Remarks on the Vol'pert theory of travelling-wave solutions for parabolic systems

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Abstract

Vol'pert and Vol'pert have shown that the parabolic system of n equations,

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(u)$$

has monotone travelling-wave solutions connecting two equilibria, S and T say, under 'bistable' and 'monostable' conditions on the function $f : \mathbb{R}^n \to \mathbb{R}^n$, when A is a positive-definite diagonal matrix. We observe that this genre of hypotheses on f yields results on the existence and properties of intermediate zeros of f between S and T. These imply the existence of a 'chain' of monotone travelling waves from S to T under 'unstable' conditions on f for which the Vol'perts have established the non-existence of a direct monotone connection. It is also proved that for wave velocities consistent with their existence theory, there are unstable and stable monotone directions at S and T respectively, and the dimensions of the unstable manifold at S and the stable

manifold at T sum to at least $2n$, the dimension of the underlying phase space. Some probabilistic applications of these results are anticipated.

1 Introduction

1.1 Background and main results

Recently, Vol'pert and Vol'pert [13,14] discussed the existence of travelling-wave solutions of the parabolic system

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(u) \tag{1.1}$$

where A is an $n \times n$ positive-definite diagonal matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function satisfying the so-called monotonicity property

$$\frac{\partial f_i(u)}{\partial u_j} > 0 \text{ for } i \neq j, \quad i, j = 1, \ldots, n \tag{1.2}$$

where f_i denotes the i^{th} component function of f. Restriction of attention to travelling-wave solutions $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}^n$ of (1.1) satisfying

$$u(t, x) = w(x - ct), \tag{1.3}$$

where $w : \mathbb{R} \to \mathbb{R}^n$ and c is the unknown constant wave velocity, leads to the system of ordinary differential equations,

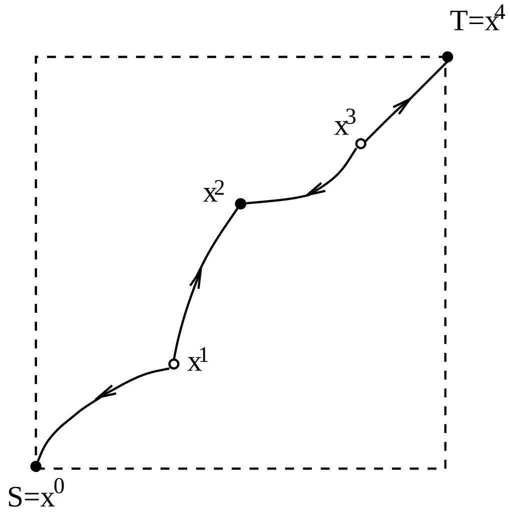
$$Aw'' + cIw' + f(w) = 0 \tag{1.4}$$

with I denoting the $n \times n$ identity matrix and the prime differentiation with respect to the independent variable.

Suppose that there exist $S, T \in \mathbb{R}^n$ with $S < T$ and $f(S) = f(T) = 0$. (For vectors $p, q \in \mathbb{R}^n$, we say that $p < q$ if $p_k < q_k$ for each k, $1 \leq k \leq n$ and p_k denotes the k^{th} co-ordinate of p. When $q < p$, (q, p) is the set $\{x \in \mathbb{R}^n : q < x < p\}$. An *equilibrium* of f is a point of $\mathbb{R}^n$ where $f = 0$.) Vol'pert and Vol'pert consider the important case of monotone solutions of (1.4) (say $w'(x) > 0$ for $x \in \mathbb{R}$) connecting two such equilibria. First, they prove the existence of a unique velocity c and corresponding solution w to (1.4), unique modulo translations, when the Fréchet derivatives of f at S and T (denoted $df[S], df[T]$) each have all their eigenvalues in the open left-half plane — the 'bistable' case [13]. Second, the situation where one of the derivatives at S and T has all its eigenvalues in the open left-half plane and the other has an eigenvalue in the open right-half plane is considered — the 'monostable' case, and existence of monotone solutions shown if and only if c exceeds a positive critical value [14]. Finally, they prove that there are no monotone solutions when $df[S]$ and $df[T]$ each have an eigenvalue in the open right-half plane — the 'unstable' case. We will classify equilibria to reflect the stability of the equation $\frac{du}{dt} = f(u)$, as follows.

Definition 1.1 *An equilibrium of f, w_0, is said to be stable if all the eigenvalues of $df[w_0]$ are in the open left-half plane, and unstable (weakly-unstable) if there is an eigenvalue of $df[w_0]$ in the open (closed) right-half plane.*

The hypothesis in the main existence theorem in the bistable case [13], is that there exists at least one equilibrium of f in the interval (S, T), and that each such equilibrium is unstable. The only purpose of this assumption is to ensure that there is no intermediate stable equilibrium of f. This enables the proof of *a priori* bounds in an appropriate function space and a use of degree theory to deduce existence of monotone solutions. For *scalar-valued* differentiable functions f, it is clearly true that the bistable conditions imply the existence of an intermediate zero of f, and it is germane to ask the following — do the eigenvalue conditions on $df[S]$ and $df[T]$ in the bistable case imply the existence of zeros of f between those at S and T for an n-dimensional system? The answer to this question



An example of multiple connections in the unstable problem, in the case when n=2 and there are 3 intermediate equilibria as shown. • and ∘ denote unstable and stable equilibria respectively. For sufficiently large positive velocities, there are monotone connections between adjacent equilibria in the directions indicated by the arrows.

Figure 1

is yes. We prove that given a continuously differentiable monotone function $f : \mathbb{R}^n \to \mathbb{R}^n$ with stable equilibria at S and T, there exists $x \in (S,T)$ with $f(x) = 0$. Moreover, if the zeros of f are isolated (for example, if f is analytic) then we show that there exists a weakly-unstable equilibrium of f in (S,T). The proof, given in Section 2, uses positive operator theory to exploit the monotonicity assumptions on f, which together with the eigenvalue conditions allow use of Brouwer degree to deduce the existence of such a zero.

Interestingly, we also show that if both S and T are *weakly-unstable* equilibria of f, then there exists $x \in (S,T)$ with $f(x) = 0$. Assuming that the zeros of f are isolated, it is proved additionally that there must be an intermediate equilibrium which is stable. In fact, given a maximal totally ordered set of equilibria in (S,T), $\{x^1, \ldots, x^k\}$, k must be odd and x^i is a stable or weakly-unstable equilibrium according to whether i is odd or even.

This has important consequences for the 'unstable' case of Vol'pert and Vol'pert (that is, when S and T are both unstable equilibria). Assume that in addition to the equilibria of f being isolated, each weakly-unstable equilibrium is in fact *unstable*. Then although no monotone connection from S to T can exist, the alternating stability of the equilibria $\{x^1, \ldots, x^k\}$ enables the existence theory for the 'monostable' case [14] to be applied to (x^i, x^{i+1}) for each i, $0 \leq i \leq k$. Thus there *are* monotone connections from S and T to the intermediate zeros x^1 and x^k respectively, and also between x^i and x^{i+1} for $1 \leq i \leq k$ if $k > 1$. For example, if there is exactly one zero x of f between S and T, then for sufficiently large positive velocities, there is an increasing monotone connection from x to T (that is, $w'(x) > 0$ for $x \in \mathbb{R}$) and a decreasing monotone connection from x to S ($w'(x) < 0$ for $x \in \mathbb{R}$). (See Figure 1.)

Our second remark concerns the interpretation in dynamical systems theory of the Vol'pert and Vol'pert results. Setting $v = w'$, the second order system of n ordinary differential equations (1.4) becomes a first order system of $2n$ equations,

$$\begin{pmatrix} w' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ -cA^{-1}v - A^{-1}f(w) \end{pmatrix} \tag{1.5}$$

whose linearization about $(v, w) = (0, w_0)$ with $f(w_0) = 0$ is

$$\begin{pmatrix} w' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A^{-1}B & -cA^{-1} \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} =: K_c(w_0) \begin{pmatrix} w \\ v \end{pmatrix} \tag{1.6}$$

where $B = df[w_0]$. Recall that in general, the behaviour of a nonlinear system at a hyperbolic equilibrium is determined by the eigenvalue structure of its linearization there. The dimensions of the unstable and stable manifolds about a hyperbolic equilibrium point

w_0 (denoted by $D_u(w_0)$ and $D_s(w_0)$, say) equal the dimensions of the unstable and stable subspaces of the linearized problem (1.6), which in turn are equal to the numbers of eigenvalues (counted according to algebraic multiplicity) of $K_c(w_0)$ in the open right- and left-half planes respectively. (See, for example, [2].) So upon manipulating (1.6), we see that the qualitative properties of the system in the vicinity of the equilibria at S and T can be gleaned from the 'eigenvalues' $\lambda \in \mathbb{C}$ and corresponding $z \in \mathbb{C}^n \backslash \{0\}$ of the nonlinear-in-λ, linear eigenvalue problem

$$(\lambda^2 A + \lambda c I + B) z = 0 \tag{1.7}$$

where B is the derivative of f at the equilibrium point w_0. (Note that these values λ are eigenvalues of $K_c(w_0)$ in the usual sense, with eigenvector $\begin{pmatrix} z \\ \lambda z \end{pmatrix} \in \mathbb{C}^{2n} \backslash \{0\}$.)

For a given velocity c, such λ and z will henceforth be referred to as *eigenvalues and eigenvectors of the travelling-wave problem linearized at* w_0. The linear space of all $z \in \mathbb{C}^n$ such that $(\lambda^2 A + \lambda c I + B) z = 0$ will be called the *eigenspace* corresponding to the travelling-wave eigenvalue λ. By the *stable (unstable) manifold at an equilibrium* w_0, we will mean the stable (unstable) manifold of the travelling-wave system (1.5) at w_0. (This use of stable/unstable should not be confused with the notion of the stability of an *equilibrium* introduced in Definition 1.1.) For brevity, we will write $M(\lambda, c) := \lambda^2 A + \lambda c I + B$, where the matrices A and B are fixed in a given context.

