{a'(X)}{a(X)} + \frac{b(X)}{2a(X)^2} + \frac{2}{a(X)A(X)} + \frac{A(X)}{12a(X)^2}\left(2a(X)b'(X) - 3b(X)a'(X)\right)\right) \\ = 2A(X)^2 D(X). \end{aligned} \tag{B.8}$$

The equations (B.6), (B.7) and (B.8) may be integrated to give the quadratures in (3.17).

It is instructive to note that the linearization performed to derive (B.2)–(B.5) is the same as that required for an eigenmode analysis. An artificial parameter ϵ could be introduced as a multiplying factor for the Ψ^1, T^1_{ij} terms in (B.1) and then (B.2)–(B.5) arise by equating terms of $O(\epsilon)$. In this sense, ϵ is being used as a small gauge. As far as the eigenmode analysis is concerned, it is worth noting that WKBJ ansatz can still be used and in the case $a(X) < 0$ we obtain from (B.6) the expression

$$A(X) = -(-a(X))^{-1/2} \int (-a(X))^{-1/2} dX.$$

Consequently the terms Ψ^1, T^1_{ij} are now exponentially large and cannot be included in the asymptotic expansion (B.1). This (in part) explains why the expansion at the wall is analytic for the case $a(X) < 0$.

For a specific example, let us see how these expressions reduce when we adopt the similarity solution for the single layer structure that occurs at a re-entrant corner of angle π/α ($1/2 \leq \alpha < 1$) given in [2]. Taking

$$\xi = \frac{Y}{X^{2-\alpha}}, \quad \Psi = X^{3-\alpha} f(\xi), \quad \bar{p} = p_0 X^{2(\alpha-1)},$$
$$\bar{T}_{11} = X^{2(\alpha-1)} t_{11}(\xi), \quad \bar{T}_{12} = X^{(\alpha-1)} t_{12}(\xi), \quad \bar{T}_{22} = t_{22}(\xi), \tag{B.9}$$

with p_0 a constant, (3.14) implies that we must take

$$a(X) = aX^{\alpha-1}, \qquad b(X) = bX^{2\alpha-3} \tag{B.10}$$

for constants a and b in order for the expansions to involve only ξ. The equations (3.14) and (3.15) then reduce to those given in [2]. Now addressing the exponentially small terms in (3.30), where we assume $a > 0$. Using (B.10) in (3.17) we have

$$A(X) = \frac{2}{(3-\alpha)a} X^{2-\alpha},$$

where the constant of integration must be taken as zero to obtain just a function of ξ in the controlling factor of the exponential and further

$$D(X) = K_1 X^{\left(\frac{1}{2}(\alpha-1)B + \frac{1}{2}(7\alpha-15) + \frac{b}{3a^2}\right)},$$
$$D_{11}(X) = X^{\left(2 + \frac{(\alpha-1)B}{2} + \frac{b}{3a^2}\right)} \left(K_2 - \frac{16K_1}{3(3-\alpha)^3 a^2} X^{\frac{3}{2}(\alpha-3)}\right),$$

where K_1, K_2 are arbitrary constants of integration. We immediately see that there are two cases to consider, namely $K_1 \neq 0, K_2 = 0$ and $K_2 \neq 0, K_1 = 0$ which give rise to independent exponentially small terms for the similarity variables.

