# A construction of all non-isotropic harmonic tori in complex projective space.*

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## 1 Introduction.

When studying harmonic maps from a Riemann surface $M$ into $\mathbf{CP}^n$ we are drawn to distinguish between (complex) isotropic maps and non-isotropic maps. The construction of isotropic maps is well understood: each isotropic map arises by homogeneous projection from a (super) horizontal holomorphic map of $M$ into the manifold of full flags in $\mathbf{C}^{n+1}$.

In neat parallel to this, Burstall [5] has recently shown that each weakly conformal non-isotropic map $\psi_0 : M \to \mathbf{CP}^n$ may be obtained by homogeneous projection from a harmonic map (called its *primitive lift*) of $M$ into a flag manifold $F^r(\mathbf{CP}^n)$ of not-necessarily-full flags. The flags in this case are only full up to dimension $m = r + 1 \leq n$ and this number turns out to be the isotropy order of $\psi_0$ (recall that $\psi_0$ is weakly conformal precisely when $m \geq 2$). These primitive lifts are 'horizontal holomorphic' in a weak sense: each is $f$-holomorphic with respect to a certain horizontal $f$-structure on the flag manifold.

When $M$ is the Riemann sphere one knows that all the harmonic maps into $\mathbf{CP}^n$ are isotropic. By contrast, in the case where $M$ is compact with genus $\geq 1$ there may be non-isotropic maps but little appears to be known apart from the case of tori [2, 5, 6, 7, 11, 15, 16, 20]. For tori it was shown in [6] (for non-conformal maps) and [5, 7] (for conformal maps) that each non-isotropic map is eventually obtainable by solving an 'algebraically completely integrable' Hamiltonian system of o.d.e's. In fact the solution of such a system produces harmonic maps $\mathbf{R}^2 \to \mathbf{CP}^n$, called *maps of finite type*, which include all doubly periodic maps i.e. all tori. 'Algebraically completely integrable' means that one expects to be able to identify each solution manifold of the system with the Jacobi variety of a complete algebraic curve. This leads to the expectation that all non-isotropic harmonic tori ought to be characterized by data including an algebraic curve. However, such a characterisation seems quite difficult to obtain from the Hamiltonian approach.

Nevertheless, such a characterisation is possible by different means. The principal aim of this article is to describe a bijective correspondence between:

**(i)** non-isotropic, linearly full harmonic maps $\psi_0 : \mathbf{R}^2 \to \mathbf{CP}^n$ of finite type, up to isometry, and

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**(ii)** triples $(X, \pi, \mathcal{L})$ consisting of a real, complete, connected algebraic curve $X$ (which we call the *spectral curve* for $\psi_0$), a rational function $\pi$ on $X$ and a line bundle $\mathcal{L}$ over $X$, all three of which are required to satisfy certain conditions.

Moreover I will present conditions on $(X, \pi, \mathcal{L})$ (for smooth $X$) sufficient to make $\psi_0$ doubly periodic: almost all of these conditions are necessary. This allows us to make a naive parameter count for the dimension of the moduli space of harmonic tori of fixed isotropy order (cf. §6 for details). From this count one expects to find tori in $\mathbf{CP}^n$ whenever the isotropy order is not too large with respect to $n$ (roughly no more than half; cf. prop. 3) and the spectral curves for such tori may have arbitrarily large genus. The formulae for such tori will involve the Riemann $\theta$-function for $X$ and therefore it is not surprising that most of these tori have never been seen.

The correspondence above is constructive: given $(X, \pi, \mathcal{L})$ this article shows how to construct $\psi_0$ and (perhaps less easily) vice versa. The construction here generalizes the one described in [20] which gave all harmonic 2-tori in $\mathbf{CP}^n$ with isotropy order $m = n$ (the *superconformal* tori). However, in [20] extensive use was made of the correspondence between superconformal tori and doubly periodic solutions of the Toda field equations, which I now believe obscures the conceptually clearer construction described here. We will see that each torus $T^2$ arises from an analytic homomorphism $L$ from its covering space $\mathbf{R}^2$ to a real subgroup $J_R(X)$ of the Jacobian $J(X)$ of $X$. Although it is not made clear in [20], this construction identifies $\mathbf{CP}^n$ with the complete linear system of $\mathcal{L}$. The conditions on $\mathcal{L}$ ensure that, for each $L(z) \in J_R(X)$, the space of global holomorphic sections $H^0(X, \mathcal{L} \otimes L(z))$ (whose projective space is the complete linear system of $\mathcal{L} \otimes L(z)$) is $n+1$-dimensional and admits a Hermitian inner product. These vector spaces are the fibres for a vector bundle $H^0(X) \to J_R(X)$. A section of the pullback $L^*H^0(X)$ over $\mathbf{R}^2$ would provide us with a map $T^2 \to H^0(X, \mathcal{L})$ if we could identify all the fibres of $H^0(X)$ with $H^0(X, \mathcal{L})$ in some smooth fashion. There is a parallel transport on $L^*H^0(X)$ which does this, but it has holonomy around the periods of $J_R(X)$. It follows that $T^2$ might not be a submanifold of the Jacobian although it will always cover such a submanifold.

To see how such a recipe can produce *harmonic* maps it is easiest to look at the construction of primitive lifts. From [5] we know that such maps arise from direct sum decomposition of the trivial bundle $\mathbf{R}^2 \times \mathbf{C}^{n+1}$ into orthogonal subbundles $\ell_0, \ldots, \ell_r, \ell_{r+1}$, where $1 \leq r \leq n$ and each $\ell_j$ is a line bundle for $0 \leq j \leq r$. When these bundles possess the property

$$\frac{\partial}{\partial z}\ell_j \subset \ell_j \oplus \ell_{j+1}, \tag{1}$$

(for $z = x + iy$ and with $j$ counted modulo $r+2$) one knows from [5] that $\ell_0$ corresponds to a harmonic map $\psi_0 : \mathbf{R}^2 \to \mathbf{CP}^n$ derived by projection from a primitive map into $F^r(\mathbf{CP}^n)$: a few extra conditions ensure that $\psi_0$ has isotropy order $r+1$. To obtain such a splitting from $H^0(X)$ one chooses ideal sheaves $\mathcal{L}_j$ of $\mathcal{L}$ such that each subspace $H^0(X, \mathcal{L}_j \otimes L(z))$ provides a fibre for $\ell_j$ by pulling it back to $H^0(X, \mathcal{L})$ via the parallel transport. To satisfy (1) the divisors defining each $\mathcal{L}_j$

must be compatible (in some sense) with $L_*(\partial/\partial z)$ so that the covariant derivative along it maps sections of $\mathcal{L}_j$ to sections of $\mathcal{L}_j(P_0) \supset \mathcal{L}_{j+1}$.

Although this construction appears to rely on the condition $m \geq 2$ it will also produce harmonic maps for $m = 1$: this is at first somewhat surprising and has to be verified by directly computing that the $(1,0)$-component of the second fundamental form of $\ell_0$ is holomorphic with respect to the Koszul-Malgrange structure. In this case the maps are all nowhere conformal (of course we expect this of the tori of isotropy order 1).

The construction of $\psi_0$ from $(X, \pi, \mathcal{L})$ is contained in §2-§6: §2 reviews [5]; §3 describes the conditions on $(X, \pi, \mathcal{L})$ and describes the Hermitian inner product on $H^0(X, \mathcal{L})$ as well as the parallel transport; and §4 constructs the harmonic maps into $F^r(\mathbf{CP}^n)$. In §5 we briefly examine extra conditions which are sufficient to get maps into spheres (via the totally geodesic immersion of $S^n$ onto $\mathbf{RP}^n$). As one might expect, this places extra constraints on the curve $X$: it is required to possess a certain holomorphic involution commuting with its real structure such that $M$ covers a subgroup of the Prym variety of $X$. The discussion in this section is not complete and I hope to return to this elsewhere. In §6 we finish this half of the construction by describing conditions on $(X, \pi, \mathcal{L})$ which gives rise to doubly periodic maps.

The reverse construction, of $(X, \pi, \mathcal{L})$ from $\psi_0$, makes extensive use of the recent work by Burstall & Pedit [8] on dressing orbits of harmonic maps. They have described an action (*the dressing action*) of a certain loop group on the space of primitive harmonic maps from $\mathbf{R}^2$ into a $k$-symmetric space. Using their results and the fact that $F^r(\mathbf{CP}^n)$ is a rank one $r+2$-symmetric space we can deduce that, after possibly an isometry, every primitive lift of finite type lies in a single dressing orbit $O_\Lambda$ (this is very much like the result in [20] that all finite type solutions of the Toda equations come from dressing the 'vacuum solution').

On this orbit one has a dynamical system whose flows (called the 'higher flows' in [8]) are generated by the action of an abelian Lie subalgebra $\mathfrak{C}_R$ of the loop algebra. The maps of finite type are characterised in [8] as being those with finite dimensional $\mathfrak{C}_R$-orbit. The stabiliser of such a point determines a maximal abelian (in some sense 'toral') subalgebra $\mathfrak{R}$ of the Lie algebra of *polynomial Killing fields* for that map (in the sense of [6, 11]). $\mathfrak{R}$ is shown to be a commutative unital $\mathbf{C}$-algebra of dimension one and $Spec(\mathfrak{R})$ is an affine curve whose completion by smooth points yields $X$. Since the Killing fields are elements of the loop algebra $\mathfrak{R}$ comes equipped with a representation: this equips $X$ with $\mathcal{L}$. Moreover, the tangent space to the $\mathfrak{C}_R$-orbit is identifiable with the tangent space to $J_R(X)$, whose dimension gives the arithmetic genus of $X$.

Finally, we examine the simplest points in $O_\Lambda$, the $\mathfrak{C}_R$-fixed points, and show that these correspond precisely to the $\mathbf{R}^2$-equivariant maps into $\mathbf{CP}^n$ (i.e. those which possess a frame which is a homomorphism $\mathbf{R}^2 \to U_{n+1}$): all such maps are of finite type. As an illustration I compute the spectral curve for any $\mathbf{R}^2$-equivariant map into $\mathbf{CP}^n$ of isotropy order $n-1$.

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# 2 Primitive maps and the harmonic sequence.

In this section we will recall the notion of the harmonic sequence for a harmonic map $\psi_0 : \mathbf{R}^2 \to \mathbf{CP}^n$ and present Burstall's characterisation [5] of a non-isotropic weakly conformal harmonic map in terms of its primitive lift into a flag manifold. From this we derive the relations (1) on the sequence of bundles $\ell_0, \ldots, \ell_{r+1}$. Following this we will briefly review the analogous theory for maps into $S^n$ via $\mathbf{RP}^n$.

## 2.1 Maps into $\mathbf{CP}^n$.

Let $z = x + iy$ be the usual complex coordinate on $\mathbf{R}^2$, then we can write a metric on $\mathbf{R}^2$ as $dz \otimes d\bar{z}$. Recall that any map $\psi_0 : \mathbf{R}^2 \to \mathbf{CP}^n$ is identifiable with the line subbundle $\ell_0 = \psi_0^* L_0 \subset \mathbf{R}^2 \times \mathbf{C}^{n+1}$, where $L_0$ is the tautological line bundle over $\mathbf{CP}^n$. Conversely, any line subbundle $\ell_0$ determines a map $\psi_0$. Every smooth vector subbundle $\ell \subset \mathbf{R}^2 \times \mathbf{C}^{n+1}$ admits a holomorphic structure (the Koszul-Malgrange structure) for which the holomorphic sections are characterised by $\pi_\ell(\partial s/\partial \bar{z}) = 0$, where $\pi_\ell$ denotes the orthogonal projection onto $\ell$. One then knows (from e.g. [9]) that $\psi_0$ is a harmonic map precisely when the map

$$A'_0 : \ell_0 \to \ell_0^\perp \quad , \quad s \mapsto \pi_0^\perp(\frac{\partial s}{\partial z}) \tag{2}$$

(which is the (1,0)-component of the second fundamental form of $\ell_0$) is holomorphic for this holomorphic structure on $\ell_0$.

It is well-known that when $\psi_0$ is harmonic and not anti-holomorphic the image $\ell_1$ of $A'_0$ is the line bundle for another harmonic map. This construction of new harmonic maps is sometimes called the *$\partial$-Gauss transform* and iteration of this transform produces the *harmonic sequence* $\ell_0, \ell_1, \ldots$ of the map $\psi_0$. [To be precise, the harmonic sequence is obtained by iterating both the $\partial$ and $\bar{\partial}$-Gauss transforms, where the latter is obtained by replacing $z$ with $\bar{z}$.] If the sequence terminates (in either direction, i.e. we arrive at either a holomorphic or anti-holomorphic map) then we say $\psi_0$ is (complex) *isotropic*. If $\psi_0$ is non-isotropic we have the notion of its *isotropy order*, given by

$$\text{isotropy order} = \max\{j : \ell_j \perp \ell_0, j > 0\} \tag{3}$$

If the isotropy order is $r+1$ then in fact all the line bundles $\ell_0, \ldots, \ell_{r+1}$ are mutually orthogonal (and, indeed, so are any $r+2$ consecutive bundles in the harmonic sequence cf. e.g. [4, 9]).

**Remark.** In [2] Bolton *et al.* define the similar notion of a '$k$-orthogonal' harmonic sequence, which is one for which $k$ is the maximum length of a consecutive sequence of mutually orthogonal line bundles in the harmonic sequence. Clearly $k$ is one more

than the isotropy order. On the other hand, Burstall [5] uses the notion of 'isotropy dimension', which is (for maps into $\mathbf{CP}^n$) one less than the isotropy order.

Recall that a harmonic map is weakly conformal if it has isotropy order $\geq 2$. Burstall [5] has shown that every non-isotropic weakly conformal harmonic map of isotropy order $r+1$ is covered by a primitive harmonic map into a certain flag manifold over $\mathbf{CP}^n$. Let us examine this result.

Let $\mathrm{pr} : F^r(\mathbf{CP}^n) \to \mathbf{CP}^n$ denote the bundle of flags in $T^{1,0}\mathbf{CP}^n$ with fibre

$$F^r_x(\mathbf{CP}^n) = \{w_1 \subset \ldots \subset w_r \subset T^{1,0}_x\mathbf{CP}^n : \dim w_j = j\}.$$

Using the identification between $T^{1,0}\mathbf{CP}^n$ and $\mathrm{Hom}(L_0, L_0^\perp)$ we can find subbundles $L_j \subset L_0^\perp$ for which each orthogonal complement $w_{j+1} \ominus w_j$ (for $j = 0, \ldots, r$ with $w_{r+1} = T^{1,0}_x\mathbf{CP}^n$) is identified with the fibre over $x$ of $\mathrm{Hom}(L_0, L_j)$. Altogether this results in the isomorphism

$$\beta : T^C F^r(\mathbf{CP}^n) \to \sum_{i,j=0, i\neq j}^{r+1} \mathrm{Hom}(L_i, L_j)$$

and we think of $\beta$ as the Maurer-Cartan form for the homogeneous space $F^r(\mathbf{CP}^n)$. This flag bundle is an $r+2$-symmetric space i.e. it is isomorphic to a homogeneous space $SU_{n+1}/H$ where $H$ is the fixed point subgroup of a periodic automorphism on $SU_{n+1}$ of order $r+2$. To be explicit, if we fix a base point $x$ on $\mathbf{CP}^n$ then $\mathbf{C}^{n+1}$ has orthogonal decomposition $\sum (L_j)_x$ and we obtain an endomorphism $\sigma$ which acts by scalar multiplication by $\omega^{-j}$ on $(L_j)_x$, where $\omega = \exp(2\pi i/(r+2))$. Clearly $\sigma$ is unitary with determinant 1 and conjugation by $\sigma$ gives a periodic automorphism of $SU_{n+1}$ with order $r+2$.

This automorphism induces one on the Lie algebra $\mathfrak{g}^C = \mathfrak{sl}_{n+1}$ and we obtain a periodic grading from the eigenspace decomposition

$$\mathfrak{g}^C = \mathfrak{g}_0 + \mathfrak{g}_{-1} + \ldots + \mathfrak{g}_{-r-1} \tag{4}$$

where $\mathfrak{g}_j$ is the eigenspace for the eigenvalue $\omega^j$. From this we deduce the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ where $\mathfrak{h}^C = \mathfrak{g}_0$ and $\mathfrak{p}^C = \sum_{j\neq 0} \mathfrak{g}_j$. We think of the image of $\beta$ as the twisted vector bundle $[\mathfrak{p}^C] = SU_{n+1} \times_H \mathfrak{p}^C$. Inside this bundle lies the subbundle

$$[\mathfrak{g}_{-1}] = \sum_{j=0}^{r} \mathrm{Hom}(L_j, L_{j+1}) \oplus \mathrm{Hom}(L_{r+1}, L_0).$$

A smooth map $\psi : M \to F^r(\mathbf{CP}^n)$ from a Riemann surface $M$ is called a *primitive map* whenever $\partial\psi/\partial z$ is a section of $[\mathfrak{g}_{-1}]$. The significance of this definition lies in the next result.

**Theorem 1 ([1, 5])** *Let $\psi : M \to F^r(\mathbf{CP}^n)$ be a primitive map, then $\psi$ is harmonic and so is $\psi_0 = \mathrm{pr} \circ \psi$.*

This has the converse result:

**Theorem 2 ([5])** *Every non-isotropic weakly conformal harmonic map $\psi_0 : M \to \mathbf{CP}^n$ of isotropy order $r+1$ is covered by a unique primitive map $\psi : M \to F^r(\mathbf{CP}^n)$.*

From the latter result we can derive a simple characterisation of the non-isotropic weakly conformal harmonic maps in terms of the sequence of $\partial$-Gauss transforms.

A map $\psi : M \to F^r(\mathbf{CP}^n)$ is the same as an orthogonal decomposition of $M \times \mathbf{C}^{n+1}$ into a sum of line bundles $\ell_0, \ldots, \ell_r$ and the orthogonal complement $\ell_{r+1}$ of this sum (which will have rank $n-r$). In this correspondence $\ell_0$ is the line bundle associated with $\psi_0$ and the flag of vector bundles $\mathrm{Hom}(\ell_0, \sum_{i=1}^{j} \ell_j)$ in $T^{1,0} = \psi_0^{-1} T^{1,0}\mathbf{CP}^n$ pulls back the image of $\psi$. Let $b = \psi^*\beta$, so that

$$b : T^C M \to \sum_{i,j=0, i\neq j}^{r+1} \mathrm{Hom}(\ell_i, \ell_j)$$

Then for any section $Z$ of $T^C M$, $b(Z)$ has its component in $\mathrm{Hom}(\ell_i, \ell_j)$ given by

$$b_{ij}(Z) : \ell_i \to \ell_j \quad , \quad s \mapsto \pi_j(Z.s)$$

where once again $\pi_j$ denotes the orthogonal projection onto $\ell_j$. By definition $\psi$ is primitive precisely when

$$b(\frac{\partial}{\partial z}) \in \sum_{j=0}^{r} \mathrm{Hom}(\ell_j, \ell_{j+1}) \oplus \mathrm{Hom}(\ell_{r+1}, \ell_0)$$

i.e. when the bundles $\ell_0, \ldots, \ell_{r+1}$ have the property (1). Provided $\pi_{j+1}(\frac{\partial}{\partial z}\ell_j) \neq 0$ for $j = 0, \ldots, r$ we see that $\ell_1, \ldots, \ell_r$ are the first $r$ consecutive bundles in the harmonic sequence of $\psi_0$. Moreover, $\pi_{r+1}(\frac{\partial}{\partial z}\ell_r)$ is a line subbundle orthogonal to $\ell_0$, so that $\psi_0$ must have isotropy order at least $r+1$. It will have precisely this isotropy order provided the next $\partial$-Gauss transform is not orthogonal to $\ell_0$. We summarise this as follows.

**Corollary 1** *Let $\ell_0 \oplus \ldots \oplus \ell_r \oplus \ell_{r+1}$ be a smooth orthogonal decomposition of $M \times \mathbf{C}^{n+1}$ into a direct sum of line bundles $\ell_0, \ldots, \ell_r$ and the complementary vector bundle $\ell_{r+1}$, which satisfy (1) and have $\pi_{j+1}(\frac{\partial \ell_j}{\partial z}) \neq 0$ for $j = 0, \ldots, r$ and $\pi_0(\frac{\partial^2 \ell_r}{\partial z^2}) \neq 0$. Then $\ell_0$ determines a non-isotropic weakly conformal harmonic map $\psi_0 : M \to \mathbf{CP}^n$ of isotropy order $r+1$.*

When the domain $M$ is a torus $T^2$ the weakly conformal maps are actually strongly conformal.

**Proposition 1** *All weakly conformal non-isotropic harmonic maps $\psi_0 : T^2 \to \mathbf{CP}^n$ are strongly conformal.*

**Proof.** To prove this we recall a result from [5]: let $\psi_0 : T^2 \to \mathbf{CP}^n$ be a weakly conformal harmonic map with primitive lift $\psi : T^2 \to G/H$, then $b(\partial/\partial z)$ takes values in a single $\mathrm{Ad}(G^{\mathbf{C}})$-orbit of a non-zero semisimple elements of $\mathfrak{g}^{\mathbf{C}}$. Now if we equip $G/H$ with the invariant metric it gets from the Killing form on $G$, then

$G/H \to \mathbf{CP}^n$ is an isometry, so it suffices to show that the primitive lift $\psi$ is strongly conformal i.e. $\langle b(\partial/\partial z), b(\partial/\partial z)\rangle > 0$. However, $b(\partial/\partial z)$ takes values in a single $\mathrm{Ad}(G^{\mathbf{C}})$-orbit and since this is the orbit of a non-zero semisimple element it is closed and does not contain the zero element. Therefore the inner product above cannot vanish.■

**Remark.** In fact this result holds for all maps $\mathbf{R}^2 \to \mathbf{CP}^n$ obtained from a weakly conformal primitive map of finite type. Thus all the maps we construct of isotropy order at least 2 will be genuinely conformal.

## 2.2 Maps into $S^n$ via $\mathbf{RP}^n$.

The two-to-one immersion $S^n \to \mathbf{RP}^n \subset \mathbf{CP}^n$ is a totally geodesic local isometry, and therefore a map into $S^n$ is harmonic if and only if it pushes down to a harmonic map into $\mathbf{CP}^n$. Thus we could study all the harmonic maps $\mathbf{R}^2 \to S^n$ via those into $\mathbf{RP}^n$ and indeed this is how we will describe harmonic maps into $S^n$ later. However, in [5] there is also a theory of primitive lifts for non-isotropic weakly conformal harmonic maps into spheres. These are maps into a flag manifold over $S^n$ which is a homogeneous space for $SO_{n+1}$ and it turns out that when we come to construct harmonic maps into the spheres it will not take much extra effort to construct their primitive lifts. So let us briefly review the theory of primitive lifts for spheres given in [5] and how it relates to harmonic sequences.

Let $\phi_0 : M \to S^n$ be a weakly conformal harmonic map and let $\psi_0 : M \to \mathbf{RP}^n \subset \mathbf{CP}^n$ be its projection. Then $\psi_0$ corresponds to a line subbundle $\ell_0 \subset M \times \mathbf{C}^{n+1}$ for which $\bar{\ell}_0 = \ell_0$. Let $\ell_0, \ell_1, \ldots$ be the $\partial$-Gauss sequence it generates, then one may readily show that $\bar{\ell}_1$ is the $\bar{\partial}$-Gauss transform of $\ell_0$ and, since $\psi_0$ is weakly conformal, $\ell_1$ is an isotropic subbundle i.e. $\ell_1 \perp \bar{\ell}_1$. We will follow [5] and define the isotropy dimension of $\phi_0$ to be

$$\text{isotropy dimension} = \max\{j : \ell_j \perp \ell_0 \text{ and } \ell_j \perp \bar{\ell}_j\}.$$

It is not hard to show that $\phi_0$ has isotropy dimension $r$ precisely when $\psi_0$ has isotropy order $2r+1$ (cf. for example [4]).

