On the existence of superconformal 2-tori in complex projective space.*

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1 Introduction.

The superconformal 2-tori in $\mathbf{CP}^n$ are (by definition, *cf.* [5]) those harmonic tori which lie in an orthogonally periodic harmonic sequence. The purpose of this article is to describe the conditions for their existence and determine the number of moduli available to satisfy these conditions.

From [5] one knows that each superconformal 2-torus corresponds to a doubly periodic solution of the elliptic Toda equations (although the reverse is not true). These are the equations

$$\frac{\partial^2}{\partial z \partial \bar{z}} \ln(s_j^2) = s_{j+1}^2 s_j^{-2} - s_j^2 s_{j-1}^{-2}, \tag{1}$$

where z is a coordinate on the uniformizing space $\mathbf{C}$ of the torus and each $s_0, \ldots, s_n$ is a strictly positive real-valued function of $z, \bar{z}$. For some choices of n some examples of superconformal 2-tori are already known. For example, the Toda equations admit an elementary doubly periodic solution (often called *the trivial solution*) for which $s_j = 1$ for each j. Bolton & Woodward [4] state that in the cases $n = 1, 2, 3, 5$ (and only in these cases) there is a flat superconformal 2-torus corresponding to this solution and they present each one explicitly. Moreover, for $n = 1$ the Toda equations reduce to the elliptic sinh-Gordon equation studied by Bobenko [2, 3] and the superconformal 2-tori are precisely the nowhere conformal 2-tori in $\mathbf{CP}^1$ (this unfortunate contradiction in nomenclature only occurs for $n = 1$; when $n \geq 2$ superconformal maps are genuinely [weakly] conformal *cf.* [5]). An account of these is given by Hitchin in [18].

To study the existence of other examples we look at the non-trivial doubly periodic solutions of the Toda equations. It turns out that all of these come from the algebro-geometric construction described in [19, 20]. By carrying this construction over to the construction of superconformal 2-tori outlined in [5] we arrive at the following result, which summarises the situation.

Theorem 1 *Each superconformal 2-torus in* $\mathbf{CP}^n$ *determines a real, irreducible, complete algebraic curve* X *(which we will call its* spectral curve*) equipped with a*

*Completed with the support of SERC research grant GR/J33043.

rational function π of degree $n+1$ possessing only one zero and one pole. Conversely, if the arithmetic genus of X is $p > 2$, the pair (X, π) determines a real $(p-2)$-dimensional analytic family of superconformal 2-tori. However, when $p \geq 2$ such a pair (X, π) typically has only $2p$ real moduli on which there are $2p+2n-4$ constraints.

More precisely, the number of moduli in (X, π) depends upon the order of those branch points of the cover $\pi : X \to \mathbf{CP}^1$ which do not lie at 0 or ∞. The case of general position, when these are all of order one, gives $2p$ moduli. The $2p + 2n - 4$ existence conditions are of two kinds. There are $2p - 4$ conditions which govern the existence of a non-trivial doubly periodic Toda solution: these remain independent as the moduli of X are varied. However, the independence of the other $2n$ conditions, which govern the double periodicity of the corresponding harmonic 2-plane in $\mathbf{CP}^n$, depends upon the location and order of the branch points of (X, π). Their number can be reduced by moving off the open set in general position. Since this also decreases the number of moduli, individual cases must be examined to see whether or not this improves the situation. It is also necessary to treat the cases where $p = 0$ or 1 individually (it is the case $p = 0$, where $X \simeq \mathbf{CP}^1$, which corresponds to the flat tori described in [4]).

Despite the fact that it is possible to write down the existence conditions quite explicitly I have not attempted to prove that these $2p + 2n - 4$ conditions are independent in general position: this difficult task will have to wait for another time. Instead I have settled for listing (in section 5) the circumstances under which the number of moduli is no less than the number of conditions given various possibilities for the location of branch points: for $n > 1$ this only occurs away from the case of general position. I show that, by a judicious choice of branch points, one can satisfy the conditions required to produce new examples of superconformal 2-tori in $\mathbf{CP}^n$ for $n = 2$ or 4. Moreover, it is possible to write down the equations for some of these spectral curves; the details are in section 5.3. These examples all have $p \leq 2$, in which case one set of conditions (those governing the periodicity of the corresponding Toda solution) disappear and the analysis simplifies slightly. Analysing which values of $p > 2$ permit 2-tori seems to be a much more difficult problem. It is comparable with the problem studied by Ercolani *et al.* [13] concerning the existence of constant mean curvature (cmc) 2-tori in $\mathbf{R}^3$. They found that for every even genus there exists a spectral curve for a cmc 2-torus: the corresponding Gauss map is then a superconformal (sic) 2-torus in $\mathbf{CP}^1$.

Now let me outline how this article proceeds. First we collect together the relevant results from [5], the principal result being that each superconformal 2-torus possesses a doubly periodic *Toda frame:* this is a map from the covering space $\mathbf{C}$ of the torus into SU_{n+1} whose left-invariant derivative only depends upon the corresponding solutions to the Toda equations (1). It is easy to see that Toda frames come in S^1-families (or *loops*), which puts us into the context of the dressing construction described in [19, 20]. One knows from there how to construct loops of Toda frames which possess a spectral curve, and the relevant part of the construction is described in section 4. This introduces the *scalar Baker functions* which may be explicitly computed. Using one of these functions we derive conditions on the pair

(X, π) which characterise the doubly periodic Toda frames. These are described in section 5, where there is also a description of the cases under which we can deduce the existence of superconformal 2-tori.

Throughout all this the fundamental assumption is that this construction gives *all* superconformal 2-tori. The justification of this is the most complicated step in proving theorem 1 and is left until section 6. It follows partly from results in [5] and recent work of F E Burstall & F Pedit [9, 10]. From these sources one learns that every superconformal 2-torus is of *finite type* (in the same sense as [8]) and that all finite type maps come from the dressing construction used in this article. It only remains for me to show that this implies each superconformal 2-torus has a spectral curve in the precise sense needed here.

Finally, there is an appendix dealing with the relationship between the construction of a spectral curve used here and that used in the article [14] which studies harmonic 2-tori in S^4 arising from the elliptic Toda equations for $so(5)$. The alternative definition used there fits in with the general program (outlined in [8]) for constructing harmonic 2-tori by solving certain 'algebraically completely integrable' o.d.e's. Since the approach I use here is less well-known it seems appropriate to provide a translation between the two approaches.

Acknowledgments. I am grateful to J Bolton, F E Burstall, F Pedit and L M Woodward for keeping me informed of their work [4, 5, 7, 9, 10] and for their enlightening remarks.

2 Superconformal 2-tori and the Toda equations.

This section summarises the results of [5] (see also [7]) which explain the link between superconformal tori and the elliptic Toda equations. This requires the notion of a *primitive* harmonic map into G/T (which is essentially the same as the term 'τ-primitive map' used in [5]) where T is the maximal torus of diagonal matrices in $G = SU_{n+1}$.

The full flag manifold G/T is an $n+1$-symmetric space *i.e.* T is the fixed point subgroup of a periodic automorphism ν on G which has period $n+1$. This automorphism, which we will call the Coxeter automorphism, is defined by $\nu(g) = \sigma g \sigma^{-1}$ where σ^{-1} is the diagonal matrix $diag(1, \omega, \ldots, \omega^n)$ and ω is the $n+1$-st root of unity $\exp(2\pi i/(n+1))$. This induces an automorphism on the Lie algebra $\mathbf{g} = su_{n+1}$ which has the same description and is completely characterized by its eigenspace decomposition of the complexification $\mathbf{g}^C = sl_{n+1}$, which we write as

$$\mathbf{g}^C = \mathbf{t}^C + \mathbf{g}_1 + \ldots + \mathbf{g}_n.$$

Here $\mathbf{t}^C$ is the complexification of $\mathbf{t}$, the Lie algebra of T. The subspaces $\mathbf{g}_k$ are the eigenspaces for eigenvalue ω^k. In particular $\mathbf{g}_1$ consists of all the matrices in sl_{n+1}

of the form

$$\begin{pmatrix} 0 & * & & & \\ & . & * & & \\ & & . & . & \\ & & & . & * \\ * & & & & 0 \end{pmatrix}$$

where a '$*$' denotes a possible non-zero entry. We will say an element of $\mathbf{g}_1$ is *cyclic* if all these entries are non-zero.

Let ξ^* denote the complex conjugate transpose of an element in $\mathbf{g}^C$. The conjugation $\xi \mapsto -\xi^*$ commutes with ν and the splitting $\mathbf{g} = \mathbf{t} + \mathbf{m}$ is reductive when $\mathbf{m}^C$ denotes $\mathbf{g}_1 + \ldots + \mathbf{g}_n$. Now recall (from *e.g.* [11]) that the Maurer-Cartan form β for G/T is the endomorphism of vector bundles which identifies the tangent bundle $T(G/T)$ with the bundle $[\mathbf{m}] = G \times_T \mathbf{m}$, where T acts adjointly on $\mathbf{m}$.

Throughout this article $\mathbf{T}^2$ will denote a 2-torus viewed as the quotient of the complex plane $\mathbf{C}$ by a lattice: since harmonic maps from a Riemann surface depend only on the conformal structure induced by the metric we need not be more precise. We will say a real-analytic map $\phi : \mathbf{T}^2 \to G/T$ is *primitive* if $\phi^*\beta^{(1,0)}$ takes values in $[\mathbf{g}_1] = G \times_T \mathbf{g}_1$. One knows from [1] that all primitive maps are equi-harmonic (*i.e.* harmonic with respect to any G-invariant metric on G/T) provided the order of ν is greater than 2 (*i.e.* $n > 1$).

Now let $\pi_k : G/T \to \mathbf{CP}^n$ be one of the $n+1$ possible homogeneous projections. Then one also knows from [1] that $\pi_k \circ \phi$ is harmonic whenever ϕ is primitive. This leads us to the following result of Bolton *et al.* [5] which effectively characterizes the superconformal tori in $\mathbf{CP}^n$.

Theorem 2 ([5]) *A harmonic torus $\phi_k : \mathbf{T}^2 \to \mathbf{CP}^n$ is superconformal if and only if $\phi_k = \pi_k \circ \phi$ for some primitive $\phi : \mathbf{T}^2 \to G/T$ for which $\phi^*\beta^{(1,0)}$ is cyclic.*

Remark. Since there are $n+1$ homogeneous projections π_k each superconformal torus belongs to a sequence which is nothing other than the orthogonally periodic harmonic sequence generated by ϕ_k.

The elliptic Toda equations are linked to primitive maps from $\mathbf{T}^2$ in the following way. Let z be a complex coordinate on $\mathbf{C}$ and let $\Phi : \mathbf{C} \to G$ satisfy the condition

$$\Phi^{-1}\partial\Phi = -s^{-1}\partial s - s^{-1}\Lambda s, \tag{2}$$

where $\partial = \frac{\partial}{\partial z}$, $s = diag(s_0, \ldots, s_n)$ and Λ is the constant cyclic matrix in $\mathbf{g}_1$ given by

$$\Lambda = \begin{pmatrix} 0 & 1 & & & \\ & . & 1 & & \\ & & . & . & \\ & & & . & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

The equation (2) and its complex conjugate provide a pair of partial differential equations in the real variables $z, \bar{z}$ which have a common solution Φ provided the s_j satisfy the elliptic Toda equations in the form (1). For Φ to be G-valued each

s_j must be a real-valued and strictly positive function of $z, \bar{z}$. The elliptic Toda equations should be thought of as the conditions under which (2) and its complex conjugate can be integrated to produce Φ in a region where every s_j is analytic.

The form of (2) is such that $\Phi^{-1}\partial\Phi$ takes values in $\mathbf{t}^C + \mathbf{g}_1$ with cyclic $\mathbf{g}_1$-component. It follows that the map $\phi : \mathbf{C} \to G/T$ defined by $\phi = \Phi T$ is primitive and each $\pi_k \circ \phi$ gives a superconformal plane in $\mathbf{CP}^n$. In [5] it was shown that in fact every doubly periodic primitive map on the plane with cyclic $\phi^*\beta^{(1,0)}$ has *doubly periodic* 'Toda frame' Φ.

Theorem 3 ([5]) *Let $\phi : \mathbf{T}^2 \to G/T$ be primitive with cyclic $\phi^*\beta^{(1,0)}$, then $\phi = \Phi T$ for some doubly periodic $\Phi : \mathbf{C} \to G$ satisfying (2) with positive definite s. The functions s_j are doubly periodic with the same periods as $\mathbf{T}^2$ but Φ may change by a factor $\omega^k I$ around these periods. Moreover, Φ is uniquely determined by ϕ and the coordinate z up to translation of z or rescaling z by some power of ω.*

We conclude from these two theorems and the earlier remark that there is essentially (*i.e.* modulo trivialities) a bijection between the class of harmonic sequences of superconformal 2-tori in $\mathbf{CP}^n$ and the class of doubly periodic maps $\Phi : \mathbf{C} \to G$ which satisfy (2) for some doubly periodic solution $\{s_j\}$ of the elliptic Toda equations (1). Moreover, it is not difficult to show (*cf.* [5]) that distinct solutions of (1) lead to distinct primitive maps ϕ.

