Quadrature Methods for Symm's Integral Equation on Polygons *

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Abstract

In a recent paper the authors obtained stability and convergence results for spline collocation methods of arbitrarily high order applied to the classical first-kind boundary integral equation with logarithmic kernel (Symm's equation) on a polygonal domain. This paper obtains the same convergence results for the case when the collocation integrals are approximated using an appropriate quadrature rule. The analysis depends on the calculus of non-standard Mellin pseudodifferential operators and on some new estimates for the kernels of these operators. Calculations using piecewise constant and piecewise linear splines are reported. These are consistent with the theoretical results and suggest some further conjectures about the performance of this method.

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1 Introduction

This work is the third in a series of papers (the others being [4], [5]) concerned with practical collocation methods for the boundary integral equation

$$-\frac{1}{\pi}\int_\Gamma \log|\boldsymbol{x}-\boldsymbol{\xi}|u(\boldsymbol{\xi})d\Gamma(\boldsymbol{\xi}) = f(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma, \tag{1.1}$$

when Γ is a closed polygon enclosing a simply connected bounded domain in $\mathbb{R}^2$. This equation arises regularly in the solution of two-dimensional potential problems. The collocation method is well-analysed when Γ is smooth. Until recently this analysis did not carry over to the case of polygonal Γ, and for this case only special results for low-order methods were known. This situation changed with the recent paper [4] which obtained stability and optimal convergence for collocation methods of arbitrarily high order applied to (1.1). The chief purpose of the present paper is to obtain analogous results for a fully discrete version of the method in [4], in which the collocation integrals are computed by an appropriate quadrature rule. This quadrature analysis is carried out using some new estimates which lead to a more general version of the results in [4]. In addition extensive numerical experiments illustrating the theoretical results are given in the final section of the paper.

The key step in obtaining the results of [4], [5] is the introduction of an appropriate nonlinear parametrization $\boldsymbol{\gamma}:[-\pi,\pi]\to\Gamma$ and use of the change of variables $\boldsymbol{x}=\boldsymbol{\gamma}(s)$, $\boldsymbol{\xi}=\boldsymbol{\gamma}(\sigma)$, to transform (1.1) into

$$Kw(s) := -\frac{1}{\pi}\int_{-\pi}^{\pi}\log|\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(\sigma)|w(\sigma)d\sigma = g(s), \quad s\in[-\pi,\pi], \tag{1.2}$$

where

$$w(\sigma) = |\boldsymbol{\gamma}'(\sigma)|u(\boldsymbol{\gamma}(\sigma)), \quad g(s) = f(\boldsymbol{\gamma}(s)). \tag{1.3}$$

The corners of Γ induce singularities in the solution u of (1.1). However, the solution w of the transformed equation (1.2) may (evidently) be made as smooth as desired on $[-\pi,\pi]$, provided $\boldsymbol{\gamma}$ is smooth and varies sufficiently slowly near the corners of Γ. In this case w may be approximated to optimal order by an arbitrarily high-order spline based on a *uniform* mesh on $[-\pi,\pi]$. The numerical method which we propose then is the classical one for integral equations of this sort: Approximate w by a smooth spline of order k on a uniform grid with mesh diameter $h=2\pi/n$, and determine the degrees of freedom in this approximation by collocating (1.2) at the mid-points of subintervals if k is odd and the break-points if k is even.

One of the chief difficulties in the numerical analysis of this method is first of all to establish the well-posedness of the transformed equation (1.2) in appropriate Sobolev spaces and to prove that indeed the solution w can be made smooth with appropriate choice of $\boldsymbol{\gamma}$. This was done for the first time in [4], with a generalisation given in [5]. The analysis proceeds by first showing that the theory of (1.2) can be reduced to the theory of a related second-kind integral equation featuring a Mellin convolution operator. In general, such operators arise frequently in the analysis of boundary integral equations in non-smooth domains. However in this case the operators which arise are quite non-standard due to the use of the nonlinear parametrization $\boldsymbol{\gamma}$. A lengthy analysis in [4] obtains the well-posedness (in an appropriate sense) of this second-kind equation. A convergence theory for the collocation method (also given in [4]) follows. It is typical of the analysis of approximation schemes for other Mellin convolution operators: stability is only proved under the assumption that the approximating space may be modified near the corners. (For the origin of this stability technique see [2]. A more comprehensive discussion is given in [3].)

The plan of the present paper is as follows. In Section 2 we describe briefly the reduction of (1.2) to a second-kind Mellin convolution equation and we give a review of the well-posedness results from [4], [5]. We also obtain some new estimates related to the kernel of the Mellin operator. These are used in Section 3 to give a much simpler proof of the convergence of the collocation method than that given in [4]. Then in Section 4 we show that the convergence results of Section 3 remain true even when the collocation integrals are approximated numerically, provided an appropriate quadrature scheme is employed. Numerical experiments in Section 5 verify the theoretical results of the paper but they also suggest that some aspects of the theory may be strengthened in future. We finish this introductory section with some preliminaries and a description of the nonlinear parametrisation $\boldsymbol{\gamma}$ which we shall use. Before doing this we mention that there is now a large literature on the numerical solution of (1.1). We do not attempt to give a review of this here but merely mention that an extensive review (mostly for the case of smooth Γ) is given in [11]. An extended discussion of the literature for polygonal Γ is given in [4].

We assume throughout that the transfinite diameter of Γ is not equal to 1. We suppose that Γ has corners $\boldsymbol{x}_0, \boldsymbol{x}_1, ..., \boldsymbol{x}_r = \boldsymbol{x}_0$ and that, for each j, the exterior angle between the sides $\boldsymbol{x}_j - \boldsymbol{x}_{j-1}$ and $\boldsymbol{x}_{j+1} - \boldsymbol{x}_j$ is $(1+\chi_j)\pi$, $0 < |\chi_j| < 1$. The side joining $\boldsymbol{x}_j$ to $\boldsymbol{x}_{j+1}$ is denoted Γ_j and $|\Gamma_j|$ denotes its length. $|\Gamma|$ is the length of Γ.

