Harmonic tori in spheres and complex projective spaces

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Introduction

A map $\phi : M \to N$ of Riemannian manifolds is *harmonic* if it extremises the *energy* functional:

$$\int |\mathrm{d}\phi|^2 \,\mathrm{d}vol$$

on every compact subdomain of M. Harmonic maps arise in many different contexts in Geometry and Physics (for an overview, see [**15**,**16**]) but the setting of concern to us is the following: take M to be 2-dimensional and N to be a Riemannian symmetric space of compact type. In this case, the energy is conformally invariant so that we may take the domain to be a Riemann surface and the methods of complex analysis may be brought to bear. Moreover, the symmetric nature of the target allows us to reformulate the harmonic map equations in a gauge-theoretic way so that harmonic maps may be viewed as simple analogues of Yang–Mills fields.

This paper treats harmonic maps of a 2-torus into a sphere S^n or a complex projective space $\mathbb{C}\mathrm{P}^n$ and makes use of the ideas and methods of two separate developments in the theory of harmonic maps. The first, and more recent, of these is the soliton-theoretic approach which has its roots in the Pinkall–Sterling classification [**23**] of constant mean curvature 2-tori in $\mathbb{R}^3$ (which are equivalent, via the Gauss map, to non-conformal harmonic 2-tori in S^2). Pinkall–Sterling showed that all such maps could be constructed from solutions to a family of finite-dimensional completely integrable Hamiltonian ordinary differential equations. There followed a rapid development and extension of these ideas [**5**,**19**] which culminated in a rather general theory of harmonic maps into symmetric spaces due to Burstall–Ferus–Pedit–Pinkall [**9**]. This theory distinguishes special harmonic maps of $\mathbb{R}^2$ into a symmetric space called *harmonic maps of finite type* which are constructed from commuting Hamiltonian flows on finite-dimensional subspaces of a loop algebra. Viewing maps of 2-tori as doubly periodic maps of $\mathbb{R}^2$, these authors prove:

Theorem *A non-conformal harmonic map of a 2-torus into a rank one symmetric space is of finite type.*

In particular, this result accounts for all non-conformal harmonic 2-tori in S^n and $\mathbb{C}\mathrm{P}^n$ but excludes the conformal harmonic (i.e., branched, minimal) tori.

The second development of importance to us is the well-established twistor theory of harmonic maps which goes back to Calabi's study [**11**,**12**] of minimal surfaces and, especially, minimal 2-spheres in S^n. Recall [**11**,**17**] that a harmonic map of a Riemann surface into

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a sphere or complex projective space has a sequence of invariants which are differentials measuring the lack of orthogonality of iterated derivatives of the map. These invariants have the following properties:

1. The first invariant is the obstruction to conformality.
2. The first non-zero invariant is a holomorphic differential.
3. There is a number N (depending on the target) such that, if the first N invariants vanish, then all the invariants vanish.

If all these invariants vanish, the map is variously called *pseudo-holomorphic* [**12**], *superminimal* [**7**], or *isotropic* [**17**]. In this case, the harmonic map is covered by a horizontal holomorphic map into an auxiliary complex manifold, a *twistor space*, and the study of isotropic harmonic maps is therefore reduced to a problem in Algebraic Geometry.

In particular, since the Riemann sphere admits no non-vanishing holomorphic differentials, any harmonic 2-sphere in S^n or $\mathbb{C}\mathrm{P}^n$ is isotropic. This is the basis of Calabi's classification of minimal 2-spheres in S^n [**11**,**12**] and the classification theorem for harmonic 2-spheres in $\mathbb{C}\mathrm{P}^n$ [**8**,**14**,**17**,**20**].

We see from these results that, as far as harmonic 2-tori are concerned, we can treat two extremes of behaviour: on the one hand, when the first invariant is non-zero, we obtain the harmonic map from soliton-theoretic ODE; on the other hand, when all invariants vanish, we obtain the harmonic map from a holomorphic curve in a twistor space. It is the purpose of this article to treat all the intermediate cases and thus account for all harmonic 2-tori in spheres and projective spaces. One can briefly describe our results as follows: a conformal non-isotropic map is covered by a map into a twistor space and this map, instead of being holomorphic, is constructed from soliton-theoretic ODE.

That such a picture obtains is already indicated by the analysis by Ferus–Pedit–Pinkall–Sterling of minimal, non-superminimal tori in S^4 [**19**] which was generalised to S^n and $\mathbb{C}\mathrm{P}^n$ by Bolton–Pedit–Woodward [**6**]. In these papers, the harmonic maps under consideration are characterised by the vanishing of all invariants except the last. In this case, the harmonic map is covered by a map into a flag manifold which can be shown to arise from commuting Hamiltonian ODE. One may view the flag manifold as a twistor space and the covering map, while no longer horizontal holomorphic, satisfies a first order condition.

The situation is similar for all non-isotropic 2-tori in spheres and complex projective spaces: any such map is covered by a "twistor lift" which is a map into a homogeneous space. Moreover, this lift can be constructed from commuting ODE and so is, in an appropriate sense, of finite type. The lift is constructed from iterated derivatives of the harmonic map by the usual method of twistor theory. The only novelty here is the nature of our twistor space and the map into it so obtained. Our twistor spaces are k-symmetric spaces which are analogues of the familiar Riemannian symmetric spaces where where the involutive isometries are replaced by isometries of finite order k [**22**]. The k-symmetric spaces form a class of reductive homogeneous spaces that includes both Riemannian symmetric spaces ($k = 2$) and flag manifolds. The use of the latter as twistor spaces for Riemannian symmetric spaces is well-known [**10**] but our use of other k-symmetric spaces in this way seems to be new.

As for the twistor lifts, these are examples of what we have called *primitive maps*: a map into a k-symmetric space, $k > 2$, is primitive if it satisfies a first-order equation of Cauchy–Riemann type which arises from the geometry of the k-symmetric space. Primitive maps are maps which are "f-holomorphic with respect to a horizontal f-structure" in the sense of Black [**4**] and, as such, enjoy a number of interesting properties: they are harmonic maps and their harmonicity is preserved under homogeneous projection. Thus primitive maps project onto harmonic maps. In our case, the converse is true: we show that any conformal harmonic map of a Riemann surface into S^n or $\mathbb{C}\mathrm{P}^n$ is covered by a primitive map into a suitable k-symmetric space.

To construct such maps from commuting ODE, we make use of the results of [**9**]: recall that a harmonic map $\phi : \mathbb{R}^2 \to G$ into a Lie group is essentially the same as a loop of flat connections $\mathrm{d} + A_\lambda$ with A_λ of the form

$$A_\lambda = (\lambda - 1)\alpha' + (\lambda^{-1} - 1)\alpha'',$$

$\lambda \in S^1$, where α' is a $\mathfrak{g}^{\mathbb{C}}$-valued $(1,0)$-form on $\mathbb{R}^2$ with complex conjugate α''. In [**9**], a method is given for constructing such loops of flat connections by solving commuting ODE on finite-dimensional subspaces of the based loop algebra. To treat harmonic maps into a Riemannian symmetric space G/K, one uses the totally geodesic Cartan embedding $G/K \hookrightarrow G$. The composition of a harmonic map into G/K with the Cartan embedding is also harmonic and so gives rise to a loop of flat connections. Moreover, if appropriate initial conditions are chosen for the commuting ODE, one constructs from the flows a harmonic maps which factors through the Cartan embedding in this way.

It transpires that we are able to treat primitive maps $\psi : \mathbb{R}^2 \to G/H$ into a k-symmetric space in a similar fashion. Firstly, there is an obvious analogue of the Cartan embedding $G/H \to G$ of any k-symmetric space into its group of isometries. For $k > 2$, this map is no longer totally geodesic so that its composition with a primitive map need not be harmonic. However, the structure equations for a primitive map are identical to those for a harmonic map into a Riemannian symmetric space and this enables us to construct a loop of flat connections. Moreover, a simple extension of the arguments of [**9**] suffices to show that commuting flows with the right initial conditions give rise to primitive maps.

We therefore arrive at the notion of a primitive map of finite type. The main result of [**9**] gives a simple criterion for a loop of flat connections on a torus to arise from commuting ODE from which we can deduce sufficient conditions for a primitive map to be of finite type. In our applications to twistor lifts, these conditions amount to the non-vanishing of one of the invariants of the underlying harmonic torus.

We have developed the theory of primitive maps of finite type in rather more generality than is necessary for our applications to harmonic tori in spheres and complex projective spaces. We have done this for two reasons: firstly, this theory provides the natural framework for our results and, secondly, because we hope it will find applications elsewhere. In this latter regard, it is already known that primitive maps into a full flag manifold are closely related to affine Toda fields [**6**].

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Notation Throughout this work, when a Lie group is denoted by an upper case letter, its Lie algebra will be denoted by the corresponding lower case gothic letter. Thus G is a Lie group with Lie algebra $\mathfrak{g}$.

1 k-symmetric spaces and harmonic maps

1.1 Preliminaries

In this paper we shall have much to do with the geometry of maps into reductive homogeneous spaces. We therefore begin by collecting some facts about these spaces. For more details, see [**10**, Chapter 1].

Let N be a manifold on which a Lie group G acts transitively and pick a base point $o \in N$ with stabiliser H so that $N \cong G/H$. N is a reductive homogeneous space if there is an $\mathrm{Ad}\, H$-invariant splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Such a splitting equips N with two kinds of extra structure. Firstly, left translation of $\mathfrak{m}$ around G gives a connection on the principal H-bundle $G \to G/H$. We call this connection the *canonical connection* and denote it by D.

Secondly, we have an isomorphism of TN with the sub-bundle $[\mathfrak{m}]$ of the trivial bundle $\underline{\mathfrak{g}} = N \times \mathfrak{g}$ given by

$$[\mathfrak{m}]_{g \cdot o} = \mathrm{Ad}\, g\, (\mathfrak{m}).$$

Indeed, for $x \in N$, the map $\mathfrak{g} \to T_x N$ given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot x,$$

restricts to give an isomorphism $[\mathfrak{m}]_x \to T_x N$. The inverse map $\beta_x : T_x N \to [\mathfrak{m}]_x \subset \mathfrak{g}$ defines a $\mathfrak{g}$-valued 1-form β on N which we call the *Maurer–Cartan form* of N (c.f., [**10**]).

In general, if V is a G-module with H-invariant subspace W, we may form the sub-bundle $[W] \subset \underline{V} = N \times V$ via $[W]_{g \cdot o} = g \cdot W$. In particular, we have an invariant splitting

$$\underline{\mathfrak{g}} = [\mathfrak{h}] \oplus [\mathfrak{m}],$$

and $[\mathfrak{h}]_x$ is the Lie algebra of the stabiliser of $x \in N$.