Taking the case $K_1 \neq 0, K_2 = 0$ first. Then

$$\Psi^1 = K_1 X^{3-\alpha} X^{\left(\frac{1}{2}(3-\alpha)B - \frac{1}{2}(1+\alpha) + \frac{b}{3a^2}\right)} \xi^{B+5} e^{-\frac{2}{(3-\alpha)a\xi}} (1 + O(\xi)),$$
$$T^1_{11} = -\frac{16K_1}{3(3-\alpha)^3 a^2} X^{2(\alpha-1)} X^{\left(\frac{1}{2}(3-\alpha)B - \frac{1}{2}(1+\alpha) + \frac{b}{3a^2}\right)} \xi^{B} e^{-\frac{2}{(3-\alpha)a\xi}} (1 + O(\xi)),$$
$$T^1_{12} = -\frac{16(2-\alpha)K_1}{3(3-\alpha)^3 a^2} X^{\alpha-1} X^{\left(\frac{1}{2}(3-\alpha)B - \frac{1}{2}(1+\alpha) + \frac{b}{3a^2}\right)} \xi^{B+1} e^{-\frac{2}{(3-\alpha)a\xi}} (1 + O(\xi)),$$
$$T^1_{22} = -\frac{4K_1}{(3-\alpha)^2 a^3} X^{\left(\frac{1}{2}(3-\alpha)B - \frac{1}{2}(1+\alpha) + \frac{b}{3a^2}\right)} \xi^{B+1} e^{-\frac{2}{(3-\alpha)a\xi}} (1 + O(\xi)).$$

Observing (B.1) and (B.9), these expressions thus collapse to be a function of the similarity variable provided B takes the value B_1 given by

$$B_1 = \frac{(1+\alpha)}{(3-\alpha)} - \frac{2b}{3(3-\alpha)a^2}. \tag{B.11}$$

Proceeding similarly for the case $K_2 \neq 0, K_1 = 0$ we obtain

$$\begin{aligned}
&\Psi^1 = 0,\\
&T_{11}^1 = K_2 X^{2(\alpha-1)} X^{\left(\frac{1}{2}(3-\alpha)B+2(2-\alpha)+\frac{b}{3a^2}\right)} \xi^B e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi)),\\
&T_{12}^1 = (2-\alpha) K_2 X^{\alpha-1} X^{\left(\frac{1}{2}(3-\alpha)B+2(2-\alpha)+\frac{b}{3a^2}\right)} \xi^{B+1} e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi)),\\
&T_{22}^1 = 0,
\end{aligned}$$

and now B must take the value B_2 given by

$$B_2 = -\frac{4(2-\alpha)}{(3-\alpha)} - \frac{2b}{3(3-\alpha)a^2}, \tag{B.12}$$

for these expressions to collapse to similarity form.

Thus the expansion (3.30) in similarity variables takes the form

$$\begin{aligned}
f &= \frac{1}{2}a\xi^2 + \frac{1}{6}b\xi^3 + \ldots + K_1 \xi^{B_1+5} e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi)),\\
t_{11} &= 2a^2 + 2a(2b - 3(\alpha-1)a^2)\xi + \ldots - \frac{16K_1}{3(3-\alpha)^3 a^2} \xi^{B_1} e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi))\\
&\qquad + K_2 \xi^{B_2} e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi)),\\
t_{12} &= a + (b - 3(\alpha-1)a^2)\xi + \ldots - \frac{16(2-\alpha)K_1}{3(3-\alpha)^3 a^2} \xi^{B_1+1} e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi))\\
&\qquad + (2-\alpha) K_2 \xi^{B_2+1} e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi)),\\
t_{22} &= -2(\alpha-1)a\xi + \ldots - \frac{4K_1}{(3-\alpha)^2 a^3} \xi^{B_1+1} e^{-\frac{2}{(3-\alpha)a\xi}} (1+O(\xi)).
\end{aligned}$$

The two sets of exponentially small terms has been treated additively here for these expansions, which is permissible since expressions for Ψ^1, T_{ij}^1 were obtained by linearisation. This general expansion thus involves the four free constants a, b, K_1, K_2. It is worth noting that these two sets of exponentially small terms was predicted by the eigenmode analysis in [2].

