A plethora of solitary gravity-capillary water waves with nearly critical Bond and Froude numbers

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Abstract

This paper considers the existence of solitary-wave solutions of the classical water-wave problem in the presence of surface tension. A region of Bond number-Froude number parameter space close to $(1/3, 1)$ is identified, at each point of which there are infinitely many distinct, multi-troughed solitary waves of depression.

The method is to study a Hamiltonian formulation of the mathematical problem for solitary waves using a centre-manifold technique valid near Bond number $1/3$ and Froude number 1. The problem is thus replaced by an equivalent problem posed on a 4-dimensional manifold. In a certain region of parameter space near $(1/3, 1)$ there is a Smale horseshoe in the dynamics on the centre manifold and therefore infinitely many distinct homoclinic orbits.

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1 Introduction

1.1 The main result

This paper considers the existence of solitary-wave solutions of the classical water-wave problem, which concerns the two-dimensional, irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The mathematical formulation of this hydrodynamic problem, which is stated in Section 2.1 (equations (12), (13), (14), (15), (16)) involves two dimensionless parameters, namely the *Froude number* $F = \sqrt{c^2/gh}$ and the *Bond number* $\tau = \sigma/gh^2$, where σ is the coefficient of surface tension, h is the asymptotic depth of the water and c is the velocity of the solitary wave. It is well known that eigenvalues ν of the linearised version of (12), (13), (14), (15) satisfy the equation

$$\nu\cos(\nu) = (\alpha - \beta\nu^2)\sin(\nu),$$

where $\alpha = 1/F^2$, $\beta = \tau/F^2$. Kirchgässner (1988) was the first to recognise the crucial role played in bifurcation theory by parameter values (β, α) at which the number of purely imaginary eigenvalues $\nu = is$ of the linearised problem changes. These values are shown in Figure 1 by the curves

$$C_2 = \left\{(\beta,\alpha) = \left(\frac{-1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)}, \frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)}\right) : s \in [0,\infty)\right\}$$

and

$$C_3 = \{(\beta,\alpha) : \beta \le 1/3, \alpha = 1\}, \qquad C_4 = \{(\beta,\alpha) : \beta \ge 1/3, \alpha = 1\}.$$

The curve

$$C_1 = \left\{(\beta,\alpha) = \left(\frac{1}{2\sin^2(\nu)} - \frac{1}{2\nu\tan(\nu)}, \frac{\nu^2}{2\sin^2(\nu)} + \frac{\nu}{2\tan(\nu)}\right) : \nu \in [0,\pi)\right\} \qquad (1)$$

depicts a different phenomenon: it is the locus of points where four non-zero, real, simple eigenvalues become complex without passing through zero. Amick & Kirchgässner (1989) considered the bifurcation of solitary waves near the curve $C_4 \setminus \{(1/3, 1)\}$ and proved the existence of a unique solitary wave of depression just above this curve. The curve $C_2 \setminus \{(1/3, 1)\}$ was studied by Iooss & Kirchgässner (1990), who found two solitary waves, one of elevation and one of depression, just to the right of the curve. A different bifurcation phenomenon near $C_3 \setminus \{(1/3, 1)\}$ was discovered independently by Beale (1991) and Sun (1991) and elaborated upon by other authors, notably Lombardi (1992) and Iooss & Kirchgässner (1992). At such points the existence of *generalised solitary waves*, which have an oscillatory tail of very small amplitude, has been established. At the time of writing it remains unknown whether there are true solitary waves for these parameter values. (By definition true solitary waves converge to the asymptotic depth h at infinity.) Finally, Mielke, Holmes & O'Reilly (1992) discussed the possibility of chaotic behaviour when $\beta < 1/3$, $\alpha \sim 1$, but found none (see Mielke (1991, p. 108)).

The above observations make it clear that $(\beta, \alpha) = (1/3, 1)$ is the critical point in parameter space. The main result of the present paper concerns bifurcation phenomena near $(\beta, \alpha) = (1/3, 1)$ and may be stated as follows. There exist real numbers $\epsilon_1, \epsilon_2 > 0$ such that in the region

$$\beta = \frac{1}{3} + d\mu^2, \quad \alpha = 1 + \frac{1}{45}\mu^4, \qquad d \in ((2-\epsilon_1)/45, 2/45), \quad 0 < |\mu| < \epsilon_2 \qquad (2)$$

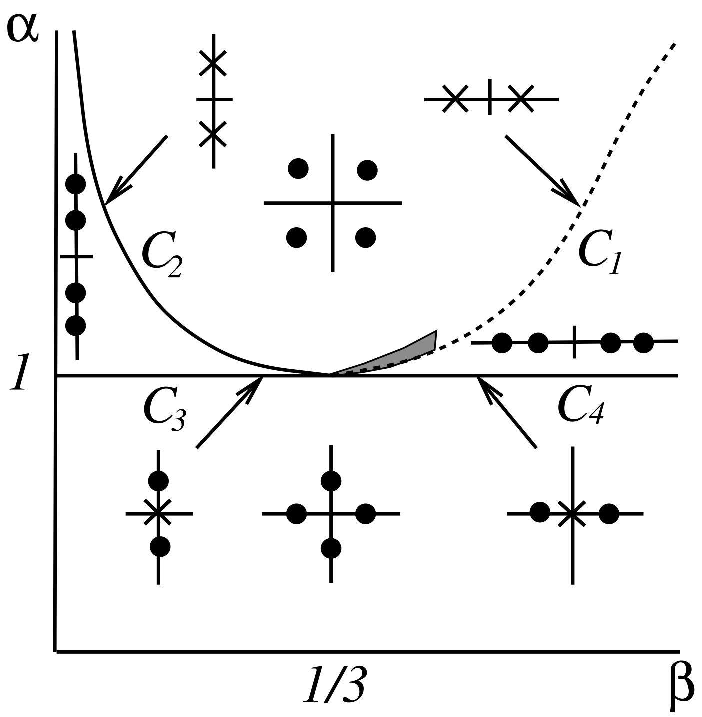

Figure 1: The curves C_2, C_3, C_4 consist of points in (β, α)-parameter space at which the number of purely imaginary eigenvalues of the linearised problem changes; at C_1 four non-zero, real eigenvalues become complex without passing through zero. Dots and crosses denote respectively simple and double eigenvalues. The shaded region is the region to be studied in this paper.

(the shaded region in Figure 1) there are infinitely many distinct solitary waves. These are waves of depression which have an arbitrary but finite number of troughs, between which there are distributed smaller local maxima and minima, and which have an exponentially decaying oscillatory tail at infinity. They are divided into an infinite number of classes, each of which contains infinitely many distinct solitary waves; the classes are characterised by the number of large troughs possessed by each of their members. The overall structure of the solution set is that associated with the presence of a Smale horseshoe in a finite-dimensional dynamical system and may be interpreted precisely in terms of associated symbolic dynamics. It is to be emphasised that these waves are true solitary-wave solutions of the exact water-wave problem which converge exponentially to the zero flow with depth h at infinity. In addition to the solitary waves, the associated symbolic dynamics ensures the existence of a countably infinite family of periodic travelling waves described by the number of large troughs in a period and the number of small oscillations between consecutive troughs.

1.2 The method

The method used is to reduce the mathematical problem for solitary waves to an equivalent system of ordinary differential equations by means of a centre-manifold reduction technique due to Mielke (1988). The first step in this method is to formulate the problem as an infinite-dimensional, quasilinear dynamical system. Mielke (1991) has examined the case when such a system is Hamiltonian and has shown that the resulting reduced system is also Hamiltonian. In the present paper Mielke's theory is applied to a Hamiltonian formulation of the mathematical problem for solitary waves discussed in Section 2. Section 3 is concerned with the statement and application of the centre-manifold reduction theorem, and the reduced system of equations is calculated in Section 4. This system is four dimensional and valid for parameters in a whole neighbourhood of $(\beta, \alpha) = (1/3, 1)$. As $\mu \to 0$, its limiting form in the parameter regime (2) is equivalent to a single forced ordinary differential equation

$$u^{(iv)} + Pu'' + u - u^2 = O(|\mu|), \qquad P \in (-2, -2 + \epsilon_1), \tag{3}$$

where the variable u is related to the dimensionless depth η of the water by the relation

$$\eta(x) = 1 - \frac{2}{135}\mu^4 u(\mu x) + O(|\mu|^7).$$

Notice that the $O(|\mu|)$ forcing in (3) and the $O(|\mu|^7)$ term in the formula for η depend upon the other three components of the flow on the centre manifold.

Equation (3) has been the subject of much recent study (see Hunt, Bolt & Thompson (1989), Hunt & Wadee (1991), Amick & Toland (1992a; 1992b), Iooss (1992), Champneys & Toland (1993), Buffoni (1993a; 1993b; 1993c), Buffoni, Champneys & Toland (1994), Buffoni & Séré (1994) and the references therein). In particular, Buffoni, Champneys & Toland (1994) have shown that when $P = -2$ the limiting form of (3) as $\mu \to 0$ has a unique non-zero homoclinic solution which is positive and even and that it has a non-trivial homoclinic solution which is the transverse intersection (relative to the zero energy surface) of the stable and unstable manifolds of its zero equilibrium for all $P \in (-2, -2+\epsilon_1)$. Since transversality is an open condition, the same is true of (3) when $|\mu| \in (0, \epsilon_2)$. For parameter values in the regime (2) the equation has a saddle-focus at 0, and it follows immediately by the work of Devaney (1976) (see Wiggins (1988, p. 275)) that there is a Smale horseshoe in the Hamiltonian dynamics on the zero energy surface. Consequently there are infinitely many distinct homoclinic solutions of the four-dimensional equations on the centre manifold.

The above theory proves a conjecture, made by Champneys & Toland (1993), that the capillary-gravity water-wave problem (12), (13), (14), (15), (16) has infinitely many solitary-wave solutions for certain values of the Bond and Froude numbers above the curve C_1. This result complements the work of Iooss & Kirchgässner (1990), who found two solitary waves near C_2, and Amick & Kirchgässner (1989), who found a solitary wave near C_4 and proved it to be unique. (In the absence of surface tension the uniqueness of small-amplitude solitary waves of elevation has been understood since the classical paper by Friedrichs & Hyers (1954) (see also Beale (1977)).) In the case studied by Iooss & Kirchgässner (1990), the linearised problem has a complex-conjugate pair of non-zero imaginary eigenvalues of geometric multiplicity 1 and algebraic multiplicity 2. The corresponding equations on the centre manifold involve a single parameter and are studied using a normal form explained in detail by Iooss & Pérouème (1993). By contrast, the analysis in the present paper concerns a linearised system with a zero eigenvalue of geometric multiplicity 1 and algebraic multiplicity 4, and the corresponding equations on the centre manifold involve two parameters. This time the essential ingredient in the study of the equations is not normal-form analysis but the Hamiltonian structure, the transversality result of Buffoni, Champneys & Toland (1994) and the work by Devaney (1976).

The present results might be regarded as mainly of mathematical interest since, for water in air at $20°$C, values of $F \sim 1$ and $B \sim 1/3$ correspond to waves on water of depth 4.5mm which travel with a velocity of 21.5cms^{-1}; in practice such waves may be dominated by viscous effects. However, there is no reason to believe that the existence of infinitely many gravity-capillary waves is restricted to the small region discussed here. Work on a model problem (Buffoni *et al.* 1994) leads to the speculation that there may be a plethora of solitary-wave solutions of the gravity-capillary water-wave problem for all values of α and β such that (β, α) lies above the curve $C_1 \cup C_2$ in Figure 1. If the conclusions for the model problem are a guide to the behaviour of large-amplitude gravity-capillary solitary waves for parameter values far from criticality, then symmetry-breaking bifurcations and coalescences are to be expected. Preliminary results in this direction are indicated by the work of Zufiria (1987) and Dias, Menasce & Vanden-Broeck (1994). The only global theory of solitary water waves currently available is that of Amick & Toland (1981b; 1981a), which gives the existence of waves of all amplitudes up to and including the Stokes' wave of greatest height in the absence of surface tension. Craig & Sternberg (1988) have shown that all such waves are necessarily symmetric; this result contrasts with present speculation concerning the existence of asymmetric solitary waves when surface tension is present.

2 The hydrodynamic problem and its Hamiltonian formulation

2.1 The steady water-wave problem

The classical water-wave problem concerns the two-dimensional, irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. Let (x, y) denote the usual Cartesian coordinates. The fluid occupies the domain $D_\eta = \{(x, y) : x \in \mathbb{R},\ y \in (0, \eta(x, t))\}$, where $\eta > 0$ is a function of the spatial coordinate x and time t. In terms of an Eulerian velocity potential $\phi(x, y, t)$, the mathematical problem is to solve Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{in} \quad D_\eta \tag{4}$$

with boundary conditions

$$\begin{aligned} \phi_y &= 0 \qquad \text{on} \quad y = 0, & (5)\\ \eta_t &= \phi_y - \eta_x\phi_x \qquad \text{on} \quad y = \eta(x, t), & (6)\\ \phi_t &= -\frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 - g\eta + \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x + B \qquad \text{on} \quad y = \eta(x, t), & (7)\end{aligned}$$

in which $\sigma \geq 0$ is the coefficient of surface tension and B is a constant called the Bernoulli constant (e.g. see Benjamin (1974, p. 5)). Equation (5) is the kinematic condition that water cannot permeate the rigid horizontal boundary at $y = 0$, while (6), (7) are respectively the kinematic and dynamic conditions at the free surface.