For future use, we recall from [**10**, Lemma 1.3] the structure equations of β:

$$\mathrm{d}\beta = (1 - \tfrac{1}{2}P)[\beta \wedge \beta] \tag{1}$$

where $P : \underline{\mathfrak{g}} \to [\mathfrak{m}]$ is projection along $[\mathfrak{h}]$.

Finally, the canonical connection induces a connection on $\underline{V} \cong G \times_H V$ which, from [**10**, Proposition 1.1], is related to flat differentiation in $\underline{V}$ by

$$\mathrm{d} = D + \beta. \tag{2}$$

All G-invariant tensors are D-parallel and so, in particular, the sub-bundles $[W]$ are all stable under D-covariant differentiation.

We now turn to the class of reductive homogeneous spaces that will most occupy us in the sequel.

1.2 k-symmetric spaces

Let G be a compact Lie group and $\tau : G \to G$ an automorphism of order $k > 1$. We let $H \subset G$ be a subgroup satisfying

$$(G^\tau)_0 \subset H \subset G^\tau$$

and consider the homogeneous space $N = G/H$. Denote by $o \in N$ the identity coset eH and let $\widehat{\tau} : N \to N$ be given by

$$\widehat{\tau}(g \cdot o) = \tau(g) \cdot o$$

so that $\widehat{\tau}$ is a (well-defined!) diffeomorphism of order k which has o as an isolated fixed point. Moreover, for $x = g \cdot o \in N$, we define $\widehat{\tau}_x : N \to N$ by

$$\widehat{\tau}_x = g \circ \widehat{\tau} \circ g^{-1} = g\tau(g)^{-1} \circ \widehat{\tau},$$

to get a diffeomorphism of order k having x as an isolated fixed point. Finally equip N with a metric for which all the $\widehat{\tau}_x$ are isometries (such metrics certainly exist). N now has the structure of a *regular k-symmetric space* and it is known that all compact regular k-symmetric spaces arise in this way [**22**]. Of course, the 2-symmetric spaces are just the familiar Riemannian symmetric spaces.

Let us examine the infinitesimal structure on TN induced by the $\widehat{\tau}_x$. The derivative of τ at $e \in G$, also denoted τ, is an order k automorphism of $\mathfrak{g}$ and, setting $\omega = e^{2\pi i/k}$, we have an eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_k} \mathfrak{g}_j$$

where $\mathfrak{g}_j$ is the ω^j-eigenspace of τ. Then $\overline{\mathfrak{g}_j} = \mathfrak{g}_{-j}$, where the conjugation is with respect to the real form $\mathfrak{g}$ and we have the relations

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

recalling throughout that $i, j \in \mathbb{Z}_k$. Setting

$$\mathfrak{h} = \mathfrak{g}_0 \cap \mathfrak{g}, \qquad \mathfrak{m} = \left(\sum_{j \in \mathbb{Z}_k \setminus \{0\}} \mathfrak{g}_j \right) \cap \mathfrak{g},$$

we see that $\mathfrak{h}$ is the Lie algebra of H and that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

is an $\mathrm{Ad}(H)$-invariant splitting of $\mathfrak{g}$. Thus N is a reductive homogeneous space and the results of Section 1.1 apply.

In particular, we have an isomorphism

$$TN^{\mathbb{C}} \overset{\beta}{\cong} [\mathfrak{m}]^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_k \setminus \{0\}} [\mathfrak{g}_j]$$

and a decomposition

$$\underline{\mathfrak{g}}^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_k} [\mathfrak{g}_j]$$

where $[\mathfrak{g}_j]_x$ is the ω^j-eigenspace of an automorphism τ_x—*the automorphism of N at x*—which is given by $\mathrm{Ad}\, g \circ \tau \circ \mathrm{Ad}\, g^{-1}$ when $x = g \cdot o$.

1.3 Primitive maps

We start with a definition:

Definition Let $\psi : M \to N$ be a map of an almost complex manifold into a k-symmetric space, $k > 2$. ψ is *primitive* if $\psi^*\beta^{(1,0)}$ takes values in $[\mathfrak{g}_1]$.

Such maps are examples of a class of f-holomorphic maps studied by Black [**4**] and have a number of interesting properties which ultimately stem from the relation

$$[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{h}^{\mathbb{C}} \tag{3}$$

For instance, Black proves

Proposition 1.1 [4] *Let $\psi : M \to N$ be a primitive map of a Hermitian manifold with co-closed Kähler form. Then ψ is harmonic with respect to all invariant metrics on N for which $[\mathfrak{g}_1]$ is isotropic.*

Remark The condition on the invariant metrics of N is very mild. Indeed, if all the irreducible subrepresentations in the H-representation on $\mathfrak{m}$ occur with multiplicity one (true, for example, if H contains a maximal torus of G) then $[\mathfrak{g}_1]$ is isotropic for *any* choice of invariant metric. In this case, ψ is *equiharmonic*, that is, harmonic with respect to all invariant metrics.

In all our applications, we shall be concerned with the case where G is semisimple and the metrics on homogeneous G-spaces are induced from the Killing inner product on $\mathfrak{g}$. To simplify the exposition, we therefore restrict attention to this case for the rest of the paper.

An important property of primitive maps is that their harmonicity is preserved under homogeneous projection. For this, let M be an almost complex manifold and N a reductive G-space with Maurer–Cartan form β. Let $\psi : M \to N$ be a map and let $b = \psi^*\beta$ with type decomposition $b = b' + b''$. Further, let $\mathrm{d} = \partial + \overline{\partial}$ be the type decomposition of the exterior derivative (we are *not* assuming that $\partial^2 = 0$!). We now have a criterion for the harmonicity of ψ:

Lemma 1.2 *Let M be almost Hermitian with co-closed Kähler form. Then ϕ is harmonic if and only if, for any unitary frame $Z_1, \ldots, Z_n$ of $T^{1,0}M$,*

$$\overline{\partial} b'(Z_i, \overline{Z}_i) = \partial b''(Z_i, \overline{Z}_i).$$

Proof From [**24**], it is known that ψ is harmonic if and only if b is co-closed which is equivalent to demanding that

$$Z_i b(\overline{Z}_i) - b(\nabla_{Z_i} \overline{Z}_i) + \overline{Z}_i b(Z_i) - b(\nabla_{\overline{Z}_i} Z_i) = 0, \tag{4}$$

where ∇ is the Levi–Civita connection on M. But the co-closure of the Kähler form means that $\nabla_{\overline{Z}_i} Z_i$ takes values in $T^{1,0}M$ so that

$$b(\nabla_{\overline{Z}_i} Z_i) = b'([\overline{Z}_i, Z_i]).$$

It follows that (4) can be written as

$$\partial b''(Z_i, \overline{Z}_i) + \overline{\partial} b'(\overline{Z}_i, Z_i) = 0$$

and the result follows. $\square$

Now suppose that N is a k-symmetric space and that $\psi : M \to N$ is primitive so that b' is a $[\mathfrak{g}_1]$-valued $(1,0)$-form on M. The structure equation (1) pulls back to give

$$\mathrm{d}b = (1 - \tfrac{1}{2}P)[b \wedge b]. \tag{5}$$

In view of (3) and the fact that ψ is primitive, we have

$$P[b' \wedge b''] = 0$$

so that the $(1,1)$-part of (5) reads

$$\overline{\partial} b' + \partial b'' = 2[b' \wedge b'']. \tag{6}$$

Now let $D = \mathrm{d} - \mathrm{ad}\, b$ be the pull-back of the canonical connection on N. In terms of D, (6) reads

$$\overline{\partial}^D b' + \partial^D b'' = 0. \tag{7}$$

However, b' and hence $\overline{\partial}^D b'$ is $[\mathfrak{g}_1]$-valued since $[\mathfrak{g}_1]$ is D-stable and, similarly, $\partial^D b''$ is $[\mathfrak{g}_{-1}]$-valued so that both summands of (7) must vanish separately:

$$\overline{\partial}^D b' = \partial^D b'' = 0. \tag{8}$$

Using this, we prove:

Proposition 1.3 *Let $H \subset K$ be closed subgroups of G with G/H k-symmetric and K τ-stable. Let $\pi : G/H \to G/K$ be the homogeneous projection. If $\psi : M \to G/H$ is a primitive map of an almost Hermitian manifold with co-closed Kähler form then $\phi = \pi \circ \psi : M \to G/K$ is harmonic.*

Proof Set $\mathfrak{p} = \mathfrak{k}^\perp$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an $\mathrm{Ad}\, K$- and τ-invariant splitting so that, in particular, G/K is reductive. Both $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ have decompositions into eigenspaces of τ so that there is an $\mathrm{Ad}\, H$-invariant orthogonal splitting $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}$, say, with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{l}$. We write

$$\mathfrak{g}_{\pm 1} = \mathfrak{l}_{\pm 1} \oplus \mathfrak{p}_{\pm 1},$$

where $\mathfrak{l}_1 = \mathfrak{l}^C \cap \mathfrak{g}_1$ and so on. Note that, from (3) and the fact that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ we get

$$[\mathfrak{l}_{\pm 1}, \mathfrak{p}_{\mp 1}] \subset \mathfrak{p}^{\mathbb{C}} \cap \mathfrak{h}^{\mathbb{C}} = \{0\} \tag{9}$$

As usual, let b be the pull-back by ψ of the Maurer–Cartan form of G/H and write

$$b = b_{\mathfrak{l}} + b_{\mathfrak{p}}$$

for its decomposition into $[\mathfrak{l}]$ and $[\mathfrak{p}]$ parts. Further, let β^K be the Maurer–Cartan form of G/K. From [**10**, lemma 1.8], we have

$$\phi^*\beta^K = b_{\mathfrak{p}}$$

so that, by (1.2), it suffices to prove that

$$\overline{\partial} b'_{\mathfrak{p}} = \partial b''_{\mathfrak{p}}.$$

For this, observe that since $\mathfrak{p}$ is Ad H-invariant, $[\mathfrak{p}]$ is D-parallel so that the $[\mathfrak{p}]$-component of (8) reads:

$$\overline{\partial}^D b'_{\mathfrak{p}} = \partial^D b''_{\mathfrak{p}} = 0.$$

Otherwise said,

$$0 = \overline{\partial} b'_{\mathfrak{p}} - [b'' \wedge b'_{\mathfrak{p}}] = \overline{\partial} b'_{\mathfrak{p}} - [b''_{\mathfrak{l}} \wedge b'_{\mathfrak{p}}] - [b''_{\mathfrak{p}} \wedge b'_{\mathfrak{p}}].$$

However, in view of (9) and the primitivity of ψ, $[b''_{\mathfrak{l}} \wedge b'_{\mathfrak{p}}]$ vanishes identically so that

$$\overline{\partial} b'_{\mathfrak{p}} = [b''_{\mathfrak{p}} \wedge b'_{\mathfrak{p}}].$$

Taking complex conjugates gives

$$\partial b''_{\mathfrak{p}} = [b'_{\mathfrak{p}} \wedge b''_{\mathfrak{p}}] = [b''_{\mathfrak{p}} \wedge b'_{\mathfrak{p}}]$$

and we are done. □

Remark Under the conditions described in the previous remark, Proposition 1.3 can be considerably improved: in fact, Black [4] proves that any homogeneous projection of an equiharmonic map is also equiharmonic.