For it to be possible to prove that there exists a travelling-wave from S to T for a given value of the velocity c, we expect that $D_u(S) + D_s(T) \geq 2n$, the dimension of the underlying phase space, and $D_u(S) + D_s(T) \geq 2n+1$ is necessary for the unstable manifold at S and the stable manifold at T to intersect transversally. Additionally, for a monotonic travelling-wave to exist, there must be unstable and stable monotone eigenvalues of the travelling-wave problem linearized at S and T respectively, in the following sense.

Definition 1.2 *An eigenvalue λ of the travelling-wave problem linearized at an equilibrium w_0, with corresponding eigenvector z, is stable (unstable) monotone if*

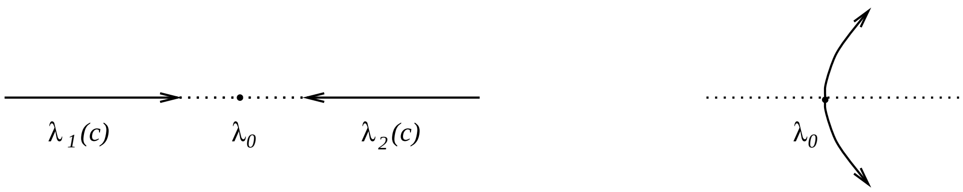
(i) *λ is real and negative (positive), and*

(ii) *all components of the vector z are of the same sign.*

There is said to be a stable (unstable) monotone direction at w_0 if there is a stable (unstable) monotone eigenvalue at w_0.

Here we show that this dimensional analysis ties in with the results of Vol'pert and Vol'pert. Decisive use is made of a result due to Cohen [3] on the convex dependence of the Perron-Frobenius eigenvalue on the diagonal of a matrix. In the bistable case, there exist both stable and unstable monotone eigenvalues at S and at T and $D_s(S) = D_s(T) = D_u(S) = D_u(T) = n$, for all velocities c. It is shown in [13] that there is a *unique* c and corresponding travelling-wave w *unique* modulo translations satisfying (1.4) in this case, so it is not surprising that the dimensions of the invariant manifolds only just 'add up' since the solution is not stable under perturbation.

In the monostable case in which S and T are stable and unstable equilibria respectively, Vol'pert and Vol'pert prove the existence of monotone travelling-waves for all $c \geq c_0 > 0$. We show that there exists some $c_0 > 0$ such that there is a unique stable monotone eigenvalue λ_0 at T for $c = c_0$, and that for $c > c_0$, there are *two* such eigenvalues, $\lambda_1(c) < \lambda_0 < \lambda_2(c)$. Moreover, λ_1 and λ_2 are decreasing and increasing continuous functions of c respectively, each of which tends to λ_0 as c is decreased to c_0. For $c < c_0$, there is *no* stable monotone eigenvalue and a pair of complex conjugate branches of eigenvalues emanates from λ_0. (See Figure 2.) Using the Cohen convexity theorem, it is proved that



(a) As c is decreased to c_0, $\lambda_1(c)$ and $\lambda_2(c)$ both tend monotonically to λ_0.

(b) As c is decreased below c_0, a pair of complex conjugate branches of eigenvalues emanate from λ_0.

Figure 2 : The dependence of eigenvalues on c near c_0.

for $c > c_0$, there can be no eigenvalues with real part between $\lambda_1(c)$ and $\lambda_2(c)$. Together with the observation that n eigenvalues tend to $-\infty$ as $c \to \infty$, this shows that for such c, $D_s(T) \geq n+1$ and hence $D_s(T) + D_u(S) \geq 2n+1$. (The structure at S follows from the bistable analysis.) The fact that we have a surplus of dimensions here is compatible with the Vol'perts' result that there is an infinite range of velocities for which monotone solutions exist.

Replacing λ and c in (1.7) by their negatives, it is clear that when the eigenvalue conditions on $df[S]$ and $df[T]$ are interchanged, there exists some $c_0 < 0$ such that there is an unstable monotone eigenvalue at S precisely when $c \leq c_0$, and that for such velocities, $D_s(T)+D_u(S) \geq 2n+1$. Hence if *both* S and T are unstable equilibria , there is no velocity such that both an unstable monotone direction at S and a stable monotone direction at T exist. The proof in [14] that no monotone connection can occur in the unstable case also rests on the observation that were there to be a monotone solution to (1.4), the structure at the two equilibria would require opposite signs of the velocity.

We make an interesting observation in the following special case of the monostable problem. Suppose that S and T are stable and unstable equilibria respectively, and that for $c = c_0$, $D_s(T) = 2n$. Then the stable manifold at T has the *full* dimension of the underlying space, and so will intersect *transversally* with the unstable manifold at S. By virtue of the fact that transversality is an open condition, there thus exists $\delta > 0$ such that for $c \in (c_0 - \delta, c_0 + \delta)$, the intersection of the unstable manifold at S and the stable manifold at T will still be transversal. Hence for $c \in (c_0 - \delta, c_0)$, there does exist a connection from S to T. But the criticality of c_0 implies that for such c, this connection must be *non-monotone*.

Note that no claims are made here about the existence or behaviour of centre manifolds. Numerical experiments have shown that there can be eigenvalues of the travelling-wave problem on the imaginary axis in the monostable case, but we are concerned simply with counting the number of eigenvalues in the open left-half plane, which give the dimension of the stable manifold, and the number of eigenvalues in the open right-half plane, which give the dimension of the unstable manifold.

Section 2 gives the statement and proof of the intermediate zeros results, whilst the theorems concerning the eigenvalues of the travelling-wave problem are presented and proved in Section 3. Section 4 comprises of a brief discussion of a generalized FKPP system of n equations, which is a specific example of the monostable case; in particular, the aspects of Section 3 important for the probabilistic treatment of this system are highlighted.

1.2 Notation and some useful theorems

Before going on to prove these remarks, let us recall here some well-known results from positive operator theory. First some notation.

Definition 1.3 **(i)** *A vector x in $\mathbb{R}^n$ is positive (non-negative) if all its components are positive (non-negative). We write $x > 0$ ($x \geq 0$).*

(ii) *K denotes the positive cone in $\mathbb{R}^n$ of non-negative vectors.*

(iii) *$P^{n\times n}$ denotes the set of all real $n \times n$ matrices with positive off-diagonal elements.*

Recall that the Fréchet derivative of a function f at w, $df[w]$, can be identified with its Jacobian matrix, the matrix of partial derivatives of f at w with respect to the standard basis of $\mathbb{R}^n$. Without further comment, we will use the notation $df[w]$ to denote the derivative or its Jacobian matrix, as appropriate.

Note that given a monotone function f, $df[w] \in P^{n\times n}$ for each $w \in \mathbb{R}^n$. The Perron-Frobenius Theorem (see, for example [12], page 3), implies the following.

Theorem 1.4 *(Perron-Frobenius). Let $M \in P^{n\times n}$. Then M has a real, simple eigenvalue $\mu_{PF}(M)$ such that an associated eigenvector is positive and every other eigenvalue of M has real part less than $\mu_{PF}(M)$. Moreover, any positive eigenvector must be a multiple of the Perron-Frobenius eigenvector.*

To exploit this result, we need a special case of a theorem due to Krasnosel'skii ([9], page 67).

Theorem 1.5 *(Krasnosel'skii). Let A be a real $n \times n$ matrix with non-negative entries such that $A^p u \geq \alpha u$ $(\alpha > 0)$ for some $p \in \mathbb{N}$ with $u \in \mathbb{R}^n \backslash \{0\}$, $-u \notin K$ and $u = v - w$ for some $v, w \in K$. Then A has a real eigenvalue λ which satisfies $\lambda \geq \sqrt[p]{\alpha} > 0$.*

The following corollary is particularly useful.

Corollary 1.6 *Let $M \in P^{n\times n}$ and suppose that there exists $u \in \mathbb{R}^n \backslash \{0\}$ with $-u \notin K$ such that $Mu > 0$. Then $\mu_{PF}(M) > 0$.*

Proof. Since $Mu > 0$, there exists some $\beta > 0$ such that $Mu > \beta u$. Also, there exists $\alpha > 0$ such that $M + \alpha I$ has positive entries, and $(M + \alpha I)u > (\alpha + \beta)u$, so by Theorem 1.5, $\mu_{PF}(M + \alpha I) \geq \alpha + \beta$.

Then clearly $\mu_{PF}(M) = \mu_{PF}(M + \alpha I) - \alpha \geq \beta > 0$.

$\square$

We will also need a related result, which is a variant of Theorem 1.6 in [12].

Theorem 1.7 *Let $M \in P^{n\times n}$ and suppose that there exists $u \in K\backslash\{0\}$ such that $Mu < 0$. Then $\mu_{PF}(M) < 0$.*

2 The existence of intermediate zeros

First some definitions and notation. For a set Ω in $\mathbb{R}^n$, $\partial\Omega$ denotes the boundary of Ω, $\overline{\Omega}$ the closure and Ω^o the interior. We write $B_\delta(x)$ for the open ball, centre x, radius δ, and C for the open n-rectangle (S, T) . Here we do not need the function $f : \mathbb{R}^n \to \mathbb{R}^n$ to be monotone in the strict sense of (1.2); it is sufficient for f to be locally monotone.

Definition 2.1 *A continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally monotone if whenever $f_k(x) = 0$ for some k, $1 \leq k \leq n$, and $x \in \mathbb{R}^n$, then*

$$\frac{\partial f_k}{\partial x_j}(x) > 0 \text{ for } j \neq k,\ l \leq j \leq n.$$

Lemma 2.2 *(Vol'pert and Vol'pert [13]). Let f be locally monotone. If, for some k, $1 \leq k \leq n$, $f_k(x) = 0$ and $y \geq x$, $y \neq x$ and $y_k = x_k$, then $f_k(y) > 0$. Also, $f_k(y) < 0$ if $y < x$, $y \neq x$ and $y_k = x_k$.*

Proof. If $\phi(t) = f_k(x(1-t) + yt)$, where x and y are as in the statement of the lemma, then $\phi'(t) \neq 0$ if $\phi(t) = 0$, by local monotonicity. Since $\phi(0) = 0$, the result is now immediate in both cases.