Waves that are steady with respect to a uniformly-translating frame of reference are described by solutions of (4), (5), (6), (7) of the special form $\eta(x, t) = \eta(x - ct)$, $\phi(x, y, t) = \phi(x - ct, y)$. Substituting this form of η, ϕ into (4), (5), (6), (7), one finds that

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{for} \quad 0 < y < \eta(x), \tag{8}$$

$$\phi_y = 0 \qquad \text{on} \quad y = 0, \tag{9}$$
$$\phi_y = \eta_x \phi_x - c\eta_x \qquad \text{on} \quad y = \eta(x), \tag{10}$$
$$\frac{1}{2}\phi_x^2 - c\phi_x + \frac{1}{2}\phi_y^2 + g\eta - \sigma \left[\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right]_x = B \qquad \text{on} \quad y = \eta(x), \tag{11}$$

where x is now a shorthand for the variable $x - ct$. A *solitary wave* is a non-trivial solution of (8), (9), (10), (11) together with the asymptotic conditions

$$\eta(x) \to h, \quad (\phi_x, \phi_y) \to (0,0) \qquad \text{as} \quad x \to \pm\infty,$$

where $h > 0$ is a constant. (Here 'trivial' means $\eta(x) = h$ for all x and $(\phi_x, \phi_y) = 0$ for all x and y.) The correct choice of the Bernoulli constant in (11) is therefore $B = gh$ when one is seeking solitary-wave solutions. Let us now introduce the dimensionless variables

$$(x', y') = \frac{1}{h}(x, y), \qquad \eta'(x') = \frac{1}{h}\eta(x), \qquad \phi'(x', y') = \frac{1}{ch}\phi(x, y),$$

in terms of which the problem for solitary waves becomes

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{for} \quad 0 < y < \eta(x), \tag{12}$$
$$\phi_y = 0 \qquad \text{on} \quad y = 0, \tag{13}$$
$$\phi_y = \eta_x \phi_x - \eta_x \qquad \text{on} \quad y = \eta(x), \tag{14}$$
$$\frac{c^2}{gh}\left(-\phi_x + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 \right) + \eta - 1 - \frac{\sigma}{gh^2}\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right]_x = 0 \qquad \text{on} \quad y = \eta(x) \tag{15}$$

with asymptotic conditions

$$\eta \to 1, \qquad (\phi_x, \phi_y) \to 0 \qquad \text{as} \quad x \to \pm\infty, \tag{16}$$

where the primes have been dropped for notational simplicity. The dimensionless parameters

$$F = \sqrt{\frac{c^2}{gh}}, \qquad \tau = \frac{\sigma}{gh^2}$$

are called respectively the *Froude* and *Bond* numbers. It is usual to regard h and σ, and hence τ, as being fixed by experimental conditions and to view F as a dimensionless parameter that measures the velocities of the waves.

Suppose that η and ϕ solve (12), (13), (14), (15) and let $\psi(x, y)$ be the harmonic conjugate of $\phi(x, y)$ with $\psi(0, 0) = 0$. An application of the Cauchy-Riemann equations to (12), (13), (14), (15) shows that ψ satisfies the equations

$$\psi_{xx} + \psi_{yy} = 0 \qquad \text{for} \quad 0 < y < \eta(x), \tag{17}$$
$$\psi(x, 0) = 0 \qquad \text{for} \quad x \in \mathbb{R}, \tag{18}$$
$$\psi(x, \eta(x)) - \eta(x) = k \qquad \text{for} \quad x \in \mathbb{R}, \tag{19}$$
$$\frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2 - \psi_y + \alpha(\eta - 1) - \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right]_x = 0 \qquad \text{on} \quad y = \eta(x), \tag{20}$$

where k is a constant and $\alpha = 1/F^2$, $\beta = \tau/F^2$. The asymptotic conditions (16) transform to

$$\eta \to 1, \qquad (\psi_x, \psi_y) \to 0 \qquad \text{as} \quad x \to \pm\infty, \tag{21}$$

and these conditions imply that $k = -1$. Similarly, suppose that η and ψ solve (17), (18), (19), (20), and let $\phi(x, y)$ be a harmonic conjugate of $\psi(x, y)$. An identical argument shows that η and ϕ solve (12), (13), (14), (15), and (16) follows from (21). Notice that (12), (13), (14), (15), (16) is equivalent to (17), (18), (19), (20), (21) with $k = -1$, but one can say only that (12), (13), (14), (15) is equivalent to (17), (18), (19), (20) with arbitary k.

This paper is concerned with the existence of solitary waves when (β, α) is near $(1/3, 1)$. The strategy is to prove the existence of nontrivial solutions of (17), (18), (19), (20) with $k = -1$ together with the asymptotic conditions (21) that are relevant for solitary waves.

2.2 *Hamiltonian formulation*

The purpose of this section is to formulate (17), (18), (19), (20) with $k = -1$ as an infinite-dimensional, quasilinear Hamiltonian system. Equations (21) are regarded as asymptotic conditions that distinguish homoclinic orbits of the Hamiltonian flow. The Hamiltonian formulation to be introduced is similar to that introduced by Mielke (1991, Chapter 9) and Baesens & MacKay (1992), which was analysed by Groves & Toland (1995). The present formulation is a variant that has advantages for the subsequent mathematical analysis concerning the linearisation of the nonlinear boundary condition. It also reflects both the velocity-potential and stream-function formulations of the steady water-wave problem, in the latter respect being specific to the case when $k = -1$ in equation (19) (see the derivation of (43) below); the corresponding solution of (12), (13), (14), (15) satisfies $\int_0^\eta \phi \, dy = 0$. This additional restriction has the effect of reducing the dimensionality of the generalised eigenspace corresponding to the zero eigenvalue of the linearised problem by 2.

The Hamiltonian system (M, Ω, H) in question is defined on the manifold

$$M = \{(\eta, \omega, \Phi, \Gamma) \in (0, \infty) \times \mathbb{R} \times H^1(0,1) \times H_0^1(0,1) : \int_0^1 \Phi \, dz = 0, |W| < \beta\}, \tag{22}$$

where

$$W = \omega + \frac{1}{\eta}\int_0^1 z\Phi_z(\Gamma_z - 1)\, dz = \omega - \frac{1}{\eta}\Phi(1) + \frac{1}{\eta}\int_0^1 z\Phi_z\Gamma_z \, dz,$$

which is an open subset of the Hilbert space

$$\mathcal{H} = \{(\eta, \omega, \Phi, \Gamma) \in \mathbb{R} \times \mathbb{R} \times H^1(0,1) \times H_0^1(0,1) : \int_0^1 \Phi \, dz = 0\}.$$

The symplectic 2-form Ω on M is defined (independently of $m \in M$) by

$$\Omega|_m((\bar\eta_1, \bar\omega_1, \bar\Phi_1, \bar\Gamma_1), (\bar\eta_2, \bar\omega_2, \bar\Phi_2, \bar\Gamma_2)) = \bar\omega_2\bar\eta_1 - \bar\eta_2\bar\omega_1 + \int_0^1 (\bar\Gamma_{2z}\bar\Phi_1 - \bar\Gamma_{1z}\bar\Phi_2)\, dz \tag{23}$$

and the Hamiltonian function is

$$H(\eta, \omega, \Phi, \Gamma, \alpha, \beta) = \frac{1}{2\eta}\int_0^1 (\Gamma_z^2 - \Phi_z^2)\, dz - \frac{1}{2}\alpha(\eta-1)^2 + \frac{1}{2}\left(\frac{1}{\eta} - 1\right) + \frac{1}{2}(\eta - 1) + \beta - \sqrt{\beta^2 - W^2}. \tag{24}$$

The domain $\mathcal{D}(v_H)$ of the Hamiltonian vector field v_H is the set of all $(\eta, \omega, \Phi, \Gamma) \in M$ such that $(\eta, \omega, \Phi, \Gamma)$ belongs to $(0, \infty) \times \mathbb{R} \times H^2(0,1) \times H^2(0,1)$ and satisfies

$$\begin{aligned} \Phi_z &= 0 \qquad \text{on} \quad z = 0, && (25)\\ \Phi_z - \frac{W(\Gamma_z - 1)}{\sqrt{\beta^2 - W^2}} &= 0 \qquad \text{on} \quad z = 1, && (26) \end{aligned}$$

and the smooth vector field v_H is defined on $\mathcal{D}(v_H)$ by

$$\begin{pmatrix} \eta \\ \omega \\ \Phi \\ \Gamma \end{pmatrix} \mapsto \begin{pmatrix} \frac{W}{\sqrt{\beta^2 - W^2}} \\ \frac{1}{2\eta^2}\int_0^1 (\Gamma_z^2 - \Phi_z^2)\,dz + \frac{W}{\eta^2\sqrt{\beta^2-W^2}}\int_0^1 z\Phi_z(\Gamma_z - 1)\,dz + \frac{1}{2\eta^2} + \alpha(\eta - 1) - \frac{1}{2} \\ \frac{\Gamma_z}{\eta} + \frac{W}{\eta\sqrt{\beta^2-W^2}}\left(z\Phi_z - \Phi(1)\right) \\ -\frac{\Phi_z}{\eta} + \frac{zW(\Gamma_z - 1)}{\eta\sqrt{\beta^2-W^2}} \end{pmatrix}.$$

A continuously-differentiable path $\gamma : (a, b) \to \mathcal{D}(v_H) \subset M$ given by

$$\gamma(x) = (\eta(x), \omega(x), \Phi(x)(\cdot), \Gamma(x)(\cdot)).$$

is therefore a trajectory of v_H if it satisfies Hamilton's equations

$$\dot{\gamma}(x) = v_H(\gamma(x)), \qquad x \in (a, b). \tag{27}$$

One would like to proceed by introducing two new variables $\tilde{\Phi}$, $\tilde{\Gamma}$ that are formally defined by $\tilde{\Phi}(x, z) = \Phi(x)(z)$, $\tilde{\Gamma}(x, z) = \Gamma(x)(z)$. The following theorem justifies this step; its proof involves functional-analytic considerations of measurability and regularity, and is given by an analysis similar to that carried out by Groves & Toland (1995, Theorems 2,3).

Theorem 1 *Suppose that $\gamma : (a, b) \to \mathcal{D}(v_H) \subset M$ is a continuously-differentiable path that satisfies Hamilton's equations for the Hamiltonian system (M, Ω, H). There exist functions $\tilde{\Phi}$, $\tilde{\Gamma}$ that are measurable on $(a, b) \times (0, 1)$ and belong to $H^2[(x_1, x_2) \times (0, 1)]$ for each x_1, x_2 with $a < x_1 < x_2 < b$. These functions satisfy the equations*

$$\begin{aligned} \tilde{\Phi}_x(x, z) &= \frac{1}{\eta(x)} Ga\tilde{m}ma_z(x, z) + \frac{\tilde{W}(x)}{\eta(x)\sqrt{\beta^2 - \tilde{W}^2(x)}}\left(z\tilde{\Phi}_z(x, z) - \tilde{\Phi}(x, 1)\right), && (28)\\ \tilde{\Gamma}_x(x, z) &= -\frac{1}{\eta(x)}\tilde{\Phi}_z(x, z) + \frac{z\tilde{W}(x)(\tilde{\Gamma}_z(x, z) - 1)}{\eta(x)\sqrt{\beta^2 - \tilde{W}^2(x)}}, && (29) \end{aligned}$$

where

$$\tilde{W}(x) = \omega(x) + \frac{1}{\eta(x)}\int_0^1 z\tilde{\Phi}_z(x, z)\tilde{\Gamma}_z(x, z)\,dz - \frac{\tilde{\Phi}(x, 1)}{\eta(x)}, \qquad |\tilde{W}| < \beta, \tag{30}$$

in the weak sense on $(x_1, x_2) \times (0,1)$ *with boundary conditions*

$$\begin{aligned}
\tilde{\Gamma}(x,0) &= 0, && (31)\\
\tilde{\Gamma}(x,1) &= 0, && (32)\\
\tilde{\Phi}_z(x,0) &= 0, && (33)\\
\tilde{\Phi}_z(x,1) &= \frac{\tilde{W}(x)(\tilde{\Gamma}_z(x,1)-1)}{\sqrt{\beta^2-\tilde{W}^2(x)}}; && (34)
\end{aligned}$$

in addition $\tilde{\Phi}$ *satisfies the condition that* $\int_0^1 \tilde{\Phi}(x,z)\,dz = 0$ *for all* $x \in (x_1, x_2)$. *The functions* η *and* ω *are twice continuously differentiable in* x *and satisfy the equations*

$$\begin{aligned}
\eta_x(x) &= \frac{\tilde{W}(x)}{\sqrt{\beta^2-\tilde{W}^2(x)}}, \qquad \eta(x) > 0, && (35)\\
\omega_x(x) &= \frac{1}{2\eta^2(x)}\int_0^1 (\tilde{\Gamma}_z^2(x,z) - \tilde{\Phi}_z^2(x,z))\,dz + \alpha(\eta(x)-1) + \frac{1}{2\eta^2(x)} - \frac{1}{2}\\
&\quad + \frac{\tilde{W}(x)}{\eta^2(x)\sqrt{\beta^2-\tilde{W}^2(x)}}\left(\int_0^1 z\tilde{\Gamma}_z(x,z)\tilde{\Phi}_z(x,z)\,dz - \tilde{\Phi}(x,1)\right). && (36)
\end{aligned}$$

The next step is to define two new variables $\phi, \psi \in H^2[\{(x,y) : x \in (x_1,x_2), y \in (0,\eta(x))\}]$ by the formulae

$$\begin{aligned}
&\phi(x, z\eta(x)) = \phi(x,y) = \tilde{\Phi}(x,z) + \int_{x_1}^{x} \left\{ \frac{\eta_x(s)}{\eta(s)}\tilde{\Phi}(s,1) + 1 - \frac{1}{\eta(s)} \right\} ds,\\
&\psi(x, z\eta(x)) = \psi(x,y) = \tilde{\Gamma}(x,z) + z(\eta(x)-1),
\end{aligned}$$

where $y = z\eta(x)$. Differentiating these equations first with respect to z and then with respect to x, one finds that

$$\begin{aligned}
\eta(x)\phi_y(x,y) &= \tilde{\Phi}_z(x,z), && (37)\\
\eta(x)\psi_y(x,y) &= \tilde{\Gamma}_z(x,z) + \eta(x) - 1, && (38)\\
\phi_x(x,y) + z\eta_x(x)\phi_y(x,y) &= \tilde{\Phi}_x(x,z) + \frac{\eta_x(x)}{\eta(x)}\tilde{\Phi}(x,1) + 1 - \frac{1}{\eta(x)}, && (39)\\
\psi_x(x,y) + z\eta_x(x)\psi_y(x,y) &= \tilde{\Gamma}_x(x,z) + z\eta_x(x). && (40)
\end{aligned}$$

Equations (29) and (35) imply that

$$\tilde{\Gamma}_x = \frac{1}{\eta}\left(-\tilde{\Phi}_z + z\eta_x\tilde{\Gamma}_z - z\eta_x\right),$$

whence, by means of (37), (38), equation (40) may be written as

$$\psi_x(x,y) + z\eta_x(x)\psi_y(x,y) - z\eta_x(x) = -\phi_y(x,y) + z\eta_x(x)\psi_y(x,y) - z\eta_x,$$

that is,

$$\psi_x(x,y) = -\phi_y(x,y). \tag{41}$$

Equation (39) implies that

$$\begin{aligned}\phi_x(x,y) + z\eta_x(x)\phi_y(x,y) &= \tilde{\Phi}_x(x,z) + \frac{\eta_x(x)}{\eta(x)}\tilde{\Phi}(x,1) + 1 - \frac{1}{\eta(x)} \\ &= \frac{1}{\eta}\tilde{\Gamma}_z(x,z) + \frac{\eta_x(x)}{\eta(x)}z\tilde{\Phi}_z(x,z) + 1 - \frac{1}{\eta(x)} \\ &= \psi_y(x,y) + z\eta_x(x)\phi_y(x,y),\end{aligned}$$

where the second line follows by (28) and (35) and the third by (37) and (38); one concludes that

$$\phi_x(x,y) = \psi_y(x,y). \tag{42}$$

Equations (41), (42) imply that ψ is harmonic in the region $\{(x,y) \in (x_1,x_2) \times (0,\eta(x))\}$ with harmonic conjugate ϕ, and because of (31), (32) it satisfies the boundary conditions

$$\psi(x,0) = 0, \quad \psi(x,\eta(x)) = \eta(x) - 1, \qquad x \in (x_1,x_2). \tag{43}$$