In our applications of Proposition 1.3, G/K will be a Riemannian symmetric space and we will view G/H as a kind of twistor space for G/K. We will show that, under auspicious circumstances, a converse to the proposition is available so that certain harmonic maps into G/K are covered by primitive maps into some G/H. It is this possibility that motivates our study of primitive maps.

1.4 A Cartan embedding

A popular device for studying harmonic maps into symmetric spaces G/K is to use the totally geodesic Cartan embedding of G/K into G (c.f., [**9**,**10**,**26**]).

We now describe a mild generalisation of this embedding to the case of k-symmetric spaces which will be useful in the sequel.

Henceforth, we shall always take G to be compact, connected and semisimple.

Definition Let G/H be a k-symmetric space with automorphism τ. The *Cartan embedding* of G/H is the map $\iota : G/H \to G$ given by

$$\iota(g \cdot o) = \tau(g)g^{-1}.$$

Remarks (i) The alert reader will have noted an abuse of terminology here: ι is in general only a finite-to-one immersion. It is an embedding precisely when $H = G^\tau$.

(ii) Of course, when $k > 2$, ι will not be totally geodesic so that post-composition by ι will not preserve harmonicity of maps. However, we shall see below that primitive maps are quite well-behaved with respect to ι.

(iii) When $G = SU(n)$ and G/H is a flag manifold, the map ι has been independently studied by Ferreira [**18**] who investigated its harmonicity.

Let θ be the (left) Maurer–Cartan form of G and β the Maurer–Cartan form of G/H. We compute $\iota^*\theta$.

Lemma 1.4 *For $x \in G/H$, $\iota^*\theta_x = \tau_x\beta_x - \beta_x$, where τ_x is the automorphism at x.*

Proof Let $X \in T_xG/H$. Then

$$X = \left.\frac{d}{dt}\right|_{t=0} \exp t\beta(X)\cdot x$$

so that

$$\iota_*(X) = \left.\frac{d}{dt}\right|_{t=0} \tau(\exp t\beta(X))\iota(x)\exp -t\beta(X) = L_{\iota(x)*}(\operatorname{Ad}\iota(x)^{-1}\tau\beta(X) - \beta(X)).$$

Thus

$$\iota^*\theta(X) = \operatorname{Ad}\iota(x)^{-1}\tau\beta(X) - \beta(X).$$

However, if $x = g\cdot o$, then

$$\operatorname{Ad}\iota(x)^{-1}\tau = \operatorname{Ad} g\tau(g)^{-1}\tau = \operatorname{Ad} g\circ\tau\circ\operatorname{Ad} g^{-1} = \tau_x$$

and we are done. □

Let us now determine the image of ι.

Proposition 1.5 *$\iota(G/H)$ is the connected component of $F = \{h \in G\colon (\tau^{-1}\operatorname{Ad} h)^k = 1\}$ containing the identity.*

Proof For convenience, set $I = \iota(G/H) \subset G$. If $h = \tau(g)g^{-1} \in I$, then

$$\tau^{-1}\operatorname{Ad} h = \tau^{-1}\operatorname{Ad}\tau(g)g^{-1} = \operatorname{Ad} g\circ\tau^{-1}\circ\operatorname{Ad} g^{-1}.$$

From this it is clear that $(\tau^{-1}\operatorname{Ad} h)^k = 1$ so that $I \subset F$ and also that $\operatorname{Ad}(I) \subset \operatorname{Aut}(\mathfrak{g})$ is the left translate by τ of the $\operatorname{Ad}(G)$-conjugacy class of τ^{-1} in $\operatorname{Aut}(\mathfrak{g})$.

On the other hand,

$$\operatorname{Ad}(F) = \{\sigma \in \operatorname{Inn}(\mathfrak{g})\colon (\tau^{-1}\sigma)^k = 1\} = \tau\{\rho \in \operatorname{Aut}(\mathfrak{g})\colon \rho^k = 1\}\cap\operatorname{Inn}(\mathfrak{g}).$$

However, $\{\rho \in \operatorname{Aut}(\mathfrak{g})\colon \rho^k = 1\}$ consists of a finite number of $\operatorname{Ad}(G)$-conjugacy classes, each of which is therefore a connected component. Thus $\operatorname{Ad}(I)$ is the connected component of $\operatorname{Ad}(F)$ containing 1.

Thus, letting Z denote the centre of G, we see that $\mathrm{Ad}^{-1}(\mathrm{Ad}(I)) = IZ$ is open and closed in $\mathrm{Ad}^{-1}(\mathrm{Ad}(F)) = F$. It therefore suffices to show that I is open and closed in IZ. For this, let z_1, $z_2 \in Z$ and suppose $Iz_1 \cap Iz_2 \neq \emptyset$. Then there are g_1, $g_2 \in G$ such that

$$\tau(g_1)g_1^{-1}z_1 = \tau(g_2)g_2^{-1}z_2,$$

so that

$$\tau(g_2^{-1}g_1)(g_2^{-1}g_1)^{-1} = z_2z_1^{-1}$$

Now set $h = g_2^{-1}g_1$ and observe that

$$\tau(g)g^{-1}z_2 = \tau(g)z_2z_1^{-1}g^{-1}z_1 = \tau(gh)(gh)^{-1}z_1.$$

From this it follows that $Iz_1 = Iz_2$ so that IZ is a finite disjoint union of translates of I. However, each Iz is closed (since compact) and so open in IZ also. In particular, I is open and closed in IZ and the proof is complete. $\square$

We use this to provide a simple criterion for when a map into G has image in $\iota(G/H)$.

Proposition 1.6 *Let $\psi : M \to G$ be a map of a manifold M such that $\psi(m) = e$ for some $m \in M$. Then ψ has image in $\iota(G/H)$ if and only if*

$$(1 + (\mathrm{Ad}\,\psi^{-1}\tau) + \cdots + (\mathrm{Ad}\,\psi^{-1}\tau)^{k-1})\psi^*\theta \equiv 0.$$

Proof By Proposition 1.5, it suffices to show that $(\tau^{-1}\,\mathrm{Ad}\,\psi)^k$ is constant. However,

$$\begin{aligned} d(\tau^{-1}\,\mathrm{Ad}\,\psi)^k &= \sum_{i=1}^{k}(\tau^{-1}\,\mathrm{Ad}\,\psi)^i \circ \mathrm{ad}\,\psi^*\theta \circ (\tau^{-1}\,\mathrm{Ad}\,\psi)^{k-i} \\ &= \sum_{i=1}^{k}(\tau^{-1}\,\mathrm{Ad}\,\psi)^k \circ \mathrm{ad}(\tau^{-1}\,\mathrm{Ad}\,\psi)^{i-k}\psi^*\theta \\ &= (\tau^{-1}\,\mathrm{Ad}\,\psi)^k \circ \mathrm{ad}\left(\sum_{i=1}^{k}(\mathrm{Ad}\,\psi^{-1}\tau)^{k-i}\psi^*\theta\right). \end{aligned}$$

Since $\mathfrak{g}$ is semisimple, ad is an isomorphism and the result follows. $\square$

2 Commuting flows

In [**9**], harmonic maps of a Riemann surface into a Lie group where produced by solving a pair of commuting ordinary differential equations. Harmonic maps into symmetric spaces were viewed as harmonic maps into Lie groups via the Cartan embedding and were obtained from the commuting flows by choosing an appropriate initial condition. It was also shown that a large class of harmonic 2-tori in symmetric spaces arose from this procedure. In this section, we adapt these methods to deal with primitive maps into k-symmetric spaces.

2.1 Zero-curvature representation

Let G/H be a k-symmetric space, $k > 2$, with Maurer–Cartan form β. Let M be a Riemann surface and $\psi : M \to G/H$ a primitive map. As usual, set $b = \psi^*\beta$ with type decomposition $b = b' + b''$.

Recall that the structure equation for b gave (8):

$$\overline{\partial}^D b' = \partial^D b'' = 0$$

which, on our Riemann surface, read

$$\mathrm{d}b' - [b' \wedge b''] = \mathrm{d}b'' - [b' \wedge b''] = 0. \tag{10}$$

Following [**26**,**29**,**30**], we introduce a spectral parameter $\lambda \in S^1$ and, for each λ, we define a $\mathfrak{g}$-valued 1-form on M by

$$A_\lambda = (\lambda - 1)b' + (\lambda^{-1} - 1)b'' \tag{11}$$

It turns out that the corresponding connections $\mathrm{d} + A_\lambda$ are all flat: indeed,

$$\begin{aligned}\mathrm{d}A_\lambda + \tfrac{1}{2}[A_\lambda \wedge A_\lambda] &= (\lambda - 1)\mathrm{d}b' + (\lambda^{-1} - 1)\mathrm{d}b'' + (\lambda - 1)(\lambda^{-1} - 1)[b' \wedge b''] \\ &= (\lambda - 1)(\mathrm{d}b' - [b' \wedge b'']) + (\lambda^{-1} - 1)(\mathrm{d}b'' - [b' \wedge b''])\end{aligned}$$

which vanishes by (10).

Observe that Lemma 1.4 allows us to relate the A_λ to the composition $\iota \circ \psi : M \to G$. In fact,

$$A_\omega = (\omega - 1)b' + (\omega^{-1} - 1)b'' = (\iota \circ \psi)^*\theta, \tag{12}$$

where, as usual, $\omega = e^{2\pi i/k}$. In the next section, we shall see how to obtain a converse to this construction and, from certain loops of flat connections $\mathrm{d} + A_\lambda$ on $\mathbb{R}^2$, produce primitive maps $\psi : \mathbb{R}^2 \to G/H$ with $(\iota \circ \psi)^*\theta = A_\omega$.