To define the parametrization $\boldsymbol{\gamma}$, choose a grading exponent $q \in \mathbb{N}$ and introduce r points $-\pi < S_0 < S_1 < ... < S_{r-1} < \pi$, given by

$$(S_{j+1} - S_j)/2\pi = |\Gamma_j|^{1/q} / \sum_{j=1}^{r-1} |\Gamma_j|^{1/q}, \quad j = 0, ..., r-1, \tag{1.4}$$

where $S_r = S_0 + 2\pi$. These will be the preimages of the corner points $\boldsymbol{x}_j$ under $\boldsymbol{\gamma}$. Further select a function ν such that

$$\nu \in \mathcal{C}^\infty[0,1], \quad \nu(0) = 0, \quad \nu(1) = 1, \quad \nu'(s) > 0, \quad 0 \le s \le 1, \tag{1.5}$$

and define, for $j = 0, ..., r-1$,

$$\boldsymbol{\gamma}(s) = \boldsymbol{x}_j + \rho_j(s)(\boldsymbol{x}_{j+1} - \boldsymbol{x}_j), \quad S_j \le s \le S_{j+1},$$

where

$$\rho_j(s) := \frac{\nu^q\left(\frac{s-S_j}{S_{j+1}-S_j}\right)}{\left\{\nu^q\left(\frac{s-S_j}{S_{j+1}-S_j}\right) + \nu^q\left(\frac{S_{j+1}-s}{S_{j+1}-S_j}\right)\right\}}, \tag{1.6}$$

where the usual periodicity convention $\boldsymbol{\gamma}(s+2\pi) = \boldsymbol{\gamma}(s)$ is adopted. The parametrization we have chosen is graded with exponent q near each corner.

Example 1.1 The simplest choice of ν satisfying (1.5) is, of course, $\nu(s) = s$. If in addition $q = 1$, we obtain from (1.6)

$$\boldsymbol{\gamma}(s) = \boldsymbol{x}_j + \frac{s - S_j}{S_{j+1} - S_j}(\boldsymbol{x}_{j+1} - \boldsymbol{x}_j), \quad S_j \leq s \leq S_{j+1},$$

and condition (1.4) means that $(S_{j+1} - S_j)/|\Gamma_j| = 2\pi/|\Gamma|$ for all j. So the corresponding parametrization is then proportional to arc-length.

Example 1.2 A more sophisticated choice of ν in (1.6) is the following cubic polynomial, see [7]

$$\nu(s) = \left(\frac{1}{q} - \frac{1}{2}\right)(1 - 2s)^3 + \frac{1}{q}(2s - 1) + \frac{1}{2},$$

where the grading exponent q is an integer ≥ 2.

2 Reduction to a second-kind equation

We introduce the operator

$$Av(s) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \log|2\sin(s-\sigma)/2| v(\sigma)d\sigma + \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\sigma)d\sigma, \tag{2.1}$$

which arises from (1.1) when Γ is the unit circle. Then $A : H^0 \to H^1$ is an isomorphism and we can rewrite (1.2) as the second kind equation

$$(I + M)w = e, \quad \text{with} \quad M = A^{-1}(K - A), \; e = A^{-1}g. \tag{2.2}$$

Here and in what follows H^t, $t \in \mathbb{R}$, refers to the periodic Sobolev space of order t on $(-\pi, \pi)$, with norm denoted by $\|\cdot\|_t$. Note that $H^0 = L^2(-\pi, \pi)$.

Recall the standard result that $A^{-1} = -HD + J$, where D is the (periodic) differentiation operator, H is the Hilbert transform and Jv is the linear functional on H^0 given by the last term on the right-hand side of (2.1). We then have

$$M = HD(A - K) + J(K - A). \tag{2.3}$$

It turns out that M is a Mellin convolution operator local to each corner; see [12] for $q = 1$ and [4] in the general case. We first recall, in Theorems 2.1, 2.2 below, two analytical results from [4], [5] concerning equations (1.2) and (2.2). Note that conditions (1.4) – (1.6) imply that, for all j,

$$\boldsymbol{\gamma}(S_j) = \boldsymbol{x}_j, \tag{2.4}$$

$$(s - S_j)^{-q}(\boldsymbol{\gamma}(s) - \boldsymbol{x}_j), \; (S_{j+1} - s)^{-q}(\boldsymbol{\gamma}(s) - \boldsymbol{x}_{j+1}) \in (\mathcal{C}^\infty[S_j, S_{j+1}])^2, \tag{2.5}$$

$$|\boldsymbol{\gamma}'(s)| > 0, \quad s \in (S_j, S_{j+1}), \tag{2.6}$$

$$\lim_{s \to S_j} |\boldsymbol{\gamma}(s) - \boldsymbol{x}_j| \, |s - S_j|^{-q} = \nu'(0)^q (2\pi)^{-q} \left(\sum_{j=0}^{r-1} |\Gamma_j|^{1/q}\right)^q. \tag{2.7}$$

Therefore, the parametrization $\boldsymbol{\gamma}$ satisfies the assumptions (A1)-(A4) of [5], and we have

Theorem 2.1 *The operator $I+M$ is (continuously) invertible on H^0 and satisfies the strong ellipticity estimate*

$$Re\langle (I+M+E)v, v\rangle \geq c\|v\|_0^2, \qquad v \in H^0, \tag{2.8}$$

where E is some compact operator on H^0 and $\langle\cdot,\cdot\rangle$ denotes the H^0 scalar product.

Remark By (2.2), this result ensures that $K : H^0 \to H^1$ is invertible, and hence (1.2) has a unique solution $w \in H^0$ for each $g \in H^1$. The regularity of w is determined by the regularity of g (equivalently the regularity of f).

Theorem 2.2 *Let $k \in \mathbb{N}$, let $f \in H^{k+5/2}(\Gamma)$ (where this denotes the usual Sobolev space on Γ) and suppose $q > (k+1/2)(1+\chi)$ with $\chi = \max_j |\chi_j|$. Then the unique solution of (1.2) satisfies $w \in H^k$. Moreover there exists $\delta < 1/2$ such that for all $m = 0, ..., k$, and $j = 0, ..., r-1$, we have the estimate:*

$$|D^m w(s)| = O(|s - S_j|^{k-m-\delta}), \qquad s \to S_j. \tag{2.9}$$

Remark Since q is assumed to be an integer the hypothesis of Theorem 2.2 implies that $q \geq k+1$.

The following result on the kernel function

$$k(s,\sigma) = \frac{1}{\pi}\log\left|\frac{\gamma(s)-\gamma(\sigma)}{2\sin(s-\sigma)/2}\right| \tag{2.10}$$

of the integral operator $A - K - J$ is of importance in our convergence analysis.

Theorem 2.3 *(i) For all $i, l \in \mathbb{N}$, with $i + l \leq q$, the derivatives $D_s^i D_\sigma^l k(s,\sigma)$ are bounded on compact subsets of $\Omega := [-\pi,\pi]\times[-\pi,\pi]\backslash\{(S_j,S_j) : j = 0,...,r-1\}$ and are 2π-periodic in s and σ.*
(ii) For each j and sufficiently small $\varepsilon > 0$, we have the following estimates, valid for $s, \sigma \in [S_j - \varepsilon,\ S_j + \varepsilon]\backslash\{S_j\}$:

$$\begin{aligned} |k(s,\sigma)| &\leq c|\log(|s-S_j| + |\sigma - S_j|)|, & (2.11)\\ |D_s^i D_\sigma^l k(s,\sigma)| &\leq c_{i,l}(|s-S_j| + |\sigma - S_j|)^{-i-l}, \quad i+l \geq 1. & (2.12)\end{aligned}$$

Proof (i) Properties (2.5)-(2.7) imply that $\boldsymbol{\gamma}$ is infinitely smooth with $\boldsymbol{\gamma}'$ $(= D\boldsymbol{\gamma})$ non-vanishing on each (S_j, S_{j+1}). Moreover $D^q\boldsymbol{\gamma}$ is bounded on $[-\pi,\pi]$ but is discontinuous at each S_j.