Finally, note from (35) that

$$\tilde{W} = \frac{\beta\eta_x}{\sqrt{1+\eta_x^2}}.$$

This observation, together with (30) and (36), implies that

$$\begin{aligned}&\left[\frac{\beta\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - \frac{d}{dx}\left(\frac{1}{\eta}\int_0^1 z\tilde{\Phi}_z(\tilde{\Gamma}_z - 1)\,dz\right) = \frac{1}{2\eta^2}\int_0^1 (\tilde{\Gamma}_z^2 - \tilde{\Phi}_z^2)\,dz \\ &+\frac{\tilde{W}}{\eta^2\sqrt{\beta^2 - \tilde{W}^2}}\int_0^1 z(\tilde{\Gamma}_z - 1)\tilde{\Phi}_z\,dz + \alpha(\eta-1) + \frac{1}{2\eta^2} - \frac{1}{2},\end{aligned}$$

which, combined with (35), means that

$$\left[\frac{\beta\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - \frac{1}{\eta}\frac{d}{dx}\left(\int_0^1 z\tilde{\Phi}_z(\tilde{\Gamma}_z - 1)\,dz\right) = \frac{1}{2\eta^2}\int_0^1 \int_0^1 (\tilde{\Gamma}_z^2 - \tilde{\Phi}_z^2)\,dz + \alpha(\eta-1) + \frac{1}{2\eta^2} - \frac{1}{2}. \tag{44}$$

Because of (37), (38), it follows that

$$\begin{aligned}\frac{1}{\eta}\frac{d}{dx}\left(\int_0^1 z\tilde{\Phi}_z(\tilde{\Gamma}_z - 1)\,dz\right) &= \frac{1}{\eta}\frac{d}{dx}\left(\int_0^\eta y\phi_y(\psi_y - 1)\,dy\right) \\ &= \eta_x\phi_y(\psi_y - 1)|_{y=\eta} + \frac{1}{\eta}\int_0^\eta [y\phi_{xy}(\psi_y - 1) + y\phi_y\psi_{xy}]\,dy \\ &= \eta_x\phi_y(\psi_y - 1)|_{y=\eta} + \frac{1}{\eta}\int_0^\eta [y\psi_{yy}(\psi_y - 1) - y\phi_y\phi_{yy}]\,dy \\ &= \eta_x\phi_y(\psi_y - 1)|_{y=\eta} + \frac{1}{2}\left((\psi_y - 1)^2 - \phi_y^2\right)\Big|_{y=\eta} \\ &\quad -\frac{1}{2\eta}\int_0^\eta [(\psi_y - 1)^2 - \phi_y^2]\,dy,\end{aligned} \tag{45}$$

where (41) and (42) have been used to obtain the third line. Equations (41) and (43) imply that

$$(\psi_y(x,\eta(x)) - 1)\eta_x(x) = -\psi_x(x,\eta(x)) = \phi_y(x,\eta(x)),$$

so that (45) becomes

$$\begin{aligned} \frac{1}{\eta}\frac{d}{dx}\left(\int_0^1 z\tilde{\Phi}_z(\tilde{\Gamma}_z - 1)\,dz\right) &= \frac{1}{2}(\psi_x(x,\eta(x)))^2 + \frac{1}{2}(\psi_y(x,\eta(x)) - 1)^2 \\ &\quad -\frac{1}{2\eta^2}\int_0^1 [(\tilde{\Gamma}_z - 1)^2 - \tilde{\Phi}_z^2]\,dz, \end{aligned} \tag{46}$$

where (37), (38) and (41) have once again been used. Combining (44) and (46) and using the fact that $\tilde{\Gamma}(x,0) = \tilde{\Gamma}(x,1) = 0$, one finds that

$$\left[\frac{\beta\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - \frac{1}{2}(\psi_x^2 + (\psi_y - 1)^2) = \alpha(\eta - 1) - \frac{1}{2} \qquad \text{on} \quad y = \eta(x).$$

One concludes that η and ψ solve (17), (18), (19), (20) with $k = -1$. Since ϕ is the harmonic conjugate of ψ (equations (41), (42)), it follows that η, ψ solve (12), (13), (14), (15). It has therefore been demonstrated that a trajectory of Hamilton's equations for the Hamiltonian system (M, Ω, H) results in a solution of the steady water-wave problem.

3 Mielke's centre-manifold reduction theorem

This section is concerned with the application of centre-manifold reduction theory to the steady water-wave problem. The key theorem (Theorem 2 below) is due to Mielke (1991, Theorem 2.1). A complete discussion of the theorem, its proof and generalisations is given by Mielke (1986; 1988; 1990).

Theorem 2 *Consider the differential equation*

$$\dot{x} = Kx + f(x,\lambda), \tag{47}$$

where x belongs to a Hilbert space X, $\lambda \in \mathbb{R}^n$ is a parameter, $K : \mathcal{D}(K) \subset X \to X$ is a closed linear operator and $f(x,\lambda_0) = O(\| x \|^2)$ as $x \to 0$, so that 0 is an equilibrium point when $\lambda = \lambda_0 \in \mathbb{R}^n$. Regarding $\mathcal{D}(K)$ as a Hilbert space equipped with the graph norm, suppose that

(H1) X admits a direct-sum decomposition $X = X_1 \oplus X_2$, where X_1, X_2 are closed, K-invariant subspaces, so that (47) can be written as the linearly-decoupled system

$$\dot{x}_1 = K_1 x_1 + f_1(x_1 + x_2, \lambda), \tag{48}$$
$$\dot{x}_2 = K_2 x_2 + f_2(x_1 + x_2, \lambda), \tag{49}$$

in which $K_i = K|_{\mathcal{D}(K)\cap X_i} : \mathcal{D}(K) \cap X_i \subset X_i \to X_i$, $i = 1, 2$;

(H2) X_1 is finite dimensional and the spectrum of K_1 lies on the imaginary axis;

(H3) The imaginary axis lies in the resolvent set of K_2 and

$$\| (K_2 - iaI)^{-1} \| \leq \frac{C}{1+|a|}, \qquad a \in \mathbb{R},$$

for some constant C that is independent of a;

(H4) There exists a natural number k and neighbourhoods $\Lambda \subset \mathbb{R}^n$ of λ_0 and $U \subset \mathcal{D}(K)$ of 0 such that f is $(k+1)$ times continuously differentiable on $U \times \Lambda$, its derivatives are bounded and uniformly continuous on $U \times \Lambda$ and

$$f(0, \lambda_0) = 0, \qquad d_1 f[0, \lambda_0] = 0.$$

(Here $d_1 f[x, \lambda]$ and $d_2 f[x, \lambda]$ denote the partial Fréchet derivatives of f with respect to x and λ respectively.)

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ of λ_0 and $\tilde{U}_1 \subset U \cap X_1$, $\tilde{U}_2 \subset U \cap X_2$ of zero and a reduction function $h : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2 \subset X_2$ with the following properties. The reduction function h is k times continuously differentiable on $\tilde{U}_1 \times \tilde{\Lambda}$, its derivatives are bounded and uniformly continuous on $\tilde{U}_1 \times \tilde{\Lambda}$ and

$$h(0, \lambda_0) = 0, \qquad d_1 h[0, \lambda_0] = 0. \tag{50}$$

The graph

$$M_C^\lambda = \{x_1 + h(x_1, \lambda) \in \tilde{U}_1 \times \tilde{U}_2 : x_1 \in \tilde{U}_1\}$$

is a centre manifold for (47), so that

(1) M_C^λ is a locally invariant manifold of (47): through every point in M_C^λ there passes a unique solution of (47) that remains on M_C^λ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$;

(2) Every small bounded solution $x(t)$, $t \in \mathbb{R}$ of (47) that satisfies $x_1(t) \in \tilde{U}_1$, $x_2(t) \in \tilde{U}_2$ lies completely in M_C^λ;

(3) Every solution $\tilde{x} : (a, b) \to \tilde{U}_1$ of the reduced equation

$$\dot{x}_1 = K_1 x_1 + f_1(x_1 + h(x_1, \lambda), \lambda) \tag{51}$$

generates a solution

$$x(t) = x_1(t) + h(x_1(t), \lambda)$$

of the full equation (47).

(There is a slight abuse of nomenclature in Theorem 2. To explain this remark, suppose that 0 is an equilibrium for all values of λ in a neighbourhood of λ_0, and notice that the hypotheses in the theorem do not ensure that the linearisation of (47) about 0 has any purely imaginary spectral values for $\lambda \neq \lambda_0$. In this case M_C^λ is a nontrivial centre manifold for (47) only when $\lambda = \lambda_0$. Nevertheless it is an invariant manifold for $\lambda \in \tilde{\Lambda} \setminus \{\lambda_0\}$ which has all the essential attributes of a centre manifold. This observation justifies the reference to M_C^λ as a centre manifold for all $\lambda \in \tilde{\Lambda}$.)

An immediate application of Theorem 2 to the system (27) to prove the existence of a centre manifold for the steady water-wave problem is ruled out by the nonlinear boundary condition (26). In order to overcome this difficulty one needs to introduce a change of variable which leads to an equivalent problem in a linear space. Before introducing the requisite change of variable, let us first make some simplifying observations.

Since the analysis in the present paper is ultimately concerned with the existence of small-amplitude solutions of the steady water-wave problem (that is, solutions with η close to unity and ω, Φ, Γ close to zero) when α is close to 1 and β is close to $1/3$, let us set

$$\eta = 1+\rho, \qquad \alpha = 1+\mu_1, \qquad \beta = 1/3+\mu_2, \tag{52}$$

and consider the manifold

$$N = \{(\rho,\omega,\Phi,\Gamma)\in\mathcal{H} : \max\{|\rho|, |\omega|, \|\, \Phi \,\|_{H^1(0,1)}, \|\, \Gamma \,\|_{H^1(0,1)}\} < 1/12\} \tag{53}$$

with $|\mu_1|, |\mu_2| < 1/12$. The manifold N is independent of α and β and on N one has that

$$\begin{aligned} |W| &< |\omega| + \frac{1}{1-|\rho|}\left(\|\, \Phi \,\|_{H^1(0,1)}\left(1+\|\, \Gamma \,\|_{H^1(0,1)}\right)\right) \\ &< \frac{1}{12}+\frac{12}{11}\left(\frac{1}{12}\left(1+\frac{1}{12}\right)\right) = \frac{2}{11} \\ &< \frac{1}{4} < \frac{1}{3}+\mu_2 = |\beta|, \end{aligned}$$

so that solutions on N of the system

$$\dot\rho = \frac{W}{\sqrt{(\frac{1}{3}+\mu_2)^2 - W^2}}, \tag{54}$$

$$\dot\omega = \frac{1}{2(\rho+1)^2}\int_0^1 (\Gamma_z^2-\Phi_z^2)\,dz + \frac{W}{(\rho+1)^2\sqrt{(\frac{1}{3}+\mu_2)^2-W^2}}\int_0^1 z\Phi_z(\Gamma_z-1)\,dz + \mu_1\rho + \frac{\rho^2(3+2\rho)}{2(1+\rho)^2}, \tag{55}$$

$$\dot\Phi = \frac{\Gamma_z}{\rho+1} + \frac{W}{(\rho+1)\sqrt{(\frac{1}{3}+\mu_2)^2-W^2}}\left(z\Phi_z - \Phi(1)\right), \tag{56}$$

$$\dot\Gamma = -\frac{\Phi_z}{\rho+1} + \frac{zW(\Gamma_z-1)}{(\rho+1)\sqrt{(\frac{1}{3}+\mu_2)^2-W^2}} \tag{57}$$

with boundary conditions

$$\Phi_z = 0 \qquad \text{on} \quad z=0, \tag{58}$$

$$\Phi_z - \frac{W(\Gamma_z-1)}{\sqrt{(\frac{1}{3}+\mu_2)^2-W^2}} = 0 \qquad \text{on} \quad z=1 \tag{59}$$

correspond to solutions on M of (27) with boundary conditions (25), (26) via the formulae (52). Let us write (54), (55), (56), (57) as

$$\dot{\gamma}_1(x) = g(\gamma_1(x), \mu_1, \mu_2),$$

where $\gamma_1(x) = (\rho(x), \omega(x), \Phi(x), \Gamma(x))$ and the domain of g is the set of $(\eta, \omega, \Phi, \Gamma, \mu_1, \mu_2)$ in $N \times \mathbb{R}^2$ such that $\Phi, \Gamma \in H^2(0,1)$ and satisfy (58), (59).