For $2r < n$ define $F^r(S^n)$ to be the bundle of isotropic flags over $S^n$ whose fibre is

$$F^r_x(S^n) = \{w_1 \subset \ldots \subset w_r \subset T^C_x S^n : \dim w_j = j, w_j \perp \bar{w}_j\}.$$

The standard identification of $T^C S^n$ with $L_0 \otimes L_0^{\perp}$ (where $L_0 \subset S^n \times \mathbf{C}^{n+1}$ is the complexification of the tautological line bundle) allows us to identify $T^C F^r(S^n)$ with

$$[\mathfrak{p}^C] = \sum_{i<j} L_i \otimes L_j + \sum_{i \neq j} \bar{L}_i \otimes L_j + \sum_{i<j} \bar{L}_i \otimes \bar{L}_j$$

where $i, j = 0, \ldots, r+1$. Here $L_{r+1} + \sum_{j=1}^r (L_j + \bar{L}_j)$ is an orthogonal decomposition of $L_0^{\perp}$ into a sum of isotropic line subbundles $L_j, \bar{L}_j$ (for $j = 1, \ldots, r$) and a real subbundle $L_{r+1}$ (i.e. $\bar{L}_{r+1} = L_{r+1}$) of rank $n-2r$. The space $F^r(S^n)$ is a homogeneous space of $SO_{n+1}$, indeed it is a $2r+2$-symmetric space. The periodic automorphism of $\mathfrak{so}_{n+1}$ it corresponds to gives each fibre of $[\mathfrak{p}^C]$ an eigenspace decomposition, where

now each eigenvalue is a power of $\omega = \exp(\pi i/(r+1))$. In particular, the subbundle of $\omega^{-1}$-eigenspaces is

$$[\mathfrak{g}_{-1}] = \sum_{i=0}^{r} \bar{L}_i \otimes L_{i+1} = \sum_{i=0}^{r} \mathrm{Hom}(L_i, L_{i+1})$$

where, to get the second equality, we have identified $L_i^*$ with $\bar{L}_i$ using the complexified inner product. As before, we say a map $\phi : M \to F^r(S^n)$ is primitive whenever $\partial\phi/\partial z$ is a section of $[\mathfrak{g}_{-1}]$.

Clearly a map $\phi : M \to F^r(S^n)$ is equivalent to an orthogonal decomposition of $M \times \mathbf{C}^{n+1}$ into a sum $\ell_0 + \sum \ell_j + \sum \bar{\ell}_j + \ell_{r+1}$ of line bundles $\ell_0, \ldots, \ell_r$ (and their conjugates) and a rank $n-2r$ subbundle $\ell_{r+1}$, where both $\ell_0$ and $\ell_{r+1}$ are real, together with a nowhere vanishing section of $\ell_0$. Given such a collection, the corresponding map $\phi$ will be primitive whenever these bundles satisfy

$$\frac{\partial}{\partial z}\ell_j \subset \ell_j \oplus \ell_{j+1} \quad j = 0, \ldots, r.$$

# 3 A Hermitian inner product on $H^0(X, \mathcal{L})$.

This section will set up the properties required of a curve $X$ and a line bundle $\mathcal{L}$ over it to view $H^0(X, \mathcal{L})$ as a Hermitian vector space of dimension $n+1$. Then we describe a parallel transport around the vector bundle $H^0(X)$ after it has been pulled back to a covering space of a real subgroup of the Jacobian $J(X)$ of $X$.

## 3.1 The curve $X$ and the line bundle $\mathcal{L}$.

First, let us fix $\mathbf{P}_\lambda$ to denote the Riemann sphere equipped with a rational coordinate $\lambda$: this is equivalent to fixing the points $\lambda = 0, 1, \infty$. For convenience set $m = r+1$. Now define $X$ to be a complete irreducible algebraic curve with arithmetic genus $p$, admitting:

- a degree $n+1$ cover $\pi : X \to \mathbf{P}_\lambda$ for which $\lambda \circ \pi$ has divisor

  $$(\lambda) = (m+1)P_0 + P_1 + \ldots + P_{n-m} - (m+1)P_\infty - Q_1 - \ldots - Q_{n-m}$$

  of non-singular points, where the points $P_j, Q_j$ for $j = 1, \ldots, n-m$ may occur in multiple degree but are different from $P_0, Q_0$,

- an anti-holomorphic involution $\rho : X \to X$ for which $\pi(\rho P) = \bar{\pi}(P)^{-1}$. We agree to index the points $P_i, Q_j$ so that $\rho(P_i) = Q_i$ (and clearly $\rho(P_0) = P_\infty$),

- a line bundle $\mathcal{L} \to X$ of degree $p+n$ satisfying

  $$\overline{\rho_*\mathcal{L}} \otimes \mathcal{L} \simeq \mathcal{O}_X(R) \tag{5}$$

  where $R$ is the ramification divisor of the covering $\pi$.

**Remark.** The conditions that $X$ be irreducible and that $\mathcal{L}$ be a line bundle are not essential. We could allow $X$ to be reducible provided each irreducible component carries at least one of the pairs of points $(P_0, P_\infty)$ or $(P_i, Q_i)$. However, it will be shown later (prop. 6) that the harmonic map we will construct is linearly full if and only if $X$ is connected. If $X$ is singular it is also possible to take $\mathcal{L}$ to be a rank 1 torsion free coherent sheaf, provided it is *maximal* i.e. it does not come by direct image from a less singular curve. This is equivalent to $\mathrm{Hom}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X$ (so that it is not fixed by the action of any non-trivial subgroup of $J(X)$). However, for ease of exposition I will keep to the simpler case of line bundles over irreducible curves.

The existence of such a triple $(X, \pi, \mathcal{L})$ is easy to demonstrate: we can construct a smooth $X$ by fixing a branch divisor $\pi_* R$ on $\mathbf{P}_\lambda$ and glueing together $n+1$ copies of $\mathbf{P}_\lambda$ in the usual fashion. Provided $\pi_* R$ has, for example, no points fixed by $\lambda \mapsto \bar{\lambda}^{-1}$ (which we will also call $\rho$) the ramification divisor will have the form $R_0 + \rho_* R_0$ for some positive divisor $R_0$ of degree $p + n$ (observe that $R$ must be $\rho$-invariant and has degree $2p + 2n$ by the Riemann-Hurwitz formula). Since the Jacobian $J(X)$ is a torus there are always solutions to $\overline{\rho_* \mathcal{L}_0} \simeq \mathcal{L}_0^{-1}$ on $J(X)$ (indeed this defines a real sub-torus of dimension $p$), hence $\mathcal{L} = \mathcal{L}_0(R_0)$ has degree $p + n$ and satisfies (5).

The principal result of this section is:

**Proposition 2** *Any line bundle $\mathcal{L}$ satisfying (5) is non-special i.e. has $h^1(X, \mathcal{L}) = 0$.*

By the Riemann-Roch theorem, this follows if we can prove that $h^0(X, \mathcal{L}) = n + 1$. Since $h^0(X, \mathcal{L})$ equals $h^0(\mathbf{P}_\lambda, \pi_* \mathcal{L})$ this follows in turn from:

**Lemma 1** *Whenever $\mathcal{L}$ satisfies (5) its direct image $\mathcal{E} = \pi_* \mathcal{L}$ is a trivial (rank $n + 1$) vector bundle over $\mathbf{P}_\lambda$.*

**Proof.** The proof of this lemma is just a more complete version of the proof of a similar result in [18]. I will break it into a number of steps, each of which is the proof of an italicised statement.

1. *Given $\mathcal{L} \otimes \overline{\rho_* \mathcal{L}} \simeq \mathcal{O}_X(R)$ there is a $\overline{\rho}_*$-equivariant isomorphism $f$ which exhibits this.* We can think of $\mathcal{L}$ as $\mathcal{O}_X(D)$ for some divisor $D$ of degree $p + n$, so that the isomorphism will be given on sections $\sigma$ by $\sigma \mapsto f\sigma$ for some meromorphic function $f$ on $X$ with divisor $R - D - \rho_* D$. Since this divisor is fixed by $\rho_*$ we see that $\overline{\rho_* f}.f^{-1}$ is a globally holomorphic function on $X$ i.e. a constant. In fact if we evaluate $\overline{\rho_* f}.f^{-1}$ at $P$ and $\rho(P)$ (for any point $P$) we see this constant must be unimodular. Therefore it is possible to rescale $f$ so that $\overline{\rho_* f} = f$.

2. *The vector bundle $\pi_*(\mathcal{L}^{-1} \otimes \mathcal{O}_X(R))$ is isomorphic to the dual $\mathcal{E}^\star$ of $\mathcal{E}$.* First we fix an open cover $A \cup A_0 \cup A_\infty$ of $\mathbf{P}_\lambda$, where $A = \mathbf{P}_\lambda - \{0, \infty\}$ and $A_0 = \rho(A_\infty)$ is an open disc about $\lambda = 0$ which contains no other branch point. Clearly this is a Leray cover for $\mathbf{P}_\lambda$ and pulls back by $\pi$ to give a Leray cover $X_A \cup X_0 \cup X_\infty$. Observe that $X_0 = \rho(X_\infty)$ is a union of open discs; we will denote the disc about $P_0$ by $U_0$ (and use $U_\infty$ for $\rho(U_0)$). In $U_0$ we fix a local parameter

$\zeta$ satisfying $\zeta^{m+1} = \lambda$ and we will have no inconsistencies if we use $\zeta^{-1}$ to denote $\overline{\zeta \circ \rho}$. In the other discs about $P_i, Q_j$ we can use $\lambda$ and $\lambda^{-1}$ for local parameters.

To prove the statement we must show there is an $\mathcal{O}_{\mathbf{P}}$-sheaf isomorphism between $\mathcal{L}^{-1} \otimes \mathcal{O}_X(R)$ and $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_{\mathbf{P}})$. On any any open subset $V \subset \mathbf{P}_\lambda$ with cover $U = \pi^{-1}(V)$ this isomorphism will be defined by mapping a local section $\sigma$ of $\mathcal{L}^{-1} \otimes \mathcal{O}_X(R)$ to the $\mathcal{O}_{\mathbf{P}}$-homomorphism $\mathrm{Tr}(\sigma\cdot) : s \mapsto \mathrm{Tr}(\sigma s)$. Here Tr is the trace homomorphism, which gives to each section of $\mathcal{O}_U(R)$ its trace as an $\mathcal{O}_V$-linear map (i.e. by multiplication) on the $\mathcal{O}_V$-module of sections of $\mathcal{O}_U(R)$. To show this is an isomorphism it is convincing enough to check this for the stalks over $\lambda = 0$ - all other calculations are a simple variation on this one.

In each stalk $\mathcal{L}_{P_i}$ fix a non-vanishing local section $\theta_i$. By definition, the stalk of the direct image over $\lambda = 0$ is

$$(\pi_*\mathcal{L})_0 = \mathcal{O}_0\langle\theta_0, \zeta\theta_0, \ldots, \zeta^m\theta_0\rangle \oplus \sum_{i=1}^{n-m} \mathcal{O}_0\langle\theta_i\rangle$$

where $\mathcal{O}_0$ is the local ring at $\lambda = 0$. The dual $\mathcal{O}_0$-module is generated by the normalised homomorphisms $f_j$ and $g_i$ which are dual to (respectively) $\zeta^j\theta_0$ and $\theta_i$. Clearly, for any local section $s$, $f_j(s) = \mathrm{Tr}(\zeta^{-j}\theta_0^{-1}s)$ and $g_i(s) = \mathrm{Tr}(\theta_i^{-1}s)$. It follows that the dual module $(\pi_*\mathcal{L})_0^\star$ is isomorphic, via Tr, to $\mathcal{L}^{-1} \otimes \mathcal{O}_X(R)|\pi^{-1}(0)$ since the ramification divisor is $mP_0$ over $\lambda = 0$.

3. *$\mathcal{L}$ admits a local trivialising section $\theta$ over $X_0 \cup X_\infty$ for which the isomorphism $\overline{\rho_*\mathcal{E}} \simeq \mathcal{E}^\star$ sends $\overline{\rho_*\pi_*\theta}$ to the canonical dual $\pi_*\theta^\star$ of $\pi_*\theta$.* Here $\pi_*\theta$ stands for the induced trivialisation of $\mathcal{E}$ over $A_0 \cup A_\infty$ described in the previous step. Set $\theta^{(0)} = \theta|X_0$ and $\theta^{(\infty)} = \theta|X_\infty$, then since the isomorphism is $\bar{\rho}_*$-equivariant it suffices to show that $\overline{\rho_*\pi_*\theta^{(\infty)}}$ is identified with $\pi_*\theta^\star$. Each of $\theta^{(0)}$ and $\theta^{(\infty)}$ should be thought of as a collection of sections over the connected components of $X_0$ and $X_\infty$, which we will denote by $\theta_j^{(0)}, \theta_j^{(\infty)}$ respectively. By the previous step $\pi_*\theta$ is represented by the frames

$$\begin{array}{rl} \theta_0^{(0)}, \zeta\theta_0^{(0)}, \ldots, \zeta^m\theta^{(0)}, \theta_1^{(0)}, \ldots, \theta_{n-m}^{(0)} & \text{over } A_0, \text{ and} \\ \zeta^{-m}\theta_0^{(\infty)}, \ldots, \zeta^{-1}\theta_0^{(\infty)}, \theta_0^{(\infty)}, \theta_1^{(\infty)}, \ldots, \theta_{n-m}^{(\infty)} & \text{over } A_\infty. \end{array}$$

To prove the result we must show that it is possible to have

$$\begin{aligned} \mathrm{Tr}(f.\zeta^k\theta_0^{(0)} \otimes \overline{\rho_*\zeta^{m-j}\theta_0^{(\infty)}}) &= 1 \\ \mathrm{Tr}(f.\theta_j^{(0)} \otimes \overline{\rho_*\theta_j^{(\infty)}}) &= 1 \text{ for } j = 1, \ldots, n-m. \end{aligned}$$

This can always be achieved, since we have the freedom to rescale $\theta$ by a non-vanishing holomorphic function over $X_0 \cup X_\infty$.

4. *$\mathcal{E}$ admits a trivial 1-cocycle for the Leray cover $A \cup A_0 \cup A_\infty$.* For this we need to recall a theorem from [18] concerning the factorisation of maps from the pair of annuli $A \cap (A_0 \cup A_\infty)$ into $GL_{n+1}(\mathbf{C})$.

**Theorem 3 ([18])** *Set $I = A_0 \cup A_\infty$ and let $g : A \cap I \to GL_{n+1}$ be holomorphic. If $\overline{\rho_* g}^{-1t} = hg$ for some holomorphic $h : A \to GL_{n+1}$ then $g = g_A g_I$ where $g_A$ extends holomorphically into $A$ and $g_I$ extends holomorphically into $I$ with $\overline{\rho_* g_I}^{-1t} = g_I$.*

To apply this theorem to the problem at hand, observe that on $A \cap I$ the vector bundle $\mathcal{E}$ has transition relations $g.\pi_*\theta = \tau$ for some holomorphic $g : A \cap I \to GL_{n+1}$ where $\tau$ is some trivialisation over $A$. It follows that $\overline{\rho_* \mathcal{E}}$ has transition relations $\overline{g \circ \rho}.\overline{\rho_* \pi_* \theta} = \overline{\rho_* \tau}$ whilst on $\mathcal{E}^\star$ we have $g^{-1t}.\pi_*\theta^\star = \tau^\star$. Thus, if $s$ is a holomorphic frame for $\mathcal{E}^\star$ over $A \cap I$, and $\tilde{s}$ the corresponding frame for $\overline{\rho_* \mathcal{E}}$, then by the previous step $\pi_*\theta(s) = \overline{\rho_* \pi_* \theta}(\tilde{s})$ and therefore $\overline{g \circ \rho} = hg^{-1t}$ for some holomorphic $h : A \to GL_{n+1}$. By theorem 3 we see that $g$ is equivalent to the trivial cocycle on the Leray cover $A \cup I$, hence the statement follows. This finishes the proof of the lemma.■

Now that we have established that $H^0(X, \mathcal{L})$ is $n+1$-dimensional, we wish to equip it with a positive definite Hermitian form. In fact we can obtain such a form from the trace mapping used in the proof of the previous lemma. Indeed, observe that, if $B$ denotes the algebra of holomorphic maps $A \to \mathbf{C}$ then the map

$$h : \Gamma(X_A, \mathcal{L}) \times \Gamma(X_A, \mathcal{L}) \to B \quad ; \quad (v, w) \mapsto \mathrm{Tr}(f.v \otimes \overline{\rho_* w})$$

has the properties that, for any $b \in B$,

$$\begin{array}{rcl} h(bv, w) & = & b.h(v, w), \\ h(v, bw) & = & \overline{\rho_* b}.h(v, w), \\ h(w, v) & = & \overline{\rho_* h(v, w)}. \end{array} \tag{6}$$

Since the trace of a global section of $\mathcal{O}_X(R)$ yields a global section of $\mathcal{O}_{\mathbf{P}}$, all of which are constant, it follows that the restriction of $h$ to $H^0(X, \mathcal{L})$ is a Hermitian symmetric form. Now we must show it is positive definite.

**Lemma 2** *There exists a frame $\tau = \{\tau_j\}$ of global sections trivialising $\mathcal{E} = \pi_*\mathcal{L}$ over $A$ for which $h(\tau_j, \tau_k) = \delta_{jk}$ (Kronecker delta).*

**Proof.** Recall that for any frame $\tau'$ over $A$ we have the transition relations $g.\pi_*\theta = \tau'$ over $A \cap I$ where $g$ satisfies the conditions of theorem 3. Thus we may factorise $g$ into $g_A g_I$. I claim that $\tau = g_A^{-1}.\tau'$. To see this, let $T : \overline{\rho_* \mathcal{E}} \to \mathcal{E}^\star$ be the isomorphism described in lemma 1. We can represent $\pi_*\theta$ by a frame $\{\sigma_J\}$, in which case, over $A \cap I$, we have $\tau_j = g_I.\pi_*\theta(\tau_j)$ represented by $\{g_I \sigma_j\}$. Thus

$$\begin{array}{rcl} T(\overline{\rho_* \tau_k}) & = & T(\overline{\rho_* g_I.\sigma_k}) \\ & = & g_I^{-1t}.T(\overline{\rho_* \sigma_k}) \\ & = & g_I^{-1t} \sigma_k^\star \text{ by step 3 in the proof of lemma 1} \\ & = & \tau_k^\star. \end{array}$$

But $T(\overline{\rho_* \tau_k})[\tau_j] = \mathrm{Tr}(f.\tau_j \otimes \overline{\rho_* \tau_k})$ by the definition of $T$, and so $h(\tau_j, \tau_k) = \delta_{jk}$ for some basis $\{\tau_j\}$ of Hol (observe that each $\tau_j$ described in the proof above is a globally holomorphic section). Therefore we have equipped $H^0(X, \mathcal{L})$ with a positive definite Hermitian inner product. ■

## 3.2 A parallel transport.

We have already observed that if $\mathcal{L}$ is real i.e. if it satisfies (5) then $\mathcal{L}\otimes L$ is also real for any $L$ in the subgroup $\{L\in J(X):\overline{\rho_* L}\simeq L^{-1}\}$: we will define $J_R(X)$ to be the connected component of the identity of this subgroup. By lemma 1 the dimension of $H^0(X,\mathcal{L}\otimes L)$ is $n+1$ for all $L\in J_R(X)$. Now we are going to fix a way of identifying all these vector spaces with $H^0(X,\mathcal{L})$ which will amount to constructing (at least locally) a parallel transport on the vector bundle $H^0(X)\to J_R(X)$ whose fibre over $L$ is $H^0(X,\mathcal{L}\otimes L)$. To be accurate, the parallel transport is only well defined on a simply connected covering space of $J_R(X)$.

To achieve this we first fix an isomorphism between $\Gamma(X_A,\mathcal{L}\otimes L)$ and $\Gamma(X_A,\mathcal{L})$ (this is possible since $X_A$ is Stein, so every line bundle over it is trivialisable). We begin by observing that every $L\in J(X)$ is obtained from a 1-cocycle $(e^a,X_A,X_I)$, where $X_I=\pi^{-1}(I)$. Moreover, this also determines trivialising sections $\theta_A$ and $\theta_I$, over $X_A$ and $X_I$ respectively, for which

$$\theta_I = e^a.\theta_A \text{ on } X_A\cap X_I.$$

Moreover, these are unique up to scaling. Let us define $\mathcal{G}=\Gamma(X_A\cap X_I,\mathcal{O}_X)$, then we have an epimorphism of groups

$$L:\mathcal{G}\to J(X) \quad ; \quad a\mapsto L(a)$$

where $L(a)$ has the transition relations above. The subgroup $J_R(X)$ is the image of the subspace $\mathcal{G}_R=\{a\in\mathcal{G}:\overline{\rho_* a}=-a\}$.

From now on we fix $\theta$ to be a trivialising section for $\mathcal{L}$ over $X_I$ satisfying $\mathrm{Tr}(f.\theta\otimes\overline{\rho_*\theta})=1$. We have already seen in step 3 of the proof of lemma 1 that such a section exists. For each $a\in\mathcal{G}_R$ we set $\theta_a=\theta_I\otimes\theta$, which is a holomorphic non-vanishing section of $\mathcal{L}\otimes L$ over $X_I$. We obtain an isomorphism

$$\iota_a:\Gamma(X_A,\mathcal{L}\otimes L(a))\to\Gamma(X_A,\mathcal{L})$$

determined by the following lemma.

**Lemma 3** *Let $\sigma_a\in\Gamma(X_A,\mathcal{L}\otimes L(a))$ and define $\iota_a(\sigma_a)=e^a(\sigma_a\theta_a^{-1})\theta$. Then $\iota_a(\sigma_a)$ belongs to $\Gamma(X_A,\mathcal{L})$.*

**Proof.** On $X_A\cap X_I$ the line bundle $\mathcal{L}\otimes L(a)$ will have transition relations $\theta_a = c.e^a\tau\otimes\theta_A$ for some fixed trivialising section $\tau$ of $\mathcal{L}$ over $X_A$, where $c$ is a non-vanishing function on $X_A\cap X_I$. Therefore

$$\begin{aligned} e^a(\sigma_a\theta_a^{-1})\theta &= e^a\sigma_a e^{-a}c^{-1}(\tau\otimes\theta_A)^{-1}\theta \\ &= \sigma_a(\tau\otimes\theta_A)^{-1}\tau. \end{aligned}$$

Since $\sigma_a(\tau\otimes\theta_A)^{-1}$ is holomorphic over $X_A$ the resulting section is holomorphic there.■

Now we show that this produces a parallel transport around the pullback bundle $L^*H^0(X)\to\mathcal{G}_R$. Let $\{\tau_j\}$ be some basis for $H^0(X,\mathcal{L})$ and define

$$\mathrm{ev}_1:\Gamma(X_A,\mathcal{L})\to H^0(X,\mathcal{L});\quad \sum\sigma_j(\lambda)\tau_j\mapsto\sum\sigma_j(1)\tau_j. \tag{7}$$

Here we are exploiting the fact that $\Gamma(X_A, \mathcal{L})$ is a free $B$-module of rank $n+1$. The composition $\mathrm{ev}_1 \circ \iota_a$ provides a linear map $H^0(X, \mathcal{L} \otimes L(a)) \to H^0(X, \mathcal{L})$ and we must show that it has trivial kernel. But this follows immediately from the fact that the global sections of $\pi_*(\mathcal{L} \otimes L)$ generate the fibre over $\lambda = 1$, which is isomorphic to the fibre of $\pi_*\mathcal{L}$ by $\iota_a$ (observe the latter is a $B$-module isomorphism).