3 1-parameter families of primitive maps.

Given solutions s_j to (1) which are real-analytic in $z, \bar{z}$ throughout the entire z-plane there is a whole family of maps $\Phi_\zeta : \mathbf{C} \to G$, parameterized by a complex parameter ζ with modulus 1, each of which satisfies

$$\begin{aligned} \Phi_\zeta^{-1}\partial\Phi_\zeta &= -s^{-1}\partial s - \zeta s^{-1}\Lambda s, \\ \Phi_\zeta^{-1}\bar{\partial}\Phi_\zeta &= s^{-1}\bar{\partial} s + \zeta^{-1} s\Lambda^* s^{-1}. \end{aligned} \tag{3}$$

The second equation is simply the complex conjugate of the first (with respect to conjugation in $\mathbf{g}^C$). Thus, for each value of ζ, the map $\Phi_\zeta T : \mathbf{C} \to G/T$ is a primitive map.

The existence of Φ_ζ follows directly from (1) since the integrability condition for Φ_ζ, on $|\zeta| = 1$, is easily seen to be the same as that for $\Phi = \Phi_1$. For each value of $z, \bar{z}$ we can think of Φ_ζ as an element of a loop group of maps $S^1 \to G$. Indeed we may always choose Φ_ζ to be *ν-equivariant*, that is, $\nu(\Phi_\zeta) = \Phi_{\omega\zeta}$, since the right hand side of (3) has this property in the loop algebra.

Notice that if s is doubly periodic it need not be that Φ_ζ is. However, if Φ_1 is doubly periodic then, by ν-equivariance, so is Φ_{ω^k} for any integer k.

My strategy for characterizing the doubly periodic maps $\Phi = \Phi_1$ is to study the monodromy of Φ_ζ around the two periods of s whenever the latter is doubly periodic. In this section I will show, under the assumption that Φ_ζ comes from the dressing construction, that the eigenvalues of the two monodromy generators can be explicitly computed in terms of a *scalar Baker function* in the sense of [21]. First we must briefly review the dressing construction of Φ_ζ (following [20]) and introduce an intermediary, the *matrix* Baker function.

3.1 Dressing and the matrix Baker function.

Let $C = C_1 \cup C_2$ be the union of two circles in the ζ-plane, centred at $\zeta = 0$ and with radii ϵ and ϵ^{-1} respectively (where $0 < \epsilon < 1$). We consider C to be the boundary of two regions on the Riemann sphere: the union of discs I and the annulus E. Define $\Lambda_C(G^C, \nu)$ to be the group of real-analytic ν-equivariant maps $C \to G^C$; we think of such a map as a pair (g_1, g_2) of ν-equivariant loops in G^C. We will be primarily interested in the subgroup

$$\Lambda(G, \nu) = \{(g_1, g_2) : g_2 = \bar{g}_1\}$$

where $\bar{g}_1$ denotes $g_1(\bar{\zeta}^{-1})^{*-1}$. It is a corollary of a result in [20] that $\Lambda(G, \nu)$ has a global decomposition (which we will call its Iwasawa decomposition) into the product of the following three subgroups

- $\Lambda_E = \{(g_1, g_2) \in \Lambda(G, \nu)$: boundaries of holomorphic maps $g : E \to G^C$ with $g = \bar{g}\}$,
- $\Lambda_I = \{(g_1, g_2) \in \Lambda(G, \nu)$: boundaries of holomorphic maps $g : I \to G^C$ with $g_2 = \bar{g}_1$ and $g_1(0) = Id = g_2(\infty)\}$,
- $D = \{(s, s^{-1}) : s \in \exp(i\mathbf{t})$ *i.e.* s is diagonal, constant and positive definite $\}$.

The decomposition asserts that every $g \in \Lambda(G, \nu)$ has a unique factorisation $g = udn$ where $u \in \Lambda_E$, $d \in D$ and $n \in \Lambda_I$.

The dressing construction uses this factorisation to 'dress up' the function $\Phi_\zeta^{(0)} = \exp(-z\zeta\Lambda + \bar{z}\zeta^{-1}\Lambda^*)$ which corresponds to the trivial solution of (1), where every $s_j = 1$.

Theorem 4 ([20]) *Let $g \in \Lambda(G, \nu)$ and consider $\Phi_\zeta^{(0)}$ as a $\Lambda(G, \nu)$-valued function of $z, \bar{z}$. Then $g\Phi_\zeta^{(0)} = \Phi_\zeta dn$ according to the Iwasawa decomposition where Φ_ζ satisfies (3) for some possibly non-trivial s given by $d = (s, s^{-1})$. The entries s_j of the diagonal matrix s satisfy (1).*

From now on our fundamental assumption, which will be justifed in §6, is that if s is doubly periodic then Φ_ζ comes from the dressing construction for some choice of ϵ. In particular, Φ_ζ is analytic in the annulus E and not just on the unit circle.

Now let me introduce the matrix Baker function. Set $\tilde{\Psi} = \Phi_\zeta s^{-1}$ and define $q = 2\partial \ln s$, then it is a simple matter to verify that

$$\tilde{\Psi}^{-1}\partial\tilde{\Psi} = -q - \zeta\Lambda. \tag{4}$$

For any matrix s of solutions to (1) there is always a formal series solution of (4) with Fourier series in ζ of the form

$$\Psi(z, \bar{z}; \zeta) = \exp(-z\zeta\Lambda)(Id + O(\zeta^{-1})) \tag{5}$$

We will call Ψ a *matrix Baker function* for q whenever it satisfies (4) and has Fourier series (5) convergent for $|\zeta| \geq \epsilon^{-1}$.

Lemma 1 *When s comes from the dressing construction there is an analytic $g_2 : C_2 \to G^C$ for which $\Psi = g_2^{-1}\Phi_\zeta s^{-1}$ is a matrix Baker function.*

Proof. Since we have assumed $g\Phi_\zeta^{(0)} = \Phi_\zeta dn$ for some g in $\Lambda(G,\nu)$ we have, on C_2, $g_2\Phi_\zeta^{(0)} = \Phi_\zeta s^{-1}n_2$ where $n = (n_1, n_2)$. Therefore $g_2^{-1}\Phi_\zeta s^{-1} = \Phi_\zeta^{(0)}n_2^{-1}$ has Fourier series of the form (5) and satisfies (4). Moreover, it is convergent for $|\zeta| \geq \epsilon^{-1}$ since n_2 is.■

There is more than one matrix Baker function for each q. Let K_2 denote the group of all G^C-valued functions γ_2 on C_2 which extend holomorphically to $|\zeta^{-1}| \leq \epsilon$, with $\gamma_2(\infty) = Id$, such that $\gamma_2\Lambda\gamma_2^{-1} = \Lambda$.

Lemma 2 *If Ψ is a matrix Baker function for q then every other matrix Baker function for q is of the form $\gamma_2\Psi$ where $\gamma_2 \in K_2$.*

Proof. Clearly any other solution of (4) for the same q equals $\gamma_2\Psi$ for some $\gamma_2 : C_2 \to G^C$. To satisfy (5) $\gamma(z) = \exp(z\zeta\Lambda)\gamma_2\exp(-z\zeta\Lambda)$ must be of the form $Id + O(\zeta^{-1})$ for all z. In that case $(\partial^k\gamma)_{z=0} = \zeta^k ad^k\Lambda.\gamma_2$ may only have terms in negative powers of ζ. Since Λ is semisimple it follows that γ_2 has no terms of any order which do not commute with Λ, thus $\gamma_2 \in K_2$.■

Finally, we will need a result from [19] concerning a space of solutions to the *complexified* Toda equations, which are obtained from (1) by replacing $-\bar{z}$ with a complex variable t independent of z. Let $\Lambda_E(G^C,\nu)$ be the subgroup of $\Lambda_C(G^C,\nu)$ consisting of boundaries of holomorphic maps $E \to G^C$ and let $\mathcal{M}$ denote the left quotient space $\Lambda_E(G^C,\nu) \setminus \Lambda_C(G^C,\nu)$. Now let $K \subset \Lambda_C(G^C,\nu)$ be the abelian subgroup of all pairs $\gamma = (\gamma_1,\gamma_2)$ for which $\gamma_j\Lambda\gamma_j^{-1} = \Lambda$ and which are the boundaries of a holomorphic map $\gamma : I \to G^C$ satisfying $\gamma(\infty) = Id$ (this group is abelian since Λ has abelian stabilizer in G^C). K has identity component $K_0 = K_1 \times K_2$ where K_2 is as above and K_1 is, likewise, the abelian group of all loops $C_1 \to G^C$ stabilizing Λ and extending holomorphically to the disc with boundary C_1. Indeed every element of K is of the form $(\omega^k\gamma_1,\gamma_2)$ for some $\gamma_j \in K_j$ and some integer k. Hence K/K_0 is isomorphic to $\mathbf{Z}_{n+1}$. Given all this, the following theorem sums up (albeit in slightly different notation) the dressing construction described in [19].

Theorem 5 ([19]) *The abelian group K acts freely on the left of $\mathcal{M}$ and the dressing construction produces the injective horizontal maps in the commutative diagram:*

$$\begin{array}{ccl} \mathcal{M}/K_0 & \longrightarrow & \{s^2 : s_j^2 \textit{ satisfy the complexified Toda equations}\} \\ \downarrow & & \downarrow \\ \mathcal{M}/K & \longrightarrow & \{q = \partial\ln(s^2)\} \end{array} \qquad (6)$$

Each fibre of the right hand vertical map, over a point in the image of the bottom horizontal map, is of the form $\{s^2, \omega s^2, \ldots, \omega^n s^2\}$. Moreover, if the double coset for some $g \in \Lambda_C(G^C,\nu)$ is mapped, under the top map, to $s^2(z,t)$ then the double coset for $g\exp(-z_0\zeta\Lambda - t_0\zeta^{-1}\Lambda^*)$ is mapped to $s^2(z+z_0, t+t_0)$.

This last fact is very important: it describes the flow governed by the Toda equations on the space $\mathcal{M}/K_0$, which we think of as a phase space containing a certain class of solutions. Each point in the space corresponds to an initial condition, while the z and t flows are generated simply by the right action of the 2-parameter abelian group $\{\exp(-z\zeta\Lambda - t\zeta^{-1}\Lambda^*) : z, t \in \mathbf{C}\}$ (which commutes with K). More generally, let Γ be the stabilizer of Λ in $\Lambda_E(G^C, \nu)$, then its natural action (on the right) on $\mathcal{M}$ covers an action on $\mathcal{M}/K_0$, since Γ commutes with K. This generates the 'higher commuting flows' in the Toda hierarchy. We will have more to say about these higher flows later.

Note: beware that in [19] the loop algebra is identified with *right* invariant vector fields on the loop group, hence the quotient spaces there are the reverse of those here.

3.2 The monodromy generators.

Now we will use the results just stated to compute the eigenvalues of the the monodromy matrices for a doubly periodic solution of (1).

Lemma 3 *The diagonal matrix s containing the solutions s_j for the elliptic Toda lattice (1) is doubly periodic if and only if q is. Moreover, they have the same periods.*

Proof. Clearly q is doubly periodic whenever s is. Conversely, by theorem 5 we know that if q is doubly periodic then each s_j^2 changes by a multiplier which is some power of ω around each period. However, each s_j is a real-valued, strictly positive function, hence the multiplier of s_j around each period is 1.■

When q is doubly periodic, with periods ξ_1 and ξ_2, the matrix Baker function will have monodromy:

$$\Psi(z + \xi_j; \zeta) = M_j(\zeta)\Psi(z; \zeta). \tag{7}$$

By the lemma, Φ_ζ will also have monodromy:

$$\Phi_\zeta(z + \xi_j) = m_j(\zeta)\Phi_\zeta. \tag{8}$$

By assumption Φ_ζ comes from the dressing construction, so $m_j(\zeta)$ is defined on the annulus E. The link between the two is given by

$$g_2 M_j g_2^{-1} = m_j \tag{9}$$

which follows directly from lemma 1. Notice that (9) will only be defined near C_2.

The necessary and sufficient condition for double periodicity of Φ_1 is $m_j(1)^r = Id$ for some positive integer r. To investigate when this occurs it is sufficient to study the eigenvalues of $m_j(\zeta)$. Since the eigenvalues are the solutions of the characteristic equation $\det(m_j(\zeta) - \mu.Id) = 0$ they should be thought of as functions on a Riemann surface which covers the annulus of definition of $m_j(\zeta)$. Thus $m_j(1)^r = Id$ only if each of these eigenvalues is an r-th root of unity at every point lying over $\zeta = 1$. With the help of (9) we can give a clear description of this condition, by first examining the eigenvalues of $M_j(\zeta)$. For this we need a result which follows from lemma 2.