The next step is to introduce a change of variables to linearise the nonlinear boundary condition (59). Define $G : N \times \mathbb{R}^2 \to \mathcal{H} \times \mathbb{R}^2$ by

$$G(\rho, \omega, \Phi, \Gamma, \mu_1, \mu_2) = (\rho, \upsilon, \Psi, \Gamma, \mu_1, \mu_2),$$

where $\upsilon = \Phi(1)$ and

$$\Psi(z) = \Phi(z) - \frac{W}{\sqrt{(\frac{1}{3} + \mu_2)^2 - W^2}} \left(\int_0^z s(\Gamma_z - 1)\, ds - \int_0^1 \int_0^z s(\Gamma_z - 1)\, ds\, dz \right).$$

Observe that G is a smooth map from $N \times \mathbb{R}^2$ into $\mathcal{H} \times \mathbb{R}^2$ with $G(0) = 0$ and its Fréchet derivative at 0 is

$$dG[0](\rho, \omega, \Phi, \Gamma, \mu_1, \mu_2) = \left(\rho, \Phi(1), \Phi + \frac{3}{2} \left(z^2 - \frac{1}{3} \right) (\omega - \Phi(1)), \Gamma, \mu_1, \mu_2 \right).$$

Clearly $dG[0]$ is a bijection which is its own inverse, that is

$$(dG[0])^{-1}(\rho, \upsilon, \Psi, \Gamma, \mu_1, \mu_2) = \left(\rho, \Psi(1), \Psi + \frac{3}{2} \left(z^2 - \frac{1}{3} \right) (\upsilon - \Psi(1)), \Gamma, \mu_1, \mu_2 \right).$$

It follows from the inverse function theorem that G is a smooth diffeomorphism from a neighbourhood $V_1 \times \Lambda_1$ of 0 onto a neighbourhood $V_2 \times \Lambda_2$ of 0 in $\mathcal{H} \times \mathbb{R}^2$. (The inverse of G is given in closed form in Section 4.2.) Also notice that G is a diffeomorphism in a neighbourhood of zero in $(\mathcal{H} \times \mathbb{R}^2) \cap (\mathbb{R}^2 \times H^2(0,1) \times H^2(0,1) \times \mathbb{R}^2)$ with respect to the usual norm on $\mathbb{R}^2 \times H^2(0,1) \times H^2(0,1) \times \mathbb{R}^2$. For each (μ_1, μ_2), it follows that $\{\gamma_1(x) : x \in (a,b)\} \subset V_1$ is a solution of (54), (55), (56), (57) if and only if $\{\gamma_2(x) : x \in (a,b)\} \subset V_2$ is a solution of

$$\dot{\gamma}_2(x) = \hat{g}(\gamma_2(x), \mu_1, \mu_2), \tag{60}$$

where

$$\hat{g}(\gamma_2(x), \mu_1, \mu_2) = d_1 G[G^{-1}(\gamma_2(x), \mu_1, \mu_2)](g(G^{-1}(\gamma_2(x), \mu_1, \mu_2)))$$

and the domain $\mathcal{D}(\hat{g})$ of $\hat{g}$ is the linear space of $(\eta, \upsilon, \Psi, \Gamma, \mu_1, \mu_2) \in V_2 \times \mathbb{R}^2$ such that $\Psi, \Gamma \in H^2(0,1)$ with

$$\Psi_z(0) = \Psi_z(1) = 0.$$

The differential equation (60) is in a form suitable for the application of Theorem 2. One may write (60) as

$$\dot{y} = Ky + f(y, \mu_1, \mu_2),$$

where $X = \mathcal{H}$, $y = (\rho, v, \Psi, \Gamma) \in \mathcal{H}$ and K is the closed linear operator defined by

$$K \begin{pmatrix} \rho \\ v \\ \Psi \\ \Gamma \end{pmatrix} = \begin{pmatrix} 3(\Psi(1) - v) \\ \Gamma_z(1) \\ \Gamma_z - \frac{3}{2}\left(z^2 - \frac{1}{3}\right)\Gamma_z(1) \\ -\Psi_z \end{pmatrix} \tag{61}$$

with

$$\mathcal{D}(K) = \{(\rho, v, \Psi, \Gamma) \in \mathcal{H} : \Psi, \Gamma \in H^2(0,1), \Psi_z(0) = \Psi_z(1) = 0\}.$$

The task in hand is therefore to verify hypotheses (H1), (H2), (H3), (H4) of Theorem 2. Notice immediately that (H4) is satisfied with $\Lambda = \Lambda_2$, $U = V_2 \cap \mathcal{D}(K)$ for any natural number k.

In order to verify the remaining hypotheses, let us introduce the operator $L : \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$L \begin{pmatrix} \rho \\ \omega \\ \Phi \\ \Gamma \end{pmatrix} = \begin{pmatrix} 3(\omega - \Phi(1)) \\ 0 \\ \Gamma_z \\ -\Phi_z - 3z(\omega - \Phi(1)) \end{pmatrix},$$

where

$$\mathcal{D}(L) = \{(\rho, \omega, \Phi, \Gamma) \in \mathcal{H} : \Phi, \Gamma \in H^2(0,1), \Phi_z(0) = \Phi_z(1) + 3(\omega - \Phi(1)) = 0\}.$$

(The operator L is the formal linearisation of $g(\rho, \omega, \Phi, \Gamma, 0, 0)$ at 0.) Observe that $d_1G_1[0]$ is a homeomorphism of $\mathcal{D}(K)$ with the graph norm onto $\mathcal{D}(L)$ with the graph norm and

$$K = d_1G_1[0]L(d_1G_1[0])^{-1}, \tag{62}$$

where

$$G(\rho, \omega, \Phi, \Gamma, \mu_1, \mu_2) = (G_1(\rho, \omega, \Phi, \Gamma, \mu_1, \mu_2), \mu_1, \mu_2).$$

It therefore suffices to carry out the necessary spectral analysis on L; the corresponding results are automatically true for K because of (62).

Let us begin by anticipating the result that $\nu \in \mathbb{C}$ is an eigenvalue of L, and hence of K, if and only if it satisfies the transcendential equation

$$3(\sin\nu - \nu\cos\nu) = \nu^2 \sin\nu. \tag{63}$$

This result is the well-known linear dispersion relation for gravity-capillary water waves with Froude number 1 and Bond number $1/3$ (see Proposition 3 below for a proof in the present context). If ν_0 is in the resolvent set of L (the existence of such a ν_0 is proved in Proposition 3), it follows from the Rellich embedding theorem that $(L - \nu_0 I)^{-1}$ is compact. The spectrum of L, and hence of K, therefore consists of isolated eigenvalues of finite multiplicity (Kato 1976, Theorem III.6.29). Because $a > \tanh a$ for all $a > 0$, one has that

$$\frac{d}{da}(a\cosh a) < \frac{d}{da}\left(\left(1 + \frac{a^2}{3}\right)\sinh a\right),$$

so that the only solution of (63) of the form $\nu = ia$, $a \in \mathbb{R}$ is $\nu = 0$. The only purely imaginary eigenvalue of L, and hence of K, is therefore 0.

Now define the spectral projection P corresponding to the subset $\{0\}$ of $\sigma(K)$ by

$$P(x) = \frac{1}{2\pi i} \int_C (K - zI)^{-1} x \, dz,$$

where C is a closed curve in the resolvent set of K containing no point of $\sigma(K)$ except 0. Defining $X_1 = P(\mathcal{H})$ and $X_2 = (I - P)(\mathcal{H})$, one finds that the hypotheses (H1), (H2) of Theorem 2 are satisfied (see Kato (1976, Theorem III.6.17)).

It remains to confirm (63) and the hypothesis (H3) in Theorem 2. This task is accomplished by the following proposition, whose proof relies upon ideas of Iooss & Kirchgässner (1992, Lemma 2.1).

Proposition 3 *A complex number ν is an eigenvalue of L if and only if*

$$\nu \cos \nu = \left(1 - \frac{\nu^2}{3}\right) \sin \nu. \tag{64}$$

Each ia with $a \in \mathbb{R}$, $|a| \geq 1$ lies in the resolvent set of L and

$$\| (L - ia)^{-1} \| \leq \frac{C}{1 + |a|}$$

for some constant C that is independent of a.

Proof First notice that $\nu = 0$ is an eigenvalue of L with eigenvector $(1, 0, 0, 0)$ and $\nu = 0$ satisfies (64). Now suppose that $\nu \neq 0$ satisfies (64). It follows that $\sin \nu \neq 0$, and it is readily confirmed that

$$\left(-\frac{3}{\nu}, 0, \frac{3}{\nu^2}\left(1 - \frac{\nu \cos(\nu z)}{\sin \nu}\right), \frac{3}{\nu}\left(z - \frac{\sin(\nu z)}{\sin \nu}\right)\right)$$

is an eigenvector of L with eigenvalue ν.

Next suppose that $\nu \in \mathbf{C}$ does not satisfy (64) and let $(\rho^\star, \omega^\star, \Phi^\star, \Gamma^\star) \in \mathcal{H}$. The equations

$$3(\omega - \Phi(1)) - \nu\rho = \rho^\star, \tag{65}$$

$$-\nu\omega = \omega^\star, \tag{66}$$

$$\Gamma_z - \nu\Phi = \Phi^\star, \tag{67}$$

$$-\Phi_z - 3z(\omega - \Phi(1)) - \nu\Gamma = \Gamma^\star, \tag{68}$$

for $(\rho, \omega, \Phi, \Gamma) \in \mathcal{D}(L)$, may be solved as follows. Equations (66), (67), (68) imply that

$$\begin{aligned} \Gamma_{zz} + \nu^2\Gamma &= \Phi^\star_z - \nu\Gamma^\star + 3z\omega^\star + 3\nu z\Phi(1) \\ &= f^\star + 3\nu z\Phi(1), \end{aligned}$$

where $f^\star(z) = \Phi^\star_z(z) - \nu\Gamma^\star(z) + 3z\omega^\star$, and the solution of this equation with $\Gamma(0) = 0$ is

$$\begin{aligned} \Gamma(z) = &-\frac{\sin \nu z}{\nu} \int_z^1 (f^\star(t) + 3\nu t\Phi(1)) \cos(\nu t) \, dt \\ &-\frac{\cos \nu z}{\nu} \int_0^z (f^\star(t) + 3\nu t\Phi(1)) \sin(\nu t) \, dt + A \sin(\nu t), \end{aligned} \tag{69}$$

for $A \in \mathbf{C}$.

If $\sin \nu = 0$, the condition $\Gamma(1) = 0$ is satisfied if and only if

$$\int_0^1 f^\star(t) \sin(\nu t)\, dt = -3\nu\Phi(1) \int_0^1 t \sin(\nu t)\, dt = 3\Phi(1) \cos \nu,$$

so that

$$\Phi(1) = \frac{1}{3 \cos \nu} \int_0^1 f^\star(t) \sin(\nu t)\, dt. \tag{70}$$

It follows from (69) that

$$\Gamma_z(1) = \nu A \cos \nu, \tag{71}$$

and (67), (70), (71) therefore imply that

$$A = \frac{\Phi^\star(1)}{\nu \cos \nu} + \frac{1}{3 \cos^2 \nu} \int_0^1 f^\star(t) \sin(\nu t)\, dt.$$

Having established the value of A, one may now substitute (69) into (67) to define Φ, and ω and ρ are defined in turn by (66), (65). It is readily verified that $(\rho, \omega, \Phi, \Gamma)$ belongs to $\mathcal{D}(L)$, and therefore $(\rho, \omega, \Phi, \Gamma)$ is the unique solution of (65), (66), (67), (68), so that ν is not an eigenvalue of L.

If $\sin \nu \neq 0$, the condition $\Gamma(1) = 0$ is satisfied if and only if

$$A\nu^2 \sin \nu - 3(\sin \nu \cos \nu - \nu \cos^2 \nu)\Phi(1) = \nu \cos \nu \int_0^1 f^\star(t) \sin(\nu t)\, dt. \tag{72}$$

Evaluated at $z = 1$, equation (67) becomes

$$A\nu^2 \cos \nu - (3\nu \sin \nu \cos \nu + \nu^2 - 3 \sin^2 \nu)\Phi(1) = \nu \left(\Phi^\star(1) - \sin \nu \int_0^1 f^\star(t) \sin(\nu t)\, dt \right). \tag{73}$$

Equations (72), (73) are two simultaneous linear equations for $(A, \Phi(1))$; the determinant of the matrix of coefficients is

$$\nu^2(3(\sin \nu - \nu \cos \nu) - \nu^2 \sin \nu),$$

which is nonzero. Equations (72), (73) therefore determine A and hence Γ, equation (67) determines Φ and equations (65), (66) determine ω and ρ.It is again readily verified that $(\rho, \omega, \Phi, \Gamma)$ belongs to $\mathcal{D}(L)$, and therefore $(\rho, \omega, \Phi, \Gamma)$ is the unique solution of (65), (66), (67), (68), so that ν is not an eigenvalue of L.

To establish that $(L - iaI)^{-1}$, $a \in \mathbb{R}$, $|a| \geq 1$ is bounded and that its norm satisfies the desired inequality, one proceeds as follows. Let $\| \cdot \|$ denote the $L^2(0, 1)$-norm and note that the norm

$$\| (\rho, \omega, \Phi, \Gamma) \|_{\mathcal{H}} = |\rho| + |\omega| + \| \Phi_z \| + \| \Gamma_z \|$$

is equivalent to the usual norm on $\mathcal{H}$ because $\Gamma \in H_0^1(0, 1)$ and $\int_0^1 \Phi\, dz = 0$. Equations (65), (66) may be rewritten as

$$\begin{pmatrix} \rho \\ \omega \end{pmatrix} = \frac{1}{a^2} \begin{pmatrix} ia(\rho^\star + 3\Phi(1)) + 3\omega^\star \\ ia\omega^\star \end{pmatrix}.$$

There is therefore a constant C_1 that

$$
\begin{aligned}
|\rho| &\leq C_1 \left\{ \frac{|\rho^\star|}{|a|} + \frac{\| \Phi_z \|}{|a|} + \frac{|\omega^\star|}{|a|^2} \right\}, && (74)\\
|\omega| &\leq \frac{C_1}{|a|} |\omega^\star| && (75)
\end{aligned}
$$

for sufficiently large $|a|$. Here the inequality

$$
|\Phi(1)| = \left| \int_0^1 z\Phi_z \, dz \right| \leq \| \Phi_z \|
$$

has been used to obtain (74). (Each term C_n appearing in the analysis below is a constant which is independent of a. In order to avoid unnecessary repetiton, this fact will not be stated explicitly each time a new C_n appears.)