The proof proceeds by writing $k(s,\sigma) = (2\pi)^{-1}\log\{h(s,\sigma)\}$, where

$$h(s,\sigma) = \sum_{j=1}^{2}\{(\Delta\gamma_i)(s,\sigma)\}^2\left\{\frac{(s-\sigma)/2}{\sin(s-\sigma)/2}\right\}^2$$

and $\Delta\boldsymbol{\gamma} = (\Delta\gamma_1, \Delta\gamma_2)$ is given by

$$(\Delta\gamma_i)(s,\sigma) = \frac{\gamma_i(s)-\gamma_i(\sigma)}{s-\sigma}\ . \tag{2.13}$$

If C is any compact subset of Ω then there exists $0 < \varepsilon < (1/2)\min_j(S_{j+1} - S_j)$ such that $(s,\sigma) \in C$ implies

$$(s,\sigma) \notin [S_j - \varepsilon, S_j + \varepsilon]^2 \quad \text{for each } j. \tag{2.14}$$

The proof is completed by showing that on C,

$$\Delta\boldsymbol{\gamma} \in \mathcal{C}^{q-1}, \qquad D_s^i D_\sigma^l \Delta\boldsymbol{\gamma} \text{ bounded for } i+l = q, \qquad \Delta\boldsymbol{\gamma} \neq 0. \tag{2.15}$$

This is done by considering those $(s,\sigma) \in C$ with

$$(s,\sigma) \in [S_j + \varepsilon/2,\ S_{j+1} - \varepsilon/2]^2, \quad \text{for some } j. \tag{2.16}$$

Then (2.13) and the properties of $\boldsymbol{\gamma}$ imply that $\Delta\boldsymbol{\gamma}$ is a smooth non-vanishing function of (s,σ). On the other hand it is easily shown that for all $(s,\sigma) \in C$ which fail to satisfy (2.16) there holds $|s-\sigma| \geq \varepsilon/2$. In this case (2.15) follows from (2.13) and the properties of $\boldsymbol{\gamma}$.

(ii) We consider the kernel k local to the jth corner and assume without loss of generality that this is situated at $\boldsymbol{x}_j = 0$ and that $S_j = 0$. Then the parametrization $\boldsymbol{\gamma}$ (possibly after rotation and dilation) takes the form

$$\boldsymbol{\gamma}(s) = \begin{cases} a(s)|s|^q(-\cos\chi_j\pi,\ \sin\chi_j\pi), & s \in [-\varepsilon, 0] \\ a(s)s^q(1,0), & s \in [0,\varepsilon], \end{cases} \tag{2.17}$$

where (cf. (1.4)-(1.6))

$$a \in \mathcal{C}^\infty([-\varepsilon,0] \cup [0,\varepsilon]), \quad \lim_{s\to 0} a(s) = 1. \tag{2.18}$$

Setting

$$k_0(s,\sigma) = \frac{(\gamma_1(s) - \gamma_1(\sigma))^2 + (\gamma_2(s) - \gamma_2(\sigma))^2}{(s-\sigma)^2}, \tag{2.19}$$

we observe that $(2\pi)^{-1}\log h_0(s,\sigma)$ differs from $k(s,\sigma)$ by a $\mathcal{C}^\infty$ function on $[-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]$. Thus, to prove (2.11) and (2.12) for $S_j = 0$, we observe that it is sufficient to prove the following estimates for $s,\sigma \in [-\varepsilon,\varepsilon]\backslash\{0\}$:

$$|h_0(s,\sigma)| \geq c(|s| + |\sigma|)^{2q-2}, \tag{2.20}$$
$$|D_s^i D_\sigma^l h_0(s,\sigma)| \leq c_{i,l}(|s| + |\sigma|)^{2q-2-i-l}, \quad i+l \geq 0. \tag{2.21}$$

To see why note that (2.11) follows from (2.20) (since $q \geq 1$). Also observe that $D_s^i D_\sigma^l \log h_0(s,\sigma)$ is a sum of products of h_0^{-p} with expressions of orders

$$O((|s| + |\sigma|)^{p(2q-2)-i-l}), \quad \text{for } p = 1, ..., i+l\ ,$$

respectively. Thus (2.12) follows from (2.20), (2.21).

To prove (2.20), we first consider the case $s \geq \sigma > 0$. Then we have in view of (2.17) and (2.18)

$$\begin{aligned} |\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(\sigma)| &= |\gamma_1(s) - \gamma_1(\sigma)| \\ &= |a(s)(s^q - \sigma^q) + (a(s) - a(\sigma))\sigma^q| \\ &= a(s)(s-\sigma)\{s^{q-1} + s^{q-2}\sigma + ... + \sigma^{q-1} + O(\sigma^q)\}, \end{aligned}$$

and since
$$s^{q-1} + s^{q-2}\sigma + ... + \sigma^{q-1} \geq s^{q-1} + \sigma^{q-1} \geq c(s+\sigma)^{q-1}$$
we obtain the result. Let now $s > 0,\ \sigma < 0$. Then, by (2.17) and (2.18)
$$\begin{aligned}|\gamma(s) - \gamma(\sigma)|^2 &= s^{2q} + \sigma^{2q} + 2s^q|\sigma|^q \cos\chi_j\pi + O(s^{2q+1}) \\ &\qquad + O(|\sigma|^{2q+1}) + O((s+|\sigma|)s^q|\sigma|^q) \\ &\geq c(s+|\sigma|)^{2q} \\ &= c(s-\sigma)^2(s+|\sigma|)^{2q-2},\end{aligned}$$
hence the result. The proof of (2.20) in the other cases is analogous.

To verify (2.21) for $s\sigma > 0$, we observe that
$$h_0(s,\sigma) = \left(\int_0^1 \gamma_1'(\sigma + t(s-\sigma))dt\right)^2 + \left(\int_0^1 \gamma_2'(\sigma + t(s-\sigma))dt\right)^2 \tag{2.22}$$
and, for $m = 1, 2$,
$$D_s^i D_\sigma^l \left\{\int_0^1 \gamma_m{}'(\sigma + t(s-\sigma))dt\right\} = \int_0^1 t^i(1-t)^l \gamma_m^{(i+l+1)}(\sigma + t(s-\sigma))dt.$$
Since
$$|D_s^i \gamma_m(s)| \leq c_i |s|^{\max(q-i,0)}, \quad i \geq 0, \quad s \in [-\varepsilon, \varepsilon]\backslash\{0\}, \tag{2.23}$$
the last expression is of order $O((|s|+|\sigma|)^{q-i-l-1})$. Differentiating (2.22) and using (2.23) again, we obtain (2.21).