Meanwhile, we may deduce from (10) that $\iota \circ \psi$, while not harmonic, satisfies a related equation. Indeed, it is easy to see that $\alpha = (\iota \circ \psi)^*\theta$ satisfies

$$\mathrm{d}^*\alpha + \frac{\mu(\omega)}{2} * [\alpha \wedge \alpha] = 0,$$

where $\mu(e^{ix}) = \operatorname{cotan}(x/2)$, which is the harmonic map equation with Wess–Zumino terms for maps $M \to G$. This is the Euler–Lagrange equation for the functional obtained by coupling the Wess–Zumino action to the energy functional with coupling constant μ. In general, as Uhlenbeck [**27**] observes, whenever $\mathrm{d} + A_\lambda$ is a loop of flat connections of the form (11) and $\psi : M \to G$ satisfies $\psi^*\theta = A_{\lambda_0}$, then ψ solves the harmonic map equation with Wess–Zumino terms with coupling constant $\mu(\lambda_0)$, (see also [**25**]).

2.2 Commuting ordinary differential equations

In [**9**], it was shown how to obtain loops of flat connections $\mathrm{d} + A_\lambda$ of the form (11) by integrating a pair of commuting ordinary differential equations. Let us briefly recall this development.

For $d \in \mathbb{N}$, define a finite-dimensional space of $\mathfrak{g}^{\mathbb{C}}$-valued Laurent polynomials by

$$\Omega_d = \{\xi(\lambda) = \sum_{0<|n|\leq d} \lambda^n \xi_n : \quad \xi_n \in \mathfrak{g}^{\mathbb{C}}, \quad \overline{\xi_n} = \xi_{-n}\}.$$

Note that the elements of Ω_d are based, $\xi(1) = 0$, and satisfy the reality condition that $\xi(\lambda) \in \mathfrak{g}$ for $\lambda \in S^1$.

Consider now vector fields X_1, X_2 on Ω_d defined by

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(1-\lambda)\xi_d],$$

where the bracket is to be interpreted point-wise. It is shown in [**9**] that the X_i are complete commuting vector fields on Ω_d so that, fixing an initial condition $\xi_0 \in \Omega_d$, we may integrate the corresponding flows to obtain a map $\xi : \mathbb{R}^2 \to \Omega_d$. Thus ξ is the unique solution of

$$\mathrm{d}\xi = [\xi, 2i(1-\lambda^{-1})\xi_d \,\mathrm{d}z - 2i(1-\lambda^{-1})\xi_{-d}\,\mathrm{d}\overline{z}] \tag{13}$$

with $\xi(0) = \xi_0$, where z is the complex co-ordinate on $\mathbb{R}^2 = \mathbb{C}$. Moreover, for such a ξ, if we define $\mathfrak{g}$-valued 1-forms A_λ on $\mathbb{R}^2$ by

$$A_\lambda = 2i(1-\lambda)\xi_d\,\mathrm{d}z - 2i(1-\lambda^{-1})\xi_{-d}\,\mathrm{d}\overline{z},$$

then the connections $\mathrm{d} + A_\lambda$ are flat. Thus, for each $\lambda \in S^1$, we integrate to get a map $\psi_\lambda : \mathbb{R}^2 \to G$, unique up to left translation by a constant, with $\psi_\lambda^*\theta = A_\lambda$.

We now show how to obtain primitive maps from such ξ. First we have a straightforward extension of [**9**, Proposition 4.3] to the k-symmetric setting:

Proposition 2.1 *Let $d \in \mathbb{N}$ satisfy $d \equiv 1 \bmod k$. Let $\xi : \mathbb{R}^2 \to \Omega_d$ be a solution of* (13) *and $\psi : \mathbb{R}^2 \to G$ satisfy*

$$\psi^*\theta = 2i(1-\omega)\xi_d\,\mathrm{d}z - 2i(1-\omega^{-1})\xi_{-d}\,\mathrm{d}\overline{z}.$$

Define $\widetilde{\xi}$ by

$$\widetilde{\xi}(\lambda) = \tau^{-1}\operatorname{Ad}\psi(\omega\lambda).$$

Then $\widetilde{\xi}$ also satisfies (13) *i.e.,*

$$\mathrm{d}\widetilde{\xi} = [\widetilde{\xi}, 2i(1-\lambda)\widetilde{\xi}_d\,\mathrm{d}z - 2i(1-\lambda^{-1})\widetilde{\xi}_{-d}\,\mathrm{d}\overline{z}].$$

Here, since $\widetilde{\xi}$ need not be based, we interpret $\widetilde{\xi}_{\pm d}$ as the coefficient of $-\lambda^{\pm d}$ in $\widetilde{\xi}$.

Proof The proof is essentially the same as that of [**9**, Proposition 4.3] but we give it here for completeness. We compute:

$$\begin{aligned}
\mathrm{d}\widetilde{\xi}(\lambda) &= \tau^{-1}\mathrm{d}(\operatorname{Ad}\psi)\xi(\omega\lambda) + \tau^{-1}\operatorname{Ad}\psi\,\mathrm{d}\xi(\omega\lambda)\\
&= \tau^{-1}\operatorname{Ad}\psi\,[\psi^*\theta, \xi(\omega\lambda)] + \tau^{-1}\operatorname{Ad}\psi\,\mathrm{d}\xi(\omega\lambda)\\
&= \tau^{-1}\operatorname{Ad}\psi\,[2i(1-\omega)\xi_d\,\mathrm{d}z - 2i(1-\omega^{-1})\xi_{-d}\,\mathrm{d}\overline{z}, \xi(\omega\lambda)]\\
&\quad + \tau^{-1}\operatorname{Ad}\psi\,[\xi(\omega\lambda), 2i(1-\omega\lambda)\xi_d\,\mathrm{d}z - 2i(1-(\omega\lambda)^{-1})\xi_{-d}\,\mathrm{d}\overline{z}]\\
&= \tau^{-1}\operatorname{Ad}\psi[\xi(\omega\lambda), 2i\omega(1-\lambda)\xi_d\,\mathrm{d}z - 2i\omega^{-1}(1-\lambda^{-1})\xi_{-d}\,\mathrm{d}\overline{z}].
\end{aligned}$$

Moreover, since $d \equiv 1 \bmod k$,

$$\widetilde{\xi}_{\pm d} = \tau^{-1}\operatorname{Ad}\psi\,\omega^{\pm 1}\xi_{\pm d}$$

and the proposition follows. □

Now let $\Lambda(\mathfrak{g},\tau)=\{\xi : S^1 \to \mathfrak{g} \colon \xi(\omega\lambda)=\tau\xi(\lambda) \text{ for all } \lambda \in S^1\}$. Thus $\xi=\sum \lambda^n \xi_n$ lies in $\Lambda(\mathfrak{g},\tau)$ precisely when, for each n, $\xi_n \in \mathfrak{g}_{n \bmod k}$. With this understood, we have

Theorem 2.2 *Let $d \in \mathbb{N}$ satisfy $d \equiv 1 \bmod k$. Let $\xi : \mathbb{R}^2 \to \Omega_d$ be a solution of* (13) *with $\xi(0)=\xi_0 \in \Omega_d \cap \Lambda(\mathfrak{g},\tau)$. Let $\psi : \mathbb{R}^2 \to G$ satisfy $\psi(0)=e$ and*

$$\psi^*\theta = 2i(1-\omega)\xi_d\,\mathrm{d}z - 2i(1-\omega^{-1})\xi_{-d}\,\mathrm{d}\overline{z}.$$

Then, ψ has image in $\iota(G/H)$ and, viewed as a map into G/H, ψ is primitive.

Proof Set $\widetilde{\xi}(\lambda)=\tau^{-1}\operatorname{Ad}\psi\,\xi(\omega\lambda)$ so that, by (2.1), $\widetilde{\xi}$ also solves (13). Moreover, since $\xi_0 \in \Lambda(\mathfrak{g},\tau)$, we have

$$\widetilde{\xi}(\lambda)(0)=\tau^{-1}\xi(\omega\lambda)(0)=\xi_0(\lambda).$$

Thus ξ and $\widetilde{\xi}$ solve the same ordinary differential equations and have the same initial condition and so must coincide. Thus

$$\xi(\omega\lambda)=\operatorname{Ad}\psi^{-1}\tau\xi(\lambda).$$

Comparing coefficients of $\lambda^{\pm d}$ in this gives

$$\omega^{\pm 1}\xi_{\pm d}=\operatorname{Ad}\psi^{-1}\tau\xi_{\pm d} \tag{14}$$

so that

$$\begin{aligned}&(1+\operatorname{Ad}\psi^{-1}\tau+\cdots+(\operatorname{Ad}\psi^{-1}\tau)^{k-1})\psi^*\theta=\\ &\quad 2i(1-\omega)(1+\omega+\cdots+\omega^{k-1})\xi_d\,\mathrm{d}z-2i(1-\omega^{-1})(1+\omega^{-1}+\cdots+\omega^{-(k-1)})\xi_{-d}\,\mathrm{d}\overline{z}\\ &\qquad = 2i(1-\omega^k)\xi_d\,\mathrm{d}z-2i(1-\omega^{-k})\xi_{-d}\,\mathrm{d}\overline{z}=0.\end{aligned}$$

We may therefore conclude from Proposition 1.6 that ψ has image in $\iota(G/H)$ and write $\psi=\iota\circ\widehat{\psi}$ with $\widehat{\psi}:\mathbb{R}^2\to G/H$.

Concerning $\widehat{\psi}$, it is easy to see that, for $x\in\mathbb{R}^2$, $\operatorname{Ad}\psi^{-1}\tau=\tau_{\widehat{\psi}(x)}$, the automorphism at $\widehat{\psi}(x)\in G/H$, so that (14) gives

$$\tau_{\widehat{\psi}}\xi_d=\omega\xi_d.$$

On the other hand, Lemma 1.4 gives

$$(\tau_{\widehat{\psi}}-1)\widehat{\psi}^*\beta'=2i(1-\omega)\xi_d\,\mathrm{d}z$$

whence $\widehat{\psi}^*\beta'=-2i\xi_d\,\mathrm{d}z$ and so is $[\mathfrak{g}_1]$-valued. Thus $\widehat{\psi}$ is primitive as required. $\square$

The upshot of all this is that, by choosing appropriate initial conditions, we may produce primitive maps $\mathbb{R}^2\to G/H$ by integrating commuting flows. Following [**9**], we baptise the primitive maps so obtained, *primitive maps of finite type*.