$\square$

For $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous, Ω an open bounded subset of $\mathbb{R}^n$ and $p \in \mathbb{R}^n \backslash f(\partial\Omega)$, let $\deg(f, \Omega, p)$ represent the Brouwer degree of f at p relative to Ω, and $\mathrm{ind}(f, x_0, p) := \deg(f, B_\delta(x_0), p)$ for $\delta > 0$ sufficiently small, the index of an isolated solution, x_0, of $f(x) = p$. If f is continuously differentiable and $df[x_0]$ is invertible at an isolated solution x_0, then $\mathrm{ind}(f, x_0, p) = \mathrm{sgn}(\det(df[x_0]))$, where $\mathrm{sgn}(\alpha) = 1$ or -1 according to whether α is positive or negative. Recall that $\det A = \Pi_{i=1}^n \lambda_i$ for $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, listing an eigenvalue k times when it has algebraic multiplicity k.

Theorem 2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable locally monotone function such that S and T are stable equilibria of f. Then there exists $x \in C$ with $f(x) = 0$.*

Proof. Let $w \in C$ be fixed and let $h : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by $h(t,x) = (1-t)(w-x) + tf(x)$.

Since $f(S) = f(T) = 0$, Lemma 2.2 gives that for every k, $1 \le k \le n$, $f_k(x) > 0$ for $x \in \partial C \backslash \{S\}$ with $x_k = S_k$ and $f_k(x) < 0$ for $x \in \partial C \backslash \{T\}$ with $x_k = T_k$. Now for each k and $x \in \{y \in \partial C : y_k = S_k, y \neq S\}$, $w_k - x_k > 0$ and $f_k(x) > 0$, so $f(x) \neq \lambda(x-w)$ for any $\lambda > 0$. Similarly, for $x \in \{y \in \partial C : y_k = T_k, y \neq T\}$, $w_k - x_k < 0$ and $f_k(x) < 0$, so $f(x) \neq \lambda(x-w)$ for any $\lambda > 0$. Hence given $x \in \partial C \backslash \{S,T\}$ and $t \in (0,1)$, $f(x) \neq \frac{1-t}{t}(x-w)$; i.e. $h(t,x) \neq 0$. Also, $h(0,x) = w - x \neq 0$ as $w \notin \partial C$ and $h(1,x) = f(x) \neq 0$ since, for some k, $f_k(x) \neq 0$. Therefore $h(t,x) \neq 0$ for $(t,x) \in [0,1] \times \{\partial C \backslash \{S,T\}\}$.

Note that $f(S) = f(T) = 0$ and $S, T \in \partial C$, so $\deg(f, C, 0)$ is not defined. We now show there are neighbourhoods N_S and N_T of S and T such that $h(t,x) \neq 0$ for $(t,x) \in [0,1] \times \{(N_S \backslash C) \cup (N_T \backslash C)\}$.

Suppose that there is a sequence $\{x_i\}$ with $\{x_i - S\} \subset \mathbb{R}^n \backslash K$, $\| x_i - S \| < 1/i$ and $(1-t_i)(w-x_i) + t_i f(x_i) = 0$ for some $t_i \in [0,1]$. Since f is differentiable at S and

$f(S) = 0$, $f(x_i) = L(x_i - S) + R(x_i)$ where $L := df[S]$ and $R : \mathbb{R}^n \to \mathbb{R}^n$ is such that $\| R(x_i) \| / \| x_i - S \| \to 0$ as $x_i \to S$. Hence

$$0 = t_i \left[L \frac{(x_i - S)}{\| x_i - S \|} + \frac{R(x_i)}{\| x_i - S \|} \right] + \frac{(1 - t_i)}{\| x_i - S \|}(w - x_i) \tag{2.1}$$

since $x_i - S \neq 0$ for each $i \in \mathbb{N}$. Now $\{s_i\} := \left\{ \frac{x_i - S}{\|x_i - S\|} \right\}$, a bounded sequence in $\mathbb{R}^n$, has a convergent subsequence, say $s_j \to s$; L is continuous, and so $Ls_j \to Ls$ as $j \to \infty$. Also, $\{t_j\}$ is bounded in $\mathbb{R}$, so has a subsequence $t_m \to t$, and since $\frac{R(x_m)}{\|x_m - S\|} \to 0$ as $m \to \infty$, we see that

$$\frac{(1 - t_m)}{\| x_m - S \|}(w - x_m) = -t_m \left[L \frac{(x_m - S)}{\| x_m - S \|} + \frac{R(x_m)}{\| x_m - S \|} \right]$$

tends to a limit as $m \to \infty$. Since $\| x_m - S \| \to 0$, it follows that $t_m \to 1$, and, writing $\beta = \lim_{m \to \infty} \frac{(1 - t_m)}{\|x_m - S\|} \geq 0$, that $0 = Ls + \beta(w - S)$ where $\beta \geq 0$ and $s \notin K^o$ has $\| s \| = 1$. But zero is not an eigenvalue of L, so $\beta > 0$ and for some $\beta(w - S) \in K^o$ and $-s \notin -K^o$, $L(-s) = \beta(w - S)$. Moreover, L is continuous and K^o is open, so there exist $\tilde{u} \notin K$ and $\tilde{v} \in K^o$ such that $L\tilde{u} = \tilde{v}$. Now for some $\gamma > 0$, $\tilde{v} \geq \gamma \tilde{u}$, and hence $L\tilde{u} \geq \gamma \tilde{u}$. Since f is locally monotone, $L \in P^{n \times n}$. So the matrix $L + \alpha I$ has positive entries for some $\alpha > 0$ and $(L + \alpha I)\tilde{u} \geq (\gamma + \alpha)\tilde{u}$ where $\tilde{u} \notin -K$. Theorem 1.5 then implies that $L + \alpha I$ has a real eigenvalue greater than $\gamma + \alpha$, whence L has a real eigenvalue greater than γ. This contradicts the hypothesis that all the eigenvalues of $L = df[S]$ are in the open left-half plane. Thus there exists $k \in \mathbb{N}$, such that for all $x - S \notin K$ with $\| x - S \| < 1/k$ and all $t \in [0, 1]$, $h(t, x) \neq 0$. A similar argument shows that k may be chosen so that, in addition, $\| x - T \| < 1/k$, $x - T \notin -K$ implies that $h(t, x) \neq 0$ for each t.

Hence there are open neighbourhoods N_S and N_T of S and T respectively such that for $x \in (\partial N_S) \cup (\partial N_T)$, $h(t, x) = 0$ only if $x \in C$, the case of $x \in \partial C$ having been previously excluded. Next note that S and T are isolated zeros of f by the Inverse Function Theorem and we can therefore take N_S and N_T sufficiently small to ensure that there are no additional zeros of f in $\overline{N_S} \cup \overline{N_T}$. Furthermore, $\operatorname{ind}(f, S, 0) = \operatorname{ind}(f, T, 0) = (-1)^n$ since for $x = S$ or T, $\operatorname{ind}(f, x, 0) = \operatorname{sgn}(\det(df[x]))$, and all the eigenvalues of $df[S]$ and $df[T]$ are in the open left-half plane.

Now let $\Omega = C \cup N_S \cup N_T$. Then $h(t, x) \neq 0$ for any $t \in [0, 1], x \in \partial\Omega$, and so since $w \in C \subset \Omega$, the homotopy property of Brouwer degree gives that $\deg(f, \Omega, 0) = \deg(w - I, \Omega, 0) = (-1)^n$, where I denotes the identity mapping. If $\Omega_1 = \Omega \backslash \{\overline{C} \cap (\partial N_S \cup \partial N_T)\}$, then $\deg(f, \Omega, 0) = \deg(f, \Omega_1, 0)$ by the excision property, so

$$\begin{aligned} (-1)^n = \deg(f, \Omega_1, 0) &= \operatorname{ind}(f, S, 0) + \operatorname{ind}(f, T, 0) + \deg(f, \Omega_1 \backslash (N_S \cup N_T), 0) \\ &= 2(-1)^n + \deg(f, \Omega_2, 0), \end{aligned}$$

where $\Omega_2 = \Omega_1 \backslash (N_S \cup N_T) \subset C$. Hence $\deg(f, \Omega_2, 0) = (-1)^{n+1} \neq 0$, and thus by the existence property of Brouwer degree, there exists $x \in \Omega_2 \subset C$ with $f(x) = 0$, as required.

□

The corresponding result in the case in which S and T are weakly-unstable equilibria is proved similarly, using Theorem 1.7 instead Theorem 1.5.

Theorem 2.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable locally monotone function such that S and T are weakly-unstable equilibria of f. Then there exists $x \in C$ with $f(x) = 0$.*

Proof. As in Theorem 2.3, the function $h : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ defined as above is non-zero on $\partial C \backslash \{S, T\}$. In this case, the existence of neighbourhoods N_S and N_T of S and T, with $h(t, x) \neq 0$ for $x \in (N_S \cup N_T) \cap C$ and $t \in [0, 1]$ is established as follows. If there exists a sequence $\{t_i, x_i\} \subset [0, 1] \times C$ with $0 < \| x_i - S \| < 1/i$ such that $t_i f(x_i) + (1 - t_i)(w - x_i) = 0$, it follows as previously that $Ls = -\beta(w - S)$, where $L := df[S]$, $s \in K$, $w - S \in K^o$ and

$\beta > 0$. Theorem 1.7 applies to give $\mu_{PF}(L) < 0$ since $Ls < 0$, which contradicts that L has an eigenvalue in the closed right-half plane. Similar arguments apply at T, so that a homotopy from f to $w - I$ on the boundary of $\Omega := C \backslash (N_S \cup N_T)$ for some sufficiently small neighbourhoods N_S and N_T, yields that $\deg(f, \Omega, 0) = \deg(w - I, \Omega, 0) = (-1)^n$. Hence there exists some $x \in \Omega \subset C$ with $f(x) = 0$.