Finally, let $s\sigma < 0$. Since the denominator in (2.19) takes the form $(|s|+|\sigma|)^2$, the proof of (2.21) now can be done by differentiating $h_0(s,\sigma)$ directly and by applying (2.23). □

3 Collocation

In this section we review some results from [4], [5] on stability and convergence of the collocation method for equation (1.2). Consider the uniform mesh
$$s_i = -\pi + ih, \quad i = 0, ..., n, \quad h = 2\pi/n. \tag{3.1}$$
The image of this mesh under $\boldsymbol{\gamma}$ is graded with exponent q to the corner points $\boldsymbol{x}_j$, but does not necessarily include them. We shall approximate (1.2) in the space V_h^k of (smoothest) periodic splines of order k (i.e. degree $k-1$) subordinate to the mesh $\{s_i\}$. Introduce the collocation points
$$\left.\begin{array}{l} t_i = s_i \ \text{ if } k \text{ is even} \\ t_i = (s_i + s_{i+1})/2 \text{ if } k \text{ is odd}\end{array}\right\} i = 0, ..., n-1. \tag{3.2}$$
Then the collocation method for (1.2) seeks $w_h \in V_h^k$ such that
$$(Kw_k)(t_i) = g(t_i), \quad i = 0, ..., n-1. \tag{3.3}$$

The convergence analysis of this method usually proceeds by rewriting (3.3) as

$$Q_h K w_h = Q_h g \tag{3.4}$$

where Q_h is a rank n operator defined on H^1 with the property that $Q_h v = 0$ if and only if $v(t_i) = 0$ for $i = 0, ..., n-1$. Usually Q_h is taken to be the interpolatory projection onto the (n dimensional) space V_h^k at the points $\{t_i : i = 0, ..., n-1\}$. This was the approach used in [4], [5]. However in this paper we take a different approach (which facilitates the analysis in §4) and we define Q_h to be the interpolatory projection at $\{t_i\}$ onto the n dimensional space $A(V_h^k)$.

It is well known that this projection is well defined. In fact, by [8, Theorem 2.22] there exists an interpolation basis (see also [10]), $\{\chi_i : i = 0, ..., n-1\}$ of $A(V_h^k)$ with the property $\chi_i(t_j) = \delta_{ij}$. Then for any continuous function v we can define

$$Q_h v = \sum_{i=0}^{n-1} v(t_i)\chi_i. \tag{3.5}$$

It follows from [8, Theorems 2.23 and 2.3] that

$$Q_h \text{ are uniformly bounded projections on } H^1. \tag{3.6}$$

Also, from [8, Theorem 2.23 and Definition 2.1(10)], we have, for any continuous function ν on $[-\pi, \pi]$

$$\|Q_h \nu\|_1^2 \le ch \left\{ \sum_{i=0}^{n-1} [|\nu(t_i)|^2 + |h^{-1}(\nu(t_{i+1}) - \nu(t_i))|^2] \right\}, \tag{3.7}$$

where $t_n = t_0$ and the constant c does not depend on n or ν.

So let us now consider (3.4) with Q_h defined by (3.5). We shall rewrite this as a projection method for (2.2). For any $v \in H^0$, let $R_h v \in V_h^k$ solve the collocation equations $Q_h A R_h v = Q_h A v$. It is well-known (see [1], [9], or [8, Chap 13]) that this prescription defines a projection $R_h : H^0 \to V_h^k$ with the property

$$\|R_h\|_0 \le c. \tag{3.8}$$

(Throughout, c denotes a generic constant independent of h.) Moreover (3.8) and standard spline approximation results ([8, Theorem 2.6]) yield

$$\|(I - R_h)v\|_0 \le ch^j \|v\|_j, \quad v \in H^j, \quad 1 \le j \le k. \tag{3.9}$$

It is then straightforward to see that (3.4) is equivalent to

$$(I + R_h M) w_h = R_h e,$$

i.e. it is a non-standard projection method for (2.2). As is usual for Mellin convolution operators, we are only able to prove stability for a slightly modified method. Introduce, for τ sufficiently small, the truncation operator

$$T^\tau v(s) = \begin{cases} 0, & s \in [S_j - \tau, S_j + \tau], \quad j = 0, ..., r-1 \\ v(s), & \text{otherwise.} \end{cases}$$

Then for any fixed integer i^* and for n sufficiently large, define

$$K^{i^*h} = A + (K - A)T^{i^*h},$$

and consider the modified collocation method

$$Q_h K^{i^*h} w_h = Q_h g, \tag{3.10}$$

which coincides with (3.4) when $i^* = 0$. Now it is easily seen that (3.10) is equivalent to

$$(I + R_h M T^{i^*h}) w_h = R_h e. \tag{3.11}$$

The next theorem describes the convergence of (3.10).

Theorem 3.1 *Suppose the hypothesis of Theorem 2.2 holds and suppose that i^* is sufficiently large. Then we have:*
(i) The method (3.10) is stable in (H^0, H^1), i.e. the estimate

$$\|Q_h K^{i^*h} v_h\|_1 \geq c\|v_h\|_0, \quad v_h \in V_h^k \tag{3.12}$$

holds for all n sufficiently large, where c does not depend on v_h or n.
(ii) The solutions of the collocation equations (3.10) converge to the exact solution of (1.2) with the error bound

$$\|w - w_h\|_k = O(h^k) \ \text{ as } n \to \infty. \tag{3.13}$$

We give the proof of this below. First we present a Lemma which is the key to the stability of (3.11) and hence of (3.10). The proof of this Lemma is based on Theorem 2.3 and is simpler than the proof of an analogous lemma in [4].

Lemma 3.2 *For fixed $q \geq 2$ and for each $\varepsilon > 0$, there exists $i^* \geq 1$ independent of n such that*

$$\|(I - R_h) M T^{i^*h}\|_0 \leq \varepsilon \tag{3.14}$$

for all n sufficiently large.