Thus a primitive map ψ is of finite type if the loop of flat connections $\mathrm{d}+A_\lambda$ constructed from $b=\psi^*\beta$ as in (11) is obtained by the analysis of Section 2.2 from a solution ξ of (13).

In case that a primitive map is doubly periodic, there is a simple criterion for detecting when it is of finite type. Indeed, Theorem 7.2 of [**9**] applies in our situation to give:

Theorem 2.3 *A doubly periodic primitive map* $\psi : \mathbb{R}^2 \to G/H$ *is of finite type if* $\psi^*\beta(\partial/\partial z)$ *takes values in a single* $\mathrm{Ad}(G^{\mathbb{C}})$*-orbit of semisimple elements of* $\mathfrak{g}^{\mathbb{C}}$.

This condition can be further simplified by recourse to a familiar argument about holomorphic differentials.

Lemma 2.4 *Let* $P : \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$ *be a homogeneous* $\mathrm{Ad}\, G^C$*-invariant polynomial of degree* l *and let* $\psi : M \to G/H$ *be a primitive map of a Riemann surface. Then* $P(\psi^*\beta)^{(l,0)}$ *is a holomorphic section of* $\otimes^l T^*_{1,0}M$.

Proof First observe that $P \circ \beta : TG/H \to \mathbb{C}$ is G-invariant and therefore parallel with respect to the canonical connection of G/H. Moreover, from (10), we have

$$\mathrm{d}^D \psi^*\beta' = 0$$

so that

$$D^{(0,1)}\psi^*\beta(\partial/\partial z) = 0,$$

whence the conclusion. $\square$

In particular, when M is a 2-torus, $T^{1,0}M$ is canonically trivial so that each $P(\psi^*\beta(\partial/\partial z))$ is constant. Thus, in this case, $\psi^*\beta(\partial/\partial z)$ takes values in the common level set of all the invariant polynomials. The structure of such level sets was elucidated by Kostant [**21**] who showed that each level set contains a single orbit of semisimple elements which is, in addition, the unique closed orbit in the level set. We may therefore conclude:

Proposition 2.5 *A doubly periodic primitive map* $\psi : \mathbb{R}^2 \to G/H$ *is of finite type if* $\psi^*\beta(\partial/\partial z)$ *is semisimple on a dense subset of* $\mathbb{R}^2$.

3 Harmonic surfaces in S^n

We now turn to our applications of the preceding general theory. In this section, we shall show how any conformal harmonic map of a Riemann surface into a sphere S^n is either superminimal or covered by a primitive map into an appropriate k-symmetric space. Moreover, in the latter case, we show that this primitive map has semisimple derivative in the sense of Proposition 2.5 so that, when the domain is a 2-torus, the primitive map is of finite type. Superminimal harmonic maps have already been classified by Calabi [**11**, **12**] while non-conformal harmonic 2-tori in S^n have been dealt with by Burstall–Ferus–Pedit–Pinkall [**9**] so that these results account for all harmonic 2-tori in S^n.

In the next section, a similar programme will be carried out for harmonic maps of a Riemann surface into $\mathbb{C}\mathrm{P}^n$.

3.1 k-symmetric spaces over S^n

The k-symmetric spaces of interest to us here are bundles of isotropic flags over a sphere S^n. We therefore begin by describing the homogeneous geometry of S^n using the formalism of Section 1.1. So let $G = \mathrm{SO}(n+1)$. Then S^n is a symmetric G-space with stabilisers conjugate to $\mathrm{SO}(n)$. Fix a base-point $m \in S^n$ with stabiliser K. We then have a symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with $\mathfrak{p} \cong T_m S^n$ via the Maurer–Cartan form of S^n. To describe this decomposition, we use the familiar isomorphism $\mathfrak{g} \cong \bigwedge^2 \mathbb{R}^{n+1}$ given by

$$(a \wedge b)(x) = (a, x)b - (b, x)a. \tag{15}$$

Let $\ell_0 = \mathrm{span}_{\mathbb{R}}\{m\}$ and let V be its perpendicular complement in $\mathbb{R}^{n+1}$. We then have

$$\mathfrak{p} = \ell_0 V, \qquad \mathfrak{k} = \bigwedge\nolimits^2 V,$$

where, here and below, juxtaposition denotes tensor product. The associated bundles $[\ell_0]$ and $[V]$ are simply the normal and tangent bundles of S^n viewed as a submanifold of $\mathbb{R}^{n+1}$ and the Maurer–Cartan form β^{S^n} of S^n provides an isomorphism $TS^n \cong [\ell_0] \otimes [V] = [\mathfrak{p}] \subset \underline{\mathfrak{g}}$.

Finally, since S^n is symmetric, its canonical connection D^{S^n} is torsion-free and so coincides with the Levi–Civita connection.

Now fix $r \in \mathbb{N}$ with $2r < n$ and let $\pi : F^r(S^n) \to S^n$ be the bundle of isotropic flags over S^n with fibre

$$F^r_x(S^n) = \{w_1 \subset \cdots \subset w_r \subset (T_x S^n)^{\mathbb{C}} : \text{each } w_j \text{ is an isotropic } j\text{-plane}\}.$$

Here, isotropy is with respect to the complexified metric on $(TS^n)^{\mathbb{C}}$.

It is easy to see that G acts transitively on $F^r(S^n)$ with stabilisers conjugate to

$$\overbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}^{r \text{ times}} \times \mathrm{SO}(n - 2r).$$

Fix a base-point $(w_1 \subset \cdots \subset w_r) \in F^r_m(S^n)$ with stabiliser H and orthogonalise to obtain isotropic lines $\ell_1, \ldots, \ell_r$ and a real subspace ℓ_{r+1} in $V^{\mathbb{C}}$ so that

$$V^{\mathbb{C}} = \sum_{i=1}^{r} (\ell_i \oplus \overline{\ell_i}) \oplus \ell_{r+1}$$

is an orthogonal decomposition and

$$w_j = \ell_0^{\mathbb{C}} \otimes (\sum_{i=1}^{j} \ell_i),$$

for $1 \leq j \leq r$.

Take $k = 2r + 2$, let ω be the usual k-th root of unity and define $Q \in \mathrm{O}(n+1)$ by

$$Q = \omega^j \quad \text{on } \ell_j.$$

Let τ be the order k automorphism of G given by conjugation by Q. The identity component of the fixed set of τ is H so that $F^r(S^n)$ is a k-symmetric space. The associated reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

is given by

$$
\begin{aligned}
\mathfrak{h}^{\mathbb{C}} &= \sum_{i=1}^{r} \overline{\ell_i}\ell_i \oplus {\textstyle\bigwedge}^2 \ell_{r+1}, \\
\mathfrak{m}^{\mathbb{C}} &= \sum_{0\le i<j\le r+1} \ell_i\ell_j \oplus \sum_{0\le i\ne j\le r+1} \overline{\ell_i}\ell_j \oplus \sum_{0\le i<j\le r+1} \overline{\ell_i\ell_j}.
\end{aligned}
$$

Moreover, it is easy to see that

$$
\mathfrak{g}_1 = \ell_0^{\mathbb{C}}\ell_1 \oplus \sum_{i=1}^{r} \overline{\ell_i}\ell_{i+1}.
$$

Finally, we have an Ad H-invariant decomposition

$$
\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}
$$

where

$$
\mathfrak{l}^{\mathbb{C}} = \sum_{1\le i<j\le r+1} \ell_i\ell_j \oplus \sum_{1\le i\ne j\le r+1} \overline{\ell_i}\ell_j \oplus \sum_{1\le i<j\le r+1} \overline{\ell_i\ell_j}.
$$

This gives rise to a global splitting of $TF^r(S^n)$ into vertical and horizontal subspaces. Indeed, the Maurer–Cartan form β^F of $F^r(S^n)$ provides an isomorphism

$$
TF^r(S^n) \cong [\mathfrak{m}] = [\mathfrak{l}] \oplus [\mathfrak{p}].
$$

Let $\beta^F_{\mathfrak{l}}$ and $\beta^F_{\mathfrak{p}}$ be the corresponding components of the Maurer–Cartan form. Then we have

$$
\beta^F_{\mathfrak{p}} = \pi^*\beta^{S^n} \tag{16}
$$

so that $[\mathfrak{l}] \cong \ker d\pi$ and $[\mathfrak{p}] \cong \pi^{-1}TS^n$. We use this last to compute the vertical part $\beta^F_{\mathfrak{l}}$ of the Maurer–Cartan form. Observe that $\mathrm{ad} : [\mathfrak{l}] \to \mathfrak{o}([\mathfrak{p}])$ is an injection while, denoting the canonical connection of $F^r(S^n)$ by D^F, we have from (2) and (16)

$$
\pi^{-1}D^{S^n} = D^F + \mathrm{ad}\,\beta^F_{\mathfrak{l}} \tag{17}
$$

on $[\mathfrak{p}]$.

Finally we note the existence of tautological isotropic sub-bundles $W_1 \subset \cdots \subset W_r \subset \pi^{-1}(TS^n)^{\mathbb{C}}$ where the fibre of W_j at $(w_1 \subset \cdots \subset w_r)$ is just w_j. If we set $L_j = [\ell_j]$, then these are given by

$$
W_j = L_0^{\mathbb{C}} \otimes \sum_{i=1}^{j} L_i.
$$

We recall for later use that both the L_j and the W_j are G-invariant and so D^F-stable.

3.2 Primitive maps to $F^r(S^n)$

Let M be a Riemann surface with local holomorphic co-ordinate z and $\psi : M \to F^r(S^n)$ a map with $\pi\circ\psi = \phi : M \to S^n$. Set $T = \phi^{-1}TS^n$. Then ψ determines (and is determined by) a flag of isotropic sub-bundles

$$
\psi^{(1)} \subset \cdots \subset \psi^{(r)} \subset T^{\mathbb{C}},
$$

where $\psi^{(j)} = \psi^{-1} W_j$.