$\square$

To comment on the stability of intermediate equilibria, additional hypotheses on f are needed. (Note that if f is analytic, its zeros are isolated.)

Theorem 2.5 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable locally monotone function whose zeros are isolated. If both S and T are stable (weakly-unstable) equilibria, then there exists a weakly-unstable (stable) equilibrium in (S, T). Furthermore, if $\{x^1, \ldots, x^k\}$ is a maximal totally ordered set of equilibria in the interval (S, T), then k is odd and x^i is a stable or weakly-unstable (weakly-unstable or stable) equilibrium according to whether i is even or odd.*

Proof. We shall prove the theorem in the case where S and T are stable. By Theorem 2.3, there exists some $x \in C$ with $f(x) = 0$, and by the hypothesis that the zeros of f are isolated, there are a finite number of such zeros in the compact set $\overline{C}$. Let $x^1, \ldots, x^k$ be a maximal totally ordered subset of these equilibria, so that $S := x^0 < x^1 < \ldots < x^{k+1} =: T$, where there are no zeros x of f with $x^i < x < x^{i+1}$ for any $i = 0, 1, \ldots, k$. Suppose, for contradiction, that x^i is a stable equilibrium for all $i = 1, \ldots, k$. Then in particular, x^1 is stable. But Theorem 2.3 then implies that there exists a zero of f in the order interval (x^0, x^1), contradicting the maximality of the totally ordered subset. Hence x^1 is a weakly-unstable equilibrium. It follows that any two equilibria $x^i < x^{i+1}$, $i = 0, 1, \ldots, k$, must be equilibria of opposite type, from which the rest of the theorem is immediate.

□

Remark. This proof uses the fact that the local monotonicity of f implies that $df[x] \in P^{n\times n}$ for each x with $f(x) = 0$, since Theorems 2.3 and 2.4 thus apply to the interval (x, y) whenever x and y are zeros of f with $x < y$. However, when S and T are both stable equilibria, the proof of Theorem 2.3 needs only conditions on f on ∂C to yield that $\deg(f, \Omega_2, 0) = (-1)^{n+1}$, where $\Omega_2 \subset C$. If the zeros of f are isolated, this implies that there is a weakly-unstable equilibrium of f between S and T. (Note that it cannot be inferred from $\mathrm{ind}(f, x, 0) = (-1)^n$ that x is a stable equilibrium of f, and hence we cannot deduce the analogous result when S and T are both weakly-unstable equilibria.) This weakening of the hypotheses is useful in the following corollary; the case when $f = \nabla\phi$.

Corollary 2.6 *Suppose that $\phi : \mathbb{R}^n \to \mathbb{R}$ is a C^2-function whose critical points are non-degenerate (that is, the Hessian of ϕ is invertible at each such point), and that*

(i) *ϕ has local maxima at S and T*

(ii) *At points x of ∂C at which a normal to ∂C exists, $\frac{\partial\phi}{\partial n} := \langle\nabla\phi, n(x)\rangle < 0$, where $n(x)$ is the outward normal to ∂C at x.*

Then there exists a critical point x of ϕ in $\overline{C}\backslash\{S, T\}$. If $x \in C$, x is not a local maximum; if additionally the space dimension n is even, then x is a saddle point.

Proof. By the continuity of $\nabla\phi$, condition (ii) implies that *non-strict* inequalities for $\frac{\partial\phi}{\partial x_k}$ hold for all $x \in \partial C$ with $x_k = S_k$ or T_k. If there are critical points other than S and T on ∂C, then the result is clearly true. Otherwise, taking $f = \nabla\phi$ in the first part of the proof of Theorem 2.3 gives that $h(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \{\partial C\backslash\{S, T\}\}$. Now (i) implies that the Hessian matrices at S and T have no eigenvalues in the open right-half plane, and (ii) yields that the off-diagonal elements of the Hessian matrices at S and T are non-negative. Following the method of the proof of Theorem 2.3, this suffices to give that there exists a critical point of ϕ in C with $\text{ind}(\nabla\phi, x, 0) = (-1)^{n+1}$, since the critical points of ϕ are isolated by the Inverse Function Theorem. Denoting the Hessian of ϕ at x by $d^2\phi[x]$, we get that $\text{sgn}(\det(d^2\phi[x])) = (-1)^{n+1}$ since $d^2\phi[x]$ is invertible. The fact that $\det A = \Pi_{i=1}^n \lambda_i$ for an $n \times n$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ gives the remaining observations.

$\square$

3 Eigenvalues of the travelling-wave problem

3.1 The bistable case

Consider first the problem described by (1.1)–(1.4) with the additional condition that S and T are stable equilibria. We have obtained two basic results; the first is on the existence of monotone eigenvalues, and the second is on the dimensions of the stable and unstable manifolds. To begin, here are some observations on the continuity of eigenvalues.

The matrix $M(\lambda, c) := \lambda^2 A + \lambda c I + B$ depends analytically on λ and on c. Thus the eigenvalues of $M(\lambda, c)$ (that is, $\gamma \in \mathbb{C}$ with corresponding $z \in \mathbb{C}^n \backslash \{0\}$ such that $M(\lambda, c)z = \gamma z$, as distinct from the travelling-wave eigenvalues λ for a given c, which satisfy $\det(M(\lambda, c)) = 0$) are clearly bounded for (λ, c) in a bounded set in $\mathbb{C} \times \mathbb{R}$ and they vary continuously with λ and with c. (See [7], pages 63-64.) Moreover, for each fixed λ and c, $M(\lambda, c) \in P^{n\times n}$ has a Perron-Frobenius eigenvalue, $\mu_{PF}(M(\lambda, c))$, which is real and *simple*. The Implicit Function Theorem for real analytic functions (see [5], page 272) thus gives that $\mu_{PF}(M(\lambda, c))$ is a real analytic function of λ for each fixed c. Clearly, $\mu_{PF}(M(\lambda, c))$ is a linear function of c for fixed λ.

Note also that the eigenvalues λ of the travelling-wave problem vary continuously with c, as the next lemma shows.

Lemma 3.1 *Let $A \in \mathbb{R}^{n\times n}$ be diagonal, $B \in \mathbb{R}^{n\times n}$ and $R > 0$. Then for $|c| \leq R$, the solutions λ of* $\det(\lambda^2 A + \lambda c I + B) = 0$ *form a bounded set in $\mathbb{C}$.*

Proof. Suppose, for contradiction, that for all $k \in \mathbb{N}$, there exists c_k with $|c_k| \leq R$, $|\lambda_k| > k$ and $z_k \in \mathbb{C}^n \backslash \{0\}$ such that $(\lambda_k^2 A + \lambda_k c_k I + B)z_k = 0$. Since $z_k \neq 0$, we can assume that $\| z_k \| = 1$. Now $\{c_k\}$, a bounded sequence in $\mathbb{R}$, has a convergent subsequence, say $c_j \to c$. Also, the corresponding subsequence $\{z_j\}$ is bounded in $\mathbb{C}^n$, so there is a further subsequence such that $c_m \to c$, $z_m \to z$, and $\| z \| = 1$ since $\| z_m \| = 1$ for each m. Since $\lambda_m \neq 0$ for each m, $(\lambda_m A + c_m I + \lambda_m^{-1} B) z_m = 0$ and thus because $\lambda_m^{-1} \to 0$ as $m \to \infty$, $\lambda_m A z_m \to cz$. As A is diagonal, $\lambda_m A_{ii}(z_m)_i \to cz_i$ for each i, $1 \leq i \leq n$, where A_{ii} denotes the i^{th} entry on the diagonal of A. Thus since $|\lambda_k| \to \infty$, we must have $(z_k)_i \to 0$. This contradicts that $z_k \to z$ with $\| z \| = 1$.

□

We can now prove the following.

Theorem 3.2 *Let $B \in P^{n\times n}$ be such that $\mu_{PF}(B) < 0$. Then for each $c \in \mathbb{R}$, there is a stable and an unstable monotone eigenvalue of the travelling-wave problem (1.7) with linearization B.*

Proof. Fix $c \in \mathbb{R}$. Observe first that $\mu_{PF}(M(0, c)) = \mu_{PF}(B) < 0$. Next, if $q := (1, \ldots, 1) \in \mathbb{R}^n$, $(A + \lambda^{-1} c I + \lambda^{-2} B) q > 0$ for $\lambda \in \mathbb{R}$ with $|\lambda|$ sufficiently large since A is a positive-definite diagonal matrix. Thus $M(\lambda, c)q > 0$ and by Corollary 1.6, $\mu_{PF}(M(\lambda, c)) > 0$. Now $\mu_{PF}(M(\cdot, c))$ is a continuous real valued function of λ , which is negative when $\lambda = 0$ and positive for $|\lambda|$ large. So the Intermediate Value Theorem gives that there exist $\lambda_1 < 0$ and $\lambda_2 > 0$ with $\mu_{PF}(M(\lambda_1, c)) = \mu_{PF}(M(\lambda_2, c)) = 0$. That the corresponding eigenvectors are positive follows immediately from the Perron-Frobenius Theorem.

□

This shows that for all $c \in \mathbb{R}$, there is an unstable monotone direction at S and a stable monotone direction at T. Now we study the dimensions of the stable and unstable manifolds at a stable equilibrium point. In the bistable case, it is straightforward to show that $D_s(S)+D_u(T) \geq 2n$, using the fact that there are no eigenvalues of the travelling-wave problem, linearized at a stable equilibrium, on the imaginary axis.