Finally, for future use we will need to prove:

**Lemma 4** *Let $\sigma_1, \sigma_2 \in H^0(X, \mathcal{L} \otimes L(a))$ and set $s_j = \iota_a(\sigma_j)$. Then $h(s_1, s_2)$ is a constant.*

**Proof.** For simplicity, let $\underline{\mathcal{L}(a)}$ denote $\mathcal{L} \otimes L(a)$. The isomorphism $\iota_a$ induces one $\kappa_a$ between $\Gamma(X_A, \underline{\mathcal{L}(a)} \otimes \overline{\rho_*\mathcal{L}(a)})$ and $\Gamma(X_A, \mathcal{L} \otimes \overline{\rho_*\mathcal{L}})$ which sends a section $\sigma$ to $\sigma(\theta_a \otimes \overline{\rho_*\theta_a})^{-1}\theta \otimes \overline{\rho_*\theta}$. Here we have used the fact that $\bar{\rho}_*$ maps $e^a$ to $e^{-a}$. Now I claim that $s_{12} = \kappa_a(\sigma_1 \otimes \overline{\rho_*\sigma_2})$ is a globally holomorphic section of $\mathcal{L} \otimes \overline{\rho_*\mathcal{L}}$. To see this, observe that each $\sigma_j$ is holomorphic throughout $X_A$, therefore so is $s_{12}$. But $s_{12} = f_1\theta \otimes \overline{\rho_*(f_2\theta)}$ in $X_I$, where $f_j = \sigma_j\theta_a^{-1}$. Since each of $f_1, f_2$ and $\theta$ is holomorphic in $X_I$ it follows that $s_{12}$ is also. Hence it is globally holomorphic.

To finish the proof we deduce that $h(s_1, s_2) = \mathrm{Tr}(f.s_{12})$ must be a global section of $\mathcal{O}_{\mathbf{P}}$ and therefore a constant.■

**Remark.** From the proof of lemma 3 one can see that the parallel transport $\iota_a(\sigma_a)$ does not depend on the choice of $\theta$, but rather on $\theta_A$. The expression given in the lemma is simply the most convenient for future calculations. If we fix $\theta_A$ so that $\theta_A \otimes \overline{\rho_*\theta_A} = 1$ (which we can do, since $L \otimes \overline{\rho_*L} \simeq \mathcal{O}_X$ for $L \in J_R(X)$) then it is not hard to see that parallel transport around $J_R(X)$ leaves $h$ invariant.

# 4 The construction of harmonic maps into $\mathbf{CP}^n$.

## 4.1 Harmonic maps with isotropy order m> 1.

Define $a : \mathbf{R}^2 \to \mathcal{G}_R$ by $a(z, \bar{z}) = z\zeta^{-1} - \bar{z}\zeta$, where, with a slight abuse of notation, $\zeta$ stands for the function on $X_A \cap X_I$ which equals $\zeta$ in the punctured discs about $P_0, P_\infty$ but equals zero elsewhere. From now on we will restrict the homomorphism $L$ to the image of $a$, so that $L(a)$ is a 2-parameter subgroup of $J_R(X)$. Thus we may pull the vector bundle $H^0(X)$ back to one over $\mathbf{R}^2$; we will denote this simply by $H^0(X) \to \mathbf{R}^2$. This $\mathbf{C}^{n+1}$-bundle is trivialisable using the parallel transport described in the previous section. Indeed we will fix an $h$-orthonormal basis $\{\tau_j\}$ for $H^0(X, \mathcal{L})$ so that we may identify $H^0(X, \mathcal{L})$ with $\mathbf{C}^{n+1}$ in such a way that $h$ is identified with the standard Hermitian inner product on $\mathbf{C}^{n+1}$. Our aim now is to describe an 'orthogonal decomposition' of $H^0(X) \to \mathbf{R}^2$ which, after all the identifications have been made, results in an orthogonal decomposition of $\mathbf{R}^2 \times \mathbf{C}^{n+1}$ which satisfies the conditions of corollary 1. The idea is to define ideal sheaves $\mathcal{L}_j$ of $\mathcal{L}$ for which the 2-parameter family of subspaces $H^0(X, \mathcal{L}_j \otimes L(a))$ is a subbundle in this decomposition.

Recall that $\lambda$ has divisor on $X$ which we may write as

$$(\lambda) = (m+1)P_0 + E_0 - (m+1)P_\infty - E_\infty$$

where $E_0 = \sum_{i=1}^{n-m} P_i$ and $E_\infty = \sum_{i=1}^{n-m} Q_i$. We define:

- $\mathcal{L}_j = \mathcal{L}(-(m-j)P_0 - jP_\infty - E_0)$ for $j = 0, \ldots, m-1$, and
- $\mathcal{L}_m = \mathcal{L}(-mP_\infty)$

thus each $\mathcal{L}_j$ is a line bundle of degree $p$ for $j = 0, \ldots, m-1$ and degree $p+n-m$ for $j = m$.

**Lemma 5** *For $j = 0, \ldots, m$ each $\mathcal{L}_j$ is non-special i.e. $h^1(X, \mathcal{L}_j) = 0$.*

**Proof.** For each $j$ let $\mathcal{I}_j = \mathcal{L}(jP_0 - jP_\infty)$. Clearly each $\mathcal{I}_j$ is real, therefore by lemma 1 each of the direct images $\pi_*\mathcal{I}_j$ is a trivial bundle. It follows that $\mathcal{F}_j = \mathcal{I}_j(-(m+1)P_0 - E_0)$ has no non-trivial holomorphic sections, since any such section is a global section of $\pi_*\mathcal{I}_j$ which vanishes at $\lambda = 0$. By the Riemann-Roch theorem we deduce that $h^1(X, \mathcal{F}_j) = 0$. Now we use the following simple fact. For any line bundle $\mathcal{L}$ and any point $P \in X$, if $\mathcal{L}$ is non-special then so is $\mathcal{L}(P)$. [We get this using Serre duality, for if $\Omega^{(1,0)} \otimes \mathcal{L}^{-1}$ has no non-trivial holomorphic sections then it certainly can't have any which vanish at $P$.] So for $j = 0, \ldots, m-1$ we apply this fact to $\mathcal{L}_j = \mathcal{F}_j(P_0)$ to show it is non-special. Similarly, a repeated application of this fact shows that $\mathcal{L}_m = \mathcal{F}_m(P_\infty + \sum Q_i)$ is also non-special.■

**Corollary 2** *For all $a \in \mathcal{G}_R$, $h^1(X, \mathcal{L}_j \otimes L(a)) = 0$.*

This follows at once by replacing $\mathcal{L}$ with $\mathcal{L} \otimes L(a)$ in the previous proof.

We see that $H^0(X) \to \mathbf{R}^2$ has a direct sum decomposition into $\sum_{j=0}^{m} H^0(X, \mathcal{L}_j \otimes L(a))$. By the Riemann-Roch theorem and the corollary we see that this is a sum of line bundles (for $j = 0, \ldots, r = m-1$) and a complementary vector bundle of rank $n - r$. Let

$$\tau^1 : \Gamma(X_A, \mathcal{L}) \to \mathbf{C}^{n+1} \tag{8}$$

be given by composing $\mathrm{ev}_1$ in (7) with the isomorphism $H^0(X, \mathcal{L}) \to \mathbf{C}^{n+1}$ fixed by our choice of basis $\{\tau_j\}$. By combining this with the parallel transport we pull the decomposition above back to the trivial bundle $\mathbf{R}^2 \times \mathbf{C}^{n+1}$ and define $\ell_j = \tau^1 \circ \iota_a(H^0(X, \mathcal{L}_j \otimes L(a)))$. We want to show that these subbundles are orthogonal for the standard Hermitian structure on $\mathbf{C}^{n+1}$. This will follow from the next result.

**Lemma 6** *For each $j$ let $\sigma_j \in H^0(\mathcal{L}_j \otimes L(a))$ (with $z, \bar{z}$ fixed). Set $s_j = \iota_a(\sigma_j)$. Then $h(s_j, s_k) = 0$ whenever $j \neq k$.*

**Proof.** By lemma 4 we know that $h(s_j, s_k)$ is a constant so it suffices to show that it vanishes somewhere on $\mathbf{P}_\lambda$ whenever $j \neq k$. If we set $f_j = \sigma_j \theta_a^{-1}$ then $s_j = e^a f_j \theta$. Observe that $f_j$ is a holomorphic function over $X_I$ with divisor at least $(m-j)P_0 + jP_\infty + E_0$ for $j = 0, \ldots, m-1$ and the divisor of $f_m$ is at least $mP_\infty$. From the definition of $h(s_j, s_k)$ we have

$$h(s_j, s_k) = \mathrm{Tr}(f.\theta \otimes \overline{\rho_*\theta}.f_j.\overline{\rho_* f_k}) = \mathrm{Tr}(r_{jk}).$$

By step 3 in the proof of lemma 1 we know that $f.\theta \otimes \overline{\rho_*\theta}$ equals $\zeta^{-m}$ in $U_0$, $\zeta^m$ in $U_\infty$ and 1 elsewhere, where we recall that $U_0, U_\infty$ are the connected components of $X_I$ about $P_0$ and $P_\infty$ respectively. Consequently, $r_{jk}$ has divisor at least:

- $(k-j)P_0 + (j-k)P_\infty + E_0 + E_\infty$ for $j,k = 0,\ldots,m-1$,
- $(m-j)(P_0 - P_\infty) + E_0$ for $j = 0,\ldots,m-1$ and $k = m$.

Now we examine the computation of $\mathrm{Tr}(r_{jk})$.

Consider $\mathrm{Tr}(r)$ where $r \in \mathcal{O}_{X_0}(R)$, the $\mathcal{O}_{A_0}$-module of meromorphic functions on $X_0$ with divisor at worst $-mP_0$. Let $1_j$ denote the restriction of the constant function 1 to the connected component of $X_0$ about $P_j$. Then

$$\mathcal{O}_{X_0}(R) \simeq \mathcal{O}_{A_0}\langle 1_0, \zeta^{-1}1_0, \ldots, \zeta^{-m}1_0, 1_1, \ldots, 1_{n-m}\rangle.$$

We may write $r$ in each component as a Laurent series to obtain the expression

$$r = \sum_{-m}^{\infty} \alpha_j \zeta^j 1_0 + \sum_{0}^{\infty} \beta_{1j}\lambda^j 1_1 + \ldots + \sum_{0}^{\infty} \beta_{n-mj}\lambda^j 1_{n-m}.$$

A simple calculation shows that, as an $\mathcal{O}_{A_0}$-endomorphism of $\mathcal{O}_{X_0}(R)$, multiplication by $r$ has trace

$$\mathrm{Tr}(r) = (m+1)\sum_{l=0}^{\infty} \alpha_{l(m+1)}\lambda^l + \sum_{k=1}^{n-m}\sum_{j=0}^{\infty} \beta_{kj}\lambda^j$$

where we have used $\zeta^{m+1} = \lambda$. A similar computation in $X_\infty$ shows that, for $0 \le j,k \le m-1$:

- for $k > j$, $\mathrm{Tr}(r_{jk})$ vanishes at $\lambda = 0$,
- for $k < j$, $\mathrm{Tr}(r_{jk})$ vanishes at $\lambda^{-1} = 0$.

Similarly, $\mathrm{Tr}(r_{jm})$ vanishes at $\lambda = 0$ for $j < m$. Therefore we deduce that whenever $j \neq k$, $h(s_j, s_k)$ vanishes somewhere and consequently is identically zero.■

Thus we see that $\ell_0 \oplus \ldots \oplus \ell_{r+1}$ is an orthogonal decomposition of $\mathbf{R}^2 \times \mathbf{C}^{n+1}$. Now we intend to show that this collection of subbundles satisfies the conditions described in corollary 1 when $r > 0$. First we require a lemma:

**Lemma 7** *Let $\sigma : \mathbf{R}^2 \to H^0(X)$ be a smooth section for which $\sigma(z,\bar{z})$ is a globally holomorphic section of $\mathcal{F} \otimes L(a)$ for some ideal sheaf $\mathcal{F}$ of $\mathcal{L}$. Let $D$ denote the covariant derivative on $H^0(X)$ induced by the parallel transport, then $D_{\partial/\partial z}\sigma$ is a holomorphic section of $\mathcal{F}(P_0) \otimes L(a)$ and $D_{\partial/\partial\bar{z}}\sigma$ is a holomorphic section of $\mathcal{F}(P_\infty) \otimes L(a)$.*

**Proof.** By definition $D_Z\sigma = \iota_a^{-1}(Z.\iota_a(\sigma))$ for any vector field $Z$. Define $s = \iota_a(\sigma)$, then this maps $\mathbf{R}^2$ into $\Gamma(X_A, \mathcal{L})$ and over $X_I$ can be written as $s = e^a f\theta$, where $f = \sigma\theta_a^{-1}$ is a map from $\mathbf{R}^2$ into the vector space $\Gamma(X_I, \mathcal{F} \otimes \mathcal{L}^{-1})$. We have

$$\begin{array}{rcl} \partial s/\partial z &=& (\zeta^{-1}f + \partial f/\partial z)e^a\theta, \\ \partial s/\partial\bar{z} &=& (\zeta f + \partial f/\partial\bar{z})e^a\theta, \end{array} \tag{9}$$

where (as earlier) $\zeta$ is thought of as zero except on $U_0, U_\infty$. This yields

$$\begin{array}{rcl} D_{\partial/\partial z}\sigma &=& (\zeta^{-1}f + \partial f/\partial z)\theta_a, \\ D_{\partial/\partial\bar{z}}\sigma &=& (\zeta f + \partial f/\partial\bar{z})\theta_a. \end{array} \tag{10}$$

These expressions show that the covariant derivatives are holomorphic sections of, respectively, $\mathcal{F}(P_0)\otimes L(a)$ and $\mathcal{F}(P_\infty)\otimes L(a)$ over $X_I$. But they are also holomorphic over $X_A$ since the $z$ and $\bar{z}$ derivatives of $s$ are. ■

**Remark.** The map $L : \mathbf{R}^2 \to J(X)$ is designed so that the covariant derivative along $L_*\partial/\partial z$ introduces a simple pole at $P_0$. This has the attractive interpretation, which I owe to [21], that the image of $L$ (at the origin) should be tangent to the image of $X$ under its Abel map into $J(X)$ taking the base point to be $P_0$. To be precise, let

$$\mathcal{A}_{P_0} : X \to J(X) \quad , \quad P \mapsto \int_{P_0}^{P} \underline{\omega}$$

where $\underline{\omega}$ is a vector containing a normalised basis of holomorphic differentials on $X$. The $(1,0)$-component of its differential at $P_0$ is readily seen to be

$$\partial\mathcal{A}_{P_0} : T^{1,0}_{P_0}X \to H^0(X,\Omega^{1,0})^* \quad , \quad z\frac{\partial}{\partial\zeta} \mapsto (\underline{\omega} \mapsto z.res_{P_0}\zeta^{-1}\underline{\omega}).$$

Clearly the one parameter (complex) subgroup tangent to $\partial\mathcal{A}_{P_0}(\partial/\partial\zeta)$ is just the family of line bundles $L(z\zeta^{-1})$ and, as the lemma shows, differentiating along this subgroup has the effect of introducing a simple pole at $P_0$ to sections of $\mathcal{L}\otimes L(z\zeta^{-1})$.

Now let $\sigma_0,\ldots,\sigma_n$ be global sections of $H^0(X) \to \mathbf{R}^2$ for which $\sigma_j(z,\bar{z}) \in H^0(X,\mathcal{L}_j\otimes L(a))$ for $j = 0,\ldots,m-1$ and $\sigma_m,\ldots,\sigma_n$ span $H^0(X,\mathcal{L}_m\otimes L(a))$ for each value of $z,\bar{z}$. Define $s_j(z,\bar{z}) = \iota_a(\sigma(z,\bar{z}))$, then it is clear that at each value of $z,\bar{z}$ these provide a free system of generators for the $B$-module $\Gamma(X_A,\mathcal{L})$. Let us define $V_j(z,\bar{z})$ to be the $B$-module generated by $s_j(z,\bar{z})$ in the case $j = 0,\ldots,m-1$ and by $s_m,\ldots,s_n$ in the case $j = m$. Then we have a 2-parameter family of direct sum decompositions:

$$\Gamma(X_A,\mathcal{L}) = \sum_0^m V_j(z,\bar{z}).$$

Indeed, by the previous lemma and the properties (6) of $h$ we see that this is an $h$-orthogonal decomposition, and we define $\Pi_j$ to be the (2-parameter family of) $h$-orthogonal projections from $\Gamma(X_A,\mathcal{L})$ onto $V_j$.

**Lemma 8** *The maps $s_j : \mathbf{R}^2 \to \Gamma(X_A,\mathcal{L})$ defined above satisfy:*

$$\begin{array}{lll} \partial s_j/\partial z & \in & V_j \oplus V_{j+1}, \quad j = 0,\ldots,m-1, \\ \partial s_k/\partial z & \in & V_m \oplus V_0, \quad k = m,\ldots,n, \end{array}$$

*and*

$$\begin{array}{lll} \Pi_{j+1}(\partial s_j/\partial z) & \neq & 0, \quad j = 0,\ldots,m-1, \\ \Pi_0(\partial^2 s_{m-1}/\partial z^2) & \neq & 0. \end{array}$$

**Proof.** As earlier, we will write $s_j = e^a f_j(z,\bar{z})\theta$ with $f_j = \sigma_j\theta_a^{-1}$. There are three cases:

1. $j = 0,\ldots,m-2$. By the previous lemma, $\iota_a^{-1}(\partial s_j/\partial z)$ is a globally holomorphic section of $\mathcal{L}_j(P_0)\otimes L(a)$. The space of such sections is generated by $\sigma_j,\sigma_{j+1}$. Thus $\partial s_j/\partial z$ belongs to $V_j \oplus V_{j+1}$. Moreover, its orthogonal projection onto $V_{j+1}$ cannot be identically zero since $\zeta^{-1}f_j\theta_a = \zeta^{-1}\sigma_j$ cannot be a holomorphic section of $\mathcal{L}_j\otimes L(a)$ over $X_I$.

2. $j = m-1$. Again, the previous lemma shows that $\iota_a^{-1}(\partial s_{m-1}/\partial z)$ is a globally holomorphic section of $\mathcal{L}_{m-1}(P_0) \otimes L(a)$, which proves the first relation. Next, we compute

$$\iota_a^{-1}(\frac{\partial^2 s_{m-1}}{\partial z^2}) = (\zeta^{-2} f_{m-1} + 2\zeta^{-1}\frac{\partial}{\partial z} f_{m-1} + \frac{\partial^2}{\partial z^2} f_{m-1})\theta_a$$

which, by similar arguments, must be a global section of $\mathcal{L}_{m-1}(2P_0) \otimes \mathrm{L}(a)$ but cannot vanish at $P_0$. We observe that $H^0(X, \mathcal{L}_{m-1}(2P_0) \otimes L(a))$ is spanned by the sections $\sigma_{m-1}, \ldots, \sigma_n, \lambda^{-1}\sigma_0$ and the only one of these which does not vanish at $P_0$ is the last. Hence the orthogonal projection of $\partial^2 s_j/\partial z^2$ onto $V_0$ is not zero.

3. $k = m, \ldots, n$. By the previous lemma $\iota_a^{-1}(\partial s_k/\partial z)$ is a globally holomorphic section of $\mathcal{L}_m(P_0) \otimes L(a)$ and the space of these is spanned by the sections $\sigma_m, \ldots, \sigma_n, \lambda^{-1}\sigma_0$. Hence $\partial s_k/\partial z$ belongs to $V_m \oplus V_0$.■

**Theorem 4** *Let $\ell_0, \ldots, \ell_r, \ell_{r+1}$ (with $r > 0$) be the subbundles of $\mathbf{R}^2 \times \mathbf{C}^{n+1}$ described above. Then $\ell_0$ determines a harmonic map $\psi_0 : \mathbf{R}^2 \to \mathbf{CP}^n$ of isotropy order precisely $r+1$.*

**Proof.** Observe that the map $\tau^1 : \Gamma(X_A, \mathcal{L}) \to \mathbf{C}^{n+1}$ projects each submodule $V_j(z, \bar{z})$ onto the line $\ell_j(z, \bar{z})$. By the previous lemma and the fact that $\partial/\partial z$ commutes with $\tau^1$ we see that the family of bundles $\ell_0, \ldots, \ell_{r+1}$ satisfy equation (1). We also observe that $\tau^1 \circ \Pi_j = \pi_j \circ \tau^1$, so $\pi_{j+1}(\partial \ell_j/\partial z) \neq 0$ for $j = 0, \ldots, r$ and $\pi_0 \partial^2 \ell_r/\partial z^2) \neq 0$. Thus the theorem follows from corollary 1.■

Of course we have also constructed the primitive lift for each of these maps: this is determined by the flag of bundles

$$\mathrm{Hom}(\ell_0, \ell_1) \subset \ldots \subset \mathrm{Hom}(\ell_0, \sum_1^r \ell_i) \subset T^{1,0}.$$

## 4.2 Harmonic maps with isotropy order 1.

Here we will prove:

**Theorem 5** *When the construction of the previous subsection is repeated with $r = 0$ (i.e. we have $\mathbf{R}^2 \times \mathbf{C}^{n+1} = \ell_0 \oplus \ell_1$) the line bundle $\ell_0$ determines a nowhere conformal harmonic map $\psi_0 : \mathbf{R}^2 \to \mathbf{CP}^n$.*

**Proof.** The proof will be broken into two parts. First we show that $\psi_0$ is harmonic, then we show it is nowhere conformal.

*Harmonic.* We will show by direct computation that $A_0'$, the $(1,0)$-component of the second fundamental form of $\ell_0$, is a holomorphic map. First we compute the holomorphic sections of $\ell_0$ and then show that $A_0'$ maps these to holomorphic sections of $\ell_0^{\perp}$.

1. Let $e_0 : \mathbf{R}^2 \to \ell_0$ be the nowhere vanishing section given by $\tau^1(s_0)$ for $s_0 = \iota_a(\sigma_0)$ as described earlier. We compute those functions $\alpha(z, \bar{z})$ for which $\alpha e_0$ is a holomorphic section of $\ell_0$ i.e. for which

$$\pi_0(\frac{\partial}{\partial \bar{z}} \alpha e_0) = 0.$$

By lemma 7 we see that

$$\iota_a^{-1}(\frac{\partial s_0}{\partial \bar{z}}) = (\zeta f_0 + \frac{\partial f_0}{\partial \bar{z}})\theta_a$$

is a globally holomorphic section of $\mathcal{L}_0(P_\infty) \otimes L(a)$. Define $\sigma_n(a)$ to be a non-trivial global section of $\mathcal{L} \otimes L(a)$ with divisor at least $P_\infty + E_\infty$ (since $m = 1$ there is only a one dimensional space of these). Then $\lambda\sigma_n(a)$ is a non-trivial global section of $\mathcal{L}_0(P_\infty) \otimes L(a)$, independent of $\sigma_0(a)$ (recall that the latter is a non-trivial global section of $\mathcal{L}_0 \otimes L(a)$). It follows that we may write

$$\frac{\partial s_0}{\partial \bar{z}} = \beta s_0 + \tilde{\beta}\lambda s_n,$$

where $s_j = \iota_a(\sigma_j)$ and $\beta, \tilde{\beta}$ are functions of $z, \bar{z}$ only. This implies

$$\zeta f_0 + \frac{\partial f_0}{\partial \bar{z}} = \beta f_0 + \tilde{\beta}\lambda f_n.$$

Here $f_0$ has divisor $P_0 + E_0$ in $X_I$ whereas $\lambda f_n$ has divisor $2P_0 - P_\infty + E_0$ there. So if we set $B = (\zeta^{-1} f_0)(P_0)$ we see that

$$\beta = \frac{\partial}{\partial \bar{z}} \ln B.$$

Since $\pi_0(\partial e_0/\partial \bar{z}) = \beta e_0$ it follows that $\alpha e_0$ is a holomorphic section of $\ell_0$ precisely when

$$\alpha(z, \bar{z}) = A(z)B^{-1}$$

for any holomorphic function $A(z)$.