Proof Let $q \geq 2$ and let ψ_j be 2π-periodic non-negative C^∞ functions with support in some neighbourhood of S_j not containing S_i for $i \neq j$ and such that $\sum_j \psi_j \equiv 1$ on $[-\pi, \pi]$. By (2.3) we then have

$$M = \sum_j HD(A - K)\psi_j + E,$$

where $E = J(K - A)$ is a compact operator on H^0. Since $\|T^{i^*h}\|_0 = 1$ and $R_h \to I$ pointwise on H^0, it is sufficient to prove (3.12) with M replaced by $HD(A - K)\psi_j$. Without loss of generality, let $S_j = 0$ and write ψ instead of ψ_j. We have to prove that for each $\varepsilon > 0$ there exists $i^* \geq 1$ independent of n such that

$$\|(I - R_h) HDBv\|_0 \leq \varepsilon \|v\|_0, \quad v \in H^0 \tag{3.15}$$

for all n sufficiently large, where $B = (A-K)T^{i^*h}\psi = (A-K)\psi T^{i^*h}$. It follows from Theorem 2.3 (i) that D^2B is a bounded integral operator on H^0 with kernel function

$$k_1(s,\sigma) = D_s^2 k(s,\sigma)(T^{i^*h}\psi)(\sigma).$$

Hence, for all $s \in [-\pi,\pi]$ we have

$$|k_1(s,\sigma)||v(\sigma)| = 0 \tag{3.16}$$

unless $\sigma \in \{\text{supp } \psi\}\backslash[-i^*h,\ i^*h]$. However, using both parts (i) and (ii) of Theorem 2.3 we also obtain

$$|D_s^2 k(s,\sigma)| \le c(|s|+|\sigma|)^{-2}, \quad s \in [-\pi,\pi], \quad \sigma \in \text{supp } \psi. \tag{3.17}$$

Combining (3.16), (3.17) we obtain the estimate valid on $(s,\sigma) \in [-\pi,\pi]$,

$$\begin{aligned} |k_1(s,\sigma)||v(\sigma)| &\le c(|s|+|\sigma|)^{-2}|v(\sigma)| \\ &\le c|\sigma|^{-1}(|s|+|\sigma|)^{-1}|v(\sigma)| \\ &\le (c/i^*h)(|s|+|\sigma|)^{-1}|v(\sigma)|. \end{aligned} \tag{3.18}$$

Furthermore, since H and D commute, (3.9) implies that

$$\|(I-R_h)HDBv\|_0 \le ch\|HDBv\|_1 = ch\|HD^2Bv\|_0 \le ch\|D^2Bv\|_0$$

and by (3.18) the last expression can be estimated by $(c/i^*)\|v\|_0$. (Note that the integral operator with kernel function $(s+\sigma)^{-1}$ is bounded on $L^2(0,\infty)$.) Choosing i^* sufficiently large gives the result. $\square$

Proof of Theorem 3.1 (i) To obtain (3.12) it is sufficient to prove the stability estimate

$$\|(I+R_hMT^{i^*h})v_h\|_0 \ge c\|v_h\|_0, \quad v_h \in V_h^k \tag{3.19}$$

for sufficiently large i^* and n. To see why this implies (3.12), let $\rho_h = Q_hK^{i^*h}v_h$. Then $\rho_h = Q_h\rho_h$ and, following the derivation of (3.11) from (3.10), we have

$$(I+R_hMT^{i^*h})v_h = R_hA^{-1}\rho_h.$$

Then, using (3.19) and the stability estimate (3.8) together with the known properties of the isomorphism A, (3.12) follows.

To prove (3.19) we use the fact that $I+M$ is invertible and strongly elliptic (Theorem 2.1). Thus we have (cf. [4])

$$\|(I+MT^{i^*h})v_h\|_0 \ge c\|v_h\|_0, \quad v_h \in V_h^k$$

for any fixed i^* and all n sufficiently large. Hence Lemma 3.2 completes the proof.
(ii) We have $\|w-w_h\|_0 \le \|(I-R_h)w\|_0 + \|w_h - R_hw\|_0$, where the first term is of order h^k by (3.9) and Theorem 2.2. Furthermore, using (3.12), (3.6) and Theorem 2.1, we obtain

$$\begin{aligned} \|w_h - R_hw\|_0 &\le c\|Q_hK^{i^*h}(w_h - R_hw)\|_1 \\ &\le c\|Q_hK(w-R_hw)\|_1 + c\|Q_h(K-A)(I-T^{i^*h})w\|_1 \\ &\le c\|(I-R_h)w\|_0 + c\|(I-T^{i^*h})w\|_0. \end{aligned}$$

The proof is complete since by (2.9) the last term is of order h^k again, cf. [4]. $\square$

4 Quadrature

To implement the collocation method (3.3), a certain amount of integration must be done. Specifically we must find the image of each basis function for V_h^k under the action of the operator K, and evaluate it at each collocation point. These integrals may be done after the integrand is made smoother by "singularity subtraction" techniques.

Let $\{\phi_h^j,\ j = 0, ..., n-1\}$ be an appropriate basis of V_h^k (usually chosen to be periodic B-splines). Then (3.3) is equivalent to seeking $w_h = \sum_{j=0}^{n-1} \alpha_j \phi_h^j$ such that

$$\sum_{j=0}^{n-1} \alpha_j K\phi_h^j(t_i) = g(t_i), \quad i = 0, ..., n-1, \tag{4.1}$$

where t_i are the collocation points (3.2). The coefficients in this linear system are integrals in which the integrand has a singularity at $s = \sigma$. This may be "removed" by the usual technique of writing

$$\begin{aligned} -\frac{1}{\pi}\log|\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(\sigma)| &= -\frac{1}{\pi}\log\left\{\frac{|\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(\sigma)|}{|s-\sigma||2\pi - s + \sigma||2\pi - \sigma + s|}\right\} \\ &\quad - \frac{1}{\pi}\{\log|s-\sigma| + \log|2\pi - s + \sigma| + \log|2\pi - \sigma + s|\} \\ &=: k_1(s,\sigma) + k_2(s,\sigma). \end{aligned}$$

Then

$$K\phi_h^j(s) = \int_{-\pi}^{\pi} \{k_1(s,\sigma) + k_2(s,\sigma)\}\phi_h^j(\sigma)d\sigma. \tag{4.2}$$

The second integral is singular but may be done analytically. The first is smoother and we approximate it by quadrature.

For this task we use a composite quadrature rule with respect to the mesh $\{s_i\}_{i=0}^n$ on $[-\pi, \pi]$. To construct this we proceed in the normal way by starting with an l point interpolatory rule on $[0,1]$

$$\int_0^1 v \approx \int_0^1 Pv = \sum_{j=1}^{l} w_j v(\xi_j), \tag{4.3}$$

where Pv is the polynomial of degree $l-1$ which interpolates v at the points ξ_j. We assume that

$$l \le k+1 \text{ and (4.3) is exact for polynomials of degree } 2k-1 \text{ (order } 2k). \tag{4.4}$$

Hence if ψ is a polynomial of degree $k-1$ and v is continuous on $[0,1]$ then $(Pv)\psi$ is of degree $2k-1$ and so

$$\int_0^1 (Pv)\psi = \int_0^1 P((Pv)\psi) = \int_0^1 P(v\psi) \tag{4.5}$$

If, in addition ϕ is a polynomial of degree k then

$$\int_0^1 (P\phi)\psi = \int_0^1 P(\phi\psi) = \int_0^1 \phi\psi. \tag{4.6}$$

The rule with the smallest number of points satisfying the exactness criterion in (4.4) is the k point Gauss-Legendre rule and thus

$$k+1 \geq l \geq k. \tag{4.7}$$

Another rule, useful for our purposes, and satisfying (4.4) is the $k+1$ point Gauss-Lobatto rule which uses the end-points 0 and 1 as well as $k-1$ interior points.