Theorem 3.3 *Let $B \in P^{n\times n}$ be such that $\mu_{PF}(B) < 0$. Then for each $c \in \mathbb{R}$, there are n eigenvalues of the travelling-wave problem in the open left- and right-half planes.*

Proof. Observe first that since A is positive-definite, $\det(\lambda^2 A + \lambda cI + B)$ is a polynomial in λ of degree $2n$ for any fixed $c \in \mathbb{R}$. So counting algebraic multiplicities, there are $2n$ solutions λ of $\det(\lambda^2 A + \lambda cI + B) = 0$ in $\mathbb{C}$.

Now zero is not an eigenvalue of the travelling-wave problem for any value of c since $\mu_{PF}(B) < 0$. Suppose that for some $c \in \mathbb{R}$, and $\alpha \in \mathbb{R}\backslash\{0\}, i\,\alpha$ is an eigenvalue of the travelling-wave problem. Then there exists $z \in \mathbb{C}^n\backslash\{0\}$ such that $(-\alpha^2 A + i\alpha cI + B)z = 0$ and $i\alpha c$ is an eigenvalue of $B - \alpha^2 A \in P^{n\times n}$. Hence $\mu_{PF}(B - \alpha^2 A) \geq 0$ and there exists $x > 0$ such that $(B-\alpha^2 A)x = \mu_{PF}(B-\alpha^2 A)x$. Therefore $Bx = \mu_{PF}(B-\alpha^2 A)x + \alpha^2 Ax > 0$. Corollary 1.6 then yields that $\mu_{PF}(B) > 0$, which is false. Thus for each c, there are no eigenvalues of the travelling-wave problem with linearization at B on the imaginary axis.

When $c = 0$, travelling-wave eigenvalues λ satisfy $\det(\lambda^2 A + B) = 0$. Then λ is a solution whenever $-\lambda$ is and the algebraic multiplicity of a solution λ is equal to that of $-\lambda$. It is known that there can be no eigenvalues on the imaginary axis, and hence counting algebraic multiplicities, there must be n in each open half plane since there are $2n$ solutions λ of $\det(\lambda^2 A + B) = 0$. All that remains is to observe that Lemma 3.1 implies that the eigenvalues vary continuously with c, and they do not cross the imaginary axis by the first part of this proof. Hence for each $c \in \mathbb{R}$, there are n eigenvalues in each open half plane.

$\square$

The dimensions of the unstable and stable manifolds at S and T for all values of c are immediate:

$$D_u(S) = D_u(T) = D_s(S) = D_s(T) = n.$$

3.2 The monostable case

Throughout this section, S is a stable equilibrium ($\mu_{PF}(df[S]) < 0$) and T is unstable ($\mu_{PF}(df[T]) > 0$). First note that Theorems 3.2 and 3.3 say that, for each c, there is an unstable monotone eigenvalue of the travelling-wave problem linearized at S and that the unstable manifold at S has dimension n. We start by discussing the existence and behaviour of stable monotone eigenvalues at an unstable equilibrium.

Lemma 3.4 *Let $B \in P^{n\times n}$ with $\mu_{PF}(B) > 0$. Then*

(i) *For $c \leq 0$, there are no stable monotone eigenvalues of the travelling-wave problem with linearization B.*

(ii) *For $c > 0$ sufficiently large, there exists a stable monotone eigenvalue of the travelling-wave problem with linearization B.*

(iii) *If there exists a stable monotone eigenvalue for $c = c^* > 0$, then for all $c \geq c^*$, a stable monotone eigenvalue exists.*

Proof. Let $x > 0$ be a Perron-Frobenius eigenvector of B and recall that $M(\lambda, c) := \lambda^2 A + \lambda c I + B$.

(i) Take $\lambda < 0$ and $c \leq 0$. Then $\lambda^2 A + B \in P^{n\times n}$ and $(\lambda^2 A + B)x = \lambda^2 Ax + \mu_{PF}(B)x > 0$, so by Corollary 1.6, $\mu_{PF}(\lambda^2 A + B) > 0$. Let $y > 0$ be a Perron-Frobenius eigenvector of $\lambda^2 A + B$. Then $(\lambda^2 A + B)y = \mu_{PF}(\lambda^2 A + B)y$ so that $(\lambda^2 A + \lambda c I + B)y = (\mu_{PF}(\lambda^2 A + B) + \lambda c)y$. Since an eigenvector which is positive is a Perron-Frobenius eigenvector, $\mu_{PF}(M(\lambda, c)) = \lambda c + \mu_{PF}(M(\lambda, 0)) > 0$. Hence given $c \leq 0$, $\mu_{PF}(M(\lambda, c)) > 0$ for $\lambda < 0$ and thus there is no stable monotone eigenvalue.

(ii) $(A + B)x = Ax + \mu_{PF}(B)x > 0$. Hence by Corollary 1.6, $\mu_{PF}(A + B) > 0$, and so for some $y > 0$, $(A + B)y = \mu_{PF}(A + B)y$. Then $(A - cI + B)y = (\mu_{PF}(A + B) - c)y$ and thus $\mu_{PF}(M(-1, c)) = \mu_{PF}(A - cI + B) = \mu_{PF}(A + B) - c$. So $\mu_{PF}(M(-1, c)) < 0$ for $c > \mu_{PF}(A + B)$. But $\mu_{PF}(M(0, c)) = \mu_{PF}(B) > 0$. Hence, by the continuous dependence of the eigenvalues of $M(\lambda, c)$ on λ, there exists λ_0 with $-1 < \lambda_0 < 0$ and $\mu_{PF}(M(\lambda_0, c)) = 0$. That is, λ_0 is a stable monotone eigenvalue.

(iii) Suppose that $\lambda < 0$ is a stable monotone eigenvalue at $c = c^*$ with corresponding eigenvector $y > 0$ so that $(\lambda^2 A + \lambda c^* I + B)y = 0$. Then for $c > c^*$, $(A\lambda^2 + \lambda c I + B)y = \lambda(c - c^*)y$. So $\mu_{PF}(M(\lambda, c)) = \lambda(c - c^*) < 0$. Again, by continuity of the dependence of $\mu_{PF}(M(\lambda, c))$ on λ, there exists λ_0 with $\lambda < \lambda_0 < 0$, and $\mu_{PF}(M(\lambda, c)) = 0$, since $\mu_{PF}(M(0, c)) > 0$. The result follows.

□

So for c sufficiently large and positive, a stable monotone eigenvalue of the travelling-wave problem linearized at T exists. In fact, the set of c for which such an eigenvalue exists is $[c_0, \infty)$ for some $c_0 > 0$.

Theorem 3.5 *Let $B \in P^{n\times n}$ with $\mu_{PF}(B) > 0$. Then there exists $c_0 > 0$ such that for each $c \geq c_0$, there is a stable monotone eigenvalue of the travelling-wave problem with linearization B, and for $c < c_0$, there is no such λ.*

Proof. It follows immediately from Lemma 3.4 (i),(ii) that the set $V := \{c \in \mathbb{R} : (A\lambda^2 + \lambda cI + B)y = 0$, for some $\lambda < 0, y > 0\}$ is non-empty and bounded below by 0, so has a non-negative infimum, say c_0. Consider sequences $c_k \to c_0$, with corresponding $\lambda_k < 0$ and $z_k > 0$ such that

$$(\lambda_k^2 A + \lambda_k c_k I + B)z_k = 0, \ \| z_k \| = 1 \tag{3.1}$$

Taking a subsequence if necessary, we can assume that $z_k \to z$ where $\| z \| = 1$, and thus, since $\lambda_k \neq 0$ and $\mu_{PF}(B) > 0$, there results that $(\lambda_k A + \lambda_k^{-1}B)z_k \to -cz$ as $k \to \infty$. So, by the argument given in Lemma 3.1, $\{\lambda_k\}$ is bounded, and hence has a convergent subsequence, $\lambda_n \to \lambda \leq 0$. Passing to the limit in (3.1) gives that $(\lambda^2 A + \lambda cI + B)z = 0$. It is immediate that $z \geq 0$; in fact, $z > 0$ because the off-diagonal elements of B are strictly positive. Now it is clear that $\lambda < 0$ since otherwise $\lambda = 0$, which contradicts $\mu_{PF}(B) > 0$. Therefore $c_0 \in V$ as required. That V contains all $c \geq c_0$ is the result of Lemma 3.4(iii).

□

Even more is true: for $c > c_0$, there are precisely two stable monotone eigenvalues, both of which depend monotonically on c. Although not directly related to the existence of monotone eigenvalues, this analysis eventually yields information on the dimension of the stable manifold at T. The first step is to recall a convexity result due to Cohen [3].

Theorem 3.6 *(Cohen). Let $M \in P^{n\times n}$ and $D \in \mathbb{R}^{n\times n}$ be diagonal. Then the Perron-Frobenius eigenvalue of $M+D$ is a convex function of D; that is, given diagonal matrices D_1 and D_2 and $0 < \alpha < 1$,*

$$\mu_{PF}(\alpha D_1 + (1-\alpha)sD_2 + M) \leq \alpha\mu_{PF}(D_1+M) + (1-\alpha)\mu_{PF}(D_2+M).$$

Particularly elementary proofs of this result are given in [6] and [10]. For the travelling-wave eigenvalue problem, it has the following elegant consequence.

Lemma 3.7 *Let $B \in P^{n\times n}$, $c \in \mathbb{R}$ and let A be a positive-definite diagonal matrix. Then the Perron-Frobenius eigenvalue $\mu_{PF}(\lambda^2 A + \lambda cI + B)$ is a strictly convex function of λ.*

Proof. Recall first that increasing any element of a matrix with positive off-diagonal elements increases its Perron-Frobenius eigenvalue ([10], Lemma 1.2). Now for $\lambda_1, \lambda_2 \in \mathbb{R}$ ($\lambda_1 \neq \lambda_2$) and $0 < t < 1$, $(t\lambda_1 + (1-t)\lambda_2)^2 < t\lambda_1^2 + (1-t)\lambda_2^2$. So since A is positive-definite,

$$\begin{aligned}
& \mu_{PF}((t\lambda_1 + (1-t)\lambda_2)^2 A + (t\lambda_1 + (1-t)\lambda_2)cI + B) \\
< \; & \mu_{PF}((t\lambda_1^2 + (1-t)\lambda_2^2)A + (t\lambda_1 + (1-t)\lambda_2)cI + B) \\
= \; & \mu_{PF}(t(\lambda_1^2 A + \lambda_1 cI) + (1-t)(\lambda_2^2 A + \lambda_2 cI) + B) \\
\leq \; & t\mu_{PF}(\lambda_1^2 A + \lambda_1 cI + B) + (1-t)\mu_{PF}(\lambda_2^2 A + \lambda_2 c + B)
\end{aligned}$$

Since $\lambda_k^2 A + \lambda_k cI$ are diagonal ($k = 1, 2$), Theorem 3.6 gives the last inequality.