Proposition 1 *For $|\zeta| \geq \epsilon^{-1}$ we have*

$$M_j(\zeta) = \exp(-\xi_j \zeta \Lambda + \sum_1^\infty a_j \zeta^{-j} \Lambda^{-j}) \tag{10}$$

for some sequence $\{a_j\}$.

Proof. When q has periods ξ_j we see that $\exp(-\xi_j \zeta \Lambda)\Psi(z + \xi_j)$ is also a matrix Baker function for q, hence (by lemma 2) it equals $\gamma_2 \Psi$ for some $\gamma_2 \in K_2$. It is easy to verify that all elements of K_2 are exponentials of power series in negative powers of $\zeta\Lambda$, hence M_j is of the form (10).■

Note: one should be aware that in the sequence $\{a_j\}$ we must have $a_{k(n+1)} = 0$ for every integer k in order for M_j to have determinant 1. This detail, which could be avoided by working in GL_{n+1}, must be kept in mind although I will rarely mention it.

Corollary 1 *The k-th eigenvalue of the monodromy matrix $M_j(\zeta)$ is $\mu_{jk}(\zeta) = \mu_j(\omega^k \zeta)$ where*

$$\mu_j(\zeta) = \exp(-\xi_j \zeta + \sum_1^\infty a_j \zeta^{-j}) \tag{11}$$

with $\{a_j\}$ the same sequence as appears in (10).

This follows at once from the fact that Λ may be diagonalised into the matrix $diag(1, \omega, \ldots, \omega^n)$.

Now we will see that $\mu_j(\zeta)$ can be computed using the scalar Baker function defined as follows. Let $\psi_0^{row}(z, \bar{z}; \zeta)$ denote the top row of the matrix Ψ^{-1}. The ν-equivariance of Ψ means that

$$\psi_0^{row}(\zeta) = (\psi_{00}(\zeta^{n+1}), \zeta\psi_{01}(\zeta^{n+1}), \ldots, \zeta^n \psi_{0n}(\zeta^{n+1})).$$

We may define the scalar Baker function $\psi(z, \bar{z}; \zeta)$ to be the sum of the entries of ψ_0^{row} (that this coincides with the usual definition in [12, 19, 20, 21] is explained in the next section). It is easily seen that it has Fourier series in ζ of the form

$$\psi(\zeta) = exp(z\zeta)(1 + O(\zeta^{-1})).$$

Moreover, one can also see that summing together the entries of the row vector $\psi_0^{row}.(\zeta\Lambda)^{-j}$ gives $\zeta^{-j}\psi$. It follows from (7), (10) and (11) that

$$\psi(z + \xi_j; \zeta) = \psi(z; \zeta)\mu_j(\zeta)^{-1}$$

We immediately deduce that $\mu_j(\zeta) = \psi(0; \zeta)\psi(\xi_j; \zeta)^{-1}$. However, we may always choose ψ so that $\psi(0; \zeta) = 1$ (see the remark below). Using this normalization (which is the standard one in *e.g.* [21]) we have

$$\mu_j(\zeta) = \psi(\xi_j; \zeta)^{-1} \tag{12}$$

Remark. That we can normalize ψ in this way follows from the fact (lemma 2) that Ψ is unique modulo elements of K_2, since this implies that ψ is unique up to factors of the form $1 + O(\zeta^{-1})$. The Fourier series for $\psi(0; \zeta)$ is of this form hence ψ can be normalized to 1 at $z = 0$.

4 The spectral curve.

In the final section I will show that when q is doubly periodic it not only comes from a dressing construction (*i.e.* from theorem 5) but in fact arises from the algebro-geometric construction given in [19, 20] which sits inside the dressing construction. In this section I will review this algebro-geometric construction. The aim here is to describe the link between the pair (q, ψ) and a complete algebraic curve X together with some other data.

First we observe (as in [20]) that if we have $n+1$ scalar functions $\psi_j(z,t;\zeta)$ satisfying a system of the type

$$\begin{aligned} \partial_z \psi_j &= q_j \psi_j + \zeta \psi_{j+1}, \\ \partial_t \psi_j &= \zeta^{-1} v_j \psi_{j-1}, \end{aligned} \tag{13}$$

for some functions $q_j(z,t)$ and $v_j(z,t)$, then the compatibility conditions for these equations imply the complexified Toda lattice in the form

$$\partial_z \partial_t \ln(v_j) = -v_{j-1} + 2v_j - v_{j+1} \tag{14}$$

(where in both (13) and (14) j is counted modulo $n+1$). The relationship between this form of Toda and the earlier form (1) (which also has $t = -\bar{z}$) is given by $v_j = s_j^2 s_{j-1}^{-2}$.

We can obtain a set of functions ψ_j satisfying (13) for some q_j, v_j after fixing a Riemann surface X (or, more generally, a complete irreducible algebraic curve) together with a line bundle $\mathcal{L}$ over X whose degree equals the arithmetic genus p of X. The curve X must admit a rational function $\pi : X \to \mathbf{CP}^1$ of degree $n+1$ with divisor of the form $(n+1)(P_0 - P_\infty)$ for two non-singular points P_0 and P_∞ on X. Using π we define local parameters ζ about P_0 and ζ^{-1} about P_∞ such that $\zeta^{n+1} = \pi$ (hence ζ^{n+1} is globally defined on X although ζ will not be).

Given $(X, \pi, \mathcal{L})$ we define the following collection of functions f_k. For simplicity we assume $\mathcal{L}$ is the divisor line bundle for a positive divisor D, of degree p, whose support does not include P_0 or P_∞. For any integer k let $f_k(z,t;P)$ be the unique function on X satisfying:

1. near P_∞ the function f_k has Fourier series of the form $\zeta^k \exp(z\zeta)(1 + O(\zeta^{-1}))$,
2. on $X - \{P_0, P_\infty\}$ it is meromorphic with divisor of poles $\leq D$,
3. at P_0 the function $\exp(-t\zeta^{-1}) f_k$ has an isolated zero of order k.

This definition is very similar to the definition of the two-point scalar Baker function given in [12]. Indeed, using standard arguments from [12] it is easily shown that the $n+1$ functions ψ_j given by (restricting f_j to a neighbourhood of P_∞ and) setting $\psi_j = \zeta^{-j} f_j$ satisfy (13), where $j = 0, \ldots, n$. The corresponding functions q_j and v_j are given by

$$q_j = a_j - a_{j+1} \qquad ; \qquad v_j = \partial_t a_j$$

where $\psi_j = \exp(z\zeta)(1 + a_j \zeta^{-1} + \ldots)$.

Remark. The uniqueness of each f_k is most easily seen as follows. We will assume that the line bundle $\mathcal{L}$ is non-special *i.e.* it admits, up to scaling, only one non-trivial globally holomorphic section φ, which will have divisor of zeroes D. We will see shortly that we need not consider the case where $\mathcal{L}$ is special. For almost all complex values of z, t it is known (from a result in [21]) that the degree p bundle $\mathcal{L}_k(z,t) = \mathcal{L}(k(P_\infty - P_0)) \otimes [e^{z\zeta}] \otimes [e^{t\zeta^{-1}}]$ is also non-special and hence (by Riemann-Roch) has a one-dimensional family of globally holomorphic sections. There is a unique one φ_k of these for which the function $\varphi_k\varphi^{-1}$ has the normalization in property 1 above (properties 2 and 3 are true for any choice of section φ_k). Here $[e^{z\zeta}]$ denotes the degree zero line bundle constructed using $e^{z\zeta}$ as a the sole transition function over a punctured disc about P_∞. Similarly $[e^{t\zeta^{-1}}]$ displays the transition function, but this time it is for a punctured disc about P_0. Further, if either P_0 or P_∞ do lie in the support of D we simply modify the behaviour of f_k about these points to take this into account.

A comparison between (4) and (13) shows that the function ψ_0 is precisely the scalar Baker function ψ for functions v_j which provide solutions to the complexified Toda lattice (14). To pass to the elliptic Toda lattice we must impose certain reality conditions on X and $\mathcal{L}$ so that each v_j is real-valued and strictly positive along $t = -\bar{z}$. From [19, 20] it is known that the appropriate conditions are:

4a. X must admit an anti-holomorphic involution ρ for which $\overline{\pi(P)}^{-1} = \pi(\rho(P))$,

4b. $\mathcal{L}$ must satisfy $\overline{\rho_*\mathcal{L}} \simeq \mathcal{O}_X(R) \otimes \mathcal{L}^{-1}$, where R is the ramification divisor of π on $X - \{P_0, P_\infty\}$ (which has degree $2p$). This implies that $\mathcal{L}$ is non-special.

It follows from [19] that under these conditions the bundle $\mathcal{L}_0(z, -\bar{z})$ satisfies 4a, 4b and, more importantly, is non-special for all complex values of z. It follows that the functions q_j, v_j are globally analytic along $t = -\bar{z}$.

For the investigation in the next section we will need to know when X admits solutions to 4b.

Lemma 4 *A necessary and sufficient condition for X to admit line bundles satisfying 4b is that the divisor R equals $R_+ + \rho_* R_+$ for some positive divisor R_+ of degree p. It follows that the branch divisor $\pi_* R$ may only have even degree at points on the unit circle.*

Proof. It is easy to see that this is sufficient, for when this happens we see that $\mathcal{L}$ satisfies 4b precisely when $\mathcal{L} \otimes \mathcal{O}_X(-R_+)$ is fixed by the anti-holomophic involution $\mathcal{L}_0 \mapsto \overline{(\rho_*\mathcal{L}_0)}^{-1}$ on $J(X)$. The fixed points of this involution form a real closed algebraic subgroup J_R of $J(X)$ (of real dimension p). Now let us see that this is necessary. The divisor R is non-special, so by Riemann-Roch its complete linear system $|R|$ has (projective) dimension p. Inside $|R|$ lies the real projective subspace $\mathcal{R}$ of positive divisors fixed by ρ_*. Now suppose there exists a solution to 4b. Since the real p-dimensional group J_R preserves 4b, and each solution is non-special, there is a real p-dimensional projective variety of positive divisors $D + \rho_* D$ with divisor class $[D + \rho_* D] = [R]$ (this variety is the image of J_R, identified with $\{(D, \rho_* D) :$

$D + \rho_* D \in |R|\}$, under the finite morphism $X^{(p)} \times X^{(p)} \to X^{(2p)}$ which maps (D, E) to $D + E$, where $X^{(p)}$ denotes the p-th symmetric product). This closed variety lies entirely in $\mathcal{R}$ and has the same dimension, so they must be equal. But $R \in \mathcal{R}$, hence it has the form $R_+ + \rho_* R_+$ for some positive divisor R_+ of degree p. ■

5 Double periodicity.

Suppose we have a pair (q, ψ) arising from the data $(X, \pi, \mathcal{L})$ as in the previous section, and let (s, Φ_ζ) denote the corresponding pair satisfying (3).

Proposition 2 *Φ_1 is doubly periodic if and only if:*

(i) *$q(z)$ is doubly periodic (let the periods be called ξ_1, ξ_2), and,*

(ii) *at every point $Q_0, \ldots, Q_n$ for which $\pi(Q_k) = 1$ we have $\psi(-r\xi_j; Q_k) = 1$ for some positive integer r (which may depend on j).*

Proof. From our earlier discussion it is clear that only (ii) requires explanation. If Φ_1 is doubly periodic then the monodromy matrices $m_j(\zeta)$ satisfy $m_j(1)^r = Id$. Each $m_j(\zeta)$ is ν-equivariant, since Φ_ζ is, which implies that $\det(m_j(\zeta) - \mu.Id)$ is an analytic function of ζ^{n+1} and μ: we will write this as $F_j(\zeta^{n+1}, \mu)$. Now consider the function $F_j(\pi(P), \psi(\xi_j; P)^{-1})$, which is defined and analytic on that part of $X - \{P_0, P_\infty\}$ corresponding to $|\zeta| \geq \epsilon$. By (9) and corollary 1 this vanishes for $|\zeta| \geq \epsilon^{-1}$, therefore it vanishes identically wherever it is defined. Thus $\psi(\xi_j; P)^{-r} = 1$ at all the points P with $\pi(P) = 1$. Clearly $\psi(\xi; P)^k = \psi(k\xi; P)$ for any integer k (by the monodromy property) which gives us the result.