2. We now want to compute, as an intermediate step, $A_0'(e_0) = \pi_0^\perp(\partial e_0/\partial z)$. As above we see that

$$\iota_a^{-1}(\frac{\partial s_0}{\partial z}) = (\zeta^{-1} f_0 + \frac{\partial f_0}{\partial z})\theta_a = \tilde{\gamma}\sigma_0 + \gamma\sigma_1$$

where $\sigma_1(z, \bar{z})$ is a global section of $\mathcal{L}(-P_\infty - E_0) \otimes L(a)$ and $\gamma, \tilde{\gamma}$ are functions of $z, \bar{z}$ only. It follows that

$$\pi_0^\perp(\frac{\partial e_0}{\partial z}) = \gamma e_1, \tag{11}$$

where $e_1 = \tau^1 \circ \iota_a(\sigma_1)$. Now we compute $\gamma$. As usual, we write $s_1 = \iota_a(\sigma_1) = e^a f_1 \theta$, then

$$\zeta^{-1} f_0 + \frac{\partial f_0}{\partial z} = \tilde{\gamma} f_0 + \gamma f_1.$$

Since $f_0$ has divisor $P_0 + E_0$ and $f_1$ has divisor $P_\infty + E_0$ it follows that $\gamma = BC^{-1}$, where $C = f_1(P_0)$.

3. Finally we must show that, for all $\alpha = A(z)B^{-1}$, the section $A_0'(\alpha e_0)$ of $\ell_0^\perp$ is a holomorphic one. This means

$$\pi_0^\perp \frac{\partial}{\partial \bar{z}}(\alpha\gamma e_1) = 0$$

with $\gamma = BC^{-1}$ given by the previous step. As before

$$\iota_a^{-1}(\frac{\partial s_1}{\partial \bar{z}}) = (\zeta f_1 + \frac{\partial f_1}{\partial \bar{z}})\theta_a = \tilde{\delta}\sigma_0 + \delta\sigma_1$$

for some functions $\delta, \tilde{\delta}$ of $z, \bar{z}$ only. It follows that $\delta$ is given by

$$\frac{\partial f_1}{\partial \bar{z}}(P_0) = \delta f_1(P_0) \quad \text{i.e.} \quad \delta = \frac{\partial C}{\partial \bar{z}}C^{-1}.$$

Therefore

$$\pi_0^\perp \frac{\partial}{\partial \bar{z}}(\alpha\gamma e_1) = \frac{\partial C^{-1}}{\partial \bar{z}}e_1 + \frac{\partial C}{\partial \bar{z}}C^{-2}e_1 = 0$$

which completes the proof that $\psi_0$ is a harmonic map.

*Nowhere conformal.* The harmonic map $\psi_0$ is non-conformal at all the points $z, \bar{z}$ for which $\langle A_0', A_0''\rangle$ is non-zero, where $\langle\cdot,\cdot\rangle$ denotes the Hermitian inner product on $\mathrm{Hom}(\ell_0, \ell_0^\perp)$. This inner product is at each point the trace of the bundle map

$$T : \ell_0 \to \ell_0 \quad , \quad s \mapsto \pi_0 \frac{\partial}{\partial z}\pi_0^\perp \frac{\partial s}{\partial z}.$$

Our aim is to show that $T$ is the identity, whence its trace never vanishes.

Let $e_0 = \tau^1(s_0)$ be the generating section for $\ell_0$ used earlier, then for any $s = \mu(z,\bar{z})e_0$ we see from (11) that

$$\pi_0^\perp \frac{\partial s}{\partial z} = \mu BC^{-1}e_1.$$

Next, we want to compute $\partial e_1/\partial z$. Observe that

$$\iota_a^{-1}(\frac{\partial s_1}{\partial z}) = (\zeta^{-1}f_1 + \frac{\partial f_1}{\partial z})\theta_a$$

is a global section of $\mathcal{L}(P_0 - P_\infty - E_0) \otimes L(a)$. We know, from the proof of lemma 8 that $H^0(X, \mathcal{L}(P_0) \otimes L(a))$ is spanned by the sections $\sigma_1, \ldots, \sigma_n, \lambda^{-1}\sigma_0$ and the last is the only one which, as a section of $\mathcal{L} \otimes L(a)$, has a pole at $P_0$. Therefore there are functions $\mu_0, \ldots, \mu_n$ of $z, \bar{z}$ only for which

$$\zeta^{-1}f_1 + \frac{\partial f_1}{\partial z} = \sum_1^n \mu_j f_j + \mu_0\lambda^{-1}f_0.$$

Taking the residue of both sides at $P_0$ gives $f_1(P_0) = \mu_0(\zeta^{-1}f_0)(P_0)$ i.e. $\mu_0 = CB^{-1}$. Thus

$$\pi_0 \frac{\partial}{\partial z}e_1 = CB^{-1}e_0$$

so that

$$\begin{aligned} T(\mu e_0) &= \pi_0 \tfrac{\partial}{\partial z}(\mu BC^{-1}e_1) \\ &= \mu BC^{-1}\pi_0 \tfrac{\partial}{\partial z}e_1 \\ &= \mu e_0. \end{aligned}$$

Hence $T$ is the identity, as required.■

# 5 Harmonic maps into $S^n$.

Since every harmonic map from $\mathbf{R}^2$ into $\mathbf{RP}^n$ lifts to one into $S^n$ we can obtain harmonic maps into the spheres by placing extra conditions on the data $X, \mathcal{L}, \pi$ which give $H^0(X, \mathcal{L})$ a real structure for which the Hermitian inner product $h$ is real-valued on the subspace of real points. In view of the theory described in section 2 we wish to fix things so that the fibres of $\ell_0$ and $\ell_0^\perp$ are preserved by this real involution. The aim of this section is to briefly explain how this can be achieved: I hope to present a more thorough discussion of these conditions elsewhere.

## 5.1 Extra conditions on $X$ and $\mathcal{L}$.

In addition to the properties required of $X$ in section 3 we insist that it possesses another anti-holomorphic involution $\nu : X \to X$ which commutes with $\rho$, covers $\rho : \lambda \to \bar{\lambda}^{-1}$ on $\mathbf{P}_\lambda$ and maps $P_0$ to $P_\infty$. Equivalently, we could ask for $X$ to have a holomorphic involution $\mu$ covering the identity, since $\mu = \nu \circ \rho$ has this property. Since $\mu$ fixes fibres of $\pi$ we deduce that: (i) $X$ is reducible whenever $n+1$ is odd, (ii) $\mu_* R = R$, (iii) $m+1$ must be even, so we write (for this section) $m+1 = 2r+2$. It follows that near $P_0$ (or $P_\infty$) we will have $\mu : \zeta \mapsto -\zeta$. When we come to construct harmonic maps, $m$ will be the isotropy order of the map into $\mathbf{CP}^n$ and $r$ will be the isotropy dimension for the corresponding map into $S^n$.

The line bundle $\mathcal{L}$ will be required to satisfy another condition in addition to the equation (5). Recall that $\mathcal{L}_0 = \mathcal{L}(-mP_0 - E_0)$ and this is the line bundle for a unique positive divisor $D_0$. We require $\mathcal{L}_0 \simeq \overline{\nu_* \mathcal{L}_0}$, which implies $\bar{\nu}_* D_0 = D_0$, and we fix an isomorphism between these line bundles. Observe that on the 2-parameter subspace $\{a(z, \bar{z}) = z\zeta^{-1} - \bar{z}\zeta\} \subset \mathcal{G}_R$ we have $\overline{\nu_* a} = a$ so that

$$\overline{\nu_* \mathcal{L}_0} \simeq \mathcal{L}_0 \Rightarrow \nu_* \overline{(\mathcal{L}_0 \otimes L(a))} \simeq \mathcal{L}_0 \otimes L(a).$$

Clearly $L(a(z, \bar{z}))$ moves along the intersection of the Prymian

$$\mathrm{Prym}(X, \mu) = \{L \in J(X) : \mu_* L \simeq L^{-1}\}$$

with the real subgroup $J_R(X)$.

Since $\mathcal{L} = \mathcal{L}_0(mP_0 + E_0)$ we obtain a conjugate linear map

$$\bar{\nu}_* : \Gamma(X_A, \mathcal{L}) \to \Gamma(X_A, \mathcal{L}) \quad , \quad \sigma \mapsto \overline{\nu_* \sigma}$$

which has the property that, for any $b \in B$, $\bar{\nu}_*(b\sigma) = \overline{\rho_* b}.\bar{\nu}_*(\sigma)$. With this we define the bilinear map

$$k : \Gamma(X_A, \mathcal{L}) \times \Gamma(X_A, \mathcal{L}) \to B \quad , \quad k(v, w) = h(v, \bar{\nu}_* w).$$

**Lemma 9** *The map $k$ is a symmetric $B$-bilinear form.*

**Proof.** That $k$ is bilinear is immediate. To demonstrate its symmetry we must show that $\mu_* f = f$ where (from the definition of $h$) $k(v, w) = \mathrm{Tr}(f.v \otimes \mu_* w)$.

But recall that $f$ has divisor $(f) = R - D - \rho_* D$ and $D = D_0 + mP_0 + E_0$. It follows that $f$ has neither poles nor zeroes at $P_0$ and $P_\infty$ and we can write $(f) = R_A - D_0 - E_0 - \rho_* D_0 - E_\infty$, where $R_A$ is the restriction of $R$ to $X_A$. But since $\mu_*$ preserves $R_A, E_0, E_\infty$ and $\rho\mu_* D_0 = D_0$ we have

$$(\mu_* f) = R_A - \mu_* D_0 - E_0 - D_0 - E_\infty = (f).$$

Hence $\mu_* f . f^{-1}$ is a constant. Now we evaluate this at $P_0$ to see that $\mu_* f . f^{-1} = 1$.■

As an immediate corollary of this we find

$$h(\bar{\nu}_* v, \bar{\nu}_* w) = \overline{\rho_* h(v, w)}. \tag{12}$$

We want to restrict everything to the finite dimensional subspace $H^0(X, \mathcal{L})$, but this is not preserved by $\bar{\nu}_*$ so we cannot do this directly. However, from lemma 6 we deduce that $H^0(X, \mathcal{L})$ has an $h$-orthogonal decomposition into the sum of subspaces $H^0(X, \mathcal{L}_0) \oplus H^0(X, \mathcal{L}(-P_\infty))$ and it is a straightforward calculation to check that

$$\lambda^{-1}\bar{\nu}_* : H^0(X, \mathcal{L}(-P_\infty)) \to H^0(X, \mathcal{L}(-P_\infty)). \tag{13}$$

With this observation we may prove:

**Lemma 10** *$\Gamma(X_A, \mathcal{L})$ has a $B$-module basis $\{\tau_j\} \subset H^0(X, \mathcal{L})$ for which $\bar{\nu}_* \tau_0 = \tau_0$, $\lambda^{-1}\bar{\nu}_* \tau_j = \tau_j$ (for $j \neq 0$) and $h(\tau_j, \tau_k) = \delta_{jk}$.*

**Proof.** Since $H^0(X, \mathcal{L}_0)$ is a one dimensional space preserved by $\bar{\nu}_*$ it has a unique element $\tau_0$ which is real for $\bar{\nu}_*$ and has $h(\tau_0, \tau_0) = 1$ (observe that by (12) we know that this must be a real number). Similarly, we can find $\tau_j \in H^0(X, \mathcal{L}(-P_\infty))$, for $j = 1, \ldots, n$, in the subspace fixed by $\lambda^{-1}\bar{\nu}_*$ satisfying the conditions required, since $h$ is positive definite and real-valued on this real subspace (to see this latter fact observe that $h(\lambda^{-1}\bar{\nu}_* v, \lambda^{-1}\bar{\nu}_* w) = h(\bar{\nu}_* v, \bar{\nu}_* w)$ and use (12)). ■

Let us fix an $h$-orthonormal $B$-module basis $\{\tau_j\}$ of the type in the lemma. With respect to this basis for $H^0(X, \mathcal{L})$ we obtain the map $\tau^1 : \Gamma(X_A, \mathcal{L}) \to \mathbf{C}^{n+1}$ defined by (8), which has the added property that

$$\tau^1(\bar{\nu}_* \sigma) = \overline{\tau^1(\sigma)}.$$

To ensure that the reality conditions are preserved by the parallel transport we must have $\overline{\nu_* \theta_A} = \theta_A$. This is always achievable by rescaling $\theta_A$ by a unimodular scalar (in which case the condition $\overline{\rho_* \theta_A} = \theta_A^{-1}$ is preserved).

As earlier we define $\ell_0$ to be the line subbundle of $\mathbf{R}^2 \times \mathbf{C}^{n+1}$ given by the image of $\iota_a(H^0(X, \mathcal{L}_0 \otimes L(a)))$ under $\tau^1$. Then $\ell_0 = \bar{\ell}_0$ and it therefore determines a map into $\mathbf{RP}^n$ which, by theorems 4 and 5, must be harmonic of isotropy order $m = 2r + 1$. Now let $\sigma_0 : \mathbf{R}^2 \to H^0(X, \mathcal{L}_0 \otimes L(a))$ be nowhere vanishing and set $e_0 = \tau^1 \circ \iota_a(\sigma_0)$, then its normalisation $\hat{e}_0 = e_0 / \|e_0\|$ maps into $S^n$. From the discussion in section 2 we deduce:

**Theorem 6** *The map $\phi_0 : \mathbf{R}^2 \to S^n$ given by $\phi_0(z, \bar{z}) = \hat{e}_0(z, \bar{z})$ is harmonic of isotropy dimension $r$.*

In the next section we will see how to construct the primitive lifts for these maps when $r > 0$.

## 5.2 Primitive lifts.

As earlier, set $\mathcal{L}_j = \mathcal{L}_0(jP_0 - jP_\infty)$ for $j = 1, \ldots, r+1$ and define

$$\mathcal{L}_{-j} = \mathcal{L}_0(jP_\infty - jP_0) = \overline{\nu_* \mathcal{L}_j},$$

for $j = 1, \ldots, r$. It is clear that $\mathcal{L}_{-j}$ is not an ideal subsheaf of $\mathcal{L}$, however it is isomorphic to the ideal sheaf $\mathcal{L}(-(j-1)P_0 - (m+1-j)P_\infty - E_\infty)$, with the isomorphism given by $\sigma \to \lambda^{-1}\sigma$. We will write this latter line bundle simply as $\mathcal{L}_{-j}(\lambda^{-1})$. Now suppose $\sigma_{-j} : \mathbf{R}^2 \to H^0(X, \mathcal{L}_{-j}(\lambda^{-1}) \otimes L(a))$ is a smooth family of non-vanishing sections and, for $-r \leq j \leq r+1$, set $s_j = \iota_a(\sigma_j)$. We may define $V_j = B\langle s_j \rangle$, whereupon we have a conjugate linear bijection of $B$-modules

$$\bar{\nu}_* : V_j(z, \bar{z}) \to V_{-j}(z, \bar{z}), \quad j = 1, \ldots, r.$$

Let us show that:

**Lemma 11** *For $j = 0, \ldots, r+1$ and $k = 1, \ldots, r$ we have $h(s_j, s_{-k}) = 0$ whenever $j \neq k$.*

**Proof.** We prove this in much the same way as lemma 6. For each $z, \bar{z}$, $s_{-k}$ equals $\iota_a(\sigma_{-k})$ for a holomorphic section $\sigma_{-k}$ of $\mathcal{L} \otimes L(a)$. Thus $h(s_j, s_{-k})$ is a constant, by lemma 4. As in the proof of lemma 6, we can write $h(s_j, s_{-k})$ as $\mathrm{Tr}(r_{jk})$, where this time

$$r_{jk} = f.\theta \otimes \overline{\rho_* \theta}.f_j.\overline{\rho_* f_{-k}}$$

for $f_{-k} = \sigma_{-k}\theta_a^{-1}$. It follows that each meromorphic function $r_{jk}$ has divisor at least $(m+1-j-k)P_0 + 2E_0$ and therefore $\mathrm{Tr}(r_{jk})$ vanishes at $\lambda = 0$ whenever $m \geq k+j$ i.e. $2r+1 \geq k+j$, which is always the case for these ranges of $k$ and $j$.■

Now define

$$\ell_j = \tau^1(V_j) \quad , \quad -r \leq j \leq r.$$

Then all these line bundles are mutually orthogonal and $\bar{\ell}_j = \ell_{-j}$. We also define

$$\ell_{r+1} = (\sum_{j=-r}^{r} \ell_j)^\perp$$

and observe that $\tau^1(V_{r+1}) \subset \ell_{r+1}$. From lemma 8 we learn that

$$\frac{\partial}{\partial z}\ell_j \subset \ell_j \oplus \ell_{j+1} \text{ for } j = 0, \ldots, r.$$

We already have a conformal harmonic map $\phi_0 : \mathbf{R}^2 \to S^n$ derived from the line bundle $\ell_0$. In view of the discussion earlier its primitive lift is determined by the flag of bundles

$$\mathrm{Hom}(\ell_0, \ell_1) \subset \ldots \subset \mathrm{Hom}(\ell_0, \sum_1^r \ell_j) \subset \mathrm{Hom}(\ell_0, \ell_0^\perp) \simeq \phi_0^{-1} T^C(S^n).$$

# 6 Harmonic tori.

In the preceding sections we have seen that, with the appropriate choice of data $X, \pi, \mathcal{L}, \theta$ and the basis $\{\tau_j\}$, we can construct harmonic maps from $\mathbf{R}^2$ into $\mathbf{CP}^n$ or $S^n$. The aim of this section is to describe conditions sufficient for these maps to be doubly periodic. In fact we will see that these conditions only depend upon the choice of $X, \pi$ and deduce from this that the tori we construct come in families parameterised by the $J_R(X)$-orbit of $\mathcal{L}$ (recall that a change of $\theta$ affects nothing in the construction, and a change of $\{\tau_j\}$ only alters $\psi_0$ by an isometry). Throughout this section I will simplify the discussion by assuming that $X$ is a smooth curve. It is possible to deal with singular curves (cf. [20] for a discussion relevant to superconformal tori) but technical problems arise which make the discussion more involved.

## 6.1 Sufficient conditions for tori.

We must begin by assuming that $X$ admits solutions to (5). To this end we will assume that the ramification divisor $R$ is of the form $R_0 + \rho_* R_0$ for some positive divisor $R_0$ (cf. section 3: in the case where $m = n$ it was shown in [20] that this is a necessary and sufficient condition for (5) to have solutions). Given this, let us address the question of the periodicity of the line bundle $\ell_0$ from which we derive our harmonic maps.

Recall $\ell_0$ has generating sections of the form $e_0 = \tau^1(s_0)$ where $s_0 = \iota_a(\sigma_0)$ for some non-vanishing family $\sigma_0(a)$ of global sections of $\mathcal{L}_0 \otimes L(a)$. Let us fix a global section $\sigma = \sigma_0(0) = s_0(0)$ of $\mathcal{L}_0$, then we can make a particular choice for $s_0(z)$ which is uniquely determined by the behaviour of the function $t_0 = s_0\sigma^{-1}$ in a neighbourhood of $P_0$. The function we obtain is the natural analogue of the 'scalar Baker function' used in [20] so we will also call $t_0$ the scalar Baker function for $X, \pi, \mathcal{L}$. Observe that $t_0(z; P)$ is a meromorphic function throughout $X_A$ for each $z \in \mathbf{C}$ and has divisor of poles given by the unique positive divisor in the class of $\mathcal{L}_0$.

**Lemma 12** *When $\sigma$ does not vanish at $P_0$ there is a unique choice of $s_0$ which is normalised so that the function $t_0 = s_0\sigma^{-1}$ has Fourier series about $P_0$ of the form $\exp(z\zeta^{-1})(1 + O(\zeta))$. Moreover, $t_0$ is uniquely determined by $X, \pi, \mathcal{L}$.*

**Proof.** It is trivial to see that this normalisation fixes $s_0$ with respect to $\sigma_0$, since $H^0(X, \mathcal{L} \otimes L(a))$ is one dimensional. Now observe that if we rescale $\sigma$ then to normalise we also have to rescale $s_0$ accordingly, so that the ratio $t_0$ remains unchanged.

**Remark.** It is possible to write down an explicit formula for $t_0$ using the Riemann $\theta$-function for $X$ (cf. for example [10]) but we will not use this formula. However, it will be useful to recognise that $t_0$ only depends upon $X, P_0, P_\infty, \mathcal{L}$ and $\partial/\partial\zeta$. In the case where $\sigma$ does vanish at $P_0$ we could equally well fix on a normalisation for $t_0$ but this case need not be addressed.

It follows that the section $s_0$ is independent of the trivialisation $\theta$ (which we can also see directly from the definition). Similarly, the extension of $s_0$ to all of

$\mathcal{G}_R$ may be normalised so that $s_0(a)\sigma^{-1} \equiv t_0(a)$ has Fourier expansion of the form $\exp(a_0(\zeta))(1 + O(\zeta^{-1}))$ about $P_0$, where $a_0(\zeta)$ denotes the restriction of $a$ to a neighbourhood of $P_0$. Moreover, for each $b \in \mathcal{G}_R$ we can replace $\mathcal{L}$ by $\mathcal{L} \otimes L(b)$ to obtain the scalar Baker function $t_b(a)$. I claim that this function is given by

$$t_b(a) = \iota_b^{-1}(s_0(a+b))\sigma_0(b)^{-1}.$$

To prove this claim we only have to check that this has the correct Fourier expansion about $P_0$. But

$$\begin{aligned} t_b(a) &= e^{-b}[e^{a+b}\sigma_0(a+b)\theta_{a+b}^{-1}\theta]\theta_b\theta^{-1}\sigma_0(b)^{-1} \\ &= s_0(a+b)/s_0(b) \\ &= t_0(a+b)/t_0(b) \end{aligned}$$

which has the correct Fourier expansion about $P_0$. Moreover, we learn from this that

$$t_0(a+b) = t_0(b)t_b(a). \tag{14}$$

In particular, since each scalar Baker function only depends upon the line bundle, whenever $\mathrm{L}(b) \simeq \mathcal{O}_X$ we have

$$t_0(a+b) = t_0(b)t_0(a). \tag{15}$$

Now we are in position to write down conditions which are together sufficient to guarantee that there exists a $\xi \in \mathbf{C}$ for which $\ell_0(z+\xi) = \ell_0(z)$ for all $z \in \mathbf{C}$.