Let $\nabla = \phi^{-1} D^{S^n}$, $D = \psi^{-1} D^F$ and set $b = \psi^* \beta^F$. Then (16) and (17) pull back to give

$$b_{\mathfrak{p}} = \phi^* \beta^{S^n} \tag{18}$$

and

$$\nabla = D + \operatorname{ad} b_{\mathfrak{p}}. \tag{19}$$

Put $\nabla' = \nabla_{\partial/\partial z}$, $\nabla'' = \nabla_{\partial/\partial \overline{z}}$. We now have a criterion for ψ to be primitive:

Proposition 3.1 *$\psi : M \to F^r(S^n)$ is primitive if and only if*

- *(i) $\phi_*(\partial/\partial z)$ is a (local) section of $\psi^{(1)}$;*
- *(ii) each $\psi^{(j)}$ is stable under ∇'';*
- *(iii) if σ is a local section of $\psi^{(j)}$, $1 \leq j < r$, then $\nabla'\sigma$ is a local section of $\psi^{(j+1)}$.*

Proof Pull back the L_j to get bundles, also called L_j, over M. We know that ψ is primitive if and only if $b(\partial/\partial z)$ takes values in

$$\psi^{-1}[\mathfrak{g}_1] = L_0^{\mathbb{C}} L_1 + \sum_{i=1}^{r} \overline{L_i} L_{i+1}.$$

In particular, the horizontal part of $\psi^{-1}[\mathfrak{g}_1]$ is $L_0^{\mathbb{C}} L_1 = \psi^{(1)}$ so that, by (18), $b_{\mathfrak{p}}(\partial/\partial z)$ takes values in $\psi^{-1}[\mathfrak{g}_1]$ if and only if $\phi_*(\partial/\partial z)$ takes values in $\psi^{(1)}$.

As for $b_{\mathfrak{k}}(\partial/\partial z)$, since each $\psi^{(j)}$ is D-stable, we see from (19) that the second and third conditions of the theorem amount to demanding that

$$\operatorname{ad} b_{\mathfrak{k}}(\partial/\partial \overline{z}) \psi^{(j)} \subset \psi^{(j)} \qquad \text{for } 1 \leq j \leq r; \tag{20}$$

$$\operatorname{ad} b_{\mathfrak{k}}(\partial/\partial z) \psi^{(j)} \subset \psi^{(j+1)} \qquad \text{for } 1 \leq j < r. \tag{21}$$

Now (20) holds precisely when $b_{\mathfrak{k}}(\partial/\partial \overline{z}) L_j \subset L_1 + \cdots + L_{j-1}$, for $1 \leq j \leq r$, which means that $b_{\mathfrak{k}}(\partial/\partial \overline{z})$ takes values in

$$\sum_{j=1}^{r} \overline{L_{j+1}} (L_1 + \cdots + L_j)$$

or, equivalently, that $b_{\mathfrak{k}}(\partial/\partial z) = \overline{b_{\mathfrak{k}}(\partial/\partial \overline{z})}$ takes values in

$$\sum_{j=1}^{r} \overline{L_j} (L_{j+1} + \cdots + L_{r+1}).$$

On the other hand, (21) holds when $b_{\mathfrak{k}}(\partial/\partial z) L_j \subset L_1 + \cdots + L_{j+1}$, for $1 \leq j < r$, so that (20) and (21) hold simultaneously precisely when $b_{\mathfrak{k}}(\partial/\partial z)$ takes values in

$$\sum_{i=1}^{r} \overline{L_i} L_{i=1}$$

which is the vertical part of $\psi^{-1}[\mathfrak{g}_1]$. $\square$

Of course, if ψ is primitive, then, by Proposition 1.3, ϕ is harmonic and is, moreover, conformal since $\phi_*(\partial/\partial z)$ takes values in the isotropic bundle $\psi^{(1)}$. Let us now turn to a converse of these constructions.

3.3 Twistor lifts

Let $\phi : M \to S^n$ be a harmonic map. We begin by recalling some results from the well-developed twistor theory of such maps. As before, let $T = \phi^{-1}TS^n$ with connection ∇ and inductively define local sections $\nabla^j\phi$ of $T^{\mathbb{C}}$ by

$$\nabla^1\phi = \phi_*\partial/\partial z, \qquad \nabla^{j+1} = \nabla'\nabla^j\phi.$$

From this we obtain (globally defined) differentials

$$\eta^j = (\nabla^j\phi, \nabla^j\phi)\,\mathrm{d}z^{2j}$$

with the following properties [**11**,**12**]:

1. η^1 is a holomorphic differential which vanishes if and only if ϕ is (weakly) conformal;
2. If $\eta^1, \dots, \eta^{j-1}$ all vanish, then η^j is a holomorphic differential.

If all the η^j vanish, ϕ is said to be *superminimal* (or *real isotropic* or *pseudo-holomorphic*). In this case, we have

$$(\nabla^i\phi, \nabla^j\phi) \equiv 0, \tag{22}$$

for all $i, j \in \mathbb{N}$. Superminimal maps were classified by Calabi [**12**] (see Section 3.4 below).

For $x \in M$, define $W_x^j \subset T_x^{\mathbb{C}}$ by

$$W_x^j = \mathrm{span}_{\mathbb{C}}\{\nabla_x^i\phi : 1 \le i \le j\}.$$

Clearly, each W_x^j is defined independently of the choice of holomorphic co-ordinate z.

If ϕ is non-constant then W_x^1 is 1-dimensional off a discrete set of points in M and ϕ is weakly conformal if and only if each W_x^1 is isotropic. With this in mind, we make the following definition.

Definition The *isotropy dimension* r of a conformal harmonic map $\phi : M \to S^n$ is given by

$$r = \max\{j : \max_x \dim_{\mathbb{C}} W_x^j = j \text{ and } W_x^j \text{ is isotropic for all } x\}.$$

We make the convention that a non-conformal map has isotropy dimension zero.

The following facts are well known (c.f. [**28**]): if ϕ has isotropy dimension $r > 0$ then

1. For $1 \le j \le r+1$, there is a bundle W^j of $T^{\mathbb{C}}$ whose fibre at x coincides with W_x^j except at a discrete set of points;
2. Each W^j is stable under ∇'';

3. For $1 \leq j \leq r$, $\operatorname{rank} W^j = j$ and $\operatorname{rank} W^{r+1} \leq r+1$.

We therefore have two possibilities: either $W^r = W^{r+1}$ or W^r has codimension 1 in W^{r+1} and W^{r+1} is not isotropic.

The first case occurs precisely when ϕ is superminimal: here W^r is isotropic and ∇'-stable so that (22) holds.

Remark It is clear that the isotropy dimension r of a map $\phi : M \to S^n$ must satisfy $2r \leq n$. In case that $2r = n$, it follows from the easily verified fact

$$(\nabla^{r+1}\phi, \nabla^j\phi) \equiv 0,$$

for $1 \leq j \leq r$, that $W^r = W^{r+1}$ so that ϕ is superminimal.

Remark When ϕ is not superminimal, we have the following alternative characterisation of the isotropy dimension r:

$$r = \max\{j : \eta^j \equiv 0\}.$$

Now let us take ϕ to be a non-superminimal harmonic map of isotropy dimension r (so that $2r < n$). Thus $W^r \neq W^{r+1}$ or, equivalently,

$$(\nabla^{r+1}\phi, \nabla^{r+1}\phi) \not\equiv 0.$$

In this case, η^{r+1} is a non-zero holomorphic differential so that $(\nabla^{r+1}\phi, \nabla^{r+1}\phi)$ is non-zero off a discrete set of points.

Consider the flag of bundles

$$W^1 \subset \cdots \subset W^r \subset T^{\mathbb{C}}.$$

This defines a map $\psi : M \to F^r(S^n)$ with $\pi \circ \psi = \phi$ and it is clear from Proposition 3.1 that ψ so defined is primitive. We have therefore almost proved:

Theorem 3.2 *A non-superminimal weakly conformal harmonic map of isotropy dimension $r > 0$ is covered by a unique primitive map $\psi : M \to F^r(S^n)$.*

Proof It only remains to prove the uniqueness assertion. However, it is clear from Proposition 3.1 that if $\psi : M \to F^r(S^n)$ is any primitive map covering ϕ then, for all $x \in M$,

$$\operatorname{span}_{\mathbb{C}}\{\nabla^i_x\phi : 1 \leq i \leq j\} \subset \psi^j_x,$$

for $1 \leq j \leq r$. However, for all but a discrete set of $x \in M$, both sides of this are j-dimensional so that $W^j = \psi^j$ for $1 \leq j \leq r$. □

The key point now is that the primitive map we have constructed satisfies the hypotheses of Proposition 2.5:

Theorem 3.3 *$\psi^*\beta^F(\partial/\partial z)$ is semisimple off a discrete set of points.*

Proof For all $x \in M$ off a discrete set, we have

1. $\nabla^1_x\phi \neq 0$;

2. $\nabla_x^j \phi$ has non-zero projection onto W^j/W^{j-1}, for $1 < j \leq r+1$;

3. $\nabla_x^{r+1}\phi$ is not isotropic.

By virtue of (19), these are equivalent to the following properties of $b_x(\partial/\partial z) \in L_0^{\mathbb{C}} L_1 + \sum_{i=1}^r \overline{L_i} L_{i+1}$:

1. $b_x(\partial/\partial z)L_0^{\mathbb{C}} \subset L_1$ is non-zero;

2. $b_x(\partial/\partial z)L_j \subset L_{j+1}$ is non-zero for $1 \leq j \leq r$;

3. $b_x(\partial/\partial z)L_r \subset L_{r+1}$ is not isotropic.

Set $L = b_x(\partial/\partial z)L_r \subset L_{r+1}$. Then the component of $b_x(\partial/\partial z)$ in $\overline{L_r}L_{r+1}$ can be written as $\overline{l_r} \otimes l$, $l_r \in L_r$, $l \in L$ with (l,l) non-zero. It then follows from (15) that

$$b_x(\partial/\partial z)l = -(l,l)\overline{l_r}$$

so that $b_x(\partial/\partial z)L$ is non-zero also.

Set $K = \ker b_x(\partial/\partial z)$. Then $K \subset L_{r+1}$ and

$$L_{r+1} = L \oplus K.$$

Moreover, the k-dimensional complement

$$S = L_0^{\mathbb{C}} \oplus \sum_{i=1}^{r}(\overline{L_i} \oplus L_i) \oplus L$$

is $b_x(\partial/\partial z)$-invariant. Let A be the restriction of $b_x(\partial/\partial z)$ to S. Clearly it suffices to prove that A is semisimple. However, we see from the above that if $l_0 \in L_0$ is non-zero then $A^j l_0$ is non-zero for all $j \in \mathbb{N}$. Moreover, $\{A^j l_0 : 0 \leq j \leq k-1\}$ is a basis for S so that

$$A^k = c\,\mathrm{id}_S$$

where c is non-zero. From this, it is easy to deduce that A has all eigenvalues distinct and so is semisimple. □

We now get the main result of this section as an immediate corollary to the above and Proposition 2.5:

Theorem 3.4 *Let $\phi : T^2 \to S^n$ be a weakly conformal non-superminimal harmonic map of a 2-torus with isotropy dimension r. Then ϕ is covered by a unique primitive map of finite type $\psi : T^2 \to F^r(S^n)$.*

In conclusion, we see that any non-superminimal harmonic 2-torus in S^n is either of finite type or becomes so after prolongation.