Conversely, suppose (i) and (ii) both hold, then (i) implies that $\Phi_\zeta^{-1}\partial\Phi_\zeta$ is doubly periodic (by lemma 3), so Φ_ζ has two monodromy generators $m_j(\zeta)$. When (ii) holds it follows (from the reasoning above) that $m_j(\zeta)^r = Id$ at *every* ζ for which $\zeta^{n+1} = 1$.■

Remark. The ν-equivariance of $m_j(\zeta)$ implies that $m_j(\zeta)^r = Id$ if and only if $m_j(\omega^k\zeta)^r = Id$ for every k. So the last statement in the proof above is not surprising.

We must now examine these conditions more closely to see how many there are. After this we can count the number of degrees of freedom available to satisfy them.

5.1 The number of constraints.

First we will examine the condition (i) in proposition 2. To begin let us recall from [19] that the diagonal matrix function $q(z, t) = diag(q_0, \ldots, q_n)$ is actually the restriction of a meromorphic function (which we shall also call q) on an infinite dimensional abelian Lie algebra $\mathcal{G}$ which we can take to be the space of holomorphic functions on $\epsilon \leq |\zeta| \leq \epsilon^{-1}$. Any element $a(\zeta)$ of $\mathcal{G}$ has Fourier series

$$a(\zeta) = \sum_{-\infty}^{\infty} t_k \zeta^k.$$

Thus q is a function of the infinitely many variables t_k: z and t are defined to be t_1 and t_{-1} respectively, while the other variables correspond to the higher flows generated by the action of the group Γ (recall the discussion after theorem 5). The connection between $\mathcal{G}$ and Γ is the following. Every element of Γ has the form $\exp(\sum t_k\zeta^k\Lambda^k)$ where $t_k = 0$ whenever $k \in (n+1)\mathbf{Z}$ (otherwise this exponential would not lie in SL_{n+1}). Hence we have a natural surjective homomorphism $\mathcal{G} \to \Gamma$ which factors through the Lie algebra of Γ and has kernel generated by $\{t_k\zeta^k : k \in (n+1)\mathbf{Z}\}$. This kernel contains the 'trivial flows' in the Toda hierarchy.

Note: beware that in [19] the symbol $\mathcal{G}$ was used to denote the Lie algebra of Γ. The position I have adopted here makes the ensuing discussion less complicated.

When q possesses a spectral curve X we may (following [19]) define a surjective linear map $\ell : \mathcal{G} \to H^1(X, \mathcal{O})$. This map ℓ assigns to each $a \in \mathcal{G}$ the cohomology class $[a_-, a_+]$ of the 1-cocycle which assigns $a_-(\zeta) = \sum_1^\infty a_{-k}\zeta^{-k}$ to an open annulus about $|\zeta| = \epsilon$ and $a_+(\zeta) = \sum_1^\infty a_k\zeta^k$ to an open annulus about $|\zeta| = \epsilon^{-1}$. It follows from a result in [19] that q is constant along cosets of the kernel of ℓ, and therefore q may be thought of as a function on $H^1(X, \mathcal{O})$ (in this case q is clearly independent of each $t_{k(n+1)}$ since ζ^{n+1} is a rational function on X, hence $\ell(\zeta^{n+1})$ is the trivial cohomology class). In fact (by another result in [19]) q pushes down onto the Jacobian $J(X)$ *i.e.* q is periodic with respect to the lattice which is the kernel of the epimorphism $H^1(X, \mathcal{O}) \to J(X)$. Recall that this kernel is identical to $H^1(X, \mathbf{Z})$ and, if X is non-singular, has $2p$ generators in the p-dimensional complex vector space $H^1(X, \mathcal{O})$.

On $\mathcal{G}$ there is a natural anti-holomorphic involution $a(\zeta) \mapsto -\overline{a(\bar\zeta^{-1})}$ whose fixed point subspace we will call $\mathcal{G}_R$. The reality condition 4a imposed upon X is chosen precisely so that under ℓ the real points of $H^1(X, \mathcal{O})$ are the image of $\mathcal{G}_R$: let us call this real subspace $H^1_\rho(X, \mathcal{O})$. When $p \geq 2$ the 2-dimensional real subgroup of $J(X)$ which consists of all bundles of the form $[e^{z\zeta}] \otimes [e^{-\bar z\zeta^{-1}}]$ is the image of the subspace $V = \{z\zeta - \bar z\zeta^{-1}\}$ of $\mathcal{G}_R$ under the composition of ℓ and the epimorphism $H^1(X, \mathcal{O}) \to J(X)$. It follows that $q(z, \bar z)$ is doubly periodic precisely when this subgroup of $J(X)$ is topologically a 2-torus *i.e.* when $\ell(V)$ contains two elements of the lattice $H^1_\rho(X, \mathbf{Z})$. When $p = 0$ we see that the function $q(z, \bar z)$ must be constant and therefore it yields the trivial solution. When $p = 1$, $q(z, \bar z)$ has one trivial period; to avoid complications this will be discussed at the end of this subsection.

Now we can write down these double periodicity conditions as a system of equations. Recall that $H^1(X, \mathcal{O})$ is dual to the space $H^0(X, \Omega)$ of globally holomorphic differentials. The real points of $H^1(X, \mathcal{O})$ correspond to those differentials ω for which $\rho^*\omega = -\bar\omega$: we will denote the subspace of these by $H^0_\rho(X, \Omega)$.

Proposition 3 *Let X have arithmetic genus $p \geq 2$. The function $q(z, \bar z)$ is doubly periodic precisely when there exist independent $\alpha_1, \alpha_2 \in H^1_\rho(X, \mathbf{Z})$ for which*

$$\langle \alpha_j, \omega \rangle = 0 \quad \textit{for all} \quad \omega \in H^0_\rho(X, \Omega(-P_0 - P_\infty)), \tag{15}$$

where $\langle \ , \ \rangle$ is the Serre duality between $H^1(X, \mathcal{O})$ and $H^0(X, \Omega)$. This gives a system of $2p - 4$ real equations for the moduli of (X, ρ).

Proof. The duality between $H^1(X,\mathcal{O})$ and $H^0(X,\Omega)$ comes from the bilinear pairing characterized by

$$\langle [\zeta^{-k},\zeta^l],\omega\rangle = res_{P_0}\zeta^{-k}\omega + res_{P_\infty}\zeta^l\omega.$$

The subspace $H^0_\rho(X,\Omega(-P_0-P_\infty)$ denotes all those real holomorphic differentials which have divisor of zeroes at least P_0+P_∞. This has real codimension 2 in $H^0_\rho(X,\Omega)$ (which has real dimension p) and is clearly the annihilator of $\ell(V)$. Hence an element α of $H^1_\rho(X,\mathcal{O})$ belongs to $\ell(V)$ precisely when it satisfies (15).■

Now consider the conditions in proposition 2(ii). Notice that $\psi(\xi_j;P)$ is unimodular over the unit circle in the ζ^{n+1}-sphere since it provides the eigenvalues for the matrix $m_j(\zeta)$, which is unitary over $|\zeta|=1$. Therefore the equations (ii) should be counted as 'real' equations for the real curve/covering (X,ρ,π) and the 'real' divisor D. Their number depends upon the number of distinct points in $\pi^{-1}(1)$, which is at most $n+1$. But there is always one relationship between the values of $\psi(\xi;P)$ at these points $Q_0,\ldots,Q_n$. Since the determinant of m_j is 1 everywhere, we deduce that

$$\psi(\xi_j;Q_0)\times\ldots\times\psi(\xi_j;Q_n)=1.$$

It follows that:

Corollary 2 *There are at most $2p+2n-4$ real conditions to be satisfied by the data $(X,\pi,\mathcal{L},\psi)$ to obtain a doubly periodic Toda frame Φ_1.*

Remark. The discussion above can be compared with the computation in §8 of [14] for the double periodicity conditions of minimal tori in S^4. There, in place of proposition 2(ii), one finds a condition involving the value of the exponential of the integral of a certain meromorphic differential on the spectral curve. This is completely analogous to the condition here, since $\psi(\xi_j;P)$ may be readily written in that form (*cf.* the formula for the scalar Baker function in, for example, [21]).

The cases p=0,1. When $p=0$ the curve X is a smooth rational curve on which the parameter ζ is globally defined. The corresponding Toda solution is the trivial solution: $q(z,\bar z)\equiv 0$. The scalar Baker function is simply $\psi(z,\bar z;\zeta)=\exp(z\zeta-\bar z\zeta^{-1})$. It is not hard to show this can have two independent periods in z at every point over $\zeta^{n+1}=1$ if and only if $n=1,2,3,5$. The corresponding 2-tori are the flat tori described in [4].

When $p=1$ the corresponding Toda solution has precisely one trivial period. In fact this characterises the doubly periodic elliptic Toda solutions which arise from curves with $p=1$. For the map $\mathbf{C}\to H^1_\rho(X,\mathcal{O})\simeq\mathbf{R}^p$ given by $z\mapsto\ell(z\zeta-\bar z\zeta^{-1})$ has a one dimensional kernel precisely when there exists $\xi\in\mathbf{C}^*$ such that $\xi\omega(P_\infty)+\bar\xi\bar\omega(P_\infty)=0$ for all real holomorphic differentials ω. But this system admits a non-zero solution if and only if $p\le 1$. Moreover, if X is smooth (and therefore an elliptic curve) the group $H_1(X,\mathbf{Z})$ has one real and one imaginary generator, hence there is a second, non-trivial real period. If X is not smooth then it turns out that there cannot be a non-trivial real period - this is explained in the next subsection in the remark on singular curves.

5.2 The number of degrees of freedom.

Here we will show that:

Proposition 4 *The number of free real parameters available to satisfy the double periodicity conditions in proposition 2 is at most 2p.*

Observe that to write down (15) we only need the real curve (X, ρ), but to construct ψ we must also have z, ζ and a positive divisor D (equivalently, a line bundle $\mathcal{L}$) of degree p satisfying the reality condition 4b. Since $\zeta^{n+1} = \pi$ the choice of π determines ζ (up to a selection of $n+1$-st root of unity). In fact this also gives us z, since it can be thought of as the linear form on $\mathcal{G}$ satisfying $z(\zeta^k) = 1$ for $k = 1$ and zero otherwise. However, we will prove shortly that the periodicity conditions on ψ are independent of the choice of D. Given this result, we only need to count the moduli available in the choice of (X, ρ, π).

The pair X, π is one of a finite number characterized by the branch divisor of π (*cf.* for example [16]). In the case of general position this amounts to a free choice of $2p$ distinct branch points $\pi(P_1), \ldots, \pi(P_{2p})$ on $\mathbf{CP}^1 - \{0, \infty\}$ (that they number $2p$ follows directly from the Riemann-Hurwitz formula). For X to admit the involution ρ these must be mapped to each other by $\pi \mapsto \bar{\pi}^{-1}$, hence we have at most $2p$ real parameters.

Now it remains to be shown that the periodicity conditions are independent of the choice of the divisor D.

Lemma 5 *Let $\psi_D(z; P)$ denote the scalar Baker function corresponding to the divisor D and let E be any positive divisor on X also of degree p and satisfying the same reality condition. Then for each of the periods ξ_j we have $\psi_E(\xi_j; P) = \psi_D(\xi_j; P)$ throughout X.*

Proof. To prove this we must think of ψ_D as a function on $\mathcal{G}$ defined as follows. For each $a \in \mathcal{G}$ we write $a = a_- + a_+$ as before and define $\psi_D(a; P)$ to be the unique function on X with the properties:

1. near P_∞ the function $\psi_D(a; P)$ has Fourier series of the form $\exp(a_+)(1 + O(\zeta^{-1}))$,
2. on $X - \{P_0, P_\infty\}$ it is meromorphic with divisor of poles $\leq D$,
3. $\exp(-a_-)\psi_D(a; P)$ is holomorphic about P_0 and non-zero at P_0.

From the earlier discussion we know that $\psi_D(a; P)$ has divisor of zeroes $D(a)$ where $D(a) - D$ lies in the divisor class of the degree zero line bundle lying beneath the point $\ell(a)$ in $H^1(X, \mathcal{O})$. Since ℓ is a surjective map every divisor class of degree p contains a positive divisor of the form $D(a)$ for some $a \in \mathcal{G}$. In that case we can write $E \equiv D(b)$ for some fixed $b \in \mathcal{G}$. Now we observe that, for all $a \in \mathcal{G}$

$$\psi_D(a + b) = \psi_D(b)\psi_E(a). \tag{16}$$

The proof is straightforward. The right hand side clearly has the same exponential behaviour about P_0 and P_∞ as the left hand side. Moreover, $\psi_D(b)$ has divisor of zeroes E (since the reality conditions mean that E is non-special and is therefore the unique positive divisor in its divisor class) hence both sides have divisor of poles D on $X-\{P_0,P_\infty\}$. The result now follows from the uniqueness of the Baker function. In particular this means, for $j=1,2$,

$$\psi_D(a+\xi_j)=\psi_D(\xi_j)\psi_D(a), \tag{17}$$

for all $a\in\mathcal{G}$ since $D(\xi_j)\equiv D$. Combining (16) with (17) results in

$$\psi_E(a-b+\xi_j)=\psi_D(\xi_j)\psi_E(a-b).$$

Setting $a=b$ in this gives us our result, since $\psi_E(0)=1$.■

As an important corollary of this lemma we have:

Corollary 3 *Let (X,π) be a pair admitting a superconformal 2-torus in $\mathbf{CP}^n$ and suppose $p\geq 2$. Then there is real $p-2$ dimensional moduli space of superconformal 2-tori possessing the same pair.*

Proof. Every superconformal 2-torus is determined by a collection of data (X,π,D) which gives us a doubly periodic Φ_1. The previous lemma shows that, if J_R denotes the real p-dimensional subgroup of the Jacobian of X whose action preserves the reality condition 4b, then every point on the J_R-orbit of D also gives rise to a superconformal 2-torus. J_R contains the 2-parameter subgroup of conformal automorphisms of the torus (these are tangent to the $z,\bar{z}$-directions in J_R) and we take the moduli space to be the quotient of J_R by this subgroup.■

Remark - Singular Curves. Most of what has been done so far will apply equally well to the case of irreducible singular curves, provided care is taken over two points.