**6a.** There exists $\xi \in \mathbf{C}$ for which $L(a(\xi)) \simeq \mathcal{O}_X$.

**6b.** Given $\xi$ there exists a non-zero constant $k$ for which $t_0(\xi; O_j) = k$ at all points $O_0, \ldots, O_n$ in $\pi^{-1}(1)$.

Let me explain this second condition. When it holds it follows from (15) that $s_0(z+\xi, O_j) = k.s_0(z; O_j)$ for all $z \in \mathbf{C}$. We write $s_0(z; P) = \sum s_{0j}(z;\lambda)\tau_j$ and deduce that $s_{0j}(z+\xi; 1) = k.s_{0j}(z; 1)$ for $j = 0, \ldots, n$. Hence the line bundle $\ell_0$ generated by $\tau^1(s_0)$ is periodic with period $\xi$.

**Remark.** These conditions are in fact necessary and sufficient to obtain maps with period precisely $\xi$. We will see later (prop. 7) that 6a is necessary and from this the necessity of 6b follows easily.

**Lemma 13** *The conditions 6a and 6b only depend upon the choice of $X, \pi$ and not the choice of $\mathcal{L}$.*

**Proof.** Condition 6a is clearly a condition on $X, \pi$ only: in fact the existence of some period $\xi$ depends only on $X, P_0, P_\infty$ (cf. lemma 14 below). Now suppose that $t_0$ satisfies 6b. For any choice $b \in \mathcal{G}_R$ we see from (14) that $t_0(a(\xi)+b) = t_0(b).t_b(a(\xi))$, whereas by (15) this equals $t_0(a(\xi)).t_0(b)$. Therefore $t_0(\xi; P) = t_b(\xi; P)$ throughout $X_A$ so that $t_b$ also satisfies 6b. ■

**Corollary 3** *Every harmonic 2-torus arising from the pair $X, \pi$ comes in a family of tori arising from the same pair and parameterised by $J_R(X)$.*

**Remarks:** **(i)** Observe however that two of these parameters arise from the conformal automorphisms of the torus, so that in effect we only expect distinct tori in the families when $p > 2$. Since the periods are unaffected by the action of $J_R(X)$ all these deformations preserve the conformal structure of the torus.
**(ii)** Notice that in fact the conditions 6b only depend upon $X, P_0, P_\infty, \partial/\partial\zeta$ and the fibre $\pi^{-1}(1)$. So it is possible that $X, \pi$ and $X, \pi'$ have the same periodicity equations. This happens whenever $\pi$ and $\pi'$ have the same fibre over $\lambda = 1$. However, the question as to when this occurs is a little delicate: it requires knowledge of the complete linear system $[O_0 + \ldots + O_n - (m+1)P_\infty]$. For simplicity I will henceforth assume that this linear system has zero dimension (i.e. $\pi$ is uniquely determined by this); for $p > n - m$ the general pair $X, \pi$ should have this property. This assumption simplifies the parameter count which follows.

## 6.2 Tori of isotropy order $m$.

In this section we will write down the equations which must be satisfied to obtain tori and count the number of degrees of freedom which are available to satsify them. From this naive parameter count we can estimate when we expect to find tori of isotropy order $m$ in $\mathbf{CP}^n$. The task of producing a rigorous investigation of the moduli space of tori is beyond the scope of this paper and I hope to return to it elsewhere.

We begin by counting the dimension of the moduli space of suitable pairs $X, \pi$. Each pair is one of a finite number determined by the branch divisor $\pi_* R$ on $\mathbf{P}_\lambda$. The constraints on the divisor $R$ are that it must be $\rho_*$-invariant, have degree $m$ at $P_0$ and $P_\infty$ (which are fixed over $\lambda = 0$ and $\infty$) and $R$ must admit a solution to (5). The first two of these, when combined with the Riemann-Hurwitz formula, tell us that in the choice of branch divisor $\pi_* R$ we have at most $2(p + n - m)$ free real parameters.

We want to interpret 6a and 6b as equations on the moduli space of pairs $X, \pi$. For 6a we can argue exactly as in [20] to get the following characterisation.

**Lemma 14** *When $X$ has (arithmetic) genus $p \geq 2$ there exist two independent periods for $L(a(z))$ precisely when there are two independent real periods $\alpha_1, \alpha_2 \in H^1(X, \mathbf{Z})$ for which*

$$\langle \alpha_j, \omega \rangle = 0 \tag{16}$$

*for all real holomorphic differentials $\omega$ in $H^0(X, \Omega^{1,0}(-P_0 - P_\infty))$. This gives a system of $2p - 4$ real equations in the moduli of $X, P_0, P_\infty$.*

Here we are making use of the surjective linear map $\mathcal{G} \to H^1(X, \mathcal{O}_X)$ through which the epimorphism $L : \mathcal{G} \to J(X)$ factors. An element of $H^1(X, \mathcal{O}_X)$ is real if it lies in the image of $\mathcal{G}_R$. Therefore the real periods are the periods of $J_R(X)$. The pairing $\langle \alpha, \omega \rangle$ is by Serre duality and a holomorphic differential $\omega$ is real whenever $\rho^* \omega = -\bar{\omega}$. The condition (16) ensures that each solution $\alpha_j$ lies in the image of $\{z\zeta^{-1} - \bar{z}\zeta : z \in \mathbf{C}\}$.

When we put the two types of conditions together, a pair $X, \pi$ will correspond to a harmonic 2-torus whenever there are two independent solutions of (16) for which

$$t_0(\xi_i; O_j)/t_0(\xi_i; O_0) = 1 \tag{17}$$

for each period $\xi_1, \xi_2$ of $L(a(z))$ and each $O_j \in \pi^{-1}(1)$. [Here we are assuming we want the periods to be precisely $\xi_1, \xi_2$, which need not be be the case if we are looking for tori of *some* period. However, the parameter count will be the same in either case, and this is all we are presently interested in.] The next lemma shows that these give a system of real equations for each period $\xi_i$.

**Lemma 15** $t_0(\xi_i; O_j)/t_0(\xi_i; O_0) = \overline{t_0(\xi_i; \rho O_0)}/\overline{t_0(\xi_i; \rho O_j)}$

**Proof.** To simplify notation, temporarily set $q(P) = t_0(\xi; P)$ for some period $\xi$. We will show that $q(P)\overline{q(\rho P)}$ is a (positive) constant throughout $X_A$, from which the lemma follows.

Observe that $\sigma_0(\xi)\sigma^{-1}$ is a global holomorphic section of $L(\xi) \simeq \mathcal{O}_X$ and is therefore constant. Now recall from section 3 that we can write $s_0(\xi) = \sigma_0(\xi)\theta_A^{-1}$ where $\theta_A \otimes \overline{\rho_* \theta_A} = 1$. Hence $s_0 \otimes \overline{\rho_* s_0}(\sigma \otimes \overline{\rho_* \sigma})^{-1}$ is constant on $X_A$ when $s_0$ is evaluated at $z = \xi$ i.e. $q(P)\overline{q(\rho P)}$ is constant.■

Observe that if every point $O_j$ is $\rho$-fixed then the ratio in (17) is unimodular. With this in hand we can estimate for which isotropy orders we expect to have parameters free after satisfying both sets of periodicity conditions.

**Proposition 3** *The net parameter count for tori of isotropy order $m$ with a spectral curve of genus $p \geq 2$ is non-negative whenever $m \leq 2 + \min([n/2], p + n - m)$ (where $[n/2]$ denotes the greatest integer not exceeding $n/2$).*

**Proof.** By the previous lemma there are $2p - 4$ conditions to be satisfied to get independent periods $\xi_1, \xi_2$ for $L(a(z))$. With these two periods there are at most $2n$ more conditions given by (17). However, if we take $X, \pi$ to be branched over $\lambda = 1$ this decreases both the number of conditions and the number of free parameters, which may improve the net parameter count. So suppose $\pi^{-1}(1)$ consists of $l + 1 \leq n + 1$ points. This puts $X, \pi$ inside a subspace of dimension $2(p + n - m) - 2k$ of the moduli space in which $\pi_* R$ has degree $2k$ at $\lambda = 1$ (the degree is even since we assume $R = R_0 + \rho_* R_0$ for some positive divisor $R_0$). Let each point $O_j$ lying over $\lambda = 1$ have ramification index $d_j$, then $2k = \sum_0^l d_j$ so that

$$n + 1 = \sum_0^l (d_j + 1) = 2k + l + 1.$$

In this subspace the two systems of equations (16) and (17) require at least $2p - 4 + 2l$ free parameters to be satisfied and we have $2(p + n - m) - 2k$ parameters available. So we must have $m \leq n + 2 - l - k = 2 + k$. But $2k \leq \min(n, 2p + 2n - 2m)$, which gives us the result.■

Although much needs to be done before we can make assertions about the existence or non-existence of tori for general choices of $p$, $n$ and $m$ we can immediately

deduce the existence of one family of examples of tori. This is the family for which $p = 2$, $n$ is even and $X$ has a ramification point of index $n$ over $\lambda = 1$ i.e. $2k = n$. In this case both of the periodicity conditions 6a,b are automatically satisfied and we have tori of isotropy order $m \leq 2 + n/2$ (cf. [20] for a discussion of this when $m = n = 2$ or 4).

The cases where $p = 0$ or 1 must be treated individually, since lemma 14 does not apply to them. I will not say much about them here; I hope to return to this study elsewhere.

**The case p=0.** In this case $L(z) \simeq \mathcal{O}_X$ for all $z \in \mathbf{C}$, from which we deduce that the map $\psi_0 : \mathbf{R}^2 \to \mathbf{CP}^n$ is $\mathbf{R}^2$-equivariant (i.e. it is framed by a homomorphism $\mathbf{R}^2 \to U_{n+1}$). In fact all $\mathbf{R}^2$-equivariant maps are of finite type (F E Burstall: private communication) and we will see later that this means they all have a spectral curve of genus zero (cf. prop. 9). Therefore from the parameter count above we expect the moduli space of $\mathbf{R}^2$-equivariant conformal maps to have dimension $2n-4$, which agrees with the dimension found by Kenmotsu [17] (see also [3]).

The double periodicity conditions may be written down fairly explicitly in this case, since it is possible to say precisely what $t_0(z;P)$ is. Let $\mu(P)$ denote the unique rational function on $X$ with divisor $P_\infty - P_0$ and normalised by $res_{P_0}\mu = 1$. Define $c = (d\zeta\mu/d\zeta)(0)$, then

$$t_0(z;P) = \exp(z(\mu - c) - \bar{z}\overline{\mu(\rho P)})$$

is the scalar Baker function for $X$. Therefore the double periodicity conditions (17) are satisfied whenever we can find independent $\xi_1, \xi_2 \in \mathbf{C}$ for which

$$\exp(\xi_k(\mu_j - \mu_0) - \bar{\xi}_k(\bar{\mu}_j - \bar{\mu}_0)) = 1,$$

where $\mu_j = \mu(O_j)$ and we have assumed $\rho O_j = O_j$ for each $j = 0, \ldots, n$. If we write each complex number $\mu_j - \mu_0$ as $c_j + is_j$ and $\xi_k = x_k + iy_k$ this simplifies to

$$s_j x_k + c_j y_k \equiv 0 \bmod \pi\,. \tag{18}$$

These equations are clearly independent of the absolute values $|\mu_j - \mu_0|$, so that whenever solutions to (18) can be found we might expect whole families of tori, depending upon how these absolute values vary with the moduli for $(X, \pi)$.

In fact, using the parameterisation of the moduli space from [17], Jensen & Liao [16] have shown that, in a neighbourhood of the Clifford solution, there are $n-4$-dimensional families of conformal tori for $n > 4$. The 'Clifford solution' corresponds to the unique case $\mu = \zeta^{-1}$ (i.e. where all the points $P_j$ equal $P_0$). We can think of a neighbourhood of this pair $(X, \pi)$ as obtained by taking all the $P_j$ to be close to $P_0$. The results of [16] show that there are $n-4$ degrees of freedom available to vary the values $|\mu_j - \mu_0|$ in this neighbourhood. Notice that in this neighbourhood there is no branch point at $\lambda = 1$, so that (17) has the maximum number of independent equations.

**The case p=1.** In this case $J(X) \simeq X$ (since $X$ is smooth) and a little thought shows that the homomorphism $L : \mathbf{R}^2 \to J_R(X)$ has one fixed period and an invariant direction (from this we can deduce that the map $\psi_0$ is $\mathbf{R}$-equivariant). The

equations (17) have a different character depending upon whether $\xi$ is the fixed period or lies in the invariant direction. In the latter case $\log(t_0(r\xi;P))$ is linear in the real variable $r$, so that the ratio in (17) may be independent of some of the moduli of $(X,\pi)$ (as in the case $p=0$). For this reason we can say little of use regarding the number of free parameters without a deeper investigation of the scalar Baker function.

# 7 Recovery of the spectral curve from the harmonic torus.

The aim of this section is to recover the data $X,\pi,\mathcal{L}$ from a given non-isotropic harmonic torus $\psi_0 : T^2 \to \mathbf{CP}^n$. Throughout this section it will be convenient to work with $G = U_{n+1}$ and $\mathfrak{g} = u_{n+1}$ so that the latter contains the identity matrix. The results we will use from section 2 do not change after this substitution.

## 7.1 Frames and loop groups.

Let $\psi : T^2 \to G/H$ be the unique primitive lift for $\psi_0$ (if $\psi_0$ is non-conformal we simply set $\psi = \psi_0$: 'primitive' will simply mean 'harmonic' for $m = 1$). We may frame this by $\Phi : \mathbf{R}^2 \to G$ over the cover $\mathbf{R}^2 \to T^2$ and normalise the frame so that $\Phi(0) = Id$. Set $\alpha = \Phi^{-1}d\Phi$, which is the pullback of the Maurer-Cartan form by $\Phi$, and write

$$\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{p}}$$

using the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ given earlier. One knows (from e.g. [7]) that if we define

$$\alpha_\zeta = \zeta^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{h}} + \zeta\alpha''_{\mathfrak{p}} \quad , \quad \zeta \in \mathbf{C}^*$$

then

$$d\alpha_\zeta + \frac{1}{2}[\alpha_\zeta \wedge \alpha_\zeta] = 0.$$

Moreover, since $\alpha'_{\mathfrak{p}}$ takes values in $\mathfrak{g}_{-1}$ this map $\alpha_\zeta : \mathbf{C}^* \times \mathbf{R}^2 \to \mathfrak{g}^{\mathbf{C}}$ is *$\nu$-equivariant* in $\zeta$ i.e. $\nu(\alpha_\zeta) = \alpha_{\omega\zeta}$ (recall $\omega$ is a primitive $m+1$-st root of unity). It follows that we can integrate to find an *extended frame* $\Phi_\zeta : \mathbf{C}^* \times \mathbf{R}^2 \to G^{\mathbf{C}}$ which satisfies

$$\alpha_\zeta = \Phi_\zeta^{-1}d\Phi_\zeta.$$

Moreover, we may always choose the constants of integration so that $\Phi_\zeta$ is $\nu$-equivariant and $\Phi_{1/\bar{\zeta}} = \Phi_\zeta^{\dagger -1}$, where '$\dagger$' denotes Hermitian transpose. We wish to view $\Phi_\zeta$ as a map from $\mathbf{R}^2$ into a certain loop group, defined as follows.

Let $C$ be the union of circles $C_1$ and $C_2$ in the $\zeta$-plane of radii $\epsilon$ and $\epsilon^{-1}$ respectively, where $0 < \epsilon < 1$. Define

$$\Lambda_C(G^{\mathbf{C}},\nu) = \{\text{real-analytic } g : C \to G^{\mathbf{C}} | \nu(g(\zeta)) = g(\omega\zeta)\}$$

and write $g_i = g|C_i$ for $i = 1, 2$. On this group there is an antiholomorphic involution given by

$$g \mapsto \bar{g} = g(\bar{\zeta}^{-1})^{\dagger - 1}$$

We will denote the fixed point subgroup of this involution by $\Lambda(G, \nu)$: it is this group (for any choice of $\epsilon$) that $\Phi_\zeta$ takes values in.

The crucial fact about this subgroup is that is admits an 'Iwasawa decomposition' in the following sense. First recall that $H^{\mathbf{C}}$ has an Iwasawa decomposition, which we will write as $H^{\mathbf{C}} = H.B_0$, inherited from $G^{\mathbf{C}}$. Here $B_0$ is a subgroup of upper triangular matrices with positive real diagonal entries and $B_0 = \exp(\mathfrak{b}_0)$, where $\mathfrak{h}^{\mathbf{C}} = \mathfrak{h} + \mathfrak{b}_0$ is the corresponding Lie algebra decomposition. Next, view $C$ as a pair of circles on the $\zeta$-sphere $\mathbf{P}_\zeta$ so that $C$ is the common boundary for a closed annulus $E$ and a union $I$ of closed discs. Now we can define the following three subgroups of $\Lambda(G, \nu)$:

$$\begin{aligned} \Lambda_E &= \{g \in \Lambda(G,\nu) : \text{boundaries of holo. maps } g : E \to G^{\mathbf{C}}\}, \\ \Lambda_I &= \{g \in \Lambda(G,\nu) : \text{boundaries of holo. maps } I \to G^{\mathbf{C}} \text{ with } g(0) = Id\}, \\ B &= \{(b, \bar{b}) : b \in B_0\}. \end{aligned}$$

Then it follows from a result in [18, 19] (essentially theorem 3 stated earlier) that every element $g \in \Lambda(G, \nu)$ has a unique factorisation $g = ubn$ where $u \in \Lambda_E$, $b \in B$ and $n \in \Lambda_I$. We will refer to this as the *Iwasawa decomposition* for $\Lambda(G, \nu)$. The corresponding Lie algebra decomposition is denoted by

$$\Lambda(\mathfrak{g}, \nu) = \Lambda_E(\mathfrak{g}, \nu) + \mathfrak{b} + \Lambda_I(\mathfrak{g}, \nu). \tag{19}$$

## 7.2 Extended frames and dressing orbits.

One of the results we will need from [8] is that, up to isometry, every non-isotropic torus possesses an extended frame belonging to a particular class which we will now describe. In the terminology of [8], these are the extended frames which lie in the dressing orbit of a vacuum solution.

For any positive integer $k$ let $\Lambda_k \subset \Lambda_E(\mathfrak{g}, \nu)$ denote the subspace of Laurent polynomials in $\zeta$ of degree $\leq k$. We define

$$\Delta = \Lambda_1 + \mathfrak{b} + \Lambda_I(\mathfrak{g}, \nu) \subset \Lambda(\mathfrak{g}, \nu).$$

Let $\Lambda_{I,B}$ denote the subgroup of $\Lambda(G, \nu)$ generated by the subgroups $B$ and $\Lambda_I$, then $\Lambda_{I,B}$ acts adjointly on the subspace $\Delta$. For $(\xi, \bar{\xi}) \in \Delta$ define

$$e(\xi) = \exp[(z\xi, \bar{z}\bar{\xi})]$$

which describes a 2-parameter subgroup of $\Lambda(G, \nu)$. Using the Iwasawa decomposition in $\Lambda(G, \nu)$ we write

$$e(\xi) = \Phi(\xi) b(\xi) n(\xi) \tag{20}$$

and observe that $\Phi(\xi)$ equals the identity at $z = 0$. It can be shown quite readily (cf. [8]) that $\Phi(\xi)$ is the extended frame for some primitive harmonic map of $\mathbf{R}^2$

into $G/H$. In fact if we use $\mathcal{F}$ to denote the set of all normalised extended frames (i.e. those with $\Phi_\zeta(0) = Id$) for primitive harmonic maps $\mathbf{R}^2 \to G/H$ then we have defined a map $\Phi : \Delta \to \mathcal{F}$. Now following [8] we observe that, for any $g \in \Lambda_{I,B}$,

$$\Phi(\mathrm{Ad}g.\xi) = g\#\Phi(\xi)$$

where $g\#\Phi(\xi)$ is the $\Lambda_E$-factor in the Iwasawa decomposition of $g\Phi(\xi)$. It is not hard to show that the latter defines an action of $\Lambda_{I,B}$ on $\mathcal{F}$, called the *dressing action*, and the equation shows that $\Phi$ intertwines the two actions. Now let $\mathcal{H}$ denote the space of all primitive maps $\mathbf{R}^2 \to G/H$ based by $\psi(0) = H$. Then $\mathcal{H} \simeq \mathcal{F}/C^\infty(\mathbf{R}^2, H)$ i.e. the space of based primitive maps is the quotient of $\mathcal{F}$ by the group of gauge transformations. From [8] we learn that the dressing action descends to $\mathcal{H}$: we will denote the gauge equivalence class of $\Phi(\xi)$ by $[\Phi(\xi)]$ and then $g\#[\Phi(\xi)] = [g\#\Phi(\xi)]$. The orbit $O_\xi = \{g\#[\Phi(\xi)] : g \in \Lambda_{I,B}\} \subset \mathcal{H}$ is called the *dressing orbit* of $[\Phi(\xi)]$. One of the principal results of [8] is:

**Theorem 7 ([8])** *Let $(\xi, \bar{\xi}) \in \Delta$ and write $\xi = \zeta^{-1}\xi_{-1} + \ldots$. When $\xi_{-1}$ is semisimple we can find (after possibly shrinking $\epsilon$) an element $g \in \Lambda_{I,B}$ for which $[\Phi(\xi)] = g\#[\Phi(\zeta^{-1}\xi_{-1})]$.*

Our aim for the rest of this section is to prove:

**Theorem 8** *Each primitive lift $\psi : T^2 \to G/H$ admits, after possibly an isometry, an extended frame $\Phi_\zeta$ given by $\Phi_\zeta = g\#\Phi(\zeta^{-1}\Lambda)$, where $g \in \Lambda_{I,B}$ and $\Lambda \in \mathfrak{g}_{-1}$ is a non-zero semisimple element fixed a priori.*

This is a actually a fact about primitive maps of *finite type*, in the sense of [5, 7], which includes all tori, thanks to [5]. The proof relies on the following results.

**Lemma 16 ([7])** *Each primitive map $\psi$ of finite type admits a $(\xi, \bar{\xi}) \in \Delta$ for which $\Phi(\xi)$ is an extended frame.*

The proof of the next lemma is in the appendix, to avoid distraction from the point at hand.

**Lemma 17** *$G/H$ is a rank one $m+1$-symmetric space i.e. every semisimple element of $\mathfrak{g}_{-1}$ is $\mathrm{Ad}(H^{\mathbf{C}})$-conjugate to some scalar multiple of a fixed non-zero semisimple $\Lambda \in \mathfrak{g}_{-1}$.*

Finally we need:

**Lemma 18 ([8])** *Let $g \in \Lambda(G, \nu)$ extend holomorphically into $I$ and define $g\#\Phi(\xi)$ to be the $\Lambda_E$-component of $g\Phi(\xi)$ in its Iwasawa decomposition. Then $\Phi(\mathrm{Ad}g.\xi) = g\#\Phi(\xi).\tilde{k}$ for some $\tilde{k} : \mathbf{R}^2 \to H$.*

Now we can prove theorem 8. Fix a non-zero semisimple $\Lambda \in \mathfrak{g}_{-1}$. By lemma 16 $\psi$ has an extended frame $\Phi(\xi)$ and this is gauge equivalent to $\tilde{g}\#\Phi(\zeta^{-1}\xi_{-1})$, for some $\tilde{g} \in \Lambda_{I,B}$, by theorem 7. By lemma 17 there is some $h \in H^{\mathbf{C}}$ for which $\xi_{-1} = \mathrm{Ad}h.\Lambda$, so that $\Phi(\mathrm{Ad}\tilde{g}h.\zeta^{-1}\Lambda)$ is an extended frame for $\psi$. By lemma 18 this is gauge

equivalent to the frame $(\tilde{g}h)\#\Phi(\zeta^{-1}\Lambda)$. Finally, we may write $\tilde{g}h = kg$ for some $k \in H$, $g \in \Lambda_{I,B}$, so that

$$(kg)\#\Phi(\zeta^{-1}\Lambda) = k(g\#\Phi(\zeta^{-1}\Lambda))$$

is an extended frame for $\psi$. Thus $g\#\Phi(\zeta^{-1}\Lambda)$ is an extended frame for the map got from $\psi$ after the isometry of $G/H$ determined by $k$ (indeed this isometry preserves the base point $\psi(0)$).