The composite rule on $[-\pi, \pi]$ is then obtained by applying (4.3) to each subinterval (s_i, s_{i+1}) yielding

$$\int_{\pi}^{\pi} v \approx h \sum_{i=0}^{n-1} \sum_{j=1}^{l} w_j v(s_i + h\xi_j) = \int_{-\pi}^{\pi} P_h v,$$

where $P_h v$ is the piecewise polynomial of degree $l-1$ which interpolates v in each $[s_i, s_{i+1}]$ at the points $s_i + h\xi_j, \quad j = 1, \cdots, l$. Corresponding to (4.5), (4.6) we then have

$$\int_{-\pi}^{\pi} (P_h v)\psi \quad = \quad \int_{-\pi}^{\pi} P_h(v\psi), \tag{4.8}$$

$$\int_{-\pi}^{\pi} (P_h \phi)\psi \quad = \quad \int_{-\pi}^{\pi} \phi\psi, \tag{4.9}$$

for all continuous v and all ϕ, ψ piecewise polynomial of degree k and $k-1$ respectively on $[-\pi, \pi]$.

Now, for any $v \in H^0$, define the quadrature approximation to Kv by

$$\tilde{K}v(s) = \int_{-\pi}^{\pi} \{(P_h k_1(s, \cdot))(\sigma) + k_2(s, \sigma)\} v(\sigma) d\sigma. \tag{4.10}$$

Replacing K by $\tilde{K}$ in (3.4) yields the quadrature-collocation scheme

$$Q_h \tilde{K} w_h = Q_h g, \tag{4.11}$$

where Q_h is the interpolatory projection defined in (3.5). Note that by (4.10) and (4.8) we have

$$\tilde{K}\phi_h^j(s) = \int_{-\pi}^{\pi} \{P_h(k_1(s, \cdot)\phi_h^j(\cdot))(\sigma) + k_2(s, \sigma)\phi_h^j(\sigma)\} d\sigma.$$

Thus the linear system corresponding to (4.11) is just the same as that of (3.4) except that the integrals corresponding to the smooth part of the kernel are obtained using our composite quadrature rule.

We shall now prove the analogue of Theorem 3.1 for (4.11). The stability theory follows the same lines as in Section 3. That is, instead of (4.11) we consider the slightly more general equation

$$Q_h \tilde{K}^{i^* h} w_h = Q_h g, \tag{4.12}$$

where

$$\begin{aligned}
\tilde{K}^{i^*h} &= \tilde{A} + (\tilde{K} - \tilde{A})T^{i^*h}, \\
\tilde{A}v(s) &= \int_{-\pi}^{\pi} \left\{ (P_h k_1^0(s,\cdot))(\sigma) + k_2(s,\sigma) + \frac{1}{2\pi} \right\} v(\sigma)d\sigma, \\
k_1^0(s,\sigma) &= -\frac{1}{\pi} \log \left\{ \frac{|2\sin(s-\sigma)/2|}{|s-\sigma||2\pi - s + \sigma||2\pi - \sigma + s|} \right\}.
\end{aligned}$$

Note that $\tilde{A}$ is the quadrature approximation of the form (4.10) applied to the operator A defined in (2.1).

Theorem 4.1 *Suppose the hypothesis of Theorem 3.1 holds and suppose the quadrature rule (4.3) satisfies (4.4). Then:*
(i) The quadrature method (4.12) is stable, i.e. the estimate

$$\|Q_h \tilde{K}^{i^*h} v_h\|_1 \geq c\|v_h\|_0, \quad v_h \in V_h^k \tag{4.13}$$

holds for all n sufficiently large, where c is independent of v_h and n.
(ii) The solutions w_h of (4.12) converge to the exact solution w of (1.2) with the error bound (3.13).

Proof (i) By virtue of Theorem 3.1 (i) it suffices to show that $Q_h(\tilde{K}^{i^*h} - K^{i^*h})$ is a small perturbation with respect to the operator norm in (H^0, H^1). From (2.10), (3.10) and (4.12) we have

$$\begin{aligned}
&(\tilde{K}^{i^*h} - K^{i^*h})v(s) \\
&\quad = (\tilde{K} - \tilde{A} - K + A)T^{i^*h}v(s) + (\tilde{A} - A)v(s) \\
&\quad = \int_{-\pi}^{\pi} ((I - P_h)k(s,\cdot))(\sigma)T^{i^*h}v(\sigma)d\sigma - \int_{-\pi}^{\pi} ((I - P_h)k_1^0(s,\cdot))(\sigma)v(\sigma)d\sigma \\
&\quad =: B_{i^*,h}v(s) + B_h v(s).
\end{aligned} \tag{4.14}$$

Thus recalling (3.6), (4.13) follows if we show that, for each $\varepsilon > 0$, there exists $i^* \geq 1$ independent of n such that

$$\|B_{i^*,h}v\|_1 \leq \varepsilon\|v\|_0, \tag{4.15}$$

$$\|Q_h B_h v\|_1 \leq \varepsilon\|v\|_0, \quad v \in H^0, \tag{4.16}$$

for all n sufficiently large. Note that B_h does not map into the periodic Sobolev space H^1, but into $H^1(-\pi,\pi)$ so we cannot directly use the stability of Q_h in (3.6) to estimate the left-hand side of (4.16).

We first prove (4.15). As in the proof of Lemma 3.2, it is sufficient to do this for the case when supp v is contained in a sufficiently small neighbourhood $[-\delta,\delta]$ of $S_j = 0$. To treat this case, let $I_i = (s_i, s_{i+1}) \cap \{(-\delta,-h) \cup (h,\delta)\}$. By the Remark following Theorem 2.2, $q \geq k+1$. Hence, using standard interpolation estimates and Theorem 2.3, we have for all i with $I_i \neq \emptyset$

$$\begin{aligned}
|((I-P_h)k(s,\cdot))(\sigma)| \quad + \quad & |((I-P_h)D_s k(s,\cdot))(\sigma)| \\
& \leq ch^k \sup_{\sigma \in I_i} \{|D_\sigma^k k(s,\sigma)| + |D_s D_\sigma^k k(s,\sigma)|\} \\
& \leq ch^k \sup_{\sigma \in I_i} (|s| + |\sigma|)^{-k-1} \\
& \leq ch^k (|s| + |\sigma|)^{-k-1}, \quad s \in [-\pi,\pi], \quad \sigma \in I_i. \qquad (4.17)
\end{aligned}$$

where c does not depend on n. For the last inequality we have used the fact that $\max_{I_i} |\sigma| \leq 2 \min_{I_i} |\sigma|$ for all i with $I_i \neq \emptyset$. Applying (4.17) and the relation