The first concerns the interpretation of the ramification divisor R. The full ramification divisor of $\pi: X\to\mathbf{CP}^1$ is best thought of as the divisor for the sheaf $\Omega_X\otimes(\pi^*\Omega_{\mathbf{CP}^1})^{-1}$, where Ω_X is the sheaf of regular differentials for X (*cf.* for example [17]). For a singular X the divisor (of points) only makes sense when Ω_X is locally free, which only happens under certain conditions on X *e.g.* when X is a complete intersection or, in particular, when X has only ordinary double points and cusps (*cf.* [22] for a discussion of this). The degree of R is still $2p$ (where p is the arithmetic genus of X) since the Riemann-Hurwitz formula still applies.

The second point concerns the Jacobian of a singular curve. It is still the case that $J(X)$ is analytically given by $H^1(X,\mathcal{O})/H^1(X,\mathbf{Z})$, but this is no longer a compact manifold (although it still has complex dimension p).For example, suppose X has only one singular point Q whose ideal sheaf is isomorphic to $\mathcal{O}_{\tilde{X}}(-D)$ for some positive divisor D with support S on the normalisation $\tilde{X}$. Then one knows (*cf.* [22]) that $H^1(X,\mathbf{Z})\simeq H_1(\tilde{X}-S,\mathbf{Z})$ and $H^1(X,\mathcal{O})$ has complex dimension given by the arithmetic genus of X. The resulting picture is that $J(X)$ is a topologically non-trivial extension of $J(\tilde{X})$ by a linear algebraic group isomorphic to a product of copies of $\mathbf{C}^*$ with copies of $\mathbf{C}$ (the precise number is determined by the nature of the singularity). If X has more than one singularity we obtain its Jacobian by successive extensions of the Jacobian of its normalisation. Consequently, we cannot necessarily assume there are *any* real periods in the period lattice for the Jacobian of X.

For example, let us show that when X is rational with genus $p = 1$ then the corresponding Toda solution cannot be doubly periodic. In this case X has either a double point or a cusp. The latter case is trivial, for the space $\tilde{X} - S$ is contractible ($\tilde{X}$ is a sphere and S is a single point). In the former case S is a pair of distinct points $P, \rho(P)$, so $J(X) \simeq \mathbf{C}^*$. But it is not hard to see that for any non-trivial cycle γ around P, $\rho(\gamma)$ has opposite orientation around $\rho(P)$ and is therefore homologous to γ. Hence the corresponding cohomology class in $H^1(X.\mathbf{Z})$ is imaginary for the real structure induced by ρ (a real class $[\gamma]$ satisfies $\rho_*[\gamma] = -[\gamma]$). Thus, although the Toda solution always has one trivial period it cannot have a non-trivial real period.

In fact this proof extends to show that there can be no real periods for the Jacobian of a rational curve with only cusps or double points. More generally it seems that the addition of a singular point cannot add a real period, although I do not have a general proof for this. If this is the case then it suggests that the occurence of a superconformal 2-torus with singular spectral curve is unlikely for $n \geq 2$.

5.3 The cases where n=2,3,4.

It is not hard to see that for $n \geq 5$ the number of conditions on (X, π) exceeds the number of moduli no matter how one distributes the branch points. Moreover, for $n = 1$ the best arrangement is the one of general position, which leaves a net count of 2 free moduli. This case has been treated in [18] and I leave the reader to consult the discussion there. However, let me point out that Hitchin's spectral curve and the one used here are related in a very simple way: his curve is an unramified double cover of the curve here, so that on his curve the variable ζ is a globally defined rational function. This explains why in [18] one can only obtain nowhere conformal 2-tori if the genus is odd, since the unramified double cover of X will have genus $2p - 1$.

For $n = 2, 3$ or 4 it is possible to place a branch point at $\pi = 1$ in such a way as to improve the net parameter count. The situation is the following:

n=2. The net count can be at best 2. This occurs when the divisor R has degree 2 over $\pi = 1$ (by lemma 4 it must have even degree over any point on the unit circle). In this case there is a single point on X covering $\pi = 1$, hence the conditions of proposition 2(ii) are automatically satisfied. This leaves $2p-2$ moduli to satisfy the $2p-4$ conditions remaining. By contrast, if $\pi = 1$ is not a branch point the number of conditions equals the number of moduli.

n=3. In this case the net count can be at best 0, which occurs when R has simple points except over $\pi = 1$, where R can take either of the forms $2.P_1$ or $P_1 + \rho_* P_1$.

n=4. The net count can be 0 if R has degree 4 over $\pi = 1$ and simple points elsewhere. Notice that in that case the conditions of proposition 2(ii) are automatically satisfied.

Examples. When X is smooth and has genus either 1 or 2 the periodicity conditions of proposition 3 are automatically satisfied. Hence we can use the observations above to write down the equations of some Riemann surfaces which must correspond to superconformal 2-tori in $\mathbf{CP}^2$ and $\mathbf{CP}^4$ whose corresponding Toda solutions are not trivial. Unfortunately it does not seem possible to decide, without doing some lengthy analysis, whether there are such examples for $n = 3$.

n=2. If we choose (X, π) so that $R = 2P_1 + P_2 + \rho P_2$, with $\pi(P_1) = 1$, we automatically satisfy both the conditions in proposition 2. In particular, the smooth genus 2 curves which have $\pi(P_2) = \pi(\rho P_2) \neq 1$ all have affine equations of the form $\mu^3 = \pi(\pi - 1)(\pi - \pi(P_2))$. Similarly, the elliptic curve with affine equation $\mu^3 = \pi(\pi - 1)$ (for which $R = 2P$ with $\pi(P) = 1$) also automatically satisfies both conditions. In this case the corresponding Toda solution has one trivial period, hence the superconformal 2-torus has an S^1-symmetry. For if ξ denotes the trivial period then the monodromy equation must have the form $\Phi_1(z+\lambda\xi) = \exp(\lambda \ln m(1))\Phi_1(z)$ where λ is a real parameter. But $\ln m(1) = 2\pi ni.Id$ for some integer n, and we see that $\Phi_1(z)$ is S^1-equivariant.

n=4. We find a curve yielding a superconformal 2-torus in $\mathbf{CP}^4$ by fixing $R = 4P_1$ where $\pi(P_1) = 1$. In this case let X be the compact Riemann surface with genus 2 corresponding to the equation $\pi(\pi - 1) = \lambda^5$. Then (X, π) satisfies all the conditions required. Indeed I conjecture that X is the only curve of genus 2 admitting a superconformal 2-torus in $\mathbf{CP}^4$.

6 Doubly periodic Toda fields possess a spectral curve.

The aim of this section is to prove the following theorem.

Theorem 6 *Every doubly periodic solution of the elliptic Toda equation (1) is given by the dressing construction of theorem 5 (for some $0 < \epsilon < 1$). Moreover, each has a corresponding spectral curve in precisely the sense described earlier.*

We will begin by establishing the first claim, that doubly periodic solutions always come from a dressing construction. This simply requires putting together some results from [5], [9] and [10].

6.1 Extended Toda frames of finite type.

Let us initially fix $0 < \epsilon < 1$ and observe that the Lie algebra $\Lambda(\mathbf{g}, \nu)$ of $\Lambda(G, \nu)$ splits into the direct (vector space) sum of Lie subalgebras

$$\Lambda(\mathbf{g}, \nu) = \Lambda_E(\mathbf{g}, \nu) + \mathbf{d} + \Lambda_I(\mathbf{g}, \nu)$$

corresponding to the Iwasawa decomposition of $\Lambda(G, \nu)$ described in §3.1. For any positive integer k we use Λ_k to denote the subspace of $\Lambda_E(\mathbf{g}, \nu)$ consisting of (pairs of) Laurent polynomials of degree $\leq k$. Now define the vector subspace

$$\Delta = \Lambda_1 + \mathbf{d} + \Lambda_I(\mathbf{g}, \nu) \subset \Lambda(\mathbf{g}, \nu)$$

If we let L_I denote the subgroup of $\Lambda(G, \nu)$ generated by D and Λ_I then Δ is preserved by the adjoint action of L_I.

For any $(\bar{\xi}, \xi)$ in Δ, define

$$e(\xi) = \exp[(-\bar{z}\bar{\xi}, -z\xi)].$$

This presents us with a 2-parameter subgroup of $\Lambda(G, \nu)$. According to the Iwasawa decomposition we may write

$$e(\xi) = \Phi(\xi)d(\xi)n(\xi) \tag{18}$$

and the last two factors take values in L_I. Observe that $\Phi(\xi)$ is trivial unless ξ has a non-trivial component in Λ_1. For *any* ξ in Δ, $\Phi(\xi)$ satisfies equations very much like (3). Let us write the Fourier series for ξ (which lives on C_2) as $\xi = \xi_1\zeta + \xi_0 + \ldots$.

Lemma 6 *For any $(\bar{\xi}, \xi) \in \Delta$, $\Phi(\xi)$ satisfies*

$$\begin{aligned} \Phi^{-1}\partial\Phi &= -s^{-1}\partial s - \zeta s^{-1}\xi_1 s, \\ \Phi^{-1}\bar{\partial}\Phi &= s^{-1}\bar{\partial}s - \zeta^{-1}s\bar{\xi}_1 s^{-1}. \end{aligned} \tag{19}$$

where s comes from the factor $d(\xi) = (s, s^{-1})$ in (18).

Proof. We only need to prove the first equation. Let us write $n(\xi) = (n_1, n_2)$, then $\Phi(\xi) = e(\xi)(n_1^{-1}s^{-1}, n_2^{-1}s)$, which gives us two expansions for $\Phi^{-1}\partial\Phi$:

1. on C_1, $\Phi^{-1}\partial\Phi = -s\partial n_1.n_1^{-1}s^{-1} - \partial s.s^{-1}$,

2. on C_2, $\Phi^{-1}\partial\Phi = -s^{-1}n_2\xi_1 n_2^{-1}s - s^{-1}\partial n_2.n_2^{-1}s + s^{-1}\partial s$.

The first shows that $\Phi^{-1}\partial\Phi$ extends holomorphically to the interior of C_1 and equals $-s^{-1}\partial s$ at $\zeta = 0$ (recall that s is diagonal). The second shows that it also extends into the disc with boundary C_2 where it has a single simple pole, at $\zeta^{-1} = 0$, with residue $-s^{-1}\xi_1 s$. Since $\Phi^{-1}\partial\Phi$ is also holomorphic between C_1 and C_2 it follows that it is of precisely the form given in (19).■

We will call any Φ_ζ satisfying equations(19) (for some $\exp(i\mathbf{t})$-valued s and some $\xi_1 \in \mathbf{g}_1$) and *extended Toda frame*: when ξ_1 is cyclic these equations are equivalent to (3), but if ξ_1 is not cyclic their compatibility equations are a variation on the finite lattice Toda equations (*cf.* for example [20]).

Let $\mathcal{F}$ denote the collection of all extended Toda frames. The group L_I acts on this set: we define, for any $g \in L_I$, $g\sharp\Phi_\zeta$ to be the Λ_E-factor in the Iwasawa decomposition of $g\Phi_\zeta$. It is elementary to show (using the arguments of the previous lemma) that $g\sharp\Phi_\zeta$ again satisfies equations of the form (19) and it is easy to see that this is leads to a group action. In fact we have constructed a map

$$\begin{aligned} \Phi : \Delta &\to \mathcal{F} \\ (\bar{\xi}, \xi) &\mapsto \Phi(\xi) \end{aligned}$$

which is L_I-equivariant *i.e.* $\Phi(\xi) = g^{-1}\sharp\Phi(\mathrm{Ad}g\xi)$ (this observation, which I learned from Burstall & Pedit [10], is straightforward to prove).