The next step is to differentiate (67), (68) with respect to z to obtain the equations

$$
\begin{aligned}
\Gamma_{zz} - ia\Phi_z &= \Phi_z^\star, \\
\Phi_{zz} + ia\Gamma_z &= -\Gamma_z^\star - 3(\omega - \Phi(1)).
\end{aligned} \tag{76}
$$

Taking absolute values, squaring, integrating over $(0,1)$ and adding, one finds that

$$
\begin{aligned}
\| \Gamma_{zz} \|^2 + \| \Phi_{zz} \|^2 + a^2(\| \Gamma_z \|^2 + \| \Phi_z \|^2) &= 2\mathrm{Re}(ia\Phi_z(1)\bar{\Gamma}_z(1)) + \| \Phi_z^\star \|^2 \\
&\quad + \| \Gamma_z^\star + 3(\omega - \Phi(1)) \|^2 .
\end{aligned} \tag{77}
$$

Observe that, for any $\epsilon \neq 0$,

$$
2\mathrm{Re}(ia\Phi_z(1)\bar{\Gamma}_z(1)) \leq a^2\epsilon^2 |\Phi_z(1)|^2 + \frac{1}{\epsilon^2}|\Gamma_z(1)|^2. \tag{78}
$$

To estimate $|\Phi_z(1)|$, use the fact that $\Phi_z(1) - 3(\omega + \Phi(1)) = 0$, so that

$$
|\Phi_z(1)| \leq 3(|w| + \| \Phi_z \|). \tag{79}
$$

To estimate $|\Gamma_z(1)|$, multiply (76) by z^k, where k is a natural number, and integrate the term on the left-hand side by parts, to find that

$$
\Gamma_z(1) = \int_0^1 kz^{k-1}\Gamma_z \, dz + \int_0^1 z^k \Phi_z^\star \, dz + ia \int_0^1 z^k \Phi_z \, dz.
$$

Using the Cauchy-Schwarz inequality, one finds that

$$
|\Gamma_z(1)|^2 \leq \frac{3k^2 \| \Gamma_z \|^2}{2k-1} + \frac{3 \| \Phi_z^\star \|^2}{2k+1} + \frac{3a^2 \| \Phi_z \|^2}{2k+1}. \tag{80}
$$

Equation (77), combined with the inequalities (78), (79), (80), shows that

$$
\begin{aligned}
&\| \Gamma_{zz} \|^2 + \| \Phi_{zz} \|^2 + a^2(\| \Gamma_z \|^2 + \| \Phi_z \|^2) \leq a^2 \left(18\epsilon^2 + \frac{3}{(2k+1)\epsilon^2} \right) \| \Phi_z \|^2 \\
&+ \frac{3k^2 \| \Gamma_z \|^2}{(2k-1)\epsilon^2} + \frac{3 \| \Phi_z^\star \|^2}{(2k+1)\epsilon^2} + \| \Phi_z^\star \|^2 + 3 \| \Gamma_z^\star \|^2 + 27 \| \Phi_z \|^2 + 18\epsilon^2 a^2 |\omega|^2 + 27|\omega|^2.
\end{aligned}
$$

Choosing ϵ small enough and k large enough so that

$$18\epsilon^2 + \frac{3}{(2k+1)\epsilon^2} \leq \frac{1}{2},$$

one finds that

$$\begin{aligned} \| \Gamma_{zz} \|^2 + \| \Phi_{zz} \|^2 + \frac{1}{2}a^2(\| \Gamma_z \|^2 + \| \Phi_z \|^2) \quad \leq \quad & C_2(|\omega^\star|^2 + \frac{|\omega^\star|^2}{|a|^2} + \| \Gamma_z \|^2 \\ & + \| \Phi_z \|^2 + \| \Gamma_z^\star \|^2 + \| \Phi_z^\star \|^2), \end{aligned}$$

where (75) has also been used. This inequality implies that

$$\| \Gamma_{zz} \|^2 + \| \Phi_{zz} \|^2 + \frac{1}{2}a^2(\| \Gamma_z \|^2 + \| \Phi_z \|^2) \leq C_3 \| (\rho^\star, \omega^\star, \Phi^\star, \Gamma^\star) \|_{\mathcal{H}}^2$$

for sufficiently large $|a|$, so that

$$\| \Gamma_z \| + \| \Phi_z \| \leq \frac{C_4}{|a|} \| (\rho^\star, \omega^\star, \Phi^\star, \Gamma^\star) \|_{\mathcal{H}} \tag{81}$$

for sufficiently large $|a|$, whereby equation (74) implies that

$$|\rho| \leq \frac{C_5}{|a|} \| (\rho^\star, \omega^\star, \Phi^\star, \Gamma^\star) \|_{\mathcal{H}} \tag{82}$$

for sufficiently large $|a|$.

The fact that $(L - iaI)^{-1} : \mathcal{H} \to \mathcal{H}$ is bounded, together with the resolvent estimate in the statement of the theorem, now follows immediately from (75), (81), (82). □

4 The Hamiltonian flow on the centre manifold

4.1 Theoretical results

The following theory uses the notation and hypotheses of Theorem 2 in the case when $\lambda_0 = 0 \in \mathbb{R}^2$, $k \geq 2$ and X_1 is finite dimensional, and is concerned with equations of the form (47) that are Hamilton's equations for a Hamiltonian system (M, Ω, H). Mielke (1991, Theorem 4.1) has shown that $M_C^{\lambda_0}$ is a symplectic submanifold of M and that the flow determined by the Hamiltonian system $(M_C^{\lambda_0}, \tilde{\Omega}, \tilde{H})$, where the tilde denotes restriction to M_C^{λ}, coincides with the flow on $M_C^{\lambda_0}$ determined by (M, Ω, H). Because X_1 is finite dimensional a simple continuity argument shows that M_C^{λ} is a symplectic submanifold of M for each $\lambda \in \tilde{\Lambda}$, and since M_C^{λ} is an invariant submanifold of M the same proof yields the following parameterised version of Mielke's theorem.

Theorem 4 *If $k \geq 2$, the centre manifold M_C^{λ} is a symplectic submanifold of M for each $\lambda \in \tilde{\Lambda}$. Moreover, the flow determined by the Hamiltonian system $(M_C^{\lambda}, \tilde{\Omega}, \tilde{H})$, where the tilde denotes restriction to M_C^{λ}, coincides with the flow on M_C^{λ} determined by (M, Ω, H).*

The centre manifold

$$M_C^\lambda = \{u_1 + u_2 \in \tilde{U}_1 \times \tilde{U}_2 : u_2 = h(u_1, \lambda)\}$$

is modelled on the single coordinate chart $\tilde{U}_1$, which is an open subset of the finite-dimensional space X_1. The coordinate map $\chi : M_C^\lambda \to \tilde{U}_1$ is defined by

$$\chi^{-1}(u_1) = u_1 + h(u_1, \lambda).$$

Let us henceforth work with these coordinates, so that M_C^λ, when regarded as a manifold in its own right, is identified with $\tilde{U}_1$, and the reduced equations (51) are Hamilton's equations for $(M_C^\lambda, \tilde{\Omega}, \tilde{H})$.

The following results, also due to Mielke (1991, Theorems 3.2, 3.3) and stated here in the notation of Theorems 2 and 4 with $\lambda_0 = 0$, is required in the subsequent calculations.

Lemma 5 *Suppose that Ω has the form*

$$\Omega|_{(m,\lambda)}(w_1, w_2) = \bar{\Omega}(w_1, w_2) + O(\| (m, \lambda) \| \| w_1 \| \| w_2 \|)$$

as $(m, \lambda) \to 0$, where $\bar{\Omega}$ is a constant symplectic 2-form. The spaces X_1 and X_2 are orthogonal with respect to $\bar{\Omega}$, that is

$$\bar{\Omega}(x_1, x_2) = 0$$

for each $(x_1, x_2) \in X_1 \times X_2$.

Lemma 6 *For each $x_1 + x_2 \in X_1 \oplus X_2$, one has that*

$$d_1^2 H[0,0](x, x) = \frac{1}{2}\langle A_1 x_1, x_1\rangle_X + \langle A_2 x_2, x_2\rangle_X,$$

where $A_i : X_i \to X_i$, $i = 1, 2$ are self-adjoint, linear operators. (Here $d_1^2 H[m, \tilde{\lambda}] : X_1 \times X_1 \to \mathbb{R}$ denotes the second partial Fréchet derivative with respect to m.)

4.2 The relevant Hamiltonian system

Let us now return to the Hamiltonian system (N, Ω, H) defined by (23), (24), (53). In keeping with the theory in Section 3, one sets

$$\alpha = 1 + \mu_1, \quad \beta = 1/3 + \mu_2, \qquad \mu_1, \mu_2 \in \Lambda_1,$$

where Λ_1 is a neighbourhood of 0 in $\mathbb{R}^2$, and attention is restricted to a neighbourhood V_1 of 0 in N. The Hamiltonian system of interest here is therefore (V_1, Ω, H_1), where

$$\begin{aligned} H_1(\rho, \omega, \Phi, \Gamma, \mu_1, \mu_2) \;\; &= \;\; \frac{1}{2(1+\rho)} \int_0^1 (\Gamma_z^2 - \Phi_z^2)\, dz - \frac{1}{2}(\mu_1 + 1)\rho^2 + \frac{1}{2}\left(\frac{1}{1+\rho} - 1\right) \\ &\quad + \frac{1}{2}\rho + \left(\frac{1}{3} + \mu_2\right) - \sqrt{\left(\frac{1}{3} + \mu_2\right)^2 - W^2}, \end{aligned}$$

in which

$$W = w - \frac{\Phi(1)}{1+\rho} + \frac{1}{1+\rho}\int_0^1 z\Phi_z\Gamma_z\,dz$$

and $(\mu_1, \mu_2) \in \Lambda_1$. The diffeomorphism G introduced in Section 3 transforms the Hamiltonian system (V_1, Ω, H_1) into the Hamiltonian system (V_2, Ω_2, H_2), where

$$H_2(\rho, \upsilon, \Psi, \Gamma, \mu_1, \mu_2) = H_1(G^{-1}(\rho, \upsilon, \Psi, \Gamma, \mu_1, \mu_2)), \qquad \mu_1, \mu_2 \in \Lambda_2,$$

and Ω_2 is given in coordinates by

$$\Omega_2|_{(m,\mu_2)}(v_1, v_2) = \Omega|_{G_1^{-1}(m,\mu_1,\mu_2)}(d_1G_1^{-1}[m, \mu_1, \mu_2](v_1), d_1G_1^{-1}[m, \mu_1, \mu_2](v_2)), \tag{83}$$

where $G^{-1}(m, \mu_1, \mu_2) = (G_1^{-1}(m, \mu_1, \mu_2), \mu_1, \mu_2)$.

The diffeomorphism $G^{-1} : V_2 \times \Lambda_2 \to V_1 \times \Lambda_1$ may be calculated explicitly as follows. Writing

$$G(\rho, \omega, \Phi, \Gamma, \mu_1, \mu_2) = (\rho, \upsilon, \Psi, \Gamma, \mu_1, \mu_2),$$

one has that

$$\begin{aligned} \upsilon &= \Phi(1), \\ \Psi &= \Phi - RA(\Gamma), \end{aligned} \tag{84}$$

where

$$R = \frac{W}{\sqrt{(\frac{1}{3} + \mu_2)^2 - W^2}}, \tag{85}$$

$$A(\Gamma) = \int_0^z s(\Gamma_z - 1)\,ds - \int_0^1\int_0^z s(\Gamma_z - 1)\,ds\,dz. \tag{86}$$

Evaluating (84) at $z = 1$, one finds that

$$R = \frac{\upsilon - \Psi(1)}{A(\Gamma)(1)}, \tag{87}$$

whence

$$\Phi = \Psi + RA(\Gamma), \tag{88}$$

where R is given by (87). Notice that $A(0)(1) = -1/3$ and because A is a continuous function of Γ, it follows that $A(\Gamma)(1) \neq 0$ for all $(\rho, \upsilon, \Psi, \Gamma)$ in a neighbourhood V_2 of the origin in $\mathcal{H}$.

Equation (84) implies that

$$\Psi_z = \Phi_z - Rz(\Gamma_z - 1), \tag{89}$$

and multiplying this equation by $z(\Gamma_z - 1)$ and integrating over $(0, 1)$, one finds that

$$\int_0^1 z(\Gamma_z - 1)\Phi_z\,dz = S, \tag{90}$$

where

$$S = R\int_0^1 z^2(\Gamma_z - 1)^2\,dz + \int_0^1 z(\Gamma_z - 1)\Psi_z\,dz. \tag{91}$$

It follows from (85) that

$$W = \frac{(\frac{1}{3} + \mu_2)R}{\sqrt{1+R^2}},$$

and by definition

$$\begin{aligned} W &= \omega + \frac{1}{1+\rho} \int_0^1 z(\Gamma_z - 1)\Phi_z \, dz \\ &= \omega + \frac{S}{1+\rho}, \end{aligned}$$

where the second line follows from (90). One concludes that

$$\omega = \frac{(\frac{1}{3} + \mu_2)R}{\sqrt{1+R^2}} - \frac{S}{1+\rho}. \tag{92}$$

Equations (87), (88), (91), (92) define $G_1^{-1} : V_2 \to V_1$ in closed form.

Straightforward calculations using the closed-form expression for G^{-1} show that

$$\begin{aligned} H_2(\rho, \upsilon, \Psi, \Gamma, \mu_1, \mu_2) &= \frac{1}{2(1+\rho)} \int_0^1 [\Gamma_z^2 - \Psi_z^2 - 2zR\Psi_z(\Gamma_z - 1) - z^2R^2(\Gamma_z - 1)^2] \, dz \\ &\quad - \frac{\rho^3}{2(1+\rho)} - \frac{1}{2}\mu_1\rho^2 + \left(\frac{1}{3} + \mu_2\right)\left(1 - \frac{1}{\sqrt{1+R^2}}\right), \end{aligned} \tag{93}$$

and that Hamilton's equations (60) for (V_2, Ω_2, H_2) are

$$\dot{\rho} = R, \tag{94}$$

$$\dot{\upsilon} = \frac{\Gamma_z(1)}{\rho+1} + \frac{R}{\rho+1}\left[R(\Gamma_z(1) - 1) - \upsilon\right], \tag{95}$$

$$\begin{aligned} \dot{\Psi} &= -\frac{A(\Gamma)(1+R^2)^{\frac{3}{2}}}{\frac{1}{3} + \mu_2}\left[\mu_1\rho + \frac{\rho^2(3+2\rho)}{2(\rho+1)^2} + \frac{R^2(\Gamma_z(1)-1)^2}{2(\rho+1)^2} + \frac{\Gamma_z(1)(\Gamma_z(1)-2)}{2(\rho+1)^2}\right] \\ &\quad + \frac{\Gamma_z}{\rho+1} + \frac{R}{\rho+1}\left[2z\Psi_z + Rz^2(\Gamma_z - 1) - \upsilon - \Psi - \Psi(1)\right] \end{aligned} \tag{96}$$

$$\dot{\Gamma} = -\frac{\Psi_z}{\rho+1} \tag{97}$$

with boundary conditions

$$\Gamma(0) = \Gamma(1) = 0, \tag{98}$$

$$\Psi_z(0) = \Psi_z(1) = 0. \tag{99}$$

(A derivation of these equations is given in the Appendix.) Because

$$d_1G_1^{-1}[0, \mu_1, \mu_2](\rho, \nu, \Psi, \Gamma) = \begin{pmatrix} \rho \\ \Psi(1) - 3\mu_2(\nu - \Psi(1)) \\ \Psi + \frac{3}{2}\left(z^2 - \frac{1}{3}\right)(\nu - \Psi(1)) \\ \Gamma \end{pmatrix},$$

it follows from equation (83) that the 2-form Ω_2 is given in coordinates by

$$\Omega_2|_{(m,\mu_2)}(w_1,w_2) = \bar{\Omega}(w_1,w_2) + \Omega_{\mu_2}(w_1,w_2) + O(\| m \|\| w_1 \|\| w_2 \|) \tag{100}$$

as $m \to 0$. The 2-forms $\bar{\Omega}$ and Ω_{μ_2} are defined by

$$\begin{aligned} \bar{\Omega}(w_1,w_2) &= \Psi_2(1)\rho_1 - \Psi_1(1)\rho_2 + \int_0^1 \Psi_1\Gamma_{2z}\,dz + \frac{3}{2}(v_1 - \Psi_1(1))B(\Gamma_2) \\ &\quad - \int_0^1 \Psi_2\Gamma_{1z}\,dz - \frac{3}{2}(v_2 - \Psi_2(1))B(\Gamma_1), \\ \Omega_{\mu_2}(w_1,w_2) &= -3\mu_2\rho_1(v_2 - \Psi_2(1)) + 3\mu_2\rho_2(v_1 - \Psi_1(1)), \end{aligned} \tag{101}$$

where $w_1 = (\rho_1, v_1, \Psi_1, \Gamma_1)$, $w_2 = (\rho_2, v_2, \Psi_2, \Gamma_2)$ and

$$B(\Gamma) = \int_0^1 z^2\Gamma_z\,dz.$$

Notice that $\bar{\Omega}$ and Ω_{μ_2} do not depend on m or μ_1 but Ω_{μ_2} depends on μ_2.