$$D_s(P_h k(s,\cdot))(\sigma) = (P_h D_s k(s,\cdot))(\sigma), \quad (s,\sigma) \neq (S_j, S_j) \text{ for all } j,$$

we now obtain, for $s \in [-\pi,\pi]$, $\sigma \in [-\delta,\delta] \backslash (-i^*h, i^*h)$,

$$\begin{aligned}
& |((I-P_h)k(s,\cdot))(\sigma) T^{i^*h} v(\sigma)| + |D_s((I-P_h)k(s,\cdot))(\sigma) T^{i^*h} v(\sigma)| \\
& \quad \leq ch^k |\sigma|^k (|s| + |\sigma|)^{-k-1} |\sigma|^{-k} |v(\sigma)| \\
& \quad \leq ch^k |\sigma|^k (|s| + |\sigma|)^{-k-1} (i^*h)^{-k} |v(\sigma)|, \\
& \quad = c(i^*)^{-k} |\sigma|^k (|s| + |\sigma|)^{-k-1} |v(\sigma)|. \qquad (4.18)
\end{aligned}$$

Since the integral operator with kernel $\sigma^k (s+\sigma)^{-k-1}$ is bounded on $L^2(0,\infty)$ (see for example [4, Appendix]), it follows from (4.14) and (4.18) that

$$\|B_{i^*h} v\|_1 \leq c(i^*)^{-k} \|v\|_0,$$

and hence (4.15) follows.

To prove (4.16), let $\nu = B_h v$. Then the estimate (3.7) yields

$$\begin{aligned}
\|Q_h B_h v\|_1^2 \quad & \leq \quad ch \sum_{i=0}^{n-1} \left\{ |\nu(t_i)|^2 + |h^{-1}(\nu(t_{i+1}) - \nu(t_i))|^2 \right\} \\
& \leq \quad ch \left\{ \sum_{i=0}^{n-1} |\nu(t_i)|^2 + \sum_{i=0}^{n-2} \sup_{s \in (t_i, t_{i+1})} |D\nu(s)|^2 \right\} \\
& \qquad + ch^{-1} \left\{ |\nu(t_{n-1})|^2 + |\nu(t_0)|^2 \right\} \qquad (4.19)
\end{aligned}$$

Next observe that k_1^0 is a $\mathcal{C}^\infty$ function on $[-\pi,\pi] \times [-\pi,\pi]$. Therefore for $s, \sigma \in [-\pi,\pi]$ we have

$$|((I-P_h)k_1^0(s,\cdot))(\sigma)| + |((I-P_h)D_s k_1^0(s,\cdot)(\sigma)| \leq ch^k,$$

and hence, for $v \in H^0$ and $s \in [-\pi,\pi]$,

$$|B_h v(s)| + |D_s B_h v(s)| \leq ch^k \|v\|_0. \qquad (4.20)$$

Combining (4.19) and (4.20) we have

$$\begin{aligned}
\|Q_h B_h v\|_1^2 \quad & \leq \quad c(h\{nh^{2k} + (n-1)h^{2k}\} + h^{2k-1}) \|v\|_0^2 \\
& \leq \quad ch^{2k-1} \|v\|_0^2, \qquad (4.21)
\end{aligned}$$

which, since $k \geq 1$, yields the result.
(ii) As in the proof of Theorem 3.1 (ii), we have $\|w-w_h\|_0 \leq \|(I-R_h)w\|_0 + \|w_h - R_h w\|_0$, where the first term is of order h^k. Using (4.13) and the uniform boundedness of $Q_h \tilde{K}^{i^*h}$ (cf. (4.14)-(4.16)), we further obtain

$$\begin{aligned}
\|w_h - R_h w\|_0 &\leq c\|Q_h \tilde{K}^{i^*h}(w_h - R_h w)\|_1 \\
&= c\|Q_h K w - Q_h \tilde{K}^{i^*h} R_h w\|_1 \\
&\leq c\|Q_h(K - \tilde{K}^{i^*h})w\|_1 + c\|Q_h \tilde{K}^{i^*h}(I - R_h)w\|_1 \\
&\leq c\|Q_h(K - \tilde{K}^{i^*h})w\|_1 + c\|(I - R_h)w\|_0.
\end{aligned}$$

Again the last term is $O(h^k)$ and it remains to estimate the second last term. Observe that by (4.14),

$$\|Q_h(K - \tilde{K}^{i^*h})w\|_1 \leq \|Q_h(K-A)(I - T^{i^*h})w\|_1 + \|Q_h B_{i^*,h} w\|_1 + \|Q_h B_h w\|_1. \quad (4.22)$$

The first term on the right side of (4.22) is of order h^k (see the proof of Theorem 3.1 (ii)).

For the second term it suffices to obtain the required estimate with w replaced by ϕw, where ϕ is a smooth cut-off function with support near $S_j = 0$. Since the estimate (2.9) ensures that $|s|^{-k}\phi w \in H^0$, we get from the first inequality in (4.18) that $\|B_{i^*,h}\phi w\|_1 \leq ch^k \||s|^{-k}\phi w\|_0$.

The third term is estimated by setting $\omega = B_h w$ in (4.19) and then using (4.20) to obtain

$$\|Q_h B_h w\|_1^2 \leq c[h^{2k}\|w\|_0^2 + h^{-1}\{|\omega(t_{n-1})|^2 + |\omega(t_0)|^2\}]. \quad (4.23)$$

To complete the estimate we now need a sharper bound than (4.20) for the last two terms in (4.23). This is obtained by recalling the property (4.9) of our quadrature rule and using the fact that $w \in H^k$. Let $\phi_{s,j}$ denote the piecewise polynomial of order j which coincides on each subinterval (s_i, s_{i+1}) with the Taylor polynomial of order j (degree $j-1$) about s_i of the function $k_1^0(s,\cdot)$. Then Taylor's theorem shows that

$$\|k_1^0(s,\cdot) - \phi_{s,j}\|_\infty \leq ch^j, \quad \text{and} \quad \|\phi_{s,j} - \phi_{s,j-1}\|_\infty \leq ch^{j-1},$$

where c is independent of s as well as h. Moreover, since P_h represents piecewise polynomial interpolation at $l \geq k$ points, we have $(I - P_h)\phi_k = 0$.