**$\Lambda$ and its centraliser.** Here we will fix a choice of $\Lambda \in \mathfrak{g}_{-1}$ explicitly. Let $\delta_j$ denote the column vector $(0, \ldots, 1, \ldots, 0)^t$ for which the '1' lies in the $j$-th place, counting $j$ from 0 to $n$. Take $\mathbf{CP}^n$ to be the $G$-orbit of the line $\langle \delta_0 \rangle$. Then $F^r(\mathbf{CP}^n)$ is the $G$-orbit of the flag

$$\langle \delta_0 \rangle \subset \langle \delta_0, \delta_1 \rangle \subset \langle \delta_0, \ldots, \delta_r \rangle \subset \mathbf{C}^{n+1}.$$

This is isomorphic to $G/H$ where $H$ is the fixed point subgroup of the automorphism $\nu : g \to \mathrm{Ad}\sigma.g$ with $\sigma$ the diagonal matrix $\mathrm{diag}(1, \omega^{-1}, \ldots, \omega^{-m}, \ldots, \omega^{-m})$ for $m = r + 1$ and $\omega = \exp(2\pi i/m + 1)$. Thus we may fix $\Lambda$ to be the rank $m + 1$ matrix

$$\Lambda = \begin{pmatrix} 0 & & & & 1 & & \\ 1 & \cdot & & & & & \\ & 1 & \cdot & & & & \\ & & \cdot & 0 & & & \\ & & & 1 & 0 & & \\ & & & & & \cdot & \\ & & & & & & 0 \end{pmatrix}$$

Later we will want to work with the centraliser $\mathfrak{z}$ of $\Lambda$ in $\mathfrak{g}^{\mathbf{C}}$ and its centre $\mathfrak{c}$. It is not hard to see that $\mathfrak{c} = \mathfrak{c}_* + \langle Id \rangle$ where

$$\mathfrak{c}_* = \langle \Lambda, \Lambda^2, \ldots, \Lambda^{m+1} \rangle.$$

It is worth noting that $\mathfrak{z}$ is a reductive Lie algebra whose single semisimple component is a subalgebra of $\mathfrak{g}_0$. It will be useful to observe that elements of $\mathfrak{z}$ have block decomposition according to the splitting $\mathfrak{z} = \mathfrak{c}_* \oplus \mathfrak{z}_*$ where $\mathfrak{z}_* \simeq \mathfrak{gl}_{n-m}$ is the subalgebra of matrices of rank $n - m$ whose non-zero entries occupy the bottom right hand corner.

**Higher flows in $O_\Lambda$.** For brevity (and consistency with previous work) we will write $\Phi^{(0)}$ for $\Phi(\zeta^{-1}\Lambda)$. As a result of theorem 8 we see that all the primitive maps of finite type $\mathbf{R}^2 \to G/H$ are, after possibly an isometry, found in the dressing orbit of the vacuum solution $[\Phi^{(0)}]$. From [8] we learn that this orbit $O_\Lambda$ is isomorphic to $\Lambda_{I,B}/\Gamma_{I,B}$, where $\Gamma_{I,B}$ is the stabiliser of $\zeta^{-1}\Lambda$ for the adjoint action of $\Lambda_{I,B}$. If we let $[g]$ denote the coset $g\Gamma_{I,B}$ this isomorphism is given by $[g] \mapsto g\#[\Phi^{(0)}]$. We will now describe (following [8]) the action of an abelian Lie subgroup of $\Lambda(G, \nu)$ on this orbit, the one-parameter subgroups of which generate the so-called 'higher flows' (the terminology comes from soliton theory, where the idea of dressing actions originated). This material will be required later on.

Observe that $\Lambda(\mathfrak{c},\nu)$ is the centre of the centraliser for $\zeta^{-1}\Lambda$ in $\Lambda(\mathfrak{g},\nu)$ and let $\mathfrak{C}_R$ denote the subalgebra of *finite order* elements of $\Lambda(\mathfrak{c},\nu)$ (i.e. those whose projection onto $\Lambda_E(\mathfrak{g},\nu)$ is a Laurent polynomial). The abelian Lie group $\exp(\mathfrak{C}_R)$ has a right action on $\Lambda_{I,B}/\Gamma_{I,B}$ defined by $\exp a.[g] = [(g\exp a)_I]$, where $(\cdot)_I$ denotes the $\Lambda_{I,B}$ factor in the Iwasawa decomposition. Notice in particular that the subgroup $\exp(\mathfrak{C}_R)\cap\Gamma_{I,B}$ acts trivially here.

It is particularly interesting to examine the action of the 2-parameter subgroup $\exp(\mathfrak{w})$ where

$$\mathfrak{w} = \{w\zeta^{-1}\Lambda + \bar{w}\zeta\bar{\Lambda} : w\in\mathbf{C}\}.$$

Clearly the $\exp(\mathfrak{w})$-orbit of $[g]$ can be written as $\{[(g\Phi^{(0)}(w))_I] : w\in\mathbf{C}\}$. The corresponding points in $O_\Lambda$ have extended frames of the form

$$\begin{aligned}(g\Phi^{(0)}(w))_I\#\Phi^{(0)}(z) &= (g\Phi^{(0)}(w))_E^{-1}.g\#\Phi^{(0)}(z+w)\\ &= \Phi_\zeta(w)^{-1}.\Phi_\zeta(z+w)\end{aligned} \tag{21}$$

for $\Phi_\zeta(z) = g\#\Phi^{(0)}(z)$. The right hand side is clearly an extended frame for the primitive map $\psi^w$ defined by $\psi^w(z) = h^{-1}\psi(z+w)$ where $hH = \psi(w)$. Thus the corresponding $\exp\mathfrak{w}$-orbit in $O_\Lambda$ is just the set of all primitive maps obtained from $\psi$ by the translation $z\mapsto z+w$. These observations will be useful when we come to discuss necessary conditions for tori later on.

## 7.3 The algebra of polynomial Killing fields.

The key to recovering the spectral curve is to understand the Lie algebras of formal and polynomial Killing fields. These give us those local deformations of an extended frame which correspond to moving along the Jacobi variety of the spectral curve. Hence we will see we can recover the tangent space $H^1(X,\mathcal{O}_X)$ to the Jacobi variety from the formal Killing fields and the polynomial Killing fields will give us the coordinate ring $H^0(X_A,\mathcal{O})$ of an affine open subset $X_A\subset X$. This is essentially the theory described in the appendix of [20]. However, since in general the Lie algebra of polynomial Killing fields is not abelian the theory requires more care.

By theorem 8 our primitive lift possesses an extended frame $\Phi_\zeta$ of the form $g\#\Phi^{(0)}$ for some $g\in\Lambda_{I,B}$. This means there is a map $\chi:\mathbf{R}^2\to\Lambda_{I,B}$ for which

$$g\Phi^{(0)} = \Phi_\zeta.\chi. \tag{22}$$

Observe that $\chi(0) = g$. From now on, for convenience, let us drop the subscript '$\zeta$' on $\alpha_\zeta$ etc. By a *real formal Killing field* for $\alpha = \Phi^{-1}d\Phi$ we will mean a map $\eta:\mathbf{R}^2\to\Lambda(\mathfrak{g},\nu)$ for which

**(i)** $d\eta = [\eta,\alpha]$

**(ii)** $\eta$ has finite order.

Recall that we say an element $\eta$ of $\Lambda(\mathfrak{g},\nu)$ has *finite order* when its projection $\eta_E$ onto $\Lambda_E(\mathfrak{g}^{\mathbf{C}},\nu)$ is a Laurent polynomial in $\zeta$ (and more generally an element of $\Lambda_C(\mathfrak{g}^{\mathbf{C}},\nu)$ will be said to have finite order if it extends meromorphically into $I$ where it has

at worst poles). A *real polynomial Killing field* is a real formal Killing field $\eta$ for which $\eta = \eta_E$. To define the space of all formal or polynomial Killing fields we take the complexification of the space of real ones: this gives us maps with values in $\Lambda_C(\mathfrak{g}^{\mathbf{C}}, \nu)$.

Let $\mathfrak{Z} = \{\text{finite order } a \in \Lambda_C(\mathfrak{g}^{\mathbf{C}}, \nu) : [a, \Lambda] = 0\}$ and observe that this is the same as the subset of $\Lambda_C(\mathfrak{z}, \nu)$ consisting of only the finite order elements. Denote $\mathfrak{Z} \cap \Lambda(\mathfrak{g}, \nu)$ by $\mathfrak{Z}_R$, then we will be interested in the subalgebra $\mathfrak{Z}_R^{pol} = \{a \in \mathfrak{Z}_R : (\mathrm{Ad}g.a)_E = \mathrm{Ad}g.a\}$ and its complexification $\mathfrak{Z}^{pol} \subset \mathfrak{Z}$. The following lemma is derived from a similar lemma in the appendix of [20]. I will not repeat the proof as it is virtually identical to the proof there.

**Lemma 19** *The map* $\mathrm{Ad}\chi : \mathfrak{Z} \to \{$*formal Killing fields*$\}$ *is an isomorphism of Lie algebras. It identifies* $\mathfrak{Z}^{pol}$ *with* $\{$*poly. Killing fields*$\}$.

Let us now recall (from [11]) how the Killing fields give us some information about the local deformations of the extended frame $\Phi$. Suppose $\Phi(t)$ is a 1-parameter family of extended frames for primitive maps $\psi(t)$ with $\Phi(0) = \Phi$. If $\xi = \Phi(t)^{-1}.d\Phi(t)/dt$ then

$$d\xi + [\alpha, \xi] = \frac{d\alpha}{dt}.$$

On the other hand, if $\eta$ is a real formal Killing field with splitting $\eta = \eta_E + \eta_I$ according to the Iwasawa decomposition then

$$d\eta_E + [\alpha, \eta_E] = -d\eta_I - [\alpha, \eta_I]. \tag{23}$$

A comparison between the two sides of this identity shows that this yields a map $\dot{\alpha} : \mathbf{R}^2 \to \Lambda_1$, so that in some formal sense (23) describes the local deformations of the 1-form $\alpha$. In particular, the polynomial Killing fields are local symmetries of this 1-form. More generally, the full complex algebra of formal Killing fields describes local deformations for which $\Phi(t)$ takes values in $\Lambda_C(G^{\mathbf{C}}, \nu)$ but not necessarily $\Lambda(G, \nu)$.

Let us view this in the following way. Define $\mathfrak{Z}_R^I = \mathfrak{Z} \cap (\mathfrak{b} + \Lambda_I(\mathfrak{g}, \nu))$. Then we have the exact sequence

$$\begin{array}{ccccccc} \mathfrak{Z}_R^I \oplus \mathfrak{Z}_R^{pol} & \to & \mathfrak{Z}_R & \stackrel{D}{\to} & \mathfrak{D} & \to & 0 \\ a \oplus b & \mapsto & b - a & & & & \end{array} \tag{24}$$

where $D = \mathrm{Ad}\chi^{-1}.[(d + \mathrm{ad}\alpha) \circ \mathrm{pr}_E]\mathrm{Ad}\chi$. Thus the cokernel $\mathfrak{D}$ is identifiable with the tangent space to the family of all deformations given by (23). In the superconformal case treated in [20], where $\mathfrak{Z}$ is abelian, it was shown that $\mathfrak{D}$ is finite dimensional and this is equivalent to the statement that $\psi$ is of finite type. In this case the group $\exp(\mathfrak{Z}_R)$ acts on the dressing orbit of $\Phi^{(0)}$ and the tangent space to the orbit of $g\#\Phi^{(0)}$ is identifiable with $\mathfrak{D}$. Moreover the complexification of the sequence (24) can also be interpreted as a computation of the cohomology space $H^1(X, \mathcal{O}_X)$ for the curve $X$ given by the completion of $\mathrm{Spec}(\mathfrak{Z}^{pol})$ by smooth points. This is how the spectral curve arises in [20].

In the general case, however, $\mathfrak{Z}$ is not abelian and, as we have seen above, the Lie group which acts on the dressing orbit $O_\Lambda$ of $\Phi^{(0)}$ is the centre $\exp(\mathfrak{C}_R)$ of $\exp(\mathfrak{Z}_R)$. Let us think of this as an action of $\mathfrak{C}_R$ and let $\mathfrak{C}_R^{[g]}$ denote the stabiliser of $[g]$ for this action: it is not hard to show that this equals $\mathfrak{C}_R \cap \mathfrak{Z}_R^I \oplus \mathfrak{Z}_R^{pol}$. Then the following result is proven in [8]:

**Lemma 20 ([8])** *$g\#\Phi^{(0)}$ is an extended frame for a primitive map of finite type if and only if $\mathfrak{C}_R/\mathfrak{C}_R^{[g]}$ is finite dimensional.*

It follows that $\psi$ is of finite type precisely when $\mathfrak{C}/\mathfrak{C}^{[g]}$ is finite dimensional.

Because of this we are interested in the vector space $\mathfrak{S}(z)$ of polynomial Killing fields given by the complexification of

$$\mathfrak{S}_R(z) = \{(\mathrm{Ad}\chi(z).c)_E : c \in \mathfrak{C}_R^{[g]}\}.$$

**Lemma 21** *For each $z$, $\mathfrak{S}(z)$ is a space of commuting elements of $\Lambda_E(\mathfrak{g}^{\mathbf{C}}, \nu)$ each of which is semisimple-valued on the unit circle.*

**Proof.** Let $c_1, c_2 \in \mathfrak{C}_R^{[g]}$, then by the definition $\mathrm{Ad}\chi^{-1}(\mathrm{Ad}\chi.c_i)_E \in \mathfrak{Z}_R$. So

$$[\mathrm{Ad}\chi.c_i, (\mathrm{Ad}\chi.c_j)_E] = 0 = [\mathrm{Ad}\chi.c_i, (\mathrm{Ad}\chi.c_j)_I]$$

since each $c_i$ is central. From this we deduce

$$\begin{aligned}[(\mathrm{Ad}\chi.c_1)_E, (\mathrm{Ad}\chi.c_2)_E] &= -[(\mathrm{Ad}\chi.c_1)_E, (\mathrm{Ad}\chi.c_2)_I] \\ &= [(\mathrm{Ad}\chi.c_1)_I, (\mathrm{Ad}\chi.c_2)_I]\end{aligned}$$

which implies that each side is identically zero, therefore elements of $\mathfrak{S}_R(z)$ (and therefore $\mathfrak{S}(z)$) commute. By definition if $\xi \in \mathfrak{S}(z)$ so does its conjugate $\bar{\xi} = -\xi(\bar{\zeta}^{-1})^\dagger$. Thus $[\xi, \bar{\xi}] = 0$, so on $|\zeta| = 1$ we see that $\xi$ is normal and therefore semisimple. ■

Now let $\mathfrak{R}(z)$ denote the $\mathbf{C}$-algebra generated by $\mathfrak{S}(z)$, and set $\mathfrak{R} = \mathfrak{R}(0)$. Observe that $\mathfrak{S}$ contains the $\mathbf{C}$-algebra $\mathfrak{B} = \mathbf{C}[\lambda^{-1}Id, \lambda Id]$, where $\lambda = \zeta^{m+1}$, so that $\mathfrak{R}$ is a commutative unital $\mathbf{C}$-algebra without nilpotents, as well as being a $\mathfrak{B}$-module.

**Lemma 22** *$\mathfrak{R}$ is a torsion free, finitely generated $\mathfrak{B}$-module.*

**Proof.** That it is torsion free is obvious. To see $\mathfrak{R}$ is finitely generated as a $\mathfrak{B}$-module we observe that

$$M = \{v : C \to \mathbf{C}^{n+1} | v(\omega\zeta) = \sigma v(\zeta) \text{ and extends to a Laurent poly. on } \mathbf{C}^*\}$$

is a rank $n+1$ $\mathfrak{B}$-module as well as a faithful $\mathfrak{R}$-module, where the operation is that of matrices on column vectors. ■

Let us define $\mathfrak{A} = \mathrm{Ad}g^{-1}.\mathfrak{R}$, which is an abelian subalgebra of $\Lambda(\mathfrak{z}, \nu)$. Using the splitting $\mathfrak{z} = \mathfrak{c}_* \oplus \mathfrak{z}_*$ we may write each $a \in \mathfrak{A}$ as $a = a_0 + a_\bullet$ where $a_0$ takes values in $\mathfrak{c}_*$ and $a_\bullet$ takes values in $\mathfrak{z}_*$: $a_\bullet$ is a function of $\zeta^{m+1}$ since $\mathfrak{z}_* \subset \mathfrak{g}_0$. For $a, b \in \mathfrak{A}$ we have

$$ab = a_0 b_0 + a_\bullet b_\bullet.$$

We will also define $W(z) = \chi^{-1}.M$ so that $W(z)$ is a faithful $\mathfrak{A}$-module (of boundaries of meromorphic maps from $I$ to $\mathbf{C}^{n+1}$, each component of which has finite order).

Finally, let us observe that for $\xi(z) \in \mathfrak{R}(z)$ there must be $a \in \mathfrak{A}$ such that $\xi(z) = \mathrm{Ad}\chi(z).a = \mathrm{Ad}\Phi(z)^{-1}g.a = \mathrm{Ad}\Phi(z)^{-1}.\xi(0)$, using $g = \chi(0)$. Therefore $\mathfrak{R}(z) = \mathrm{Ad}\Phi^{-1}(z).\mathfrak{R}$, a property we will find useful later.

## 7.4 The affine spectral curve.

By the previous remarks we have a commutative unital $\mathbf{C}$-algebra $\mathfrak{R}$ and therefore an affine variety $\mathrm{Spec}(\mathfrak{R})$. Our aim is to show that, when $\psi_0$ is linearly full (i.e its image does not lie in a hyperplane of $\mathbf{CP}^n$), this is an affine algebraic curve whose completion $X$ by non-singular points has the properties required to be the spectral curve. That is, we shall show that $X$ admits a rational function $\pi$ of the right type and a line bundle (or rank 1 torsion free coherent sheaf) $\mathcal{L}$ from which we recover $\psi_0$. In this subsection we will discuss the construction over the affine curve and show that for $\psi_0$ to be full the curve must be connected. This will allow us to complete the construction over $X$ in the following subsection.

As a corollary of lemma 22 we deduce that $\mathfrak{R}$ is an integral extension of $\mathfrak{B}$, therefore $\mathrm{Spec}(\mathfrak{R})$ is an affine curve and the inclusion $\mathfrak{B} \to \mathfrak{R}$ is dual to a finite morphism

$$\pi : \mathrm{Spec}(\mathfrak{R}) \to \mathrm{Spec}(\mathfrak{B}) \simeq \mathbf{C}^*_\lambda.$$

Since $\mathfrak{B}$ is a principal ideal domain $\mathfrak{R}$ is actually a free $\mathfrak{B}$-module and its rank $k$ equals the degree of $\pi$. Since $M$ has rank $n+1$ over $\mathfrak{B}$ we see that $k \leq n+1$. Now let $X$ denote the completion of $X_A \equiv \mathrm{Spec}(\mathfrak{R})$ by smooth points.

**Proposition 4** *The curve $X$ is real and admits a rational function $\pi : X \to \mathbf{P}_\lambda$ with a zero $P_0$ of degree $m+1$. Thus $\pi$ has degree $k \geq m+1$.*

**Proof.** The algebra $\mathfrak{R}$ possesses the involution $\xi \mapsto \bar{\xi}$ which is dual to a real involution on $\mathrm{Spec}(\mathfrak{R})$ and this extends to $X$. Now we must show that the fibre of the morphism $\pi$ over $\lambda = 0$ contains a point $P_0$ with ramification index $m$.

We take the point of view that each smooth point of $X$ corresponds to a valuation on some subfield of the ring of fractions $\mathfrak{F} = S^{-1}\mathfrak{A}$, where $S = \mathfrak{A} - \{0\}$. Because $\mathfrak{A}$ need not be an integral domain $\mathfrak{F}$ need not itself be a field. However, each smooth point corresponds to a surjection $\nu : \mathfrak{F}^* \to \mathbf{Z}$ which is a multiplicative homomorphism (i.e. $\nu(ab) = \nu(a) + \nu(b)$) and has $\nu(a+b) \geq \min(\nu(a), \nu(b))$. The subring $I = \{a \in \mathfrak{F} : \nu(a) \geq 0\} \cup \{0\}$ is easily seen to be a discrete valuation ring of its field of fractions. We will describe one of the valuations covering the point $\lambda = 0$.

For each $a = (a_1, a_2)$ in $\mathfrak{A}$ the component $a_1$ has a block decomposition into the sum

$$a_{10}(\zeta) + a_{1\bullet}(\zeta^{m+1})$$

derived from the decomposition of $a$ described earlier. This provides a grading $o_0 : \mathfrak{A} \to \mathbf{Z}$ for the ring $\mathfrak{A}$, defined by taking $o_0(a)$ to be the order of $a_{10}$ in $\zeta^{-1}$.

With this we define a multiplicative homomorphism

$$\nu_0 : \mathfrak{F}^* \to \mathbf{Z} \quad , \quad \nu_0(r/s) = o_0(s) - o_0(r).$$

I claim that if we can show that this is surjective then we are done, for in that case $I_0 = \{a \in \mathfrak{F} : \nu_0 \geq 0\} \cup \{0\}$ is clearly isomorphic to the regular local ring $\mathcal{O}_\zeta$. About the corresponding smooth point $P_0$ the map $\pi$ behaves like $\zeta \mapsto \zeta^{m+1}$.

To show $\nu_0$ is onto we use lemma 20. First observe that $\mathfrak{A}$ contains, by definition, $\mathrm{Ad}\chi^{-1}.\mathfrak{S}(z)$ and that for any $c \in \mathfrak{C}_R^{[g]}$

$$\mathrm{Ad}\chi^{-1}(\mathrm{Ad}\chi.c)_E \in c_E + \mathfrak{Z}_R^I.$$

Now observe that $o_0 : \mathfrak{C}_R \to \mathbf{Z}$ is onto. It follows from lemma 20 that $o_0(\mathfrak{C}_R^{[g]})$ contains all but a finite number of positive integers and therefore so does $\mathrm{Ad}\chi^{-1}.\mathfrak{S}(z) \subset \mathfrak{A}$. So for every integer $k$ there exist $r, s \in \mathfrak{A}$ for which $\nu_0(r/s) = k$.■

**Remark.** We see from this proof that with respect to the isomorphism $\mathbf{C}[X_A] \simeq \mathfrak{A}$ a regular function on $X_A$ vanishes on the irreducible component carrying $P_0$ precisely when the corresponding component $a_{10}$ is identically zero.

It is not hard to see that $\pi$ intertwines the real involution on $X$ with the map $\zeta \mapsto \bar{\zeta}^{-1}$ and so it has a pole $P_\infty$ of order $m+1$. Now let us show that the irreducible component $X_0$ of $X$ which carries the points $P_0, P_\infty$ is the completion of $\mathrm{Spec}(\mathfrak{A}_0)$, where

$$\mathfrak{A}_0 = \{a_0 : a \in \mathfrak{A}\}.$$

Thus $\mathfrak{A}_0$ is a subalgebra of $\Lambda_C(\mathfrak{c}_*, \nu)$ and a quotient algebra of $\mathfrak{A}$.