4.3 Estimates

The task in hand is to write down Hamilton's equations for the Hamiltonian system $(M_C^\lambda, \tilde{\Omega}, \tilde{H})$, where $\tilde{\Omega}$, $\tilde{H}$ are the restrictions of Ω_2, H_2 to the centre manifold M_C^λ for (94), (95), (96), (97), (98), (99), which are Hamilton's equations for (V_2, Ω_2, H_2). Recall that V_2 is an open subset of the Hilbert space $X = \mathcal{H} = X_1 \oplus X_2$, where $X_1 = PX$ is the spectral projection of X corresponding to the subset $\{0\}$ of $\sigma(K)$, that is, the generalised eigenspace of K corresponding to the zero eigenvalue. (The operator K is defined by (61).) This generalised eigenspace is 4-dimensional and has a basis consisting of the generalised eigenvectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ -\frac{1}{3} \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{6}(z - z^3) \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ -\frac{1}{45} \\ \frac{1}{24}\left(z^4 - 2z^2 + \frac{7}{15}\right) \\ 0 \end{pmatrix}, \tag{102}$$

in which $Ke_i = e_{i-1}$, $i = 2, 3, 4$ and $Ke_1 = 0$. (Note that e_4 does not lie in the range $\mathcal{R}$ of K because each $(\eta, \rho, \Psi, \Gamma) \in \mathcal{R}$ satisfies $\Psi(1) = 0$ and e_4 does not have this property.) The coordinate chart $\tilde{U}_1$ for the centre manifold M_C^λ is a neighbourhood of 0 in X_1.

The first step in the calculation of the equations governing the flow on M_C^λ is undertaken by the following lemmas, which provide estimates on the reduction function h, the reduced Hamiltonian $\tilde{H}$ and the reduced 2-form $\tilde{\Omega}$.

Lemma 7 *In the coordinates $u_1 = (\rho_1, v_1, \Psi_1, \Gamma_1) \in \tilde{U}_1$, the reduction function $h(u_1, \mu_1, \mu_2)$ for equations (60) satisfies*

$$h(u_1,\mu_1,\mu_2) = O(\| u_1 \|^2) + O(|\mu_1||\rho_1|) + O(|(\mu_1,\mu_2)| \, \| (v_1,\Psi_1,\Gamma_1) \|) + O(|(\mu_1,\mu_2)|^2 \, \| u_1 \|)$$

and

$$d_1h[u_1,\mu_1,\mu_2] = O(\| u_1 \|) + O(|(\mu_1,\mu_2)|)$$

as $(u_1, \mu_1, \mu_2) \to 0$.

Proof Writing (94), (95), (96), (97) in the notation of Theorem 2, that is in the form of equation (47) with K defined by (61), one finds that $f(0,\lambda)=0$ for all $\lambda=(\mu_1,\mu_2)\in\Lambda=\Lambda_2$. One may therefore suppose without loss of generality that $h(0,\lambda)=0$ for all $\lambda\in\tilde{\Lambda}$. It follows that all the partial derivatives of h with respect to λ vanish at $(u_1,\lambda)=(0,0)\in\tilde{U}_1\times\tilde{\Lambda}$. Taking equation (50) into account, one concludes that

$$\begin{aligned} h(u_1,\mu_1,\mu_2) &= O(\| u_1 \|^2)+O(|\lambda|\,\| u_1 \|), \\ d_1h[u_1,\mu_1,\mu_2] &= O(\| u_1 \|)+O(|\lambda|) \end{aligned} \tag{103}$$

as $(u_1,\lambda)\to 0$.

To refine the estimate (103), observe from (48), (49) that h satisfies the functional equation

$$d_1h[u_1,\lambda](K_1u_1+f_1(u_1+h(u_1,\lambda),\lambda))=K_2h(u_1,\lambda)+f_2(u_1+h(u_1,\lambda),\lambda).$$

Differentiating this expression with respect to u_1 at $u_1=0$ one finds that

$$d_1h[0,\lambda](K_1u_1+d_1f_1[0,\lambda](u_1+d_1h[0,\lambda](u_1)))=K_2d_1h[0,\lambda](u_1)+d_1f_2[0,\lambda](u_1+d_1h[0,\lambda](u_1))$$

for each $(u_1,\lambda)\in X_1\times\tilde{\Lambda}$. A further differentiation with respect to λ at $\lambda=0$ yields the equation

$$d_{12}h[0,0](K_1u_1,\lambda)=K_2d_{12}h[0,0](u_1,\lambda)+d_{12}f_2[0,0](u_1,\lambda). \tag{104}$$

(In keeping with the usual notation, the second Fréchet derivative $d_{12}h[u_1,\tilde{\lambda}]:X_1\times\Lambda\to X_1\times\Lambda$ is defined by the formula $d_{12}h[u_1,\tilde{\lambda}](x_1,\lambda)=d_1[d_2h[u_1,\tilde{\lambda}](\lambda)](x_1)$.) Equation (104) leads to the recursion formulae

$$\begin{aligned} d_{12}h[0,0](e_1,\lambda) &= -K_2^{-1}d_{12}f_2[0,0](e_1,\lambda), \\ d_{12}h[0,0](e_i,\lambda) &= K_2^{-1}(d_{12}h[0,0](e_{i-1},\lambda)-d_{12}f_2[0,0](e_i,\lambda)), \qquad i=2,3,4, \end{aligned} \tag{105}$$

where the e_i are defined by (102).

A straightforward calculation using (94), (95), (96), (97) shows that

$$d_{12}f[0,0](x,\lambda)=\left(0,0,\frac{3}{2}\left(z^2-\frac{1}{3}\right)(\mu_1\rho+3\mu_2\Gamma_z(1)),0\right),$$

where $x=(\rho,v,\Psi,\Gamma)$ and $\lambda=(\mu_1,\mu_2)$. Equation (105) implies that

$$d_{12}h[0,0](\rho_1,0,0,0,\mu_1,\mu_2)=-K_2^{-1}(I-P)\left(0,0,\frac{3}{2}\left(z^2-\frac{1}{3}\right)\mu_1\rho_1,0\right), \tag{106}$$

where $P:X\to X_1$ is the spectral projection defined in Section 3. Because $\langle(\rho_1,0,0,0),e_1\rangle=\rho_1$ and $\langle(\rho_1,0,0,0),e_i\rangle=0$, $i=2,3,4$, it follows from (103) and (106) that

$$h(u_1,\mu_1,\mu_2)=O(\| u_1 \|^2)+O(|\mu_1||\rho_1|)+O(|(\mu_1,\mu_2)|\,\| (v_1,\Psi_1,\Gamma_1) \|)+O(|(\mu_1,\mu_2)|^2\,\| u_1 \|)$$

as $(u_1,\mu_1,\mu_2)\to 0$. $\square$

Lemma 8 *The reduced Hamiltonian* $\tilde{H}$ *for* (V_2, Ω_2, H_2) *satisfies the estimate*

$$\tilde{H}(u_1, \mu_1, \mu_2) = H_2(u_1 + 0, \mu_1, \mu_2) + O((|(\mu_1, \mu_2)| \parallel u_1 \parallel + \parallel u_1 \parallel^2) \parallel h(u_1, \mu_1, \mu_2) \parallel)$$

as $(u_1, \mu_1, \mu_2) \to 0$.

Proof Writing $x \in V_2$ as $x = (\rho, v, \Psi, \Gamma)$, one finds from (93) that

$$H_2(x, \mu_1, \mu_2) = H_2(x, 0, 0) - \frac{1}{2}\mu_1 \rho^2 + \mu_2 \left(1 - \frac{1}{\sqrt{1 + R^2(x)}}\right).$$

When $x = u_1 + h(u_1, \mu_1, \mu_2)$, with $u_1 = (\rho_1, v_1, \Psi_1, \Gamma_1)$, it follows that

$$R^2(u_1 + h(u_1, \mu_1, \mu_2)) = R^2(u_1) + O(\parallel u_1 \parallel \parallel h(u_1, \mu_1, \mu_2) \parallel)$$

and

$$\frac{1}{2}\rho^2 = \frac{1}{2}\rho_1^2 + O(\parallel u_1 \parallel \parallel h(u_1, \mu_1, \mu_2) \parallel)$$

as $(u_1, \mu_1, \mu_2) \to 0$, whereby

$$\begin{aligned}
\tilde{H}(u_1, \mu_1, \mu_2) &= H_2(u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2) \\
&= H_2(u_1 + h(u_1, \mu_1, \mu_2), 0, 0) - \frac{1}{2}\mu_1 \rho_1^2 + \mu_2 \left(1 - \frac{1}{\sqrt{1 + R^2(u_1)}}\right) \\
&\quad + O(|(\mu_1, \mu_2)| \parallel u_1 \parallel \parallel h(u_1, \mu_1, \mu_2) \parallel) \\
&= H_2(u_1, \mu_1, \mu_2) + H_2(u_1 + h(u_1, \mu_1, \mu_2), 0, 0) - H_2(u_1, 0, 0) \\
&\quad + O(|(\mu_1, \mu_2)| \parallel u_1 \parallel \parallel h(u_1, \mu_1, \mu_2) \parallel)
\end{aligned}$$

as $(u_1, \mu_1, \mu_2) \to 0$. Lemma 6 implies that

$$\begin{aligned}
H_2(u_1 + h(u_1, \mu_1, \mu_2), 0, 0) - H_2(u_1, 0, 0) &= \langle A_2 h(u_1, \mu_1, \mu_2), h(u_1, \mu_1, \mu_2) \rangle \\
&\quad + N(u_1 + h(u_1, \mu_1, \mu_2)) - N(u_1),
\end{aligned}$$

where $N(u_1) = O(\parallel u_1 \parallel^3)$ as $u_1 \to 0$, so that

$$N(u_1 + h(u_1, \mu_1, \mu_2)) = N(u_1) + O(\parallel u_1 \parallel^2 \parallel h(u_1, \mu_1, \mu_2) \parallel)$$

as $(u, \mu_1, \mu_2) \to 0$. Since $\parallel h(u_1, \mu_1, \mu_2) \parallel = O(\parallel u_1 \parallel)$ as $u_1 \to 0$ (Lemma 7), one concludes that

$$\tilde{H}(u_1, \mu_1, \mu_2) = H_2(u_1 + 0, \mu_1, \mu_2) + O((|(\mu_1, \mu_2)| \parallel u_1 \parallel + \parallel u_1 \parallel^2) \parallel h(u_1, \mu_1, \mu_2) \parallel)$$

as $(u, \mu_1, \mu_2) \to 0$. $\square$

Lemma 9 *The reduced symplectic* 2*-form* $\tilde{\Omega}$ *for* (V_2, Ω_2, H_2) *takes the form*

$$\begin{aligned}\tilde{\Omega}|_{(u_1,\mu_1,\mu_2)}(w_1, w_2) &= \bar{\Omega}(w_1 + 0, w_2 + 0) + \Omega_{\mu_2}(w_1 + 0, w_2 + 0) \\ &\quad + O((|(\mu_1, \mu_2)|^2 + \| u_1 \|) \| w_1 \| \| w_2 \|)\end{aligned}$$

as $(u_1, \mu_1, \mu_2) \to 0$.

Proof The 2-form $\tilde{\Omega}$ is defined in coordinates by

$$\begin{aligned}\tilde{\Omega}|_{(u_1,\mu_1,\mu_2)}(w_1, w_2) &= \Omega_2|_{(u_1+h(u_1,\mu_1,\mu_2),\mu_2)}(w_1 + d_1h[u_1, \mu_1, \mu_2](w_1), w_2 + d_1h[u_1, \mu_1, \mu_2](w_2)) \\ &= \bar{\Omega}(w_1 + d_1h[u_1, \mu_1, \mu_2](w_1), w_2 + d_1h[u_1, \mu_1, \mu_2](w_2)) \\ &\quad + \Omega_{\mu_2}(w_1 + d_1h[u_1, \mu_1, \mu_2](w_1), w_2 + d_1h[u_1, \mu_1, \mu_2](w_2)) \\ &\quad + O(\| u_1 \| \| w_1 \| \| w_2 \|)\end{aligned}$$

as $u_1 \to 0$, where the second line follows by (100). Observe that

$$\begin{aligned}&\bar{\Omega}(w_1 + d_1h[u_1, \mu_1, \mu_2](w_1), w_2 + d_1h[u_1, \mu_1, \mu_2](w_2)) \\ = {} & \bar{\Omega}(w_1, w_2) + \bar{\Omega}(d_1h[u_1, \mu_1, \mu_2](w_1), w_2) \\ & + \bar{\Omega}(w_1, d_1h[u_1, \mu_1, \mu_2](w_2)) + \bar{\Omega}(d_1h[u_1, \mu_1, \mu_2](w_1), d_1h[u_1, \mu_1, \mu_2](w_2)). \qquad (107)\end{aligned}$$

The second and third terms in this expression vanish because of Lemma 5, so that

$$\begin{aligned}&\bar{\Omega}(w_1 + d_1h[u_1, \mu_1, \mu_2](w_1), w_2 + d_1h[u_1, \mu_1, \mu_2](w_2)) \\ = {} & \bar{\Omega}(w_1, w_2) + O(\| d_1h[u_1, \mu_2, \mu_2 \|^2 \| w_1 \| \| w_2 \|) \\ = {} & \bar{\Omega}(w_1, w_2) + O((|(\mu_1, \mu_2)|^2 + \| u_1 \|) \| w_1 \| \| w_2 \|) \qquad (108)\end{aligned}$$

as $(u_1, \mu_1, \mu_2) \to 0$.