□

Now we can determine the exact number of stable monotone eigenvalues for a given velocity c.

Lemma 3.8 *Let $B \in P^{n\times n}$ with $\mu_{PF}(B) > 0$ and let c_0 be the critical velocity in Theorem 3.5. Then at $c = c_0$, there is exactly one stable monotone eigenvalue, and for $c > c_0$, there are precisely two.*

Proof. Fix $c > c_0$. As in Lemma 3.4(iii), there exists $\lambda_0 < 0$ with $\mu_{PF}(M(\lambda_0, c)) < 0$. Now $\mu_{PF}(B) > 0$ and by the argument in Theorem 3.2, $\mu_{PF}(M(\lambda, c)) > 0$ for $|\lambda|$ large. For a given c, $\mu_{PF}(M(\cdot, c))$ is a continuous function of λ, so there exist $\lambda_1 < \lambda_0 < \lambda_2 < 0$ such that $\mu_{PF}(M(\lambda_1, c)) = \mu_{PF}(M(\lambda_2, c)) = 0$. Since $\mu_{PF}(M(\cdot, c))$ is a strictly convex function of λ, there can be at most two real solutions λ of $\mu_{PF}(M(\lambda, c)) = 0$, and hence there are precisely two. For $c = c_0$ there is at least one stable monotone eigenvalue and at most two by convexity. If there were two, say $\lambda_1 < \lambda_2$, then for $\lambda_1 < \lambda < \lambda_2$, $\mu_{PF}(M(\lambda, c_0)) < 0$ and hence for $c < c_0$ sufficiently close to c_0, $\mu_{PF}(M(\lambda, c)) < 0$. But as $\mu_{PF}(M(0, c)) = \mu_{PF}(B) > 0$, this implies that there exists λ_0 with $\lambda < \lambda_0 < 0$ and $\mu_{PF}(M(\lambda_0, c)) = 0$, which contradicts the minimality of c_0. Hence it follows that there is exactly one stable monotone eigenvalue at c_0.

□

To determine the dimension of the stable manifold at T from the behaviour of the monotone eigenvalues, we need to relate the other eigenvalues of the travelling-wave problem to the stable monotone ones. Vol'pert and Vol'pert [14] have shown the following useful result, the proof of which is included for completeness.

Lemma 3.9 *(Vol'pert and Vol'pert). Let $M \in P^{n \times n}$ with $\mu_{PF}(M) < 0$ and S be a complex diagonal $n \times n$ matrix. If the diagonal elements of S have non-positive real parts then all eigenvalues of the matrix $M + S$ have negative real parts.*

Proof. Let $d > 0$ be a Perron-Frobenius eigenvector of M and D be a diagonal matrix with the elements of the vector $d = (d_1, \ldots, d_n)$ on the diagonal. Then the sum of the elements of the rows of the matrix $D^{-1}MD$ is negative. Now the matrices $M + S$ and $D^{-1}(M + S)D = D^{-1}MD + S$ have the same eigenvalues. By Gerschgorin's theorem (see, for example, [15], page 71), the eigenvalues of the matrix $D^{-1}MD+S$ lie in disks with centres at the points $M_{ij}+S_{ij}$ and radii $\sum_{k\neq j} d_j^{-1}M_{jk}d_k$, where M_{ij}, S_{ij} are the elements of M and S respectively. Since $\sum_{k\neq j} d_j^{-1}M_{jk}d_k + \mathrm{Re}\, S_{ij} < 0$, all the Gerschgorin disks are in the left-half plane.

□

Theorem 3.10 *Let B and c_0 be as in Lemma 3.8 and denote the two stable monotone eigenvalues at $c > c_0$ by λ_1 and λ_2, where $\lambda_1 < \lambda_2 < 0$. Then there are no eigenvalues of the travelling-wave problem λ with $\lambda_1 < \mathrm{Re}\ \lambda < \lambda_2$.*

Proof. For $\gamma \in \mathbb{R}$ with $\lambda_1 < \gamma < \lambda_2$, $\mu_{PF}(M(\gamma, c)) < 0$ by the convex dependence of $\mu_{PF}(M(\lambda, c))$ on λ. Suppose that $\gamma + i\beta$ is an eigenvalue of the travelling-wave problem for some $\beta \in \mathbb{R}$. Then there exists $z \in \mathbb{C}^n \backslash \{0\}$ such that

$$((\gamma + i\beta)^2 A + (\gamma + i\beta)cI + B)z = 0.$$

But then zero is an eigenvalue of the matrix $(M(\gamma, c) - \beta^2 A + i\beta(2\lambda A + C))$, which contradicts Lemma 3.9.

□

Remark. Suppose that zero is an eigenvalue of the matrix $M(\lambda, c)$ for fixed λ and c, in the sense that there exists $z \in \mathbb{C}^n \backslash \{0\}$ such that $M(\lambda, c)z = \gamma z$ where $\gamma = 0$. Then λ is an eigenvalue of the travelling-wave problem for velocity c, as defined following (1.7) in Section 1. Since the eigenspace of zero as an eigenvalue of the matrix $M(\lambda, c)$ is the same as the eigenspace of the travelling-wave eigenvalue λ for velocity c, the dimensions of these two spaces are the same. In other words, the *geometric* multiplicities of the two eigenvalues coincide. However, the *algebraic* multiplicity of zero as an eigenvalue of $M(\lambda, c)$ may not equal the algebraic multiplicity of λ as a solution of the equation $\det(\lambda^2 A + \lambda c I + B) = 0$. In particular, $\mu_{PF}(M(\lambda_0, c_0)) = 0$, where λ_0 is the unique stable monotone eigenvalue at $c = c_0$,, and hence zero is a *simple* eigenvalue of $M(\lambda_0, c_0)$. Thus the geometric and algebraic multiplicities of zero as an eigenvalue of $M(\lambda_0, c_0)$ are both one. But the next result shows that two travelling-wave eigenvalues converge to λ_0 as c tends to c_0. So the algebraic multiplicity of λ_0 must be at least two.

Theorem 3.11 *Let B and c_0 be as in Lemma 3.8 and denote the two stable monotone eigenvalues for $c > c_0$ by $\lambda_1(c)$ and $\lambda_2(c)$ where $\lambda_1(c) < \lambda_2(c) < 0$. Let λ_0 denote the unique stable monotone eigenvalue when $c = c_0$. Then λ_1 is a continuous, decreasing function of c and λ_2 is a continuous, increasing function of c. Moreover, as $c \to \infty, \lambda_1(c) \to -\infty$ and $\lambda_2(c) \to 0$. As $c \to c_0$, $\lambda_1(c) \to \lambda_0$ and $\lambda_2(c) \to \lambda_0$ and as c is decreased below c_0, a pair of complex conjugate branches of eigenvalues emanate from λ_0.*

Proof. It is easy to show that $\lambda_1(c)$ and $\lambda_2(c)$ are widely separated for large c. For $\gamma < 0$ let $y > 0$ be such that $(\gamma^2 A + B)y = \mu_{PF}(M(\lambda, 0))y$. Then $(\gamma^2 A + \gamma c I + B)y = (\mu_{PF}(M(\gamma, 0)) + \gamma c)y$, so for c sufficiently large, $\mu_{PF}(M(\gamma, c)) < 0$. But $\mu_{PF}(M(0, c)) > 0$ for each c and it was shown in Theorem 3.2 that $\mu_{PF}(M(\lambda, c)) > 0$ for fixed c, $|\gamma|$ sufficiently large. So $\lambda_1(c) < \gamma < \lambda_2(c)$ for c large enough. Since $\gamma < 0$ was arbitrary, $\lambda_1(c) \to -\infty$ and $\lambda_2(c) \to 0$ as $c \to \infty$.

Now let $c > c_0$. By the convex dependence of $\mu_{PF}(M(\lambda, c))$ on λ for fixed c, there exists ϵ_0 with $0 < \epsilon_0 < (\lambda_0 - \lambda_1(c))$ such that $\mu_{PF}(M(\lambda, c)) > 0$ for $\lambda \in (\lambda_1(c) - \epsilon_0, \lambda_1(c))$ and $\mu_{PF}(M(\lambda, c)) < 0$ for $\lambda \in (\lambda_1(c), \lambda_1(c) + \epsilon_0)$. By algebra, $\mu_{PF}(M(\lambda_1(c), c - \delta) = -\delta\lambda_1(c) > 0$ for all $\delta > 0$. Furthermore, for $\epsilon \in (0, \epsilon_0)$ and fixed $\delta > 0$ sufficiently small, $\mu_{PF}(M(\lambda_1(c) + \epsilon, c - \delta)) = \mu_{PF}(M(\lambda_1(c) + \epsilon, c) - \delta(\lambda_1(c) + \epsilon) < 0$. Thus by the continuous dependence of $\mu_{PF}(M(\lambda, c))$ on λ for fixed c, given $\epsilon \in (0, \epsilon_0)$, there exists $\delta_\epsilon > 0$ such that for each $\delta \in (0, \delta_\epsilon)$, there exists some $\tilde{\epsilon} \in (0, \epsilon)$ such that $\mu_{PF}(M(\lambda_1(c) + \tilde{\epsilon}, c - \delta) = 0$. By choice of ϵ_0, $\lambda_1(c) + \tilde{\epsilon} < \lambda_0$, and hence $\lambda_1(c - \delta) = \lambda_1(c) + \tilde{\epsilon} > \lambda_1(c)$. Therefore, the stable monotone eigenvalue further from 0 increases as c decreases. Together with the analogous argument for $\lambda_1(c + \delta)$, $\delta > 0$ small , this also shows that λ_1 is a continuous function of c. Similarly, λ_2 is a continuous, increasing function of c.