Remark Special cases of these results have already appeared in the literature [**6**,**19**]. To put these cases into our context, recall that a non-superminimal harmonic map into S^n must have isotropy dimension r with $2r < n$. Suppose $n = 2m$ and consider such maps of maximal isotropy dimension $m-1$. These maps may be alternatively characterised by the condition that $\eta^1, \dots, \eta^{m-1}$ vanish while η^m does not (if, in fact, η^m also vanishes then the map is superminimal). Again, if $n = 2m-1$, the maximal isotropy dimension is $m-1$ and this case reduces to the previous one by viewing S^n as an equator of S^{2m}.

In this setting, our main results are due to Ferus–Pedit–Pinkall–Sterling [**19**] when $n = 4$ and Bolton–Pedit–Woodward [**6**] in general. In both these papers, advantage is taken of the fact that, in this case, the primitive maps under discussion have a framing which arises from a solution to the affine Toda field equations.

Remark What can Theorem 3.2 tell us about non-superminimal harmonic maps from surfaces of higher genus? The problem here is that, on such surfaces, the holomorphic differential η^{r+1} must necessarily vanish on a non-empty discrete set and it is clear from the argument of Theorem 3.3 that, at these points, $b_x(\partial/\partial z)$ is nilpotent. As a consequence, $b(\partial/\partial z)$ can never lie in a single $\mathrm{Ad}\, G$-orbit and so cannot arise from the constructions of this paper.

The problem of finding a systematic approach to harmonic maps of higher genus surfaces remains one of the most tantalising in the area.

3.4 Superminimal maps

For completeness of exposition and to show how the results compare with those we have obtained above, we briefly sketch the analysis of superminimal harmonic maps into S^n due to Calabi, Chern and others [**3**,**11**,**12**,**13**].

So let $\phi : M \to S^n$ be a superminimal harmonic map of isotropy dimension r. Thus $W^r = W^{r+1}$ so that W^r is ∇'-stable. Since W^r is automatically ∇''-stable, it is parallel and then

$$(W^r \oplus \overline{W^r}) \cap T$$

is a parallel sub-bundle of T with (real) rank $2r$. It follows that ϕ factors through an equatorial $2r$-sphere in S^n so that, without loss of generality, we may take $2r = n$.

Consider now the bundle of flags $F^r(S^{2r})$ defined as before with fibre

$$F^r_m(S^{2r}) = \{w_1 \subset \cdots \subset w_r \subset (T_m S^{2r})^{\mathbb{C}} : \text{each } w_j \text{ is an isotropic } j\text{-plane}\}.$$

Since w_r is a maximal isotropic subspace of $(T_m S^{2r})^{\mathbb{C}}$, it determines an almost complex structure on $T_m S^{2r}$ and so an orientation. As a consequence, $F^r(S^{2r})$ has two connected components and $\mathrm{SO}(2r+1)$ acts transitively on each of these with stabilisers conjugate to the maximal torus of $\mathrm{SO}(2r+1)$:

$$\overbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}^{r \text{ times}}.$$

Each component is therefore a realisation of the full flag manifold of $\mathrm{SO}(2r+1)$ and, as such, is a homogeneous Kähler manifold. In fact, under the usual isomorphisms, the $(1,0)$ tangent spaces for this Kähler structure are given by

$$L_0^{\mathbb{C}} \otimes (\sum_{i=1}^{r} L_i) \oplus \sum_{1 \le i < j \le r} L_i L_j \oplus \sum_{1 \le i < j \le r} \overline{L_i} L_j.$$

In the same way as before, we can equip each component with a canonical k-symmetric structure but this time $k = 2r + 1$ (as $L_{r+1} = \{0\}$) and

$$[\mathfrak{g}_1] = L_0^{\mathbb{C}} L_1 \oplus \sum_{i=1}^{r-1} \overline{L_i} L_{i+1}.$$

We now see how this situation differs from that for non-superminimal maps. Observe that $[\mathfrak{g}_1]$ contains no semisimple elements at all: in fact it consists entirely of nilpotents. Indeed, $[\mathfrak{g}_1] \subset T^{1,0}F^r(S^{2r})$ and is the *superhorizontal* distribution in the sense of Burstall–Rawnsley [**10**].

However, the analysis of the preceding sections goes through to show that $W_1 \subset \cdots \subset W_r$ defines a primitive map ψ into a component of $F^r(S^{2r})$ which, in this setting, amounts to saying that ψ is a superhorizontal holomorphic map.

We therefore conclude that a superminimal map $\phi : M \to S^{2r}$ of isotropy dimension r is covered by a superhorizontal holomorphic map $\psi : M \to F^r(S^{2r})$. We apply this result in two ways. Firstly recall that the twistor space Z of S^{2r} may be viewed as the bundle of isotropic r-planes in $(TS^{2r})^{\mathbb{C}}$. Again Z has two components each of which can be realised as the generalised flag manifold $\mathrm{SO}(2r+1)/\mathrm{U}(r)$. There is a homogeneous projection $F^r(S^{2r}) \to Z$ given by $(w_1 \subset \cdots \subset w_r) \mapsto w_r$ and this map is holomorphic and preserves superhorizontal distributions. As a consequence, we see that W_r defines a horizontal holomorphic map into Z and we have proved the following theorem of Calabi [**12**]:

Theorem 3.5 *A superminimal map $M \to S^{2r}$ is covered by a horizontal holomorphic map $M \to Z$.*

In fact we can say more: we also have a holomorphic projection from $F^r(S^{2r})$ to the quadric $\mathrm{SO}(2r+1)/\mathrm{SO}(2) \times \mathrm{SO}(2r-1)$ under which ψ projects to the holomorphic map $M \to \mathbb{C}P^{2r}$ given by $\overline{L_r}$. This map is the *directrix curve* defined by Chern [**13**] and is *totally isotropic* in the sense that its Frenet frame $\overline{L_r}, \overline{L_{r-1}}, \ldots, \overline{L_1}$ consists of isotropic lines. Clearly, we may recover both ψ and our original harmonic map ϕ from this Frenet frame, the first as the span of the L_i and the second as the perpendicular complement of the L_i and $\overline{L_i}$. We have therefore proved the following result of Chern [**13**] (see also [**3**]):

Theorem 3.6 *To a superminimal map $\phi : M \to S^{2r}$ is associated a totally isotropic holomorphic map $M \to \mathbb{C}P^{2r}$ from whose Frenet frame ϕ may be reconstructed.*

4 Harmonic surfaces in $\mathbb{C}\mathrm{P}^n$

We now show that the salient features of our analysis of non-superminimal harmonic surfaces in S^n also obtain when we take a complex projective space $\mathbb{C}\mathrm{P}^n$ as target. In particular, any such harmonic map $M \to \mathbb{C}\mathrm{P}^n$ is covered by a primitive map which is of finite type when M is a 2-torus.

The ideas and methods here are very similar to those of Section 3 and consequently we shall adopt a brisker style of exposition.

4.1 Flag manifolds as k-symmetric spaces

Let $G = \mathrm{SU}(n+1)$. The complex projective space $\mathbb{C}\mathrm{P}^n$ is a Hermitian symmetric G-space with stabilisers conjugate to $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$. Fix a base-point $\ell_0 \in \mathbb{C}\mathrm{P}^n$ with stabiliser K and view ℓ_0 as a line in $\mathbb{C}^{n+1}$. Then in the symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we have

$$\mathfrak{p}^{\mathbb{C}} = \mathrm{Hom}(\ell_0, \ell_0^{\perp}) \oplus \mathrm{Hom}(\ell_0^{\perp}, \ell_0)$$

and $\mathrm{Hom}(\ell_0, \ell_0^{\perp}) \cong T^{1,0}_{\ell_0}\mathbb{C}\mathrm{P}^n$ via the Maurer–Cartan form. The associated bundles $L_0 = [\ell_0]$ and $L_0^{\perp} = [\ell_0^{\perp}]$ are just the tautological bundle of $\mathbb{C}\mathrm{P}^n$ and its perpendicular complement. The Maurer–Cartan form then gives the familiar isomorphism

$$T^{1,0}\mathbb{C}\mathrm{P}^n \cong \mathrm{Hom}(L_0, L_0^{\perp}).$$

Now fix $r \in \mathbb{N}$ with $r < n$ and let $\pi : F^r(\mathbb{C}\mathrm{P}^n) \to \mathbb{C}\mathrm{P}^n$ be the bundle of flags over $\mathbb{C}\mathrm{P}^n$ with fibre

$$F^r_x(\mathbb{C}\mathrm{P}^n) = \{w_1 \subset \cdots \subset w_r \subset T^{1,0}_x\mathbb{C}\mathrm{P}^n : \text{each } w_j \text{ is a } j\text{-plane}\}.$$

G acts transitively on $F^r(\mathbb{C}\mathrm{P}^n)$ with stabilisers conjugate to

$$\mathrm{S}(\overbrace{\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)}^{r \text{ times}}\times\mathrm{U}(n-r))$$

so that $F^r(\mathbb{C}\mathrm{P}^n)$ is a flag manifold.

However, the important point for us is that $F^r(\mathbb{C}\mathrm{P}^n)$ admits a k-symmetric structure. For this, fix a base-point $(w_1 \subset \cdots \subset w_r) \in F^r(\mathbb{C}\mathrm{P}^n)$ with stabiliser H and orthogonalise to get lines $\ell_1, \dots, \ell_r$ and an $(n-r)$-plane ℓ_{r+1} so that

$$\sum_{i=0}^{r+1} \ell_i = \mathbb{C}^{n+1}$$

is an orthogonal decomposition and

$$w_j = \mathrm{Hom}(\ell_0, \sum_{i=1}^{j} \ell_i),$$

for $1 \le j \le r$. Take $k = r+2$, let ω be the usual k-th root of unity and define $Q \in \mathrm{U}(n+1)$ by

$$Q = \omega^j \quad \text{on } \ell_j.$$

Let τ be the order k automorphism of G given by conjugation by Q. The fixed set of τ is H so that $F^r(\mathbb{C}\mathrm{P}^n)$ is a k-symmetric space. In the corresponding reductive decomposition we have

$$\mathfrak{m}^{\mathbb{C}} = \sum_{i \neq j} \mathrm{Hom}(\ell_i, \ell_j)$$

and

$$\mathfrak{g}_1 = \sum_{i=0}^{r} \mathrm{Hom}(\ell_i, \ell_{i+1}) \oplus \mathrm{Hom}(\ell_{r+1}, \ell_0).$$

Remark This is an example of a rather general construction: any generalised flag manifold $G^{\mathbb{C}}/P$, where P is a parabolic subgroup of the complexification of a compact semi-simple group G, has a canonical k-symmetric structure [**10**, p. 52]. In case that G is simple and P is a Borel subgroup, the corresponding automorphism τ is the Coxeter–Killing automorphism.