Appendix C. Limiting behaviour of the inner equations (3.18)–(3.21)

Seeking solutions in the form

$$\Psi = \hat{c}_0(X)Y^q, \quad \bar{T}_{11} = \hat{c}_{11}(X)Y^{-2}, \quad \bar{T}_{12} = \hat{c}_{12}(X)Y^{-1}, \quad \bar{T}_{22} = -1 + \hat{c}_{22}(X), \tag{C.1}$$

then (3.19)–(3.21) give

$$q\hat{c}_0(X)\hat{c}_{11}' - 2(q-1)\hat{c}_0'(X)\hat{c}_{11} - 2q(q-1)\hat{c}_0(X)\hat{c}_{12} = 0, \tag{C.2}$$

$$q\hat{c}_0(X)\hat{c}_{22}' + 2q\hat{c}_0'(X)\hat{c}_{22} + 2\hat{c}_0''(X)\hat{c}_{12} = 0, \tag{C.3}$$

$$q\hat{c}_0(X)\hat{c}_{12}' + \hat{c}_0'(X)\hat{c}_{12} + \hat{c}_0''(X)\hat{c}_{11} - q(q-1)c_0(X)\hat{c}_{22} = 0. \tag{C.4}$$

Here $'$ denotes d/dX. The momentum equation (3.18) gives $\hat{c}_{12}(X) = \hat{c}'_{11}(X)$ and thus we have

$$\hat{c}_{11}(X) = \hat{c}_1 \hat{c}_0(X)^{\frac{2(q-1)}{q(3-2q)}}, \tag{C.5}$$

$$\hat{c}_{12}(X) = \frac{2(q-1)}{q(3-2q)} \frac{\hat{c}'_0(X)}{\hat{c}_0(X)} \hat{c}_{11}(X), \tag{C.6}$$

$$\hat{c}_{22}(X) = \hat{c}_{11}(X) \left(\frac{1}{q(q-1)(3-2q)} \frac{\hat{c}''_0(X)}{\hat{c}_0(X)} + \frac{2(q-1)(2q-1)}{q^2(3-2q)^2} \frac{\hat{c}'_0(X)^2}{\hat{c}_0(X)^2} \right), \tag{C.7}$$

for $q > 1$ and $q \neq 3/2$, where $\hat{c}_1$ is an arbitrary constant and $\hat{c}_0(X)$ satisfies

$$(3-2q)^2 q^2 \hat{c}_0^2 \hat{c}'''_0 + \hat{c}'_0 \left((3-2q)(6q^2 - 11q + 6)\hat{c}_0 \hat{c}''_0 + 4(q-1)^3(2q-1)\hat{c}'^2_0 \right) = 0. \tag{C.8}$$

In the specific case $\hat{c}_0(X) = \hat{C}_0 X^{\hat{m}q}$, we obtain from (C.8) the admissible values $\hat{m} = 0, 3 - 2q, \frac{3-2q}{2-q}$.

When $q = 3/2$ then $\hat{c}_0(X) = \hat{C}_0$ a constant and

$$\hat{c}_{11}(X) = \hat{C}_1 + \hat{C}_2 X + \hat{C}_3 X^2, \quad \hat{c}_{12}(X) = \hat{C}_2 + 2\hat{C}_3 X, \quad \hat{c}_{22}(X) = 4\hat{C}_3, \tag{C.9}$$

When $q = 1$ then $\hat{c}_0(X) = \hat{C}_0 X + \hat{D}_0$ and

$$\hat{c}_{11}(X) = \hat{C}_1, \quad \hat{c}_{12}(X) = 0, \quad \hat{c}_{22}(X) = \frac{\hat{C}_2}{(\hat{C}_0 X + \hat{D}_0)^2}, \tag{C.10}$$

where $\hat{C}_0, \hat{D}_0, \hat{C}_1, \hat{C}_2$ are arbitrary constants.