To see that $\lambda_1(c) \to \lambda_0$ and $\lambda_2(c) \to \lambda_0$ as $c \to c_0$, first note that the proof of Lemma 3.8 gives that at $c = c_0$, $\mu_{PF}(M(\lambda, c_0)) > 0$ for $\lambda \in \mathbb{R}\backslash\{\lambda_0\}$ whilst $\mu_{PF}(M(\lambda_0, c_0)) = 0$. The preceding argument yields that there exists $\epsilon_0 > 0$ such that, given $\epsilon \in (0, \epsilon_0)$, there exists $\delta_\epsilon > 0$ such that for $\delta \in (0, \delta_\epsilon)$, $\mu_{PF}((M(\lambda_0, c_0 + \delta)) = \delta\lambda_0 < 0$, $\mu_{PF}(M(\lambda_0 + \epsilon, c_0 + \delta)) > 0$ and $\mu_{PF}(M(\lambda_0 - \epsilon, c_0 + \delta)) > 0$. Hence $\lambda_0 - \epsilon < \lambda_1(c_0 + \delta) < \lambda_0 < \lambda_2(c_0 + \delta) < \lambda_0 + \epsilon$.

The last step is to note that after 'coalescence' at $c = c_0$, the extensions of the continuous branches of stable monotone eigenvalues cannot remain real, and hence must become a complex conjugate pair. Suppose to the contrary. Now zero is the Perron-Frobenius eigenvalue of the matrix $\lambda_0^2 A + \lambda_0 c_0 I + B$, and is therefore simple – in particular, it has geometric multiplicity one. (See the remark preceding this theorem.) Therefore, for some $c < c_0$, there exists $\lambda < 0$ with corresponding real eigenvector which is a perturbation of the eigenvector $z_0 > 0$ satisfying $(\lambda_0^2 A + \lambda_0 c_0 I + B)z_0 = 0$ (see [7], page 91). But a sufficiently small real perturbation of z_0 will also be positive, contradicting the fact that c_0 is the least velocity for which there exists a stable monotone eigenvalue. So the extensions of the eigenvalue branches for $c < c_0$ must become complex. The fact that eigenvalues of a real matrix must occur in complex conjugate pairs yields the required result.

$\square$

This gives a complete picture of the behaviour of the stable monotone eigenvalues at c is varied. To infer that the dimension of the stable manifold at T is at least $n + 1$ for

$c > c_0$, we show that for large c, n eigenvalues of the travelling-wave problem tend to $-\infty$. That there are at least $n+1$ eigenvalues of the travelling-wave problem in the left-half plane is then immediate by the continuous dependence of the eigenvalues on the velocity c, and the fact that no eigenvalue can cross the gap between the two stable monotone eigenvalues by Theorem 3.10.

Theorem 3.12 *Let $B \in P^{n\times n}$ with $\mu_{PF}(B) > 0$. Then as $c \to \infty$, n eigenvalues of the travelling-wave problem tend to $-\infty$.*

Proof. Suppose that $c > 0$, $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n \backslash \{0\}$ satisfy $(\lambda^2 A + \lambda c I + B)z = 0$. Dividing by c^2 and setting $\mu = \lambda c^{-1}$ and $\epsilon = c^{-2}$ leads to $(\mu^2 A + \mu I + \epsilon B)z = 0$ and as in (1.5), this can be written in the form

$$\begin{pmatrix} 0 & I \\ -\epsilon A^{-1}B & -A^{-1} \end{pmatrix} \begin{pmatrix} z \\ \mu z \end{pmatrix} = \mu \begin{pmatrix} z \\ \mu z \end{pmatrix}.$$

When $\epsilon = 0$, the set of eigenvalues μ is comprised of zero, with algebraic multiplicity n, and the n eigenvalues of $-A^{-1}$. Thus by the continuous dependence of the eigenvalues μ on the parameter ϵ, n eigenvalues μ must tend to finite negative limits as $\epsilon \downarrow 0$ since A is positive-definite. By definition of μ and ϵ, the corresponding λ tend to $-\infty$ as $c \to \infty$.

□

Thus for $c > c_0$, the stable manifold at T has dimension at least $n+1$, so $D_s(T) + D_u(S) \geq 2n+1$. For completeness, the behaviour of the eigenvalues of the travelling-wave problem having real part greater than the stable monotone eigenvalue closer to zero for large c can be elucidated under some additional restrictions on the matrix B. By the proof of Theorem 3.12, n eigenvalues of the travelling-wave problem tend to $-\infty$ as $c \to \infty$, and there are n eigenvalues λ such that $\lambda/c \to 0$ as $c \to \infty$. But the exact number of eigenvalues in the open left-half plane for c sufficiently large in terms of the number of eigenvalues of B in the right-half plane can be obtained from the Implicit Function Theorem.

Theorem 3.13 *Let $B \in P^{n\times n}$ with $\mu_{PF}(B) > 0$ have only simple eigenvalues, none of which is imaginary, and k of which are in the open right-half plane. Then for c sufficiently large, there are $n + k$ eigenvalues of the travelling-wave problem in the left-half plane.*

Proof. Let β be an eigenvalue of B with corresponding eigenvector x, $\| x \|= 1$. If $c > 0, \lambda$ and z satisfy

$$(\lambda^2 A + \lambda c I + B)z = 0, \tag{3.2}$$

set $\epsilon = c^{-2}$ and $\mu = \lambda c^{-1}$ so that

$$(\mu^2 A + \mu I + \epsilon B)z = 0. \tag{3.3}$$

Now for $\epsilon \neq 0$, define $\delta \in \mathbb{C}$ via $\mu = \epsilon(-\beta + \delta)$. Then (3.3) becomes

$$(\epsilon^2(-\beta + \gamma)^2 A + \epsilon(-\beta + \delta)I + \epsilon B)z = 0$$

and cancelling ϵ gives

$$(\epsilon(-\beta + \gamma)^2 A + (-\beta + \delta)I + B)z = 0. \tag{3.4}$$

Note that (3.4) is satisfied when $\epsilon = \delta = 0$ and $z = x$.

Since (3.4) is equivalent to (3.2) when $\epsilon \neq 0$, the Implicit Function Theorem applied to (3.4) gives the behaviour of an eigenvalue λ at large c.

Let $F : \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n$ be defined by

$$F(\epsilon, \delta, z) = \begin{cases} \langle z, x\rangle - 1 \\ (\epsilon(-\beta + \delta)^2 A + (-\beta + \delta)I + B)z. \end{cases}$$

Then $F(0,0,x) = 0$, F is continuously differentiable and

$$d_{(\delta,z)}F[0,0,x](\delta,z) = \begin{cases} \langle z, x \rangle \\ \\ (B - \beta I)z + \delta x. \end{cases}$$

If $d_{(\delta,z)}F[0,0,x](\delta,z) = 0$, then $(B - \beta I)z + \delta x = 0$. If $\delta \neq 0$, $x \in \text{range}(B - \beta I)$. But since $x \in \ker(B - \beta I)$, this contradicts the fact that β is a simple eigenvalue of B. Also $(B - \beta I)z = 0$, $z \neq 0$, means z is a non-zero multiple of x, which contradicts $\langle z, x \rangle = 0$. Hence $(\delta, z) = (0,0)$ and thus $d_{(\delta,z)}F[0,0,x]$ is injective and thus invertible. The Implicit Function Theorem then implies that $F(\epsilon, \delta, z) = 0$ defines δ and z as continuously differentiable functions of ϵ on a neighbourhood of $\delta = 0, z = x, \epsilon = 0$. In particular, for $\epsilon > 0$ small, δ is close to zero so that for the corresponding λ and c satisfying (3.2), $\lambda = 0$ when $1/c = 0$ and $\lambda = \frac{1}{c}(-\beta + \delta)$ is approximately a small multiple of $-\beta$ for $1/c > 0$ and small; that is, λ moves into the opposite half plane to β as c decreases from infinity.

It follows that, for c sufficiently large, in a small neighbourhood of the origin, there are exactly k eigenvalues of the travelling-wave problem which have negative real part. In addition to the n eigenvalues that we know from Theorem 3.12 to tend to $-\infty$, this gives $n + k$ eigenvalues in the left-half plane.

$\square$

Remark. Eigenvalues of the travelling-wave problem *can* cross the imaginary axis as c is varied in the monostable case. Examples of this have been observed in numerical experiments.

4 A special case of the monostable problem

4.1 Some background

Let a and r be positive constants. The Fisher-Kolmogorov-Petrowski-Piscounov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} a \frac{\partial^2 u}{\partial x^2} + r(u^2 - u) \tag{4.1}$$

where $u = u(t, x)$ is a scalar valued function on $[0, \infty) \times \mathbb{R}$, has been extensively studied. It has been shown that for $c \geq \sqrt{2ar}$, equation (4.1) has a monotone travelling-wave solution of speed c which is unique modulo translations. A generalized FKPP system of two equations has been considered in [1], where it is shown that for $c \geq c_0 > 0$, the system has a monotone travelling-wave solution, again unique modulo translations. Here we show that the natural generalisation to an n-dimensional system falls into the monostable case discussed in Section 3. Hence the results concerning the existence of monotone eigenvalues and dimensions of stable and unstable manifolds pertain.