Following [5] (*cf.* also [8, 9]) we say an extended Toda frame $\Phi(\xi)$ is of *finite type* if there exists an $(\bar{\eta}, \eta) : \mathbf{R}^2 \to \Lambda_m$, for some $m \equiv 1 \bmod (n+1)$, such that:

$$\begin{array}{rrcl} & d\eta & = & [\eta, \Phi_\zeta^{-1} d\Phi_\zeta] \\ \text{with} & -i\eta & = & \zeta^m s^{-1}\xi_1 s + \zeta^{m-1}\partial \ln s + \ldots \end{array} \tag{20}$$

The next theorem puts together results from [5] (§3) and [9] (§4).

Theorem 7 *Let s be a doubly periodic solution of the elliptic Toda lattice, then:*

(i) [5] *there is a corresponding solution Φ_ζ of (3) of finite type,*

(ii) [9] *consequently $\Phi_\zeta = \Phi(\xi)$ for some $(\bar{\xi}, \xi)$ in Δ with ξ_1 cyclic.*

In fact the proof of (ii) simply consists of verifying that for Φ_ζ one can take $\Phi(\xi)$ with $i\xi = \zeta^{1-m}\eta_0$, where η_0 is the value of η at $z = 0$ if $(\bar{\eta}, \eta)$ satisfies (20) for this choice of s. It follows that ξ_1 is $\exp(i\mathbf{t})$-conjugate to Λ, since s takes values in $\exp(i\mathbf{t})$.

Now observe that if $0 < \epsilon' < \epsilon$ there is a natural inclusion $\Delta(\epsilon) \to \Delta(\epsilon')$ (where $\Delta(\epsilon)$ denotes Δ defined for loops of radii ϵ and ϵ^{-1}) given by restricting meromorphic functions on I to the circles of radius ϵ' and ϵ'^{-1}. The following theorem, due to Burstall & Pedit [10], shows that by taking ϵ small enough we can view every extended Toda frame as being given by some dressing construction, provided ξ_1 is semisimple (which in our case is synonymous with saying ξ_1 is cyclic).

Theorem 8 *If $(\bar{\xi}, \xi) \in \Delta(\epsilon)$ has cyclic ξ_1 then there exists some $0 < \epsilon' < \epsilon$ and some $g \in L_I(\epsilon')$ for which $\phi(\xi) = \Phi(\mathrm{Ad}g\xi_1) = g\sharp\Phi(\xi_1)$.*

Now we only have a few loose ends to tie up to prove:

Corollary 4 *For every doubly periodic solution of the elliptic Toda lattice there is a solution Φ_ζ of (3) and an ϵ, with $0 < \epsilon < 1$, such that $\Phi_\zeta = g\sharp\Phi_\zeta^{(0)}$ for some g in L_I. Hence Φ_ζ comes from the dressing construction.*

Proof. By theorem 7 (and the subsequent remarks) $\Phi_\zeta = \Phi(\xi)$ where $(\bar{\xi}_1, \xi_1)$ is L_I-conjugate to $(\bar{\Lambda}, \Lambda)$. By theorem 8 and the L_I -equivariance of the map $\Phi : \Delta \to \mathcal{F}$ we can always choose ϵ so that there is a $g \in L_I$ with $\Phi_\zeta = g\sharp\Phi(\zeta\Lambda)$. Now all we have to show is that $g\sharp\Phi(\zeta\Lambda)$ equals the Λ_E-factor in the Iwasawa decomposition of $g\Phi_\zeta^{(0)}$, since this is how the dressing construction is defined. But clearly it is sufficient to simply show that $\Phi(\zeta\Lambda) = \Phi_\zeta^{(0)}$ (here we are thinking of the latter as restricted to the union of circles C).

By definition $\Phi(\zeta\Lambda)$ is the Λ_E-factor of $\exp[(\bar{z}\zeta^{-1}\Lambda^*, -z\zeta\Lambda)]$. Now observe that

$$(\bar{z}\zeta^{-1}\Lambda^*, -z\zeta\Lambda) = (-z\zeta\Lambda + \bar{z}\zeta^{-1}\Lambda^*, -z\zeta\Lambda + \bar{z}\zeta^{-1}\Lambda^*) + (z\zeta\Lambda, -\bar{z}\zeta^{-1}\Lambda^*).$$

The right hand side gives the Iwasawa decomposition in $\Lambda(\mathbf{g}, \nu)$, and since the two terms commute $\Phi(\zeta\Lambda)$ is the exponential of the first term (since this belongs to $\Lambda_E(\mathbf{g}, \nu)$). Clearly this gives $\Phi(\zeta\Lambda) = \Phi_\zeta^{(0)}$.$\blacksquare$

This corollary establishes the first part of theorem 6. For the second part we will make use of the 'finite type' property of Toda tori to show that they must correspond to finite dimensional Γ-orbits in the space $\mathcal{M}/K_0$ occuring in theorem 5. We will then see that all such solutions of the Toda equations possess a spectral curve of precisely the type described in §4.

6.2 Finite type solutions and their spectral curves.

Following [5] we observe that on the finite dimensional manifold Λ_m we can define two real o.d.e's

$$\begin{aligned} \partial\eta &= [\eta, i(\zeta\eta_m + \eta_{m-1})], \\ \bar{\partial}\eta &= [\eta, -i(\zeta^{-1}\eta_{-m} + \eta_{1-m})], \end{aligned} \tag{21}$$

where each $(\bar{\eta}, \eta)$ in Λ_m is identified with the Laurent polynomial η. These o.d.e's are in fact Hamiltonian and their flows commute (*cf.* [5]). Together the pair have a finite dimensional space of solutions $(\bar{\eta}, \eta) : \mathbf{R}^2 \to \Lambda_m$, each determined uniquely by its initial conditions. Given a doubly periodic solution matrix s for the elliptic Toda equations theorem 7 says there is a solution η to (21) satisfying $i\eta_m = -s^{-1}\Lambda s$ and $i\eta_{m-1} = -\partial \ln s$. In particular, these solutions of the o.d.e's determine s uniquely up to a scaling (indeed, up to a scaling by some power of ω, in view of theorem 5). Hence only a finite dimensional family of doubly periodic solutions of the Toda equations may correspond to each $m \equiv 1 \bmod (n+1)$. Now we prove:

Lemma 7 *Let $s(z, \bar{z})$ provide a doubly periodic solution to the elliptic Toda lattice and let $\tilde{s}$ be the point in $\mathcal{M}/K_0$ it comes from according to theorem 5. Then $\tilde{s}$ has a finite dimensional Γ-orbit in $\mathcal{M}/K_0$.*

Proof. By the previous corollary we know that this point $\tilde{s}$ exists. It is enough to show that the Γ_R-orbit of $\tilde{s}$ is finite dimensional, where Γ_R is real subgroup of Γ with respect to the real involution on $\Lambda_C(G^C, \nu)$. (this is because both $\mathcal{M}$ and K_0 are preserved by this involution and therefore the Γ-orbit of a real point inherits a real involution whose fixed manifold is the Γ_R-orbit). By theorem 5 the Γ_R-orbit of $\tilde{s}$ corresponds to a smooth family $s(z, \bar{z}, \mathbf{t})$ of solutions to the elliptic Toda equations, parameterised by real variables corresponding to the higher flows in the Toda hierarchy. Since all these flows commute (Γ_R is abelian) every solution in this family is also doubly periodic. But according to the discussion above each solution in this family corresponds to a solution of the o.d.e's (21) above, for some fixed $m \equiv 1 \bmod (n+1)$. Since the space of these solutions is finite dimensional so is the family $s(z, \bar{z}, \mathbf{t})$. It follows that the Γ_R-orbit of $\tilde{s}$ must also be finite dimensional.■

From now on let us use $\mathcal{N}$ to denote the quotient space $\mathcal{M}/K$ occuring in theorem 5. The aim is to prove the following result:

Proposition 5 *Every finite dimensional Γ-orbit in $\mathcal{N}$ is identifiable with the Jacobian of a complete algebraic curve of the type described in §4.*

One then knows from [19, 20] that this curve is precisely the spectral curve in the sense used here. The outline of the proof goes as follows. If we replace G^C by GL_{n+1}

and repeat the construction before theorem 5 which gave us $\mathcal{N}$ we obtain a space $\tilde{\mathcal{M}}/\tilde{K}$ which is isomorphic to $\mathcal{N}$ (provided we only work in the identity component of $\Lambda_C(GL_{n+1},\nu)$). This is because the centre of $\Lambda_C(GL_{n+1},\nu)$ factors out when we take its double quotient by $\Lambda_E(GL_{n+1},\nu)$ on the left and $\tilde{K}$ on the right (where the latter is the centralizer of K in $\Lambda_I(GL_{n+1},\nu)$). Now let $\mathcal{A}\supset\mathcal{G}$ denote the abelian Lie algebra of real-analytic maps $C\to\mathbf{C}$. This new construction of $\mathcal{N}$ allows us to define an action of $\exp(\mathcal{A})$, and therefore of $\mathcal{A}$, on $\mathcal{N}$ whose orbits are identical to the Γ-orbits. For any point x in $\mathcal{N}$ its isotropy group $\mathcal{I}_x$ belongs to an exact sequence

$$1\to\mathcal{I}_x\to\exp(\mathcal{A})\to\mathcal{J}_x\to 1 \tag{22}$$

for which $\mathcal{J}_x$ is finite dimensional precisely when it may be identified with $J(X)$ for some curve X of the right type. The result will follow immediately from this.

Let us begin by observing that $\mathcal{A}$ is isomorphic, as an abelian Lie algebra, to the centralizer of the Lie algebra of Γ in $\Lambda_C(gl_{n+1},\nu)$. Every element of this centralizer is a pair of power series' in $\zeta\Lambda$ with no constraints on the coefficients (except that the series' converge). The isomorphism replaces $\zeta\Lambda$ by ζ to get an element of $\mathcal{A}$ (which preserves the convergence since Λ and all its powers are simultaneously diagonalisable into some permutation of $diag(1,\omega,\ldots,\omega^n)$). It follows that every Γ-orbit in $\mathcal{N}$ is contained inside an $\exp(\mathcal{A})$-orbit, where the action of $\exp(\mathcal{A})$ is via its identification with the centralizer of Γ. But $\mathcal{A}$ splits into the direct Lie algebra sum $\mathcal{A}_I\oplus\mathcal{G}$, where $\mathcal{A}_I$ denotes the subalgebra of boundaries of holomorphic maps $I\to\mathbf{C}$ which vanish at $\zeta=\infty$. The splitting is explicitly given by

$$(a,b)=(a_-+b_+,a_-+b_+)+(a_+-b_+,b_--a_-)$$

where the Fourier series for a_+ (resp. a_-) consists only of non-negative (resp. negative) powers of ζ. It is not hard to check that $\exp(\mathcal{A}_I)$ becomes identified with $\tilde{K}$, therefore its action on $\mathcal{N}$ is trivial. Moreover, the action of $\exp(\mathcal{G})$ is identical to that of Γ since the only difference between them lies in the extra 'trivial flows' in $\exp(\mathcal{G})$, which leave every point of $\mathcal{N}$ fixed. Thus we have proved:

Lemma 8 *Every Γ-orbit in $\mathcal{N}$ is identical to an $\exp(\mathcal{A})$-orbit.*

Now we must develop an alternative description for points in the homogeneous space $\tilde{\mathcal{M}}=\Lambda_E(GL_{n+1},\nu)\backslash\Lambda_C(GL_{n+1},\nu)$ from which we can extract the correct interpretation of the sequence (22). The idea is to identify each point in $\tilde{\mathcal{M}}$ with a space of sections of a vector bundle over the Riemann sphere and show that when $\mathcal{J}_x$ is finite dimensional this bundle is the direct image of a line bundle over a covering $X\to\mathbf{CP}$ for which $\mathcal{J}_x\simeq J(X)$.

First we recall (from *e.g.* [20]) that $\Lambda_C(GL_{n+1},\nu)$ is isomorphic to the group $\Lambda_S GL_{n+1}$, where $S=S_1\cup S_2$ is the union of circles with radii ϵ^{n+1} and ϵ^{-n-1} respectively. The former is sometimes called the *principal realisation* of the latter and the isomorphism can be described as follows. Let $g:C\to GL_{n+1}$ be ν-equivariant and let $\kappa=diag(1,\zeta,\ldots,\zeta^n)$. It is easy to check that $(\kappa g\kappa^{-1})(\omega\zeta)=(\kappa g\kappa^{-1})(\zeta)$ and therefore it is a function of ζ^{n+1}: call this $h(\zeta^{n+1})$. Then $h(\zeta)$ belongs to $\Lambda_S GL_{n+1}$.

Clearly this identification is invertible and defines a homomorphism between these groups.