**Lemma 23** *$\mathfrak{A}_0$ is an integral domain.*

**Proof.** Observe that each element of $\Lambda_C(\mathfrak{c}_*, \nu)$ can be written as a Fourier series in $\zeta^{-1}\Lambda$ and we obtain an injective $\mathbf{C}$-algebra morphism of $\mathfrak{A}_0$ into $\mathrm{Hol}(I^*, \mathbf{C})$ (where $I^* = I \cap A$) given by $a_0(\zeta^{-1}\Lambda) \mapsto a_0(\zeta^{-1})$. The image consists solely of holomorphic maps which, unless they are identically zero, do not vanish on either of the connected components of $I^*$ and therefore $a_0 b_0 = 0$ if and only if one or the other is the zero element. ■

It follows from previous remark that $\mathrm{Spec}(\mathfrak{A}_0)$ is the irreducible component of $\mathrm{Spec}(\mathfrak{A}) \simeq X_A$ carrying the point $P_0$ and its conjugate $P_\infty$. Although $X_A$ need not be itself irreducible we will shortly see that when $\psi_0$ is full it must be connected. First we must introduce the sheaf over $X_A$ whose sections provide the harmonic map.

Recall the $\mathfrak{R}$-module $M$ which is torsion free as a $\mathfrak{B}$-module. Since $\mathfrak{B}$ is an integral domain it is elementary to show that $M$ must also be torsion free over $\mathfrak{R}$, so that it determines a torsion free coherent sheaf $\mathcal{L}_A$ over $X_A$. We want to show that its restriction to $X_{0A} = \mathrm{Spec}(\mathfrak{A}_0)$ is rank one. This is easiest to see by looking at the equivalent picture of $W(0) = \chi(0)^{-1}M$ as an $\mathfrak{A}$-module. For any $f = (f_0, \ldots, f_n)^t \in W(0)$ let us write

$$f = f^0 + f_\bullet = (f_0, \ldots, f_m, 0, \ldots, 0)^t + (0, \ldots, 0, f_{m+1}, \ldots, f_n)^t.$$

So for $a \in \mathfrak{A}$ we clearly have $(af)^0 = a_0 f^0$. Therefore the vector space $W^0(0) = \{f^0 : f \in W(0)\}$ is an $\mathfrak{A}_0$-module, clearly torsion free. Now observe that the injection

$$\Sigma : W^0(0) \to \mathrm{Hol}(I^*, \mathbf{C}) \quad ; \quad f^0 \mapsto f_0 + \ldots + f_m,$$

is an $\mathfrak{A}$-module morphism (one readily verifies that $\Sigma(a_0 f^0) = a_0(\zeta^{-1})\Sigma(f_0)$, using the representation $a_0(\zeta^{-1}\Lambda) \mapsto a_0(\zeta^{-1})$). Therefore in a smooth neighbourhood of $P_0$ each stalk of $\mathcal{L}_A$ is a module of regular $\mathbf{C}$-valued functions of $\zeta$ with respect to the appropriate regular local ring, so that $\mathcal{L}_A$ has rank one in this neighbourhood. Thus we have shown:

**Proposition 5** *The restriction of $\mathcal{L}_A$ to $X_{0A}$ (and therefore to the connected component of $X_A$ containing it) is a rank one torsion free coherent sheaf. In particular, when $X_A$ is connected $\mathcal{L}_A$ has rank one.*

We can easily repeat the previous results for each $z$, replacing $\mathfrak{R}$ by $\mathfrak{R}(z)$ and $W(0)$ by $W(z)$. This gives us, for each $z$, a sheaf $\mathcal{L}_A(z)$ over $X_A$ whose restriction to $X_{0A}$ has rank one and whose direct image under $\pi$ is the rank $n+1$ vector bundle $\mathcal{E}_A$ corresponding to the $\mathfrak{B}$-module $M$. This vector bundle comes equipped with a trivialisation determined by the isomorphism

$$\begin{array}{rcl} \Gamma(A, \mathcal{E}_A) \simeq M & \longrightarrow & \{f : A \to \mathbf{C}^{n+1} | \text{ Laurent poly.}\} \\ f(\zeta) & \mapsto & (\kappa f)(\lambda) \end{array}$$

where $A = \mathbf{C}^*_\lambda$ and $\kappa = \mathrm{diag}(1, \zeta, \ldots, \zeta^m, \ldots, \zeta^m)$, so that $\kappa(\omega\zeta) = \kappa(\zeta)\sigma^{-1}$ and therefore $\kappa f$ is a function of $\lambda$.

The effect this isomorphism is to remove the $\nu$-equivariance which has permeated the construction so far. Because of this we will find it most convenient to remove the effects of the $\nu$-equivariance from all the objects we are dealing with. To this end we will *redefine*, for this subsection and the next,

$$\begin{array}{rcl} M & \equiv & \{\text{Laurent poly. } f : A \to \mathbf{C}^{n+1}\} \\ \mathfrak{A} & \equiv & \mathrm{Ad}\kappa.\mathfrak{A} \\ \Phi(z) & \equiv & \mathrm{Ad}\kappa.\Phi(z) \\ \chi(z) & \equiv & \mathrm{Ad}\kappa.\chi(z) \end{array}$$

and so forth. Observe that in particular $\Phi$ and $\chi$ are still holomorphic in their respective domains $E$ and $I$. With these redefined we can introduce the trivial vector bundle $\mathcal{E}(z)$ over $\mathbf{P}_\lambda$ characterised by the transition relations

$$\chi(z).\hat{\tau}_z = \tau_z \quad , \text{ on } A \cap I \tag{25}$$

where $\tau_z$ is a trivialisation over $A$, $\hat{\tau}_z$ a trivialisation over $I$. Further, we can always choose $I$ so that it contains no branch points of $\pi$ other than 0 and $\infty$ and $\pi^{-1}(I)$ contains only smooth points.

Eventually we will be able show that when $\psi_0$ is full $\mathcal{E}(z)$ is the direct image of a rank one torsion free coherent sheaf $\mathcal{L}(z)$ over $X$ obtained by moving $\mathcal{L}$ (the extension of $\mathcal{L}_A$ to $X$) linearly around the Picard variety. Before we can do this we must establish that, for $\psi_0$ to be full, the curve $X$ must be connected, so that $\mathcal{L}_A$ must be rank one by the previous proposition.

**Proposition 6** *$X$ is disconnected if and only if $\psi_0$ is not full.*

**Proof.** First we will show that if $X$ is disconnected then $\Phi_1.\delta_0$ (i.e. $\Phi.\delta_0$ evaluated at $\lambda = 1$) takes values in a proper subspace of $\mathbf{C}^{n+1}$.

When $X$ is disconnected we may write $X = Y + Z$ where $Y$ is the connected component carrying the irreducible component $X_0$. For each $z$ there must be $\varepsilon_Y(z), \varepsilon_Z(z) \in \mathfrak{R}(z)$ representing the globally regular characteristic functions $1_Y, 1_Z$ on $X$ (i.e. $1_Y$ is identically one on $Y$ and identically zero on $Z$). Each of these is clearly independent of $\lambda$. Thus we obtain a direct sum decomposition

$$M = \varepsilon_Y(z).M \oplus \varepsilon_Z(z).M$$

corresponding to a global decomposition of $\mathcal{E}(z)$ into a sum of trivial bundles.

Consider now the global section $\sigma_0(z)$ of $\mathcal{E}(z)$ defined by $\tau_z(\sigma_0(z)) = \delta_0$. Since $\mathfrak{R}(z) = \mathrm{Ad}\Phi(z)^{-1}.\mathfrak{R}$ we deduce $\Phi\tau_z = \tau_0$ and therefore $\Phi.\delta_0 = \tau_0(\sigma_0(z))$. The proof that $\psi_0$ is not full will be finished if we can show

$$\varepsilon_Z(z)\tau_z(\sigma_0(z)) = 0 \tag{26}$$

since then $\varepsilon_Z(0)\Phi.\delta_0 = 0$ and therefore $\Phi_1.\delta_0$ takes values in the kernel of a matrix. But (26) is equivalent to showing $a_Z.\chi^{-1}\delta_0 = 0$, where $a_Z = \mathrm{Ad}\chi^{-1}.\varepsilon_Z$ represents $1_Z$ in $\mathfrak{A}$. Since $1_Z$ vanishes identically on $X_0$ we know $(a_Z)_0 = 0$ (recall an earlier remark). From the Fourier series for $\chi^{-1}$ on the circle $C_1$ we see that $\chi^{-1}\delta_0$ has the form

$$\chi^{-1}\delta_0 = (\alpha_0, 0, \ldots, 0) + O(\lambda)$$

about $\lambda = 0$. Therefore $a_Z.\chi^{-1}\delta_0$ vanishes at $\lambda = 0$, whence it vanishes everywhere since it represents a global section of $\mathcal{E}(z)$.

Now let us show that when $\psi_0$ is not full the algebra $\mathfrak{R}$ possesses an idempotent $\varepsilon$ different from the identity. For then $\varepsilon, Id - \varepsilon$ are a pair of 'orthogonal idempotents' (cf. for example [13]) whence $\mathrm{Spec}(\mathfrak{R})$ is disconnected.

We may assume without loss of generality that $\psi_0$ is full in the projective $k$-plane of points whose last $n - k$ coordinates vanish. From the preceding theory $\psi_0$ determines a subalgebra $\mathfrak{R}_k \subset \Lambda_C(\mathfrak{gl}_{k+1}, \nu)$ which must contain the identity matrix in $\mathfrak{gl}_{k+1}$: we take this for $\varepsilon$. Now observe $\mathfrak{R}$ is a unital subalgebra of $\Lambda_C(\mathfrak{gl}_{n+1}, \nu)$ containing $\mathfrak{R}_k$ and therefore $\varepsilon \in \mathfrak{R}$ satisfies the conditions required.■

## 7.5 Completion of the affine curve and the line bundle.

From now on we will assume that $\psi_0$ is full and thus, by the previous result, $X$ is connected. It follows that $\mathcal{L}_A(z)$ has rank one and therefore $\pi$ has degree $n+1$. Using these facts we will describe the extension $\mathcal{L}(z)$ of $\mathcal{L}_A(z)$ to all of $X$ and produce from it global sections which reconstruct the map $\psi$. We aim to prove:

**Theorem 9** *$\mathcal{L}(z)$ has degree $p + n$, is real (i.e. satisfies (5)) and moves linearly with $z, \bar{z}$ around the Picard variety of $X$. The primitive lift $\psi$ determines (and is determined by), up to scaling, global sections $\sigma_0(z), \ldots, \sigma_m(z)$ for which $\sigma_j(z)$ has divisor of zeroes at least $(m-j)P_0 + jP_\infty + E_0$ (where $(m+1)P_0 + E_0$ is the divisor of zeroes of $\pi$).*

**Remark.** In the statement of this theorem I am using 'Picard variety' to mean the moduli space of maximal rank 1 torsion free coherent sheaves when this is relevant (we will see that $\mathcal{L}$ must be maximal much later, after prop. 8).

Before this theorem can be proved we must describe $\mathcal{L}(z)$. First, recall that $I \subset \mathbf{P}_\lambda$ has been chosen so that $X_I = \pi^{-1}(I)$ consists of smooth points and only has ramification points over $\lambda = 0, \infty$. Using the direct image isomorphism $\mathrm{Hol}(X_I, \mathbf{C}) \to \mathrm{Hol}(I, \mathbf{C}^{n+1})$ (cf. for example [12]) it is easy to see that $\mathrm{Hol}(X_I, \mathbf{C})$ can be represented, as an algebra of endomorphisms on $\mathrm{Hol}(I, \mathbf{C}^{n+1})$, by $\mathfrak{g}^{\mathbf{C}}$-valued functions on $I$ which are diagonalisable at each value of $\lambda$ (indeed for $\lambda \neq 0, \infty$ the eigenvalues give the $n+1$ values of the function on $X_I$).

We deduce from these remarks that for each $\lambda \in I - \{0, \infty\}$ the commutative Lie algebra $\mathfrak{A}_\lambda \subset \mathfrak{z}$ got by evaluating elements of $\mathfrak{A}$ at $\lambda$ consists of semisimple elements. Moreover, $\mathfrak{A}$ has rank $n+1$ as a $\mathfrak{B}$-module and therefore $\mathfrak{A}_\lambda$ is a maximal toral subalgebra of semisimple elements - a Cartan subalgebra for $\mathfrak{g}^{\mathbf{C}}$. We also deduce that we must be able to complete this holomorphic family at $\lambda = 0, \infty$. Since all Cartan subalgebras of $\mathfrak{g}^{\mathbf{C}}$ are conjugate and ours all lie in $\mathfrak{z}$ there is a holomorphic map $\gamma : I \to Z$ (where the Lie subgroup $Z \subset G^{\mathbf{C}}$ is the stabiliser of $\Lambda$) for which $\mathrm{Ad}\gamma_\lambda^{-1}.\mathfrak{A}_\lambda = \mathfrak{a}$, where $\mathfrak{a}$ can be any fixed Cartan subalgebra in $\mathfrak{z}$. In particular, we will fix $\mathfrak{a} = \mathfrak{c}_* + \mathfrak{d}$, where $\mathfrak{d}$ is the torus of diagonal matrices in $\mathfrak{z}_*$.

Therefore, let us define $\tilde{\mathfrak{A}} = \mathrm{Ad}\gamma^{-1}.\mathfrak{A}$. Then each element of $\tilde{\mathfrak{A}}$ takes values in $\mathfrak{a}$, so that it has a block decomposition

$$a = a_0 + a_1 + \ldots + a_{n-m}$$

where $a_0$ takes values in $\mathfrak{c}_*$ and $a_j$, $j \neq 0$, is a diagonal matrix function whose only non-zero entry lies in the $m+j$-th place (counting places on the diagonal from 0 to $n$). In the proof of proposition 4 we saw how to use the components $a_0$ to obtain a discrete valuation ring in the ring of fractions $\mathfrak{F}$, and this provided the ramification point $P_0$. To get all the points lying over $\lambda = 0$ it is not hard to show that one defines gradings $o_j : \tilde{\mathfrak{A}} \to \mathbf{Z}$ for which $o_j(a)$ is the order of $a_{1j}$ in $\lambda^{-1}$ (for $a = (a_1, a_2)$ we define $(a_{1j}, a_{2j}) = a_j$ using the block decomposition above). We use these to define valuations of the ring of fractions $\tilde{\mathfrak{F}}$ of $\tilde{\mathfrak{A}}$ in the manner described earlier.

Next, we wish to describe fairly explicitly the extension $\mathcal{L}(z)$ to $\mathcal{L}_A(z)$ to all of $X$. For ease of exposition let us suppose we are dealing with the generic case in which $\pi^{-1}(0)$ has $n+1-m$ distinct points. First define $\tilde{W}(z) = \gamma^{-1}W(z)$ so that this represents the module of sections of $\mathcal{L}_A(z)$ over $X_A \simeq \mathrm{Spec}(\tilde{\mathfrak{A}})$. Now define $\mathcal{W}$ to be the $\tilde{\mathfrak{F}}$-module of fractions $\tilde{S}^{-1}\tilde{W}(z)$ (where $\tilde{S} = \tilde{\mathfrak{A}} - \{0\}$). For each point $P$ in $X - X_A$ we can construct an $\mathcal{O}_P$-module $\mathcal{W}_P \subset \mathcal{W}$ which is torsion free (and therefore free since $\mathcal{O}_P$ is a discrete valuation ring).This will be the stalk for $\mathcal{L}(z)$ over $P$. For the points over $\lambda = 0$, for example, we get each $\mathcal{W}_P$ in the following manner.

On $\tilde{W}(z)$ define a map $o_P : \tilde{W}(z) - \{0\} \to \mathbf{Z}$ such that: (i) $o_P(w_1 + w_2) \geq \min(o_P(w_1), o_P(w_2))$ and, (ii) $o_P(aw) = \nu_P(a) + o_P(w)$ whenever $a \in \tilde{\mathfrak{A}}$, where $\nu_P$ is the valuation for $\mathcal{O}_P$. This extends to give $o_P : \mathcal{W} - \{0\} \to \mathbf{Z}$ by setting

$o_P(w/a) = o_P(w) - \nu_P(a)$. So we define

$$\mathcal{W}_P = \{w \in \mathcal{W} - \{0\} : o_P(w) \geq 0\} \cup \{0\}.$$

This is clearly a torsion free $\mathcal{O}_P$-module since $\tilde{W}(z)$ is torsion free. Now we need to describe precisely what the map $o_P$ is.

Let $w \in \tilde{W}(z)$ and write its restriction $w_1$ to $C_1$ as the column vector

$$w_1 = (f_0(\lambda), \ldots, f_n(\lambda))^t.$$

For $j = 1, \ldots, n-m$ define

$$o_j(w) = -(\text{order of pole of } f_{m+j} \text{ at } \lambda = 0).$$

For any $a \in \tilde{\mathfrak{A}}$ it is easy to see that $w \mapsto aw$ multiplies each $f_{m+j}$ by a function of $\lambda$ whose order is $\nu_j(a)$. It follows that $o_j$ satisfies both properties (i) and (ii) above. For $j = 0$ define

$$f(\zeta) = \zeta^{m+1} \sum_{j=0}^{m} \zeta^{-j} f_j(\zeta^{m+1})$$

and

$$o_0(w) = -(\text{order of pole of } f(\zeta) \text{ at } \zeta = 0).$$

This clearly satisfies property (ii), so we only have to check for (i). Recall that the $\nu$-equivariant representation of $a$, that is $\alpha = \mathrm{Ad}\kappa^{-1}.a$, has a block decomposition $\alpha_0 + \alpha_1 + \ldots + \alpha_{n-m}$ for which the restriction $\alpha_{10}$ of $\alpha_0$ to $C_1$ has Fourier expansion of the form

$$\alpha_{10} = \sum \beta_j \zeta^{-j} \Lambda^j$$

with only finitely many negative powers of $\zeta$. Let us define

$$\beta(\zeta) = \sum \beta_j \zeta^{-j}$$

then I claim that

$$o_0(aw) = -(\text{order of } \beta f \text{ at } \zeta = 0) = \nu_0(a) + o_0(w).$$

[Proof: Set $\mathbf{1} = (1, \ldots, 1, 0, \ldots, 0)^t$ with $m+1$ ones. Then $f(\zeta) = \zeta^{m+1} \mathbf{1}^t \kappa^{-1} w_1$. We compute

$$\begin{aligned} \mathbf{1}^t \kappa^{-1} a_1 w_1 &= \mathbf{1}^t \alpha_{10} \kappa^{-1} w_1 \\ &= \mathbf{1}^t (\textstyle\sum \beta_j \zeta^{-j} \Lambda^j) \kappa^{-1} w_1 \\ &= \beta f \end{aligned}$$

where we have used the fact that $\mathbf{1}^t \Lambda^j = \mathbf{1}^t$. The claim follows at once.]

By construction the direct image $\pi_* \mathcal{L}(z)$ over $\mathbf{P}_\lambda$ has transition relations

$$\tilde{\chi}.\tilde{\tau}_z = \tau_z \tag{27}$$

where $\tilde{\chi} = \chi\gamma$ and $\tilde{\tau}_z = \gamma.\hat{\tau}_z$. The advantage of this change of trivialiation over $I$ is that $\tilde{\tau}_z$ now comes by direct image from a trivialisation $\theta_z$ for $\mathcal{L}(z)$ over $X_I$ (we will

write $\tilde{\tau}_z = \pi_* \theta_z$). To see this let $s$ be a section of $\mathcal{L}(z)$ over $\pi^{-1}(I)$ (and identify it with the corresponding section of $\mathcal{E}(z)$). Then

$$\tilde{W}(z) \ni \tilde{\tau}_z(s) = \begin{cases} (f_0, \ldots, f_n)^t \text{ about } \lambda = 0 \\ (h_0, \ldots, h_n)^t \text{ about } \lambda^{-1} = 0 \end{cases} \tag{28}$$

In the construction of $\mathcal{L}(z)$ over $X_I$ we used the one-to-one correspondence

$$\begin{array}{rcl} (f_j) & \leftrightarrow & (f, f_{m+1}, \ldots, f_n) \text{ where } f(\zeta) = \zeta^{m+1} \sum_0^m f_j(\zeta^{m+1}) \zeta^{-j} \\ (h_j) & \leftrightarrow & (h, h_{m+1}, \ldots, h_n) \text{ where } h(\zeta) = \sum_0^m h_j(\zeta^{m+1}) \zeta^{-j} \end{array} \tag{29}$$

The right hand side gives us a trivialisation over $X_I$, since this is a union of $n+1-m$ distinct pairs of discs. Moreover, the definitions of $f(\zeta)$ and $h(\zeta)$ describe the passage of direct image about a point of ramification (cf. for example [12]).

**Lemma 24** *Let $[e^a]$ denote the degree zero line bundle over $X$ determined by the 1-cocycle $(e^a, X_A, X_I)$. Then $\mathcal{L}(z) \otimes \mathcal{L} \simeq [e^a]$ for $a = z\zeta^{-1} - \bar{z}\zeta$*

**Proof.** We have already observed that $\Phi \tau_z = \tau_0$, from which it follows (using (27) and (22)) that $\Phi^{(0)}.\pi_* \theta_z = \pi_* \theta_0$. It suffices to show that this implies $\theta_z = e^a \theta_0$, where we will think of $\theta_z$ as a non-vanishing local section of $\mathcal{L}(z)$. This equation follows if we can show that

$$\pi_* \theta_z(\zeta^{-1} s) = \Lambda . \pi_* \theta_z(s) \tag{30}$$

for any local section $s$, where $\zeta$ is defined to zero except in the discs about $P_0$ and $P_\infty$. For then we see $\pi_* \theta_z(e^a s) = \Phi^{(0)} \pi_* \theta_z(s) = \pi_* \theta_0(s)$ whence $e^a s/\theta_z = s/\theta_0$. So let us prove (30) about $\lambda = 0$ - the calculation about $\lambda^{-1} = 0$ is much the same.

From (29) we know that if $\pi_* \theta_z(s) = (f_0, \ldots, f_n)^t$ about $\lambda = 0$ then $s/\theta_z = \zeta^{m+1} \sum_0^m \zeta^{-j} f_j(\zeta^{m+1})$ about $P_0$. Now a simple calculation shows that

$$\pi_* \theta_z(\zeta^{-1} s) = (\lambda^{-1} f_m, f_0, \ldots, f_{m-1}, 0, \ldots, 0)^t = \Lambda . \pi_* \theta_z(s)$$

as required.■

Now we can prove theorem 9.

**Proof of theorem 9** Since $\pi_* \mathcal{L}(z)$ is trivial $\mathcal{L}(z)$ must have degree $p+n$ and the transition relations (27) show that $\mathcal{L}(z)$ is real (since $\tilde{\chi}$ is real, cf. the proof of lemma 1). Also, by the previous lemma, $\mathcal{L}(z) \simeq \mathcal{L} \otimes [e^a]$ moves linearly around the Picard variety.