One also has that

$$\begin{aligned}&\Omega_{\mu_2}(w_1 + d_1h[u_1, \mu_1, \mu_2](w_1), w_2 + d_1h[u_1, \mu_1, \mu_2](w_2)) \\ = {} & \Omega_{\mu_2}(w_1, w_2) + \Omega_{\mu_2}(d_1h[u_1, \mu_1, \mu_2](w_1), w_2) \\ & + \Omega_{\mu_2}(w_1, d_1h[u_1, \mu_1, \mu_2](w_2)) + \Omega_{\mu_2}(d_1h[u_1, \mu_1, \mu_2](w_1), d_1h[u_1, \mu_1, \mu_2](w_2)).\end{aligned}$$

Here the second and third terms are $O(|\mu_2| \, \| d_1h[u_1, \mu_1, \mu_2] \| \| w_1 \| \| w_2 \|)$ as $(u_1, \mu_1, \mu_2) \to 0$ (see equation (101)), so that

$$\begin{aligned}&\Omega_{\mu_2}(w_1 + d_1h[u_1, \mu_1, \mu_2](w_1), w_2 + d_1h[u_1, \mu_1, \mu_2](w_2)) \\ = {} & \Omega_{\mu_2}(w_1, w_2) + O((|(\mu_1, \mu_2)|^2 + \| u_1 \|) \| w_1 \| \| w_2 \|) \qquad (109)\end{aligned}$$

as $(u_1, \mu_1, \mu_2) \to 0$. The result now follows immediately from (107), (108), (109). $\square$

4.4 Calculations

Using the estimates established in the previous section, one may now calculate approximations to $\tilde{\Omega}$ and $\tilde{H}$. Observe that

$$\bar{\Omega}(e_4, e_3) = 2/945, \quad \bar{\Omega}(e_4, e_1) = 1/45, \quad \bar{\Omega}(e_3, e_2) = -1/45$$

and that all other combinations $\bar{\Omega}(e_i, e_j)$ are zero. Elements of the new basis

$$e_1' = 3\sqrt{5}\left(e_4 - \frac{2}{21}e_2\right), \quad e_2' = 3\sqrt{5}e_2, \quad e_3' = 3\sqrt{5}e_1, \quad e_4' = 3\sqrt{5}e_3$$

for X_1 therefore satisfy the relations

$$\bar{\Omega}(e_1', e_3') = \bar{\Omega}(e_2', e_4') = 1$$

and all other combinations $\bar{\Omega}(e_i', e_j')$ are zero. Writing $x_1 = q_1e_1' + q_2e_2' + p_1e_3' + p_2e_4'$, one has that

$$\bar{\Omega}(\hat{w}, \check{w}) = \bar{\Omega}((\hat{q}_1, \hat{q}_2, \hat{p}_1, \hat{p}_2), (\check{q}_1, \check{q}_2, \check{p}_1, \check{p}_2)) = \hat{q}_1\check{p}_1 + \hat{q}_2\check{p}_2 - \check{q}_1\hat{p}_1 - \check{q}_2\hat{p}_2$$

and

$$\Omega_{\mu_2}(\hat{w}, \check{w}) = -45\mu_2\check{p}_1\hat{q}_2 + 45\mu_2\hat{p}_1\check{q}_2 - \frac{30}{7}\mu_2\hat{p}_1\check{q}_1 + \frac{30}{7}\mu_2\check{p}_1\hat{q}_1.$$

It follows from Lemma 9 that

$$\tilde{\Omega}_{(q_1,q_2,p_1,p_2,\mu_1,\mu_2)}(\hat{w}, \check{w}) = \Upsilon_{\mu_2}(\hat{w}, \check{w}) + \zeta|_{(q_1,q_2,p_1,p_2,\mu_1,\mu_2)}(\hat{w}, \check{w}), \tag{110}$$

where

$$\Upsilon_{\mu_2}(\hat{w}, \check{w}) = \hat{q}_1\check{p}_1 + \hat{q}_2\check{p}_2 - \check{q}_1\hat{p}_1 - \check{q}_2\hat{p}_2 - 45\mu_2\check{p}_1\hat{q}_2 + 45\mu_2\hat{p}_1\check{q}_2 - \frac{30}{7}\mu_2\hat{p}_1\check{q}_1 + \frac{30}{7}\mu_2\check{p}_1\hat{q}_1$$

and

$$\zeta|_{(q_1,q_2,p_1,p_2,\mu_1,\mu_2)}(\hat{w}, \check{w}) = O((|(\mu_1, \mu_2)|^2 + |(q_1, q_2, p_1, p_2)|) \parallel \hat{w} \parallel\parallel \check{w} \parallel) \tag{111}$$

as $(q_1, q_2, p_1, p_2) \to 0$.

Lemmas 7 and 8 assert that

$$\begin{aligned}\tilde{H}(q_1, q_2, p_1, p_2, \mu_1, \mu_2) &= H_2(q_1e_1' + q_2e_2' + p_1e_3' + p_2e_4', \mu_1, \mu_2) \\ &+O((|(\mu_1, \mu_2)||(q_1, q_2, p_1, p_2)| + |(q_1, q_2, p_1, p_2)|^2) \parallel h(q_1, q_2, p_1, p_2, \mu_1, \mu_2) \parallel)\end{aligned} \tag{112}$$

with

$$\begin{aligned}h(q_1, q_2, p_1, p_2, \mu_1, \mu_2) \quad &= \quad O(|(q_1, q_2, p_1, p_2)|^2) + O(|\mu_1||p_1|) + O(|(\mu_1, \mu_2)||(q_1, q_2, p_2)|) \\ &\quad +O(|(\mu_1, \mu_2)|^2|(q_1, q_2, p_1, p_2)|)\end{aligned} \tag{113}$$

as $(q_1, q_2, p_1, p_2, \mu_1, \mu_2) \to 0$. A straightforward calculation based upon (93) shows that

$$\begin{aligned}H_2 \quad = \quad &\frac{1}{2}(1 - \rho)\int_0^1 \left(\Gamma_z^2 - \Psi_z^2\right)\, dz + 3(v - \Psi(1))\int_0^1 z\Psi_z\Gamma_z\, dz - \frac{1}{2}\mu_1\rho^2 - \frac{1}{2}\rho^3 \\ &-3(1 - \rho + 3B(\Gamma))\Psi(1)(v - \Psi(1)) + \left(\frac{3}{2}\rho + 9B(\Gamma) + \frac{9}{2}\mu_2 + 27\mu_2B(\Gamma)\right)(v - \Psi(1))^2 \\ &+O(\parallel (\rho, v, \Psi, \Gamma) \parallel^4)\end{aligned} \tag{114}$$

as $(\rho, v, \Psi, \Gamma, \mu_1, \mu_2) \to 0$. Writing

$$(\rho, v, \Psi, \Gamma) = q_1 e_1' + q_2 e_2' + p_1 e_3' + p_2 e_4'$$

and using (112) and (114), one finds that

$$\begin{aligned}\tilde{H} \;\; = \;\; & \frac{p_2^2}{2} + \frac{q_1^2}{21} - q_1 q_2 - \frac{45}{2}\mu_1 p_1^2 + \frac{10}{49}\mu_2 q_1^2 - \frac{30}{7}\mu_2 q_1 q_2 + \frac{45}{2}\mu_2 q_2^2 - \frac{135\sqrt{5}}{2} p_1^3 - 6\sqrt{5} p_2 q_2^2 \\ & - \frac{3\sqrt{5}}{2} p_1 p_2^2 + \frac{3\sqrt{5}}{49} p_1 q_1^2 + \frac{45\sqrt{5}}{2} p_1 q_2^2 - \frac{9\sqrt{5}}{7} p_1 q_1 q_2 + \frac{7}{\sqrt{5}} p_2 q_1 q_2 - \frac{58}{147\sqrt{5}} p_2 q_1^2 \\ & - \frac{8\sqrt{5}}{49}\mu_2 p_2 q_1^2 - 18\sqrt{5}\mu_2 p_2 q_2^2 + \frac{24\sqrt{5}}{7}\mu_2 p_2 q_1 q_2 \\ & + O((|(\mu_1,\mu_2)||(q_1,q_2,p_1,p_2)| + |(q_1,q_2,p_1,p_2)|^2) \parallel h(q_1,q_2,p_1,p_2,\mu_1,\mu_2) \parallel) \end{aligned} \tag{115}$$

as $(q_1, q_2, p_1, p_2, \mu_1, \mu_2) \to 0$. (This calculation was performed by hand and checked by machine with the software package *Mathematica* (Wolfram 1991)).

The next step is to examine the Hamiltonian vector field $\tilde{v}$ for $(M_C^\lambda, \tilde{\Omega}, \tilde{H})$. Working in the coordinates $x_1 = (q_1, q_2, p_1, p_2)$, observe that the 2-form Υ_{μ_2} may be written as

$$\Upsilon_{\mu_2} = \langle \kappa_{\mu_2} \hat{w}, \check{w} \rangle_{X_1},$$

where $\kappa|_{\mu_2} : X_1 \to X_1$ is the linear homeomorphism defined by

$$\kappa_{\mu_2} = \begin{pmatrix} 0 & 0 & -1 - \frac{30}{7}\mu_2 & 0 \\ 0 & 0 & 45\mu_2 & -1 \\ 1 + \frac{30}{7}\mu_2 & -45\mu_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

it follows that $\Upsilon|_{\mu_2}$ is nondegenerate. Because $\zeta|_{(u_1,\mu_1,\mu_2)} : X_1 \times X_1 \to \mathbb{R}$ is a bounded bilinear operator there exists a bounded linear operator $\tau_{(u_1,\mu_1,\mu_2)} : X_1 \to X_1$ such that

$$\zeta|_{(u_1,\mu_1,\mu_2)}(\hat{w}, \check{w}) = \langle -\tau|_{(u_1,\mu_1,\mu_2)} \hat{w}, \check{w} \rangle_{X_1}$$

(the minus sign is chosen for convenience in the following calculations). The nondegeneracy of $\tilde{\Omega}_2|_{(u_1,\mu_1,\mu_2)}$, together with equation (110), implies that $\kappa_{\mu_2} - \tau|_{(u_1,\mu_1,\mu_2)} : X_1 \to X_1$ is a linear homeomorphism, and (111) shows that

$$\parallel \tau|_{(u_1,\mu_1,\mu_2)} \parallel = O(|(\mu_1,\mu_2)|^2 + \parallel u_1 \parallel) \tag{116}$$

as $u_1 \to 0$.

By the definition of $\tilde{v}$, one has that

$$\tilde{\Omega}|_{(u_1,\mu_1,\mu_2)}(\tilde{v}, v) = d_1 \tilde{H}[q_1, q_1, p_1, p_2, \mu_1, \mu_2](v)$$

for all $v \in X_1$. It follows that

$$(\kappa_{\mu_2} - \tau|_{(u_1,\mu_1,\mu_2)})(\tilde{v}) = \nabla_1 \tilde{H}[u_1, \mu_1, \mu_2], \tag{117}$$

where $\nabla_1 \tilde{H}[u_1,\mu_1,\mu_2]$ is the unique element of X_1 such that

$$d_1\tilde{H}[u_1,\mu_1,\mu_2](v) = \langle \nabla_1 \tilde{H}[u_1,\mu_1,\mu_2], v\rangle_{X_1}$$

for all $v \in X_1$. Equation (117) implies that

$$\kappa_{\mu_2}(1-\kappa_{\mu_2}^{-1}\tau|_{(u_1,\mu_1,\mu_2)})(\tilde{v}) = \nabla_1 \tilde{H}[u_1,\mu_1,\mu_2],$$

whereby

$$\begin{aligned}
\tilde{v} &= \sum_{\ell=0}^{\infty} \left[\kappa_{\mu_2}^{-1}\tau|_{(u_1,\mu_1,\mu_2)}\right]^{\ell} \kappa_{\mu_2}^{-1}\nabla_1\tilde{H}[u_1,\mu_1,\mu_2] \\
&= \kappa_{\mu_2}^{-1}\nabla_1\tilde{H}[u_1,\mu_1,\mu_2] + \sum_{\ell=1}^{\infty} \left[\kappa_{\mu_2}^{-1}\tau|_{(u_1,\mu_1,\mu_2)}\right]^{\ell} \kappa_{\mu_2}^{-1}\nabla_1\tilde{H}[u_1,\mu_1,\mu_2].
\end{aligned}$$

Hamilton's equations for $(M_C^\lambda, \tilde{\Omega}, \tilde{H})$ are therefore

$$\dot{u}_1 = \kappa_{\mu_2}^{-1}\nabla_1\tilde{H}[u_1,\mu_1,\mu_2] + \sum_{\ell=1}^{\infty} \left[\kappa_{\mu_2}^{-1}\tau|_{(u_1,\mu_1,\mu_2)}\right]^{\ell} \kappa_{\mu_2}^{-1}\nabla_1\tilde{H}[u_1,\mu_1,\mu_2], \tag{118}$$

where $u_1(t) = (q_1(t), q_2(t), p_1(t), p_2(t))$, $t \in (a,b)$.