Using this isomorphism we determine an action of $\Lambda_C(GL_{n+1},\nu)$ on the Hilbert space $H^{(n+1)} = L^2(S,\mathbf{C}^{n+1})$ via the natural action of $\Lambda_S GL_{n+1}$ given by $h : H^{(n+1)} \to H^{(n+1)}\, ; f \mapsto fh^{-1}$ for each $h \in \Lambda_S GL_{n+1}$ (here we think of each element f of $H^{(n+1)}$ as a row vector of L^2-functions on S). Inside $H^{(n+1)}$ lies the subspace $H_U^{(n+1)}$ of boundaries of holomorphic functions $f : U \to \mathbf{C}^{n+1}$ where U is the annulus with boundary S.

Lemma 9 *There is a one-to-one correspondence between the homogeneous space $\tilde{\mathcal{M}}$ and the set $\{g^{-1} \circ H_U^{(n+1)} : g \in \Lambda_C(GL_{n+1},\nu)\}$, where $g \circ H_U^{(n+1)} = \{g \circ f : f \in H_U^{(n+1)}\}$ and $g \circ f$ denotes the action of $\Lambda_C(GL_{n+1},\nu)$ on $H^{(n+1)}$ via the isomorphism $\Lambda_C(GL_{n+1},\nu) \simeq \Lambda_S GL_{n+1}$.*

The proof follows at once from the fact that under the isomorphism $\Lambda_E(GL_{n+1},\nu)$ is mapped onto $\Lambda_U GL_{n+1}$.

Each subspace $W = g^{-1} \circ H_U^{(n+1)}$ can be thought of as representing some space of boundaries of sections of a holomorphic $\mathbf{C}^{n+1}$-bundle $\mathcal{E}$ over $\mathbf{CP}^1$. This bundle has transition function g between the trivial bundles over U and the closure V of its complement. As such, $\mathcal{E}$ comes equipped with a trivialisation φ_V over V (to be precise, over some open neighbourhood of V) and W represents the L^2-closure of boundaries of holomorphic sections over U seen via φ_V.

Now observe that W can be thought of as a module over a certain subalgebra $\mathcal{C}$ of $\mathcal{G}$. This is the subalgebra of elements of $\mathcal{G}$ which are functions of ζ^{n+1} only. It corresponds to the subalgebra of $\Lambda_C gl_{n+1}$ of all elements which are boundaries of functions $U \to gl_{n+1}$ of the form $a(\zeta)Id$; the correspondence is, of course, $a(\zeta^{n+1}) \mapsto a(\zeta)Id$. Clearly $a.W \subseteq W$, since this is just scalar multiplication, thus W is a $\mathcal{C}$-module. This reflects our description of W in terms of $\mathcal{E}$: think of $\mathcal{C}$ as the 'coordinate ring' of U and then the module W becomes the module of sections for a locally free sheaf over U (which will clearly have rank $n+1$). Now our aim is to examine the maximal subalgebra of $\mathcal{A}$ which preserves W. This should give us the coordinate ring of some branched cover of U. In that case $\mathcal{E}$ will be the direct image of a lower rank bundle on this cover: the rank is given by $(n+1)/m$ where m is the degree of the cover. The extreme case is when this cover has degree $n+1$ and so $\mathcal{E}$ is the direct image of a line bundle.

However, it is not easy to describe directly how elements of $\mathcal{A}$ outside $\mathcal{C}$ act on $H^{(n+1)}$. The standard way to do this (*cf.* for example [19, 21]) is to use the interleaving Fourier series isomorphism between $H^{(n+1)}$ and $H = L^2(C,\mathbf{C})$. We will use

$$\begin{array}{ccc} H^{(n+1)} & \longrightarrow & H \\ ((f_0,\ldots,f_n) & \mapsto & \sum_0^n \zeta^{k-n} f_k(\zeta^{n+1}). \end{array} \tag{23}$$

The key property of this map (23) (which readers are invited to check for themselves) is that, for any $a \in \mathcal{A}$,

$$e^a \circ f^{(n+1)} \mapsto e^{-a} f \quad \text{whenever} \quad f^{(n+1)} \mapsto f$$

where $e^a \circ f^{(n+1)}$ denotes the action of $exp(\mathcal{A})$ on $H^{(n+1)}$ via its identification with a subgroup of $\Lambda_S GL_{n+1}$. From now on we will identify points in $\tilde{\mathcal{M}}$ with subspaces $W \subset H$ according to lemma 9 followed by the map (23). The equivalence class of W in $\mathcal{N}$ will be denoted simply by $[W]$.

Now let us examine the subalgebra $\mathcal{A}_W \subset \mathcal{A}$ defined by

$$\mathcal{A}_W = \{a \in \mathcal{A} : aW \subseteq W\}.$$

From the earlier discussion we know $\mathcal{C}$ is a subalgebra of $\mathcal{A}_W$. Moreover, $\tilde{K} \simeq exp(\mathcal{A}_I)$ acts freely on $\tilde{\mathcal{M}}$ (this follows from the result in [19] that K acts freely on $\mathcal{M}$), hence $\mathcal{A}_W \cap \mathcal{A}_I = \{0\}$.

Lemma 10 *Let $\mathcal{I}_{[W]}$ denote the isotropy group of $[W]$ for the action of* $\exp(\mathcal{A})$ *on $\mathcal{N}$. Its Lie algebra equals the direct sum $\mathcal{A}_I \oplus \mathcal{A}_W$.*

Proof. An element a of $\mathcal{A}$ belongs to the Lie algebra of $\mathcal{I}_{[W]}$ precisely when there exists, for each $t \in \mathbf{C}$, some $b_t \in \mathcal{A}_I$ depending smoothly on t such that $\exp(ta - b_t)W = W$. Taking derivatives at $t = 0$ shows that that $a \in A_I \oplus A_W$. Moreover, it is clear from this argument that $A_I \oplus A_W$ is contained in the Lie algebra of $\mathcal{I}_{[W]}$, which proves the lemma.■

As a consequence we have the exact sequence of abelian Lie algebras

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_I \oplus \mathcal{A}_W & \to & \mathcal{A} & \to & \mathcal{H}_{[W]} & \to & 0 \\ & & (b, a) & \mapsto & a - b & & & & \end{array} \tag{24}$$

corresponding to the exact sequence of Lie groups (22), where $\mathcal{H}_{[W]}$ is the Lie algebra of $\mathcal{J}_{[W]}$. Now we must use a little algebraic geometry to show that when $\mathcal{H}_{[W]}$ is finite dimensional this sequence (24) describes $H^1(X, \mathcal{O})$ for some complete algebraic curve X.

We will say an element $(a_1, a_2) \in \mathcal{A}$ has *finite order* if $a_1(\zeta)$ extends meromorphically to a neighbourhood of $\zeta = 0$ *and* a_2 extends meromorphically to a neighbourhood of $\zeta^{-1} = 0$. The collection of all finite order elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{alg}$. On $\mathcal{A}^{alg}$ we have two gradings $o_j : \mathcal{A}^{alg} \to \mathbf{Z}$, with $j = 1, 2$, for which o_1 records the order of poles of a_1 at 0 while o_2 records the order of poles of a_2 at ∞. Likewise, we use W^{alg} to denote the finite order elements of $W \subset H$. We can identify W^{alg} with the subspace of sections of $\mathcal{E}$ over $\mathbf{CP}^1$ which only have poles over $0, \infty$.

Lemma 11 *For any $W \in \tilde{\mathcal{M}}$, $\mathcal{A}_W^{alg}$ is a finitely generated free $\mathcal{C}^{alg}$-module with rank $m \leq n + 1$. Moreover, the ring A_W^{alg} is an integral domain.*

Proof. First observe that $\mathcal{A}_W^{alg}$ has no zero-divisors, for any such element must be of the form either $(a_1, 0)$ or $(0, a_2)$. But this implies that W also possesses elements of this form, which is not possible if it comes from $g^{-1} \circ H_U^{(n+1)}$ for some invertible g. It follows that W^{alg} is a torsion free $\mathcal{A}_W^{alg}$-module, since for any non-zero $a \in \mathcal{A}_W^{alg}$ and $w \in W^{alg}$ the leading order terms in either component of aw cannot be zero. Clearly W^{alg} is a rank $n + 1$ $\mathcal{C}^{alg}$-module and therefore $\mathcal{A}_W^{alg}$ is a finitely generated

$\mathcal{C}^{alg}$-module which requires no more than $n+1$ generators. But since W^{alg} is faithful the ring $\mathcal{A}_W^{alg}$ is an integral extension of $\mathcal{C}^{alg}$. Since the latter is a PID we conclude that $\mathcal{A}_W^{alg}$ is actually a free $\mathcal{C}^{alg}$-module. ■

Since $\mathcal{A}_W^{alg}$ is an integral extension of $\mathcal{C}^{alg}$ there is a finite, surjective morphism $\pi : Spec(\mathcal{A}_W^{alg}) \to Spec(\mathcal{C}^{alg})$. Clearly $\mathcal{C}^{alg}$ is isomorphic, as an algebra, to $\mathbf{C}[\zeta^{n+1}, \zeta^{-n-1}]$ and may be thought of as the coordinate ring of the ζ^{n+1}-sphere punctured at 0 and ∞, hence we have a finite degree covering of this punctured sphere. Clearly the degree of this covering can be at most $n+1$. This curve $Spec(\mathcal{A}_W^{alg})$ is irreducible (by the previous lemma) and may be completed by adding non-singular points over each of 0 and ∞. We call the completion X: when the degree of the covering is $n+1$ it has the properties described in §4.

Example. Suppose we take for W the image of $H_U^{(n+1)}$ after the mapping (23). A simple calculation shows that this is $\{\zeta^{-n}(f, f) : f \text{ extends holomorphically into } E\}$. Therefore in this case $\mathcal{A}_W$ is simply $\mathcal{G}$ and the spectral curve is a smooth curve of genus 0 which has ramification points at P_0 amd P_∞ only.

To prove proposition 5 we first prove:

Lemma 12 *The vector space $\mathcal{H}_{[W]}$ is finite dimensional if and only if the $\mathcal{C}^{alg}$-module A_W^{alg} has rank $n+1$. In that case $\mathcal{H}_{[W]}$ is identifiable with $H^1(X, \mathcal{O})$.*

Proof. Each grading $o_j : \mathcal{A}^{alg} \to \mathbf{Z}$ is clearly a *surjective* homomorphism (for the multiplicative structure on $\mathcal{A}^{alg}$). Suppose A_W^{alg} has rank $m < n+1$, then we will show that there are infinitely many integers which do not lie in the image of $\mathcal{A}_I^{alg} \oplus \mathcal{A}_W^{alg}$ under either of these maps o_j. From this we will deduce that $\mathcal{H}_{[W]}$ must be infinite dimensional.

Recall that $\mathcal{C}^{alg}$ is identifiable with the algebra of Laurent polynomials in ζ^{n+1}. Therefore $o_j(\mathcal{C}^{alg}) = (n+1)\mathbf{Z}$ for either $j = 1, 2$. Now let $w_1, \ldots, w_m$ be a system of generators for $\mathcal{A}_W^{alg}$, then $\{o_j(w_k) \bmod n+1 : k = 1, \ldots, m\}$ is a proper subset of $\mathbf{Z}_{n+1}$, hence there is a least one integer value in each interval $[l(n+1), (l+1)(n+1)]$ (for $l \in \mathbf{Z}$) not taken by o_j on $\mathcal{A}_W^{alg}$. In particular, there are infinitely many positive integers not in $o_j(\mathcal{A}_W^{alg})$. Moreover, $o_j(\mathcal{A}_I^{alg})$ contains only non-positive integers (in both gradings), so there are infinitely many positive integer values not taken by elements of the direct sum $\mathcal{A}_I^{alg} \oplus \mathcal{A}_W^{alg}$ under either of the maps o_j. Thus there are infinitely many $\mathbf{C}$-linearly independent elements of $\mathcal{A}$ not in $\mathcal{A}_{[W]}$ *i.e.* $\mathcal{H}_{[W]}$ is infinite dimensional.