Hence with ψ_k denoting any piecewise polynomial of order k, we have for any $s \in [-\pi, \pi]$,

$$\begin{aligned}
\omega(s) = B_h w(s) &= \int_{-\pi}^{\pi} (I - P_h) k_1^0(s,\cdot) w \\
&= \int_{-\pi}^{\pi} (I - P_h)(k_1^0(s,\cdot) - \phi_{s,k+1}) w \\
&\quad + \int_{-\pi}^{\pi} (I - P_h)(\phi_{s,k+1} - \phi_{s,k})(w - \psi_k) \\
&\quad + \int_{-\pi}^{\pi} (I - P_h)(\phi_{s,k+1} - \phi_{s,k})\psi_k. \quad (4.24)
\end{aligned}$$

The last term in (4.24) vanishes by (4.9). Since P_h is uniformly bounded with respect to the uniform norm, the first two terms in (4.24) are trivially bounded by

$$|\int_{-\pi}^{\pi}(I-P_h)(k_1^0(s,\cdot)-\phi_{s,k+1})w| \leq ch^{k+1}\|w\|_0,$$

and

$$|\int_{-\pi}^{\pi}(I-P_h)(\phi_{s,k+1}-\phi_{s,k})(w-\psi_k)| \leq ch^k\|w-\psi_k\|_0 \leq ch^{2k}\|w\|_k.$$

Substitution of these last two estimates into (4.24) completes the proof. □

5 Numerical Results

In this section we describe a number of experiments illustrating the solution of equation (1.1) when Γ is given by

$$\boldsymbol{\gamma}_0(s) = \sin s \ \ (\cos(1-\chi)s \ , \ \ \sin(1-\chi)s)^T \ , \quad s \in [0,\pi] \ , \ \ \chi \in (0,1).$$

In this example (suggested by K.E. Atkinson and also described in [4]), Γ is the boundary of a "teardrop–shaped" region with a single corner at $s = 0$ (or $s = \pi$) and smooth elsewhere. The exterior angle between the tangents at $s = 0$ and $s = \pi$ is $(1+\chi)\pi$. Since it will be a straightforward technical matter to extend the results of this paper to the case of a curvilinear polygon, this example provides a reasonable test of our theoretical results. Clearly this parametrization could be recast to the domain $[-\pi,\pi]$ using a simple linear change of variable. However we work here numerically with the domain $[0,\pi]$ for simplicity.

To reformulate (1.1) in terms of a solution which is sufficiently smooth at 0 and π, we choose a grading exponent $q \in \mathbb{N}$, and set

$$\boldsymbol{\gamma}(s) = \boldsymbol{\gamma}_0(\omega(s)), \quad \text{where} \quad \omega(s) = \left(\frac{\nu^q(s)}{\nu^q(s)+\nu^q(\pi-s)}\right)\pi,$$

where, following the examples at the end of §1, $\nu(s) = s$, when $q = 1$ and

$$\nu(s) = ((1/q-1/2)(1-2s/\pi)^3 + (1/q)(2s/\pi-1)+1/2)\pi, \quad \text{when} \quad q \geq 2.$$

With this parametrization we put (1.1) in the form (1.2) and solved the latter equation numerically using the quadrature-collocation scheme (4.12). We took the right-hand side f to be

$$f(\boldsymbol{x}) = \exp(x_1)\cos(x_2) + \mathrm{Re}\{(x_1+ix_2)^{1/(1-\chi)}\}. \tag{5.1}$$

Then f is the Dirichlet data for a harmonic function in the interior of Γ which has the expected (weak) singularity induced by the corner in Γ. Nevertheless the solution of (1.1) will have the stronger singularity corresponding to the worst of the two singuarities appearing in the exterior and interior harmonic boundary value problems for this domain.

Special care must be taken in the implementation of (4.12), in particular in the application of numerical quadrature to the integrals

$$\int_{-\pi}^{\pi} k_1(s,\sigma)\phi_h^j(\sigma)d\sigma, \tag{5.2}$$

where (working over the domain $[0,\pi]$), k_1 is the smooth function given by

$$k_1(s,\sigma) = \log\left\{\frac{|\gamma(s)-\gamma(\sigma)|}{|s-\sigma||\pi-s+\sigma||\pi-\sigma+s|}\right\}. \tag{5.3}$$

The evaluation of the quotient in the right-hand side of (5.3) when the numerator and denominator are both close to zero involves, in principle, an appropriately high order Taylor expansion and thus requires the calculation of derivatives of γ – obviously a practical drawback. In practice it turns out that we can (almost) avoid this difficulty by always choosing quadrature rules whose points do not collide with the underlying collocation points. Then, even for moderately large n and q , catastrophic cancellation never arises in the evaluation of $k_1(s,\sigma)$ except in the extreme case when s and σ are near the end-points of $[0,\pi]$. In this case further evasive action must be taken. If (for example) both s and σ are near 0 (but not necessarily coincident) then, for large grading exponent q and for large n, the numerator on the right-hand side of (5.3) becomes much closer to 0 than the denominator and numerically (5.3) becomes $\log(0)$. Similar difficulties arise when s is near 0 or π and σ is near 0 or π.

To avoid these difficulties we use an appropriate limiting approximation which we will describe only for the case $s\approx 0$, $\sigma\approx 0$. The other difficulties are avoided analogously. In the present case we make the approximation

$$\log\left\{\frac{|\gamma(s)-\gamma(\sigma)|}{|s-\sigma||\pi-s+\sigma||\pi-\sigma+s|}\right\}$$
$$\approx \log|\gamma'(\max\{s,\sigma\})| + \log\left\{\frac{1}{|\pi-s+\sigma||\pi-\sigma+s|}\right\}. \tag{5.4}$$

(Note that, provided the quadrature and collocation points do not collide, we will always have $\max\{s,\sigma\}>0$.) This does not yet solve the problem since for large q and n we may well still have $|\gamma'(\max\{s,\sigma\})|$ very small (i.e. numerically zero). To cope with this (for $q\geq 2$) we observe that, for small positive t, $|\gamma'(t)|$ is the product of a smooth function of t (which does not vanish at 0) with $\nu^{q-1}(t)$. Evaluating the smooth factor at $t=0$ leads to the approximation $|\gamma'(t)|\approx q\pi^{1-q}(3-4/q)\nu^{q-1}(t)$, which yields

$$\log|\gamma'(\max\{s,\sigma\})| \approx \log(q\pi^{1-q}(3-4/q)) + (q-1)\log(\nu(\max\{s,\sigma\})).$$

We substitute this into the right-hand side of (5.4) and thus obtain a formula suitable for computation.

Theorem 4.1 implies that the numerical solution w_h will converge to the true solution with rate

$$\|w-w_h\|_0 = O(h^k), \tag{5.5}$$

provided

$$q > (k+1/2)(1+\chi), \tag{5.6}$$

and provided the quadrature rule satisfies condition (4.4). In all the experiments described in this section, no modification of the collocation method was found necessary for stability and so throughout we have set $i^* = 0$. As a test of Theorem 4.1 we solved (1.2) numerically in the cases $k = 1$ (piecewise constant mid-point collocation) and $k = 2$ (continuous piecewise linear break-point collocation). All of the experiments shown here are for $\chi = 0.76$. Thus for the rate of convergence (5.5) it is sufficient to take

$$q = 3 \quad (\text{when } k = 1) \quad \text{and} \quad q = 5 \quad (\text{when } k = 2). \tag{5.7}$$

Empirically determined rates of convergence are