Globalising matters, we have tautological bundles $L_j = [\ell_j]$ on $F^r(\mathbb{C}\mathrm{P}^n)$ and the Maurer–Cartan form β^F of $F^r(\mathbb{C}\mathrm{P}^n)$ gives an isomorphism

$$TF^r(\mathbb{C}\mathrm{P}^n)^{\mathbb{C}} \cong \sum_{i\neq j} \mathrm{Hom}(L_i, L_j).$$

Again we have a decomposition into horizontal and vertical bundles with

$$\pi^{-1}T^{1,0}\mathbb{C}\mathrm{P}^n \cong \sum_{i=1}^{r+1} \mathrm{Hom}(L_0, L_i)$$

and the vertical part $\beta^F_{\mathfrak{k}}$ of β^F is determined by

$$\pi^{-1}D^{\mathbb{C}\mathrm{P}^n} = D^F + \mathrm{ad}\,\beta^F_{\mathfrak{k}} \tag{23}$$

on $\pi^{-1}T^{1,0}\mathbb{C}\mathrm{P}^n$.

Now let $\psi : M \to F^r(\mathbb{C}\mathrm{P}^n)$ be a map of a Riemann surface with $\pi \circ \psi = \phi : M \to \mathbb{C}\mathrm{P}^n$. Set $T^{1,0} = \phi^{-1}T^{1,0}\mathbb{C}\mathrm{P}^n$. The ψ determines (and is determined by) a flag of sub-bundles

$$\psi^{(1)} \subset \cdots \subset \psi^{(r)} \subset T^{1,0}$$

Let $\nabla = \phi^{-1}D^{\mathbb{C}\mathrm{P}^n}$, the pull-back of the Levi–Civita connection on $\mathbb{C}\mathrm{P}^n$ and, as usual, set $\nabla' = \nabla_{\partial/\partial z}$, $\nabla'' = \nabla_{\partial/\partial\bar{z}}$. Define local sections δ', δ'' of $T^{1,0}$ by

$$\phi_*\partial/\partial z = \delta' + \overline{\delta''}.$$

We may then argue as in Proposition 3.1, using (23), to get:

Proposition 4.1 *$\psi : M \to F^r(\mathbb{C}\mathrm{P}^n)$ is primitive if and only if*

(i) δ' is a local section of $\psi^{(1)}$ and δ'' is perpendicular to $\psi^{(r)}$;

(ii) each $\psi^{(j)}$ is stable under ∇'';

(iii) if σ is a local section of $\psi^{(j)}$, $1 \leq j < r$, then $\nabla'\sigma$ is a local section of $\psi^{(j+1)}$.

The only new ingredient here is the condition on δ'': this arises since we must have $\phi_*\partial/\partial z$ taking values in the horizontal part of $\psi^{-1}[\mathfrak{g}_1]$ which is

$$\mathrm{Hom}(L_0, L_1) \oplus \mathrm{Hom}(L_{r+1}, L_0).$$

Thus $\overline{\delta''}$ must be a local section of $\mathrm{Hom}(L_{r+1}, L_0)$ so that δ'' takes values in $\mathrm{Hom}(L_0, L_{r+1}) = (\psi^{(r)})^{\perp}$ (recall that complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ is given by $\xi \mapsto -\xi^*$).

With all this in place, we can now turn to the twistor theory of harmonic maps $M \to \mathbb{C}\mathrm{P}^n$.

4.2 Twistor lifts over $\mathbb{C}\mathrm{P}^n$

Let $\phi : M \to \mathbb{C}\mathrm{P}^n$ be a harmonic map. As before, let $T^{1,0} = \phi^{-1}T^{1,0}\mathbb{C}\mathrm{P}^n$ with connection ∇ and define local sections $\nabla^j\phi$ of $T^{1,0}$ by

$$\nabla^{-1}\phi = \delta'', \qquad \nabla^1\phi = \delta', \qquad \nabla^{j+1}\phi = \nabla'\nabla^j\phi, \quad \text{for } j > 1.$$

Using the Hermitian inner product $\langle\ ,\ \rangle$ on $T^{1,0}$, we obtain globally defined differentials

$$\gamma^j = \langle \nabla^j\phi, \nabla^{-1}\phi\rangle\, \mathrm{d}z^{j+1},$$

$j \geq 1$, with the now familiar properties [**17**]:

1. γ^1 is a holomorphic differential which vanishes if and only if ϕ is weakly conformal;
2. If $\gamma^1, \ldots, \gamma^{j-1}$ all vanish then γ^j is a holomorphic differential.

If all the γ^j vanish, ϕ is said to be (complex) isotropic [**17**]. The isotropic harmonic maps were independently classified by Eells–Wood and others [**8**,**14**,**17**,**20**]: they all arise as legs of the Frenet frame of a holomorphic map $M \to \mathbb{C}\mathrm{P}^n$ (equivalently, they are projections of superhorizontal, holomorphic maps into the full flag manifold of $\mathrm{SU}(n+1)$).

For $x \in M$, $j \in \mathbb{N}$, define $W^j_x \subset T^{1,0}_x$ by

$$W^j_x = \mathrm{span}_{\mathbb{C}}\{\nabla^i_x\phi : 1 \leq i \leq j\}$$

and, in addition, define $W^{-1}_x \subset T^{1,0}_x$ by

$$W^{-1}_x = \mathrm{span}_{\mathbb{C}}\{\nabla^{-1}_x\phi\}.$$

When ϕ is weakly conformal, $W^1_x \perp W^{-1}_x$, for all $x \in M$, and when, in addition, ϕ is not $\pm$holomorphic both W^1_x and W^{-1}_x are 1-dimensional off a discrete set of points.

Definition The *isotropy dimension* r of a non-$\pm$holomorphic, weakly conformal harmonic map $\phi : M \to \mathbb{C}\mathrm{P}^n$ is given by

$$r = \max\{j \in \mathbb{N} : \max_x \dim_{\mathbb{C}} W^j_x = j \text{ and } W^j_x \perp W^{-1}_x \text{ for all } x\}.$$

If ϕ has isotropy dimension $r > 0$ then Wood proves [**28**]:

1. For $1 \leq j \leq r+1$, there is a bundle W^j of $T^{1,0}$ whose fibre at x coincides with W^j_x except at a discrete set of points;
2. Each W^j is stable under ∇'';
3. For $1 \leq j \leq r$, $\operatorname{rank} W^j = j$ and $\operatorname{rank} W^{r+1} \leq r+1$.

Again we have two possibilities: either $W^r = W^{r+1}$ or not.

The first case occurs precisely when ϕ is isotropic: here W^r is orthogonal to W^{-1} and stable under ∇' so that

$$\langle \nabla^j\phi, \nabla^{-1}\phi\rangle \equiv 0,$$

for all $j \in \mathbb{N}$.

Remark In [**1**], Bahy-El-Dien–Wood define the *isotropy order* of a non-isotropic harmonic map $M \to \mathbb{C}P^n$. One can easily show that such a map has isotropy dimension r if and only if it has isotropy order $r+1$.

Let $\phi : M \to \mathbb{C}P^n$ be a non-isotropic conformal harmonic map of isotropy dimension r. Then $W^r \neq W^{r+1}$ and we have

$$\langle \nabla^{r+1}\phi, \nabla^{-1}\phi \rangle \not\equiv 0.$$

Now γ^{r+1} is a non-zero holomorphic differential so that $\langle \nabla^{r+1}\phi, \nabla^{-1}\phi \rangle$ is non-zero off a discrete set of points.

The bundle of flags

$$W^1 \subset \cdots \subset W^r \subset T^{1,0}$$

defines a map $\psi : M \to F^r(\mathbb{C}P^n)$ with $\pi \circ \psi = \phi$ and, by Proposition 4.1, we see that ψ is primitive. We now argue as in Theorem 3.2 to conclude:

Theorem 4.2 *A non-isotropic weakly conformal harmonic map of isotropy dimension $r > 0$ is covered by a unique primitive map $\psi : M \to F^r(\mathbb{C}P^n)$.*

As for the semisimplicity of $\psi^*\beta^F$, let b_j denote the component of $\psi^*\beta^F$ in $\mathrm{Hom}(L_j, L_{j+1})$, $1 \leq j \leq r$, and let b_{r+1} be that in $\mathrm{Hom}(L_{r+1}, L_0)$. We use (23) to see that all the b_j are non-zero, off a discrete set of points. Moreover, whenever $\langle \nabla^{r+1}\phi, \nabla^{-1}\phi \rangle$ is non-zero, we can see that $b_{r+1} \circ b_r$ is non-zero and then argue as in Theorem 3.3 to prove:

Theorem 4.3 *$\psi^*\beta^F(\partial/\partial z)$ is semisimple off a discrete set of points.*

This, together with Proposition 2.5, immediately yields the main result of the section:

Theorem 4.4 *Let $\phi : T^2 \to \mathbb{C}P^n$ be a weakly conformal non-isotropic harmonic map of a 2-torus with isotropy dimension r. Then ϕ is covered by a unique primitive map of finite type $\psi : T^2 \to F^r(\mathbb{C}P^n)$.*

Thus we conclude that any non-isotropic harmonic 2-torus in $\mathbb{C}P^n$ is either of finite type or becomes so after prolongation.

Remark It is clear that a non-isotropic harmonic map $\phi : M \to \mathbb{C}P^n$ can have isotropy dimension at most $n-1$. Non-isotropic maps with this maximal isotropy dimension were called *superconformal* by Bolton–Pedit–Woodward [**6**] and, for such maps, the main result of this section is due to those authors.

Remark It is natural to ask whether the results of this paper can be extended to treat harmonic 2-tori in other rank one symmetric spaces. Certainly, non-conformal harmonic 2-tori in such a space are of finite type. What is lacking is a suitable prolongation of conformal harmonic maps in such spaces. This deficiency is present even in the twistor theory of harmonic 2-spheres which is not nearly so well-developed for quaternionic projective spaces, say, as it is for the cases we have been considering (see, however, [**2**]).

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