Now suppose $m = n+1$. Then $X_{aff} = Spec(\mathcal{A}_W^{alg})$ is an $n+1$ degree cover of $\mathbf{CP}^1 - \{0, \infty\}$. I claim that the smooth completion X of X_{aff} has a single point above each of 0 and ∞. To see this let $\mathcal{K}$ denote the field of fractions of $\mathcal{A}_W^{alg}$. It inherits two valuations $\nu_j : \mathcal{K} \to \mathbf{Z}$ from the homomorphisms o_j in the reverse of the obvious way: for each fraction a/s we set $\nu_j(a/s) = o_j(s) - o_j(a)$. Since $m = n+1$ each of these valuations is surjective. [For otherwise the subgroup of $\mathbf{Z}_{n+1}$ generated by $o_j(w_k) \bmod n+1$ would be proper. But then it is straightforward to show that the $n+1$ generators w_k are not $\mathcal{C}^{alg}$-independent.] The corresponding discrete valuation rings $\mathcal{A}_1$ and $\mathcal{A}_2$ are therefore the local rings for ramification points P_0 and P_∞ each of index n. It follows that $\pi : X \to \mathbf{CP}^1$ is a degree $n+1$

covering, represented in a closed disc X_∞ about $P_\infty = \pi^{-1}(\infty)$ by $\zeta^{-1} \mapsto \zeta^{-n-1}$ and in a closed disc X_0 about $P_0 = \pi^{-1}(0)$ by $\zeta \mapsto \zeta^{n+1}$. Hence $\mathcal{A}_I$ is the algebra of real-analytic maps $C \to \mathbf{C}$ which extend holomorphically into $X_0 \cup X_\infty$ and equal 1 over P_∞. By construction, $\mathcal{A}_W$ is the algebra of real-analytic maps $C \to \mathbf{C}$ which extend holomorphically into $X - (X_0 \cup X_\infty)$. From our earlier discussion (§4) we know that under these circumstances there is a surjective linear map $\mathcal{G} \to H^1(X, \mathcal{O})$ which considers each element of $\mathcal{G}$ as a 1-cocyle for some open cover of X based upon its cover by X_0, X_∞ and the closure of $X - (X_0 \cup X_\infty)$. Clearly this extends to a surjective linear map $\mathcal{A} \to H^1(X, \mathcal{O})$ whose kernel is $\mathcal{A}_I \oplus \mathcal{A}_W$, since this is the subspace of coboundaries in this cohomology. Hence $\mathcal{H}_{[W]} \simeq H^1(X, \mathcal{O})$ whence it is finite dimensional.■

Now proposition 5 follows from the observation that $[W]$ uniquely determines (and is determined by) the data $(X, \pi, \mathcal{L})$ where $\mathcal{L}$ is a line bundle over X (or, if X is singular, perhaps merely a rank 1 torsion free coherent sheaf). The bigraded $\mathcal{A}_W^{alg}$-module W^{alg} determines $\mathcal{L}$ equipped with trivialisations over P_0 and P_∞: passing to $[W]$ corresponds to discarding these trivialisations. From [19] we know that the correspondence between these two identifies the exp($\mathcal{A}$)-action with the $J(X)$-action on the Picard variety of X (or, more generally, the moduli space of rank 1 torsion free coherent sheaves over X). If $\mathcal{L}$ is a line bundle we know immediately that its $J(X)$-orbit is identifiable with $J(X)$ itself. More generally, the $J(X)$-orbit is identifiable with $J(X')$, where X' is the least singular curve from which $\mathcal{L}$ could come by direct image. However, the previous lemma shows that the dimension of $J(X')$ equals that of $J(X)$, hence $X \simeq X'$.

7 Appendix: spectral curves and Killing fields.

In this section I aim to explain the connection between the spectral curve used here and alternative definition given in [14]. There one finds a concrete construction for an algebraic curve given a harmonic 2-torus in a symmetric space, and it is clear how these constructions should be adapted to our case. As in [8], the emphasis is placed on *polynomial Killing fields*. So my first task is to explain how these fit into our picture.

Let us fix a superconformal 2-torus in $\mathbf{CP}^n$ with extended Toda frame Φ, and set $\alpha = \Phi^{-1} d\Phi$ (we will drop the subscript ζ in this discussion). By corollary 4 we may always choose Φ to be given by a factorisation $g\Phi^{(0)} = \Phi\chi$ for some $g \in \Lambda(G, \nu)$ (here χ denotes the product called $d.u$ in the earlier sections). We will say an element of $\Lambda_C(\mathbf{g}^C, \nu)$ has *finite order* if its component in $\Lambda_E(\mathbf{g}^C, \nu)$ (given the usual splitting) is a Laurent polynomial in ζ. Throughout this section we work with $\mathbf{g}^C = gl_{n+1}$. A *formal Killing field* for α is an analytic function $\eta(z, \bar{z})$ which satisfies

$$d\eta = [\eta, \alpha] \tag{25}$$

and is of finite order for every $z, \bar{z}$. Notice that η is not restricted to be real here (in this section by *real* I will mean with respect to the antiholomorphic involution on $\Lambda_C(\mathbf{g}^C, \nu)$). We will say η is a *polynomial* Killing field whenever it takes values in $\Lambda_E(\mathbf{g}^C,\nu)$ (which is precisely the case for which it is a Laurent polynomial, by our definition). One of the principal results of [8], when adapted to this context (as in [5]), shows that the existence of a formal Killing field implies the existence of real polynomial Killing fields satisfying the extra condition in (20) which guarantees that Φ is of finite type.

It is elementary to show that the formal Killing fields form a Lie algebra. What is not clear is that they form a commutative Lie algebra which is isomorphic to $\mathcal{A}^{alg}$, the subalgebra of finite order elements in $\mathcal{A}$. Here we view $\mathcal{A}$ as the centralizer in $\Lambda_C(\mathbf{g}^C,\nu)$ of $\zeta\Lambda$. Under this isomorphism, the polynomial Killing fields correspond to $\mathcal{A}_W^{alg}$, where W now denotes the coset $\Lambda_E(G^C,\nu).g$. This isomorphism is given by

$$\begin{array}{rcl} Ad\chi:\mathcal{A}^{alg} & \rightarrow & \{\text{formal Killing fields}\} \\ a & \mapsto & \eta = Ad\chi.a \end{array} \tag{26}$$

In order to prove these statements it is useful to have the following lemma. Let $\alpha^{(0)}=\Phi^{(0)-1}d\Phi^{(0)}$.

Lemma 13 *Let $a(z,\bar{z})$ be a formal Killing field for $\alpha^{(0)}$. Then $a\in\mathcal{A}^{alg}$ (in particular $da=0$).*

Proof. Write $a=(a_1,a_2)$. We will show that $da_1=0$ and a_1 commutes with Λ. A similar procedure will also work for a_2, proving the lemma.

Let us write $a_1=\sum_{-\infty}^{k}a_{1j}\zeta^{-j}$. From the Killing field equation we see that

$$\partial a_{1k}=0,\quad \partial a_{1j}=[\Lambda,a_{1j-1}],\quad \bar{\partial}a_{1j+1}=[a_{1j},\Lambda^*],\quad [a_{1k},\Lambda^*]=0, \tag{27}$$

where $j<k$. The proof requires the observation that the image and kernel of $ad\Lambda$ equal those of $ad\Lambda^*$ and these intersect only on $\{0\}$. This follows from the fact that Λ is semisimple. Using this observation it follows that $\bar{\partial}a_{1k}=0$, and therefore a_{1k} is constant and commutes with Λ. This means that the equations above are also true with k replaced by $k-1$. By induction we see that for every j, a_{1j} is constant and commutes with Λ.■

Now let us prove:

Proposition 6 *The map (26) is an isomorphism of Lie algebras which identifies $\mathcal{A}_W^{alg}$ with the subalgebra of polynomial Killing fields for α.*

Proof. Using $\chi=\Phi^{-1}g\Phi^{(0)}$ we see that, for each $a\in\mathcal{A}$, $Ad\chi.a=Ad\Phi^{-1}.\eta_0$ where $\eta_0=Adg.a$ is independent of $z,\bar{z}$. Therefore

$$d(Ad\chi.a)=Ad\Phi^{-1}.([\Phi d\Phi^{-1},\eta_0]+d\eta_0)=[Ad\chi.a,\alpha]. \tag{28}$$

Moreover, $Ad\chi.a$ has finite order whenever a does, since χ extends holomorphically into I. Conversely, given a formal Killing field η one easily shows that $d(Ad\Phi.\eta)=0$ and therefore $d(Ad\Phi^{(0)}.a)=0$ for $a=Ad\chi.\eta$. When this is expanded we see that a is a formal Killing field for $\alpha^{(0)}$. By the previous lemma this implies $a\in\mathcal{A}^{alg}$.

Now let us see that η takes values in $\Lambda_E(\mathbf{g}^C,\nu)$ if and only if $a\in\mathcal{A}_W^{alg}$. We know $a\in\mathcal{A}_W$ precisely when $We^{ta}=W$ for all $t\in\mathbf{C}$. But since e^{ta} commutes with $\Phi^{(0)}$

this is equivalent to $\Lambda_E(G^C,\nu)\chi e^{ta} = \Lambda_E(G^C,\nu)\chi$. Therefore $a \in \mathcal{A}_W$ if and only if $\exp(tAd\chi.a)$ takes values in $\Lambda_E(G^C,\nu)$ for all t, which proves the result.■

It is worth noting that since χ is real η is also real if and only if a is.

It is now possible to refer to the commutative algebra of polynomial Killing fields, where the ring product is ordinary matrix multiplication. However, since each Killing field is a function of $z,\bar{z}$ it is more convenient to look at this as a parameterized family $R(z,\bar{z})$ of commutative algebras, each isomorphic to $\mathcal{A}^{alg}$. In particular, we will use R to denote $R(0,0)$: the algebra of polynomial Killing fields evaluated at $z,\bar{z}=0$.

Let us now look at two alternative definitions for the spectral curve. To each polynomial Killing field η we can assign the affine plane curve

$$Y_\eta = \{(\zeta,\mu) \in \mathbf{C}^* \times \mathbf{C} : \det(\eta(\zeta) - \mu.Id) = 0\}. \tag{29}$$

It is a standard observation about Lax equations, of which the Killing field equations (25) are an example, that Y_η is independent of z and $\bar{z}$. Moreover, the ν-equivariance of $\eta(\zeta)$ implies this curve admits the fixed-point free automorphism $(\zeta,\mu) \mapsto (\omega\zeta,\mu)$. When η is additionally an *adapted* polynomial Killing field (*i.e.* satisfies (20)) we could probably use the theory in [15] (following [6]) to show that the corresponding flows linearise on the Jacobian of the smooth completion of the normalisation of Y_η. However, there is more than one choice of adapted η for any superconformal 2-torus, and we are faced with the problem of which to choose.

An alternative definition, used in [14] to overcome the problem of which η to choose, is to take any $\eta_0 \in R$ and define Y to be the image of the map

$$\begin{array}{ccc} Y_\eta & \to & \mathbf{C}^* \times \mathbf{CP}^n \\ (\zeta,\mu) & \mapsto & (\zeta,[v]) \end{array} \tag{30}$$

where $\eta_0(\zeta)v = \mu v$ and $[v]$ denotes the line in $\mathbf{C}^{n+1}$ generated by the vector v. The spectral curve is then defined to be the compactification $\hat{Y}$ of Y in $\mathbf{CP}^1 \times \mathbf{CP}^n$. Since elements of R are simultaneously diagonalisable we see that Y is independent of the choice of η_0. However, at first sight it is not clear that this map is well-defined throughout Y_η: problems might arise at points where the eigenspaces coincide. In [14] attention was restricted to the generic case, where eigenspaces coincide at worst in pairs *i.e.* where the ramification points of Y over $\mathbf{CP}^1$ (under projection on the first factor) are all simple.

Now let us see the connection between X_{aff} and Y. The ν-equivariance of η_0 induces a fixed-point free automorphism (which we shall also call ν) on Y inherited from the automorphism $(\zeta,[v]) \mapsto (\omega\zeta,[\sigma v])$ of $\mathbf{C}^* \times \mathbf{CP}^n$. It can be shown quite easily that R is the coordinate ring for Y/ν, since R is ring of all ν-equivariant regular endomorphisms of the 'eigenspace line bundle' over Y (this is the pull-back of the tautological line bundle over $\mathbf{CP}^n$). Therefore $X_{aff} \simeq Y/\nu$. Moreover, a little thought shows that the R-module $M = \{f : \mathbf{C}^* \to \mathbf{C}^{n+1}$: Laurent poly. with $f(\omega\zeta) = \sigma f(\zeta)\}$ represents the module of regular ν-equivariant sections of the dual of this line bundle: this R-module M corresponds to the $\mathcal{A}_W^{alg}$-module W^{alg}.

In principle there is no reason why we cannot work with $\tilde{Y}$ as opposed to X. It turns out that $X \simeq \tilde{Y}/\nu$ and therefore one knows (from *e.g.* [22]) that the subgroup $J(\tilde{Y})_\nu$ of ν-fixed points in $J(\tilde{Y})$ is a finite cover of $J(X)$ with covering group $\mathbf{Z}_{n+1}$. It represents the ν-equivariant line bundles of degree zero over $\tilde{Y}$. It was shown in [19] that, whereas $J(X)$ is identifiable with a finite dimensional orbit in $\mathcal{M}/K$, this group $J(\tilde{Y})_\nu$ is identifiable with the corresponding orbit in $\mathcal{M}/K_0$, which is the dressing space of Toda solutions. The convenience of working with X is that one does not have to keep in mind that everything in sight is ν-equivariant. In particular, this considerably simplifies the periodicity conditions for the Toda frame.

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