Consider the system of equations

$$\frac{\partial u}{\partial t} = \frac{1}{2} A \frac{\partial^2 u}{\partial x^2} + R(u^2 - u) + \theta Q u \tag{4.2}$$

where u is a vector valued function from $[0, \infty) \times \mathbb{R}$ to $\mathbb{R}^n$, A and R are positive-definite diagonal matrices, $\theta \in \mathbb{R}$ is positive and $Q \in P^{n \times n}$ has the sum of the elements in each row equal to zero. In particular, the diagonal elements of Q are negative and the off-diagonal elements are positive. For this section, set $S = (0, \ldots, 0)$ and $T = (1, \ldots, 1)$. Then in the notation of equation (1.1), $f(u) = R(u^2 - u) + \theta Q u$, and $f(S) = f(T) = 0$.

Proposition 4.1 *Let $f(u) = R(u^2 - u) + \theta Q u$ where R, θ and Q are as above. Then f is monotone (in the sense of (1.2)) and S and T are stable and unstable equilibria respectively. This system satisfies the conditions for the monostable theory in Section 3.*

Proof. Let $R = \operatorname{diag}(r_1, \ldots, r_n)$ and denote by Q_{ij} the entries of Q. Then for $i \neq j$, $\frac{\partial f_i}{\partial u_j} = \theta Q_{ij} > 0$, so f is monotone. Observe that for $q = T, Qq = 0$ since Q has zero row sums, and hence $\mu_{PF}(Q) = 0$. Now $df[T] = R + \theta Q$ and $df[S] = -R + \theta Q$. Thus since $r_1, \ldots, r_n > 0$, $\mu_{PF}(df[T]) > 0$ and $\mu_{PF}(df[S]) < 0$ because increasing (decreasing) elements of a Perron-Frobenius matrix increases (decreases) the Perron-Frobenius eigenvalue.

□

By Proposition 4.1, we know that for each $\theta > 0$, there is some $c > 0$ depending on θ, say $c(\theta)$, such that for $c \geq c(\theta)$, there is a stable monotone eigenvalue of the travelling-wave problem corresponding to (4.2) linearized at T. Using the Cohen convexity result, it is not difficult to prove the following n-dimensional analogue of a result of [1].

Proposition 4.2 *For $\theta > 0$, let $c(\theta)$ be the critical velocity above which there is a stable monotone eigenvalue of the travelling-wave problem parametrised by θ. Then $c(\cdot)$ is a non-increasing function of θ.*

Proof. Take $\theta_0 > 0$. Then by Lemma 3.8, for $c = c(\theta_0)$, $\mu_{PF}(\frac{1}{2}\lambda^2 A + \lambda c I + R + \theta_0 Q) \geq 0$ for all $\lambda \in \mathbb{R}$ and equality holds at precisely one point, say λ_0, with $\lambda_0 < 0$. Define $D = \frac{1}{2}\lambda_0^2 A + \lambda_0 c(\theta_0) I + R$ and let Z denote the zero matrix. Note that for any $\theta > 0$, $\mu_{PF}(D + \theta Q) = \theta \mu_{PF}\left(\frac{1}{\theta} D + Q\right)$. In particular, they have the same sign and $\mu_{PF}\left(\frac{1}{\theta_0} D + Q\right) = 0$.

Now for $\theta > \theta_0$, $\frac{1}{\theta} D = \left(\frac{\theta_0}{\theta}\right)\left(\frac{1}{\theta_0} D\right) + \left(1 - \frac{\theta_0}{\theta}\right) Z$ where $0 < \frac{\theta_0}{\theta} < 1$. Hence by the convex dependence of μ_{PF} on the diagonal,

$$\mu_{PF}\left(\frac{1}{\theta} D + Q\right) \leq \left(\frac{\theta_0}{\theta}\right) \mu_{PF}\left(\frac{1}{\theta_0} D + Q\right) + \left(1 - \frac{\theta_0}{\theta}\right) \mu_{PF}(Q) = 0$$

and thus $\mu_{PF}(D + \theta Q) \leq 0$. Since $\mu_{PF}(R + \theta Q) > 0$, this implies that there exists $\lambda_* < 0$ with $\mu_{PF}(\frac{1}{2}\lambda_*^2 A + \lambda_* c(\theta_0) I + R + \theta Q) = 0$. Hence there is a stable monotone eigenvalue of the travelling-wave problem with parameter θ when $c = c(\theta_0)$. By definition of $c(\cdot)$, $c(\theta) \leq c(\theta_0)$.

$\square$

4.2 A springboard for probability

In [1], probabilistic methods are used to show that the 2-dimensional system (4.2) has a monotone travelling-wave solution connecting S and T for each $c \geq c(\theta) > 0$. These methods establish uniqueness of the monotone solutions (modulo translations) for a given velocity, and also provide a large-deviation explanation for the value $c(\theta)$. To facilitate the transference of the probabilistic techniques to the generalisations of these results to the n-dimensional case, we deduce the following lemmata from the material in Section 3.

Lemma 4.3 *Suppose that $c > c(\theta)$, and let $\lambda_2(c)$ denote the stable monotone eigenvalue closer to zero of the travelling-wave problem linearized at T. Then*

(i) *for $\nu < \lambda_2(c)$ with ν sufficiently close to $\lambda_2(c)$,*

$$\mu_{PF}(\frac{1}{2}\nu^2 A + R + \theta Q) = -\nu\tilde{c}(\nu) \text{ for some } \tilde{c}(\nu) < c$$

and

(ii) *as $c \downarrow c(\theta)$,*

$$\lambda_2(c)^{-1}\mu_{PF}(\frac{1}{2}\lambda_2(c)^2 A + R + \theta Q) \to -c(\theta)$$

and

$$\left[\frac{\partial}{\partial\lambda}\mu_{PF}(\frac{1}{2}\lambda^2 A + R + \theta Q)\right]_{\lambda=\lambda_2(c)} \to -c(\theta).$$

Proof. Let λ_0 denote the unique stable monotone eigenvalue when $c = c(\theta)$.

(i) $\lambda_2(\cdot)$ is a continuous function of c, and $\lambda_2(c) \downarrow \lambda_0$ as $c \downarrow c(\theta)$ by Theorem 3.11. So for $c > c(\theta)$ and $\lambda_0 < \nu < \lambda_2(c)$, there exists some $\tilde{c}(\nu)$ such that $c(\theta) < \tilde{c}(\nu) < c$ and $\lambda_2(\tilde{c}(\nu)) = \nu$ and thus $\mu_{PF}(\frac{1}{2}\nu^2 A + R + \theta Q) = -\nu\tilde{c}(\nu)$.

(ii) The first part follows from the fact that $(\lambda_2(c))^{-1}\mu_{PF}(\frac{1}{2}\lambda_2(c)^2 A + R + \theta Q) = -c$. By the Implicit Function Theorem, $\mu_{PF}(\frac{1}{2}\lambda^2 A + \lambda c I + R + \theta Q)$ is an analytic function of λ for each fixed c and $\mu_{PF}(\frac{1}{2}\lambda^2 A + R + \theta Q)$ an analytic function of λ. Thus because $\lambda_2(c) \to \lambda_0$ as $c \downarrow c(\theta)$, $[\frac{\partial}{\partial\lambda}\mu_{PF}(\frac{1}{2}\lambda^2 A + R + \theta Q)]_{\lambda=\lambda_2(c)} \to [\frac{\partial}{\partial\lambda}\mu_{PF}(\frac{1}{2}\lambda^2 A + R + \theta Q)]_{\lambda=\lambda_0}$ as $c \downarrow c(\theta)$. Now fix $c = c(\theta)$. Then

$$\frac{\partial}{\partial\lambda}\mu_{PF}(\frac{1}{2}\lambda^2 A + R + \theta Q) = \frac{\partial}{\partial\lambda}\mu_{PF}(\frac{1}{2}\lambda^2 A + \lambda c(\theta) I + R + \theta Q) - c(\theta).$$

Moreover, $\mu_{PF}(\frac{1}{2}\lambda^2 A + \lambda c(\theta) I + R + \theta Q)$ attains a local minimum at $\lambda = \lambda_0$ since $\mu_{PF}(\frac{1}{2}\lambda_0^2 A + \lambda_0 c(\theta) I + R + \theta Q) = 0$, and $\mu_{PF}(\frac{1}{2}\lambda^2 A + \lambda c(\theta) I + R + \theta Q) > 0$ for all λ in a deleted neighbourhood of λ_0 (Lemma 3.8). So $[\frac{\partial}{\partial\lambda}\mu_{PF}(\frac{1}{2}\lambda^2 A + \lambda c(\theta) I + R + \theta Q)]_{\lambda=\lambda_0} = 0$, and hence $[\frac{\partial}{\partial\lambda}\mu_{PF}(\frac{1}{2}\lambda^2 A + R + \theta Q)]_{\lambda=\lambda_0} = -c(\theta)$, as required.

$\square$

Lemma 4.4 *Suppose that $c > c(\theta)$, and $\lambda_1(c)$ denotes the stable monotone eigenvalue of the travelling-wave problem linearized at T further from zero. Then, for $\nu > \lambda_1(c)$ sufficiently close to $\lambda_1(c)$, the only non-negative vector g such that*

$$(\frac{1}{2}\nu^2 A + \nu c I + R + \theta Q) g \geq 0$$

is the zero vector, $g = 0$.

Proof. Let λ_0 be as in Lemma 4.3. Then for ν with $\lambda_1(c) < \nu < \lambda_0$, the convex dependence of μ_{PF} on ν (Lemma 3.7) gives that $\mu_{PF}(\frac{1}{2}\nu^2 A + \nu c I + R + \theta Q) < 0$.

Suppose that for such ν, there exists some non-negative g with $g \neq 0$ and $(\frac{1}{2}\nu^2 A + \nu c I + R + \theta Q)g \geq 0$. Then $\mu_{PF}(\frac{1}{2}\nu^2 A + \nu c I + R + \theta Q) \geq 0$ by Theorem 1.5 since $-g \notin K$. This is a contradiction because $\mu_{PF}(\frac{1}{2}\nu^2 A + \nu c I + R + \theta Q) < 0$.

$\square$

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