Now define a global section $\sigma_j(z)$ of $\mathcal{L}(z)$ by $\delta_j = \tau_z(\sigma_j(z))$: thus $\tau_0(\sigma_j(z)) = \Phi . \delta_j$ arises from the map $\psi$. We must show that the global sections $\sigma_j(z)$ have divisor of zeroes $(m-j)P_0 + jP_\infty + E_0$. Near $\lambda = 0$ and $\infty$ we have

$$\pi_* \theta_z(\sigma_j(z)) = \tilde{\chi}(z)^{-1} \delta_j$$

An examination of the leading order terms in the two Fourier series' for $\tilde{\chi}$ (one on $C_1$ and the other on $C_2$) shows that

$$\tilde{\chi}(z)^{-1} \delta_j = \begin{cases} (a_0, \ldots, a_j, 0 \ldots, 0)^t + O(\lambda) \text{ about } \lambda = 0 \\ (0, \ldots, 0, b_j, \ldots, b_n)^t + O(\lambda^{-1} \text{ about } \lambda^{-1} = 0 \end{cases} \tag{31}$$

It follows from (29) that

$$\sigma_j(z)/\theta_z^0 = \begin{cases} \zeta^{m+1}\sum_0^j a_i\zeta^{-i} + O(\zeta^{m+1}) \text{ about } P_0 \\ 0 + O(\lambda) \text{ about } P_k \in X_0, \quad k \neq 0 \\ \sum_j^m b_i\zeta^{-i} + O(\zeta^{-m-1}) \text{ about } P_\infty \end{cases}$$

This shows that $\sigma_j(z)$ has divisor of zeroes at least $(m-j)P_0 + jP_\infty + E_0$.■

## 7.6 Periodicity conditions and higher flows.

Recall that the dressing orbit $O_\Lambda$ is isomorphic to $\Lambda_{I,B}/\Gamma_{I,B}$ and therefore every primitive map in this orbit corresponds to a coset $g\Gamma_{I,B}$. An examination of the definition of the ring $\mathfrak{R}$ (via $\mathfrak{S}$) shows that it depends only upon this coset and not the particular choice of $g$. Given $\mathfrak{R}$ the module $M$ fixes $\mathcal{L}$ and we deduce that the results above provide a bijective correspondence between (i) 'full' primitive maps $\mathbf{R}^2 \to G/H$ of finite type, based by $\psi(0) = H$, up to base point preserving isometry, and (ii) triples $(X, \pi, \mathcal{L})$ satisfying the conditions described at the beginning of section 3 (including the possibility that $X$ is singular or reducible).

Recall also that the group of higher flows $\exp(\mathfrak{C}_R)$ acts on $O_\Lambda$ and, in particular, its two parameter subgroup $\exp(\mathfrak{w})$ induces the translation flow $\psi \mapsto \psi^w$ corresponding to the translation $z \mapsto z + w$ in $\mathbf{R}^2 \simeq \mathbf{C}$. One readily sees (from the previous subsection) that in terms of the triple $(X, \pi, \mathcal{L})$ this action fixes $X, \pi$ and maps $\mathcal{L}$ to $\mathcal{L}(w)$. In particular, whenever $\psi(z)$ is doubly periodic (and therefore of finite type) with periods $z_1, z_2$ we must have $\psi^{z_j} = \psi$ and therefore $\mathcal{L}(z_j) \simeq \mathcal{L}$. We deduce:

**Proposition 7** *A necessary condition for $(X, \pi, \mathcal{L})$ to correspond to a harmonic 2-torus is that there exist independent $z_1, z_2 \in \mathbf{C}$ for which $\mathcal{L}(z_j) \simeq \mathcal{L}$.*

**Remark.** This is not a sufficient condition. An examination of (21) shows that $\psi^w = \psi$ is not sufficient to imply $\psi(z + w) = \psi(z)$: they will in general differ by a factor depending upon the extended frame $\Phi_\zeta(w)$. We expect this from the discussion of periodicity conditions in §6: the periodicity of $\mathcal{L}(z)$ is condition 6a while the extra condition involving the extended frame is 6b.

It should be possible (with some extra work) to exhibit an analytic isomorphism between the $\exp(\mathfrak{C}_R)$-orbit of a map of finite type and the $J_R(X)$-orbit of the corresponding $\mathcal{L}$: we have already seen this partly reflected in corollary 3. Since the $J_R(X)$-orbit of (a maximal sheaf) $\mathcal{L}$ is isomorphic to $J_R(X)$ itself this would equate the arithmetic genus $p$ of $X$ with the dimension of the $\exp(\mathfrak{C}_R)$-orbit. But we can get this useful result more quickly from:

**Proposition 8** $H^1(X, \mathcal{O}_X) \simeq \mathfrak{C}/\mathfrak{C}^{[g]}$

**Proof.** As earlier, let $U_0, U_\infty$ be the open discs about $P_0, P_\infty$ obtained from $X_I$. Let $U$ denote their union and set $U^* = U - \{P_0, P_\infty\}$, $X^* = X - \{P_0, P_\infty\}$. Since $X$ is connected $X^*$ is a Stein manifold and therefore the sequence

$$0 \longrightarrow \mathrm{Hol}(U, \mathbf{C}) + \mathrm{Hol}(X^*, \mathbf{C})^{alg} \longrightarrow \mathrm{Hol}(U^*, \mathbf{C})^{alg} \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

is exact, where the superscript '$alg$' denotes functions with only finite order poles at $P_0, P_\infty$. Now observe that $\mathfrak{C}_* = \Lambda_C(\mathfrak{c}_*, \nu)^{alg}$ is isomorphic to $\mathrm{Hol}(U^*, \mathbf{C})$. The isomorphism is the one we have already seen: $c(\zeta^{-1}\Lambda) \mapsto c(\zeta^{-1})$. Moreover, we observe that $\mathfrak{C} = \hat{\mathfrak{B}} \oplus \mathfrak{C}_*$, where $\hat{\mathfrak{B}}$ is the subspace of all multiples of the identity matrix and this subspace is clearly contained in $\mathfrak{C}^{[g]}$. Therefore $\mathfrak{C}/\mathfrak{C}^{[g]} \simeq \mathfrak{C}_*/\mathfrak{C}_*^{[g]}$, where $\mathfrak{C}_*^{[g]} = \mathfrak{C}_* \cap \mathfrak{C}^{[g]}$. So it remains to show that $\mathfrak{C}_*^{[g]}$ corresponds to the kernel of the exact sequence above.

Clearly $\mathrm{Hol}(X^*, \mathbf{C})^{alg}$ is isomorphic to $\mathbf{C}[X^*]$ which we can realise as the subalgebra $\tilde{\mathfrak{A}}^0$ of $\tilde{\mathfrak{A}}$ of all those elements which extend to be holomorphic at every point except at $P_0, P_\infty$ i.e. in the decomposition $a = a_0 + a_1 + \ldots a_{n-m}$ each $a_j$ for $j \neq 0$ extends holomorphically into $I$. Let $\mathfrak{C}_*^0$ denote the image of this subspace after the projection of $\Lambda_C(\mathfrak{a}, \nu)^{alg}$ onto $\mathfrak{C}_*$. I claim this identifies these two spaces. For the kernel is $\{a \in \tilde{\mathfrak{A}}^0 : a_0 = 0\}$, each element of which represents a regular function on $X^*$ which vanishes on the irreducible component $X_0 \cap X^*$, where $X_0$ carries $P_0$. But such a function must be identically zero since its restriction to the other irreducible components of $X^*$ (which are complete subvarieties) must be globally regular. Now take $\mathfrak{C}_*^I$ to be the complexification of $\Lambda(\mathfrak{c}_*, \nu) \cap (\mathfrak{b} + \Lambda_I(\mathfrak{g}, \nu))$. A little thought shows that $\mathfrak{C}_*^I$ is identified with $\mathrm{Hol}(U, \mathbf{C})$ in the isomorphism $\mathfrak{C}_* \simeq \mathrm{Hol}(U^*, \mathbf{C})$. So it remains for us to show that $c \in \mathfrak{C}_*^{[g]}$ if and only if $c = a_0 + b_0$ where $a_0 \in \mathfrak{C}_*^0$ and $b_0 \in \mathfrak{C}_*^I$. In fact it suffices to prove this when $c$ is real.

First observe that $c = c_0$. When $c = a_0 + b_0$ we can define $b = b_0 - a_1 - \ldots - a_{n-m}$ so that $c = a + b$. This exhibits $c$ as an element of $(\mathfrak{C}_*^{[g]})_R = \mathfrak{C}_* \cap \mathfrak{Z}_R^{pol} \oplus \mathfrak{Z}_R^I$. Conversely, if $c \in (\mathfrak{C}_*^{[g]})_R$ then $c = a + b$ where we know $a \in \tilde{\mathfrak{A}}_R$ and $b \in \mathfrak{Z}_R^I$ commutes with every element of $\tilde{\mathfrak{A}}_R$ by the proof of lemma 21. Since $X$ is connected $\tilde{\mathfrak{A}}$ is a rank $n+1$ $\mathfrak{B}$-module, so elements of $\tilde{\mathfrak{A}}$ have maximal rank almost everywhere. Thus $b$ must take values in $\mathfrak{a}$ so that $c = a_0 + b_0$. ■

**Remark.** We deduce from this that $\mathcal{L}$ must be maximal, which is only an issue in the case where $X$ is singular, since then $\mathcal{L}$ might not be locally free.

It is particularly interesting to look at the fixed points of the $\exp(\mathfrak{C}_R)$-action. Every fixed point in $O_\Lambda$ is a map of finite type, since a result of [8] tells us that maps of finite type are precisely those with finite dimensional orbits. We can characterize the fixed points as follows.

**Proposition 9** *The spectral curve for $\psi$ has $p = 0$ precisely when the map $\psi$ is $\mathbf{R}^2$-equivariant.*

Recall that a map $\mathbf{R}^2 \to G/H$ is $\mathbf{R}^2$-equivariant when there exists a homomorphism $\varphi : \mathbf{R}^2 \to G$ for which $\psi(z) = \varphi(z)\psi(0)$.

**Proof.** First suppose $p = 0$. Then $\psi$ must be a fixed point of, in particular, the $\exp(\mathfrak{w})$-action i.e. for all $w \in \mathbf{C}$ we have $\psi^w(z) = \psi(z)$. An examination of (21) shows that the extended frame $\psi_\zeta$ satisfies $\Phi_\zeta(z+w) = \Phi_\zeta(z)\Phi_\zeta(w)$ and therefore $\Phi_1$ is the required homomorphism.

Conversely, suppose $\psi$ is $\mathbf{R}^2$-equivariant. Then

$$\psi^w = \varphi(w)^{-1}\varphi(z+w)\psi(0) = \psi(z).$$

Therefore $\mathcal{L}(z) \simeq \mathcal{L}$ for all $z$, which means the 1-cocycle $\exp(z\zeta^{-1} - \bar{z}\zeta)$ is trivial for all $z$. It follows that the class $[\zeta]$ in $H^1(X, \mathcal{O}_X)$ is trivial i.e. there exist $f \in H^0(U, \mathbf{C})$ and $g \in H^0(X^*, \mathbf{C})$ such that $\zeta = f + g$. This means $g$ is a rational function on $X$ with divisor $P_0 - P_\infty$, so this is a principal divisor. Therefore every divisor $k(P_0 - P_\infty)$ is principal, for any $k \in \mathbf{Z}$, which means that every class $[\zeta^k]$ is trivial. It follows that $H^1(X, \mathcal{O}_X)$ itself is trivial, whence $p = 0$. ■

**Example.** Here we will compute the spectral curve for any $\mathbf{R}^2$-equivariant map of isotropy order $n-1$ in $\mathbf{CP}^n$ (for $n \geq 2$). Recall $\delta_0 = (1, 0, \ldots, 0)^t$ and write $\psi_0 = \langle \varphi.\delta_0 \rangle$ where

$$\varphi(z, \bar{z}) = \exp(zA - \bar{z}A^\dagger).$$

The matrix $A$ is not uniquely fixed by $\psi_0$: we will compute a unitary matrix $A$ using the harmonic sequence. First observe that the harmonic sequence for $\psi_0$ is completely determined by the lines at $z = 0$, which are

$$\langle \delta_0 \rangle, \langle A\delta_0 \rangle, \ldots, \langle A^j \delta_0 \rangle, \ldots.$$

Since the isotropy order is $n-1$ we know the inner product $(A^j\delta_0, A^k\delta_0)$ vanishes for $j < k$ whenever $k - j \leq n - 1$. To compute $A$ we take an orthonormal basis $\{e_0, \ldots, e_n\}$ for $\mathbf{C}^{n+1}$ given by $e_j = A^j\delta_0$ for $j = 0, \ldots, n-1$ and

$$e_n = r^{-1}(A^n e_0 - a e_0)$$

where $a = (A^n e_0, e_0)$, $r^2 = \|A^n e_0 - a e_0\| = 1 - |a|^2$. With respect to this basis it is straghtforward to show that $A$ is represented by the matrix

$$\begin{pmatrix} 0 & 0 & . & . & a & br^{-1} \\ 1 & 0 & . & . & 0 & 0 \\ 0 & 1 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & 0 & 0 \\ 0 & 0 & . & . & r & -\bar{a}br^{-2} \end{pmatrix}$$

where $b = (A^n e_0, e_0)$ (it follows that $|b|^2 = r^2$). Since $A$ is unitary it commutes with $-A^\dagger$ and therefore $\psi$ is a homomorphism which clearly frames $\psi_0$ (and its unique primitive lift, by construction). In fact we can easily compute the corresponding extended frame: observe that

$$A_\zeta = \zeta^{-1} \begin{pmatrix} 0 & 0 & . & . & a & br^{-1} \\ 1 & 0 & . & . & 0 & 0 \\ 0 & 1 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & 0 & 0 \\ 0 & 0 & . & . & r & -\bar{a}br^{-2} \end{pmatrix}$$

is $\nu$-equivariant and commutes with $\bar{A}_\zeta$ $(= -A^\dagger_{\bar{\zeta}^{-1}})$. Therefore $\varphi_\zeta = \exp(zA_\zeta + \bar{z}\bar{A}_\zeta)$ is an extended frame for $\psi_0$ (as well as the primitive lift).

Now let us show that, provided $b \neq 0$, the characteristic polynomial of $A_\zeta$ describes the affine spectral curve $X_A$.

First observe that for $m = n-1$ the semisimple matrix $\Lambda$ is regular and therefore $\mathfrak{Z}$ is abelian. It follows that $\mathfrak{Z}^{pol} = \mathfrak{R}$. For any $\eta_\zeta \in \mathfrak{R}$ which is regular with non-zero eigenvalues for almost all $\zeta$ I claim the subalgebra $\mathbf{C}[\lambda Id, \eta_\zeta] \subset \mathfrak{R}$ is isomorphic to $\mathbf{C}[\lambda,\mu]/\langle \det(\eta_\zeta - \mu Id)\rangle$. To see this we observe that: (i) since $\eta_\zeta$ is $\nu$-equivariant its characteristic polynomial is a function of $\lambda = \zeta^n$ and $\mu$; (ii) the minimal polynomial of $\eta_\zeta$ for almost all $\zeta$ is the characteristic polynomial. Moreover, since $\eta_\zeta$ is regular almost everywhere the corresponding planar affine curve $X_\eta$ is a degree $n+1$ cover of $\mathbf{C}^*_\lambda$. Therefore the finite morphism $X_A \to X_\eta$ dual to the inclusion of algebras has degree 1 i.e. it is a birational map. We call $X_\eta$ a *planar model* for $X_A$.

Now let us see that we can take $A_\zeta$ in place of $\eta_\zeta$. Although $\varphi_\zeta$ might not be the canonical extended frame $\Phi_\zeta$ it is certainly gauge equivalent to it. This means the polynomial Killing fields for $\varphi_\zeta$ are conjugate (by an element of the gauge group) to the polynomial Killing fields for $\Phi_\zeta$, which are the elements of $\mathfrak{R}$ (since every Killing field will be independent of $z, \bar{z}$). Set $\beta_\zeta = \varphi_\zeta^{-1} d\varphi_\zeta$, then it is trivial to show $dA_\zeta = [A_\zeta, \beta_\zeta]$ i.e. $A_\zeta$ is a polynomial Killing field for $\varphi_\zeta$. Thus there is a $\eta_\zeta \in \mathfrak{R}$ with the same characteristic polynomial as $A_\zeta$.

Now let us examine the planar model corresponding to $A_\zeta$. We compute

$$\det(\mu.Id - A_\zeta) = \mu^n(\mu + \bar{a}br^{-2}) - \zeta^{-n}(a\mu + br^{-2})$$

and therefore, for $b \neq 0$ (which is precisely the case for which $A_\zeta$ has distinct non-zero eigenvalues almost everywhere) $X_\eta$ is just $\mathbf{C}^*_\mu$ equipped with the rational function

$$\lambda = \mu^{-n}\left(\frac{a\mu + br^{-2}}{\mu + \bar{a}br^{-2}}\right).$$

Since $X_\eta$ is smooth its smooth completion $\mathbf{P}_\mu$ must be the spectral curve $X$ and $\pi : \mathbf{P}_\mu \to \mathbf{P}_\lambda$ is given by $\lambda$. This has degree $n+1$ with divisor $(n-1)P_0 + P_1 - (n-1)P_\infty - Q_1$ where $P_0, P_\infty, P_1, Q_1$ are the values $\mu = 0, \infty, -ba^{-1}r^{-2}, -\bar{a}br^{-2}$. It is easy to check that $\pi(\bar{\mu}^{-1}) = \overline{\pi(\mu)}^{-1}$, so $X, \pi$ fit precisely into the description we expect for the spectral curve with $m = n-1$.

## 7.7 Appendix: the proof of lemma 17.

Recall that $F^{m-1}(\mathbf{CP}^n) \simeq G/H$ where $H$ is the fixed point subgroup for the periodic automorphism $\nu$ of order $m+1$. The induced automorphism $\nu$ on $\mathfrak{g}^{\mathbf{C}}$ gives rise to the eigenspace decomposition (4). In particular $\mathfrak{h}^{\mathbf{C}} = \mathfrak{g}_0$ is the complexified Lie algebra of $H$. Clearly $\mathfrak{g}_{-1}$ is a representation space for $\mathrm{Ad}H^{\mathbf{C}}$ and it is a theorem of Vinberg [22] that all the semisimple elements of $\mathfrak{g}_{-1}$ lie in the $\mathrm{Ad}H^{\mathbf{C}}$-orbit of a maximal abelian subspace $\mathfrak{c} \subset \mathfrak{g}_{-1}$ all of whose elements are semisimple. Vinberg calls such a subspace a *Cartan subspace* for the pair $(\mathfrak{g}^{\mathbf{C}}, \nu)$ and defines the (graded) *rank* of $(\mathfrak{g}^{\mathbf{C}}, \nu)$ to be the dimension of $\mathfrak{c}$. We will also call this the rank of $G/H$. By the result of Kac [14] the decomposition (4) is determined by a non-positive (in our convention) integer labelling of the extended Dynkin diagram for $\mathfrak{g}^{\mathbf{C}} = \mathfrak{sl}_{n+1}$. Equally, we can

assign a non-positive integer to each root $\alpha_0, \ldots, \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are simple roots for $\mathfrak{g}^{\mathbf{C}}$ (with respect to a fixed Cartan subalgebra $\mathfrak{a}$ in $\mathfrak{h}^{\mathbf{C}}$) and $\alpha_0$ is minus the maximal root $\sum_1^n \alpha_j$. The labelling we require gives weight -1 to each $\alpha_j$ for $j = 1, \ldots, m+1$ and weight zero to the rest. [Here we are assuming the simple roots are ordered in the usual way on the Dynkin diagram.] One readily checks that $\mathfrak{g}_0 \simeq \mathfrak{sl}_{n-m} \oplus \mathbf{C}^{m+1}$ for this automorphism, and that $G/H$ is isomorphic to the flag manifold $F^{m-1}(\mathbf{CP}^n)$.

To prove the lemma we will present a semisimple $\Lambda \in \mathfrak{g}_{-1}$ for which

$$\dim[\Lambda, \mathfrak{g}_0] = \dim \mathfrak{g}_{-1} - 1. \tag{32}$$

It follows that the Cartan subspace containing $\Lambda$ must be one dimensional. Now we must fix some notation.

Set

$$\delta = \alpha_0 + \sum_{m+1}^{n} \alpha_j.$$

Then $\delta$ also has weight $-1$. Now let $e_1, \ldots, e_m, e_\delta$ be non-zero root vectors for $\alpha_1, \ldots, \alpha_m, \delta$ and define

$$\Lambda = \sum_{1}^{m} e_j + e_\delta$$

Then it is not hard to show that $\Lambda$ is semisimple and by construction it lies in $\mathfrak{g}_{-1}$. We will now show that (32) holds for this $\Lambda$.

The subspace $\mathfrak{g}_{-1}$ is generated by all the root vectors for all the roots of weight $-1$. Every such root is readily seen to lie in one of the following three lists:

1. $\alpha_1, \ldots, \alpha_m, \delta$,
2. $\alpha_{m+1}, \alpha_{m+1} + \alpha_{m+2}, \ldots, \alpha_{m+1} + \ldots + \alpha_n + \alpha_0$,
3. $\alpha_1 + \alpha_0, \alpha_1 + \alpha_0 + \alpha_n, \ldots, \alpha_1 + \alpha_0 + \ldots + \alpha_{m+2}$.

For lists 2 and 3 let us show that for every root $\alpha$ in the list there is a root $\beta$ of weight zero, with root vector $e_\beta$, such that $[\Lambda, e_\beta]$ is a non-zero root vector for $\alpha$. This will show that $[\Lambda, \mathfrak{g}_0]$ contains all the root spaces $\mathfrak{g}^\alpha$ for the roots in lists 2 and 3. Indeed this reduces to the following observations. Let $\alpha$ belong to list 2, then

$$\alpha = \sum_{m}^{l} \alpha_j = \delta - \alpha_0 - \sum_{l+1}^{n} \alpha_j$$

for some $l \geq m+1$. So if $-\beta = \alpha_0 + \sum_{l+1}^n \alpha_j$ (which has weight zero for $l \geq m+1$) then $\alpha = \beta + \delta$ and $\alpha_j + \beta$ is not a root for $j = 1, \ldots, m$. Therefore, for any root vector $e_\beta$ for $\beta$, $[\Lambda, e_\beta] = [e_\delta, e_\beta]$, which is a non-zero root vector for $\alpha$.

Similarly, when $\alpha$ belongs to list 3 we write

$$\alpha = \alpha_1 + \alpha_0 + \sum_{l}^{n} \alpha_j = \alpha_1 + \beta, \qquad \text{for } l \geq m+2,$$

where once again $\beta$ has weight zero. Notice that $\alpha_j + \beta$ is not a root for $j = 2, \ldots, m$ and neither is $\delta + \beta$. Therefore $[\Lambda, e_\beta] = [e_1, e_\beta]$, which is a non-zero root vector for $\alpha$.

It remains to show that, if $\mathfrak{v}$ denotes the subspace of $\mathfrak{g}_{-1}$ generated by all root spaces for roots in list 1 then $[\Lambda, \mathfrak{g}_0] \cap \mathfrak{v}$ has codimension 1 in $\mathfrak{v}$. First observe that $[\Lambda, \mathfrak{g}_0] \cap \mathfrak{v} = [\Lambda, \mathfrak{a}]$. Now it is an elementary exercise to show that $[\Lambda, \mathfrak{a}]$ consists all elements of the form

$$\sum_1^m \lambda_j e_j + \lambda_{m+1} e_\delta$$

where $\lambda_1, \ldots, \lambda_{m+1}$ are complex numbers not all of which are equal. Hence $[\Lambda, \mathfrak{a}]$ has codimension 1 in $\mathfrak{v}$. This finishes the proof.

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