Finally, let us work in the parameter regime

$$\mu_1 = c\mu^4, \quad \mu_2 = d\mu^2, \qquad \mu \in \tilde{\Lambda},$$

where c and d are constants, and introduce new independent and dependent variables in Hamilton's equations (118) defined by

$$Q_1(X) = \frac{1}{\mu^7}q_1(x), \quad Q_2(X) = \frac{1}{\mu^5}q_2(x), \quad P_1(X) = \frac{1}{\mu^4}p_1(x), \quad P_2(X) = \frac{1}{\mu^6}p_2(x)$$

with $X = \mu x$. Equation (113) shows that $h = O(|\mu|^7)$ as $\mu \to 0$, whereby (115) implies that

$$\tilde{H}(Q_1,Q_2,P_1,P_2,c\mu^4,d\mu^2) = \mu^{12}\left[\frac{P_2^2}{2} - Q_1Q_2 - \frac{45}{2}cP_1^2 - \frac{135\sqrt{5}}{2}P_1^3 + \frac{45}{2}dQ_2^2\right] + O(|\mu|^{13})$$

as $\mu \to 0$, and therefore

$$\nabla_1\tilde{H}[u_1,c\mu^4,d\mu^2] = \begin{pmatrix} -\mu^5 Q_2 + O(|\mu|^6) \\ \mu^7(45dQ_2 - Q_1) + O(|\mu|^8) \\ -\mu^8(45cP_1 + \frac{405\sqrt{5}}{2}P_1^2) + O(|\mu|^9) \\ \mu^6 P_2 + O(|\mu|^7) \end{pmatrix}$$

as $\mu \to 0$. An easy calculation shows that

$$\begin{aligned}
\kappa_{\mu_2}^{-1} &= \frac{1}{1+\frac{30}{7}d\mu^2}\begin{pmatrix} 0 & 0 & 1+\frac{30}{7}d\mu^2 & 45d\mu^2 \\ 0 & 0 & 0 & 1 \\ -1-\frac{30}{7}d\mu^2 & 0 & 0 & 0 \\ -45d\mu^2 & -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 45d\mu^2 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -45d\mu^2 & -1 & 0 & 0 \end{pmatrix} + O(\mu^4)
\end{aligned} \tag{119}$$

as $\mu \to 0$, so that

$$\kappa_{d\mu^2}^{-1} \nabla_1 \tilde{H}[u_1, c\mu^4, d\mu^2] = \begin{pmatrix} \mu^8(45dP_2 - 45cP_1 - \frac{405\sqrt{5}}{2}P_1^2) + O(|\mu|^9) \\ \mu^6 P_2 + O(|\mu|^7) \\ \mu^5 Q_2 + O(|\mu|^6) \\ \mu^7 Q_1 + O(|\mu|^8) \end{pmatrix} \tag{120}$$

as $\mu \to 0$. Equation (116) states that

$$\| \tau|_{(u_1, c\mu^4, d\mu^2)} \| = O(|\mu|^4)$$

as $\mu \to 0$, which, together with (119) and (120), implies that

$$\sum_{\ell=1}^{\infty} \left[\kappa_{d\mu^2}^{-1} \tau|_{(u_1, c\mu^4, d\mu^2)} \right]^{\ell} \kappa_{d\mu^2}^{-1} \nabla_1 \tilde{H}[u_1, c\mu^4, d\mu^2] = \begin{pmatrix} O(|\mu|^9) \\ O(|\mu|^7) \\ O(|\mu|^6) \\ O(|\mu|^8) \end{pmatrix} \tag{121}$$

as $\mu \to 0$. Clearly

$$\dot{u}_1 = \begin{pmatrix} \mu^8 Q_1' \\ \mu^6 Q_2' \\ \mu^5 P_1' \\ \mu^7 P_2' \end{pmatrix}, \tag{122}$$

where the dot denotes differentiation with respect to x and the prime denotes differentiation with respect to X. Combining (120), (121) and (122), one finds that equations (118) are

$$\begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 45dP_2 - 45cP_1 - \frac{405\sqrt{5}}{2}P_1^2 \\ P_2 \\ Q_2 \\ Q_1 \end{pmatrix} + O(|\mu|) \tag{123}$$

as $\mu \to 0$.

4.5 *A plethora of solitary waves*

In the limit $\mu \to 0$ the system (123) is equivalent to an ordinary differential equation for P_1, namely

$$P_1^{(iv)} - 45dP_1'' + 45cP_1 + \frac{405\sqrt{5}}{2}P_1^2 = 0.$$

Writing

$$c = \frac{1}{45}, \quad d = -\frac{P}{45}, \quad u = -\frac{405\sqrt{5}}{2}P_1,$$

one finds that the equation for u is

$$u^{(iv)} + Pu'' + u - u^2 = 0 \tag{124}$$

This equation has been extensively studied (see the references in Section 1.2). In particular Buffoni, Champneys & Toland (1994) have shown that when $P = -2$ there is a unique non-zero homoclinic solution of (124), which is positive and even, and that there exists a real number $\epsilon_1 > 0$ such that for each $P \in (-\infty, -2 + \epsilon_1)$, equation (124) has a non-zero homoclinic solution which is the transverse intersection (relative to the zero energy surface) of the stable and unstable manifolds of its zero equilibrium. Since transversality is an open condition, it follows that the same is true of the system (123) for $c = 1/45$, $d \in ((2 - \epsilon_1)/45, 2/45)$ and sufficiently small μ. With these values of c, d and μ, the system (123) is a four-dimensional Hamiltonian system which has a saddle-focus at 0, and the work of Devaney (1976) (see Wiggins (1988, p. 275)) therefore implies that there is a Smale horseshoe in the dynamics. Consequently there are infinitely many distinct homoclinic solutions of the four-dimensional equations on the centre manifold, and therefore, by Theorem 2, of the full water-wave problem. Standard theory states that solutions of an ordinary differential equation that converge to a hyperbolic fixed point as $t \to \infty$ do so exponentially. The homoclinic solutions of the reduced problem on the centre manifold therefore decay exponentially to zero as $x \to \infty$, and Theorem 2 implies that the corresponding solitary-wave solutions of the full water-wave problem decay exponentially to the trivial flow as $x \to 0$. These solitary waves are divided into an infinite number of classes, each of which contains infinitely many distinct solitary waves; the classes are characterised by the number of large troughs possessed by each of their members. The overall structure of the solution set may be interpreted precisely in terms of symbolic dynamics (e.g. see Wiggins (1988, p. 275), Buffoni, Champneys & Toland (1994)). Finally, equation (1) shows that the curve C_1 has the form

$$\begin{aligned} \beta &= \frac{1}{3} + \frac{2}{45}\mu^2 + O(\mu^4), \\ \alpha &= 1 + \frac{1}{45}\mu^4 + O(\mu^6) \end{aligned}$$

near and to the right of the point $(\beta, \alpha) = (1/3, 1)$. This observation confirms the location of the solitary waves in Figure 1: just to the left of C_1 near and to the right of the point $(\beta, \alpha) = (1/3, 1)$.

An alternative way of interpreting the flow on the centre manifold is to use the theory of reversible systems in four dimensions due to Champneys (1994). (The flow generated by Hamilton's equations (94), (95), (96), (97) for (V_2, Ω_2, H_2) is reversible under the transformation

$$x \to -x, \qquad (\rho, \upsilon, \Psi, \Gamma) \to (\rho, -\upsilon, -\Psi, \Gamma),$$

and a result due to Mielke (1991, §5.5) shows that the flow on the centre manifold inherits this reversibility.) Although the Hamiltonian structure is not required in Champneys' theory, a precise description of the flow on the centre manifold is just as necessary as in the present work.

5 Appendix: Derivation of Hamilton's equations for (V_2, Ω_2, H_2)

This Appendix describes the derivation of Hamilton's equations (94), (95), (96), (97) for the Hamiltonian system (V_2, Ω_2, H_2) from (54), (55), (56), (57) using the closed-form expressions for G and G^{-1} given in Sections 60 and 4.2.

It follows immediately from (54), (85), (87) that

$$\dot{\rho} = R, \tag{125}$$

where R is given by (87). By the definition of G, one has that $v = \Phi(1)$, so that

$$\dot{v} = \dot{\Phi}(1) = \frac{\Gamma_z(1)}{\rho+1} + \frac{R}{\rho+1}(\Phi_z(1) - \Phi(1)), \tag{126}$$

where the second equality follows by (56) evaluated at $z = 1$. Evaluating (89) at $z = 1$ and using the fact that $\Psi_z(1) = 0$, one finds that

$$\Phi_z(1) = R(\Gamma_z(1) - 1). \tag{127}$$

Equations (126), (127) imply that

$$\dot{v} = \frac{\Gamma_z(1)}{\rho+1} + \frac{R}{\rho+1}[R(\Gamma_z(1) - 1) - v], \tag{128}$$

and (57) may be written as

$$\begin{aligned} \dot{\Gamma} &= -\frac{\Phi_z}{\rho+1} + \frac{Rz(\Gamma_z - 1)}{\rho+1} \\ &= -\frac{\Psi_z}{\rho+1}, \end{aligned} \tag{129}$$

where the second line follows by (89).

It follows from (84) and (86) that

$$\begin{aligned} \dot{\Psi} &= \dot{\Phi} - \dot{R}A(\Gamma) - R\left[\int_0^z s\dot{\Gamma}_z\,ds - \int_0^1\int_0^z s\dot{\Gamma}_z\,ds\,dz\right] \\ &= \frac{\Gamma_z}{\rho+1} + \frac{R}{\rho+1}(z\Phi_z - \Phi(1)) - \dot{R}A(\Gamma) \\ &\quad + \frac{R}{\rho+1}\left[\int_0^z s\Psi_{zz}\,ds - \int_0^1\int_0^z s\Psi_{zz}\,ds\,dz\right], \end{aligned} \tag{130}$$

where the second line follows from (56) and (129) differentiated with respect to z. Using (89) and the fact that $v = \Phi(1)$, one finds from (130) that

$$\begin{aligned} \dot{\Psi} &= \frac{\Gamma_z}{\rho+1} + \frac{R}{\rho+1}\left[z\Psi_z + Rz^2(\Gamma_z - 1) - v\right] - \dot{R}A(\Gamma) \\ &\quad + \frac{R}{\rho+1}\left[\int_0^z s\Psi_{zz}\,ds - \int_0^1\int_0^z s\Psi_{zz}\,ds\,dz\right]. \end{aligned} \tag{131}$$

An integration by parts shows that

$$\int_0^z s\Psi_{zz}\,ds = [s\Psi_z]_0^z - \int_0^z \Psi_z\,dz = z\Psi_z(z) - \Psi(z) + \Psi(0)$$

and therefore

$$\int_0^1\int_0^s s\Psi_{zz}\,ds\,dz = \int_0^1 z\Psi_z\,dz - \int_0^1 \Psi\,dz + \Psi(0) = \Psi(1) + \Psi(0),$$

so that (131) becomes

$$\dot{\Psi} = \frac{\Gamma_z}{\rho+1} + \frac{R}{\rho+1}[2z\Psi_z + Rz^2(\Gamma_z - 1) - \upsilon - \Psi - \Psi(1)] - \dot{R}A(\Gamma). \tag{132}$$

The next step is to differentiate (85) to find that

$$\begin{aligned} \dot{R} &= \frac{\dot{W}}{\sqrt{(\frac{1}{3}+\mu_2)^2 - W^2}} + \frac{W^2\dot{W}}{[(\frac{1}{3}+\mu_2)^2 - W^2]^{\frac{3}{2}}} \\ &= \frac{(1+R^2)^{\frac{3}{2}}}{\frac{1}{3}+\mu_2}\dot{W}. \end{aligned} \tag{133}$$

By the definition of W and the fact that $\upsilon = \Phi(1)$, one has that

$$W = w + \frac{1}{\rho+1}\int_0^1 z\Phi_z\Gamma_z\,dz - \frac{\upsilon}{\rho+1},$$

whereby

$$\begin{aligned} \dot{W} &= \dot{w} - \frac{\dot{\rho}}{(\rho+1)^2}\int_0^1 z\Phi_z\Gamma_z\,dz + \frac{\dot{\rho}}{(\rho+1)^2}\upsilon - \frac{\dot{\upsilon}}{\rho+1} \\ &\quad + \frac{1}{\rho+1}\int_0^1 z\dot{\Phi}_z\Gamma_z\,dz + \frac{1}{\rho+1}\int_0^1 z\Phi_z\dot{\Gamma}_z\,dz. \end{aligned} \tag{134}$$

One finds expressions for $\dot{\Phi}_z$ and $\dot{\Gamma}_z$ by differentiating (56), (57), and hence obtains the relationship

$$\begin{aligned} &\int_0^1 z\dot{\Phi}_z\Gamma_z\,dz + \int_0^1 z\Phi_z\dot{\Gamma}_z\,dz \\ &= \frac{1}{\rho+1}\left\{\int_0^1 z\Gamma_z\Gamma_{zz}\,dz + R\int_0^1 z\Gamma_z(z\Phi_z)_z\,dz - \int_0^1 z\Phi_z\Phi_{zz}\,dz \right. \\ &\left. + R\int_0^1 z\Phi_z(z\Gamma_z)_z\,dz - R\int_0^1 z\Phi_z\,dz\right\} \\ &= \frac{1}{\rho+1}\left\{-\int_0^1 \frac{\Gamma_z^2}{2}\,dz + R\left[z^2\Gamma_z\Phi_z\right]_0^1 + \left[\frac{z\Gamma_z^2}{2}\right]_0^1 + \int_0^1 \frac{\Phi_z^2}{2}\,dz - \left[\frac{z\Phi_z^2}{2}\right]_0^1 - R\upsilon\right\} \\ &= \frac{1}{\rho+1}\left\{\int_0^1 \frac{1}{2}(\Phi_z^2 - \Gamma_z^2)\,dz - R\upsilon + \frac{1}{2}(\Gamma_z^2(1) - \Phi_z^2(1)) + R\Phi_z(1)\Gamma_z(1)\right\}, \end{aligned} \tag{135}$$

where the fact that $\int_0^1 z\Phi_z\,dz = \Phi(1) = \upsilon$ has been used. Equation (55) implies that

$$\begin{aligned} \dot{\omega} &= \frac{1}{2(\rho+1)^2}\int_0^1 (\Gamma_z^2 - \Phi_z^2)\,dz + \mu_1\rho + \frac{\rho^2(3+2\rho)}{2(1+\rho)^2} \\ &\quad + \frac{R}{(\rho+1)^2}\int_0^1 z\Phi_z\Gamma_z\,dz - \frac{R\upsilon}{(\rho+1)^2}, \end{aligned} \tag{136}$$

and combining (134), (135), (136) and using (125), one finds that

$$\begin{aligned}\dot{W} &= \mu_1\rho + \frac{\rho^2(3+2\rho)}{2(\rho+1)^2} - \frac{\dot{v}}{(\rho+1)} - \frac{Rv}{(\rho+1)^2} \\ &\quad + \frac{1}{(\rho+1)^2}\left\{\frac{1}{2}(\Gamma_z^2(1) - \Phi_z^2(1)) + R\Phi_z(1)\Gamma_z(1)\right\}. \end{aligned} \tag{137}$$

This formula is further simplified by a substitution of $\dot{v}$ from (128) and $\Phi_z(1)$ from (127). The result is

$$\dot{W} = \mu_1\rho + \frac{\rho^2(3+2\rho)}{2(\rho+1)^2} + \frac{R^2(\Gamma_z(1)-1)^2}{2(\rho+1)^2} + \frac{\Gamma_z(1)(\Gamma_z(1)-2)}{2(\rho+1)^2}, \tag{138}$$

and (132), (133), (138) show that

$$\begin{aligned}\dot{\Psi} &= -\frac{A(\Gamma)(1+R^2)^{\frac{3}{2}}}{\frac{1}{3}+\mu_2}\left[\mu_1\rho + \frac{\rho^2(3+2\rho)}{2(\rho+1)^2} + \frac{R^2(\Gamma_z(1)-1)^2}{2(\rho+1)^2} + \frac{\Gamma_z(1)(\Gamma_z(1)-2)}{2(\rho+1)^2}\right] \\ &\quad + \frac{\Gamma_z}{\rho+1} + \frac{R}{\rho+1}[2z\Psi_z + Rz^2(\Gamma_z - 1) - v - \Psi - \Psi(1)]. \end{aligned} \tag{139}$$

Equations (125), (128), (129), (139) are Hamilton's equations for (V_2, Ω_2, H_2).

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