Appendix D. Limiting behaviour of the inner equations (3.32)–(3.35)

Seeking solutions in the form

$$\Psi = \bar{c}_0(X) Y, \quad \bar{T}_{11} = \bar{c}_{11}(X) Y^r, \quad \bar{T}_{12} = \bar{c}_{12}(X) Y^{r+1}, \quad \bar{T}_{22} = \bar{c}_{22}(X) Y^{r+2}, \tag{D.1}$$

with r arbitrary, then (3.33)–(3.35) give

$$\bar{c}_0(X)\bar{c}'_{11} + (1 - (2+r)\bar{c}'_0(X))\bar{c}_{11} = 0, \tag{D.2}$$

$$\bar{c}_0(X)\bar{c}'_{22} + (1 - r\bar{c}'_0(X))\bar{c}_{22} + 2\bar{c}''_0(X)\bar{c}_{12} = 0, \tag{D.3}$$

$$\bar{c}_0(X)\bar{c}'_{12} + (1 - (1+r)\bar{c}'_0(X))\bar{c}_{12} + \bar{c}''_0(X)\bar{c}_{11} = 0. \tag{D.4}$$

Here $'$ denotes d/dX. In this case we have the exact solution

$$\bar{c}_{11}(X) = \bar{C}_1 \bar{c}_0(X)^{r+2} e^{-\bar{I}(X)}, \tag{D.5}$$

$$\bar{c}_{12}(X) = \left(-\bar{C}_1 \bar{c}'_0(X) + \bar{C}_2 \right) \bar{c}_0(X)^{r+1} e^{-\bar{I}(X)}, \tag{D.6}$$

$$\bar{c}_{22}(X) = \left(\bar{C}_1 \bar{c}'_0(X)^2 - 2\bar{C}_2 \bar{c}'_0(X) + \bar{C}_3 \right) \bar{c}_0(X)^r e^{-\bar{I}(x)}, \tag{D.7}$$

where

$$\bar{I}(X) = \int \bar{c}_0(X)^{-1} dX \tag{D.8}$$

and $\bar{C}_1, \bar{C}_2, \bar{C}_3$ are arbitrary constants. The momentum equation (3.32) becomes

$$0 = -\frac{d\bar{p}}{dX} + Y^r \left(\bar{c}_{11}' + (1+r)\bar{c}_{12}\right).$$

Thus, for a non-trivial pressure gradient we require $r = 0$, in which case this equation simplifies to

$$0 = -\frac{d\bar{p}}{dX} + \bar{c}_0(X)e^{-\bar{I}(x)} \left(\bar{C}_1(\bar{c}_0'(X) - 1) + \bar{C}_2\right).$$

We note that when $\bar{c}_0(X)$ takes the form (3.44) then (D.5)–(D.7) with $r = 0$ give

$$\bar{c}_{11}(X) = \frac{1}{4}\bar{C}_1 c_0(X)^{\frac{2}{n}}, \tag{D.9}$$

$$\bar{c}_{12}(X) = -\frac{1}{4}\bar{C}_1 \frac{c_0(X)^{\frac{2}{n}-1}}{n} c_0'(X) + \left(-\frac{1}{2}\bar{C}_1 + \bar{C}_2\right) \frac{c_0(X)^{\frac{1}{n}}}{2I(X)}, \tag{D.10}$$

$$\begin{aligned}\bar{c}_{22}(X) = \frac{1}{4}\bar{C}_1 \frac{c_0(X)^{\frac{2}{n}-2}}{n^2} c_0'(X)^2 + \left(\frac{1}{2}\bar{C}_1 - \bar{C}_2\right) \frac{c_0(X)^{\frac{1}{n}-1}}{nI(X)} c_0'(X) \\ + \left(\frac{1}{4}\bar{C}_1 - \bar{C}_2 + \bar{C}_3\right) \frac{1}{I(X)^2},\end{aligned} \tag{D.11}$$

with $I(X)$ as in (A.17). These expressions then recover a reduced form of the $c_{ij}(x)$ given in (A.14)–(A.16) upon using the identification $\bar{C}_1 = 4c_1$ (as well as $x = X$) and taking $c_2 = c_3 = 0$ together with $\bar{C}_2 = \frac{1}{2}\bar{C}_1, \bar{C}_3 = \frac{1}{4}\bar{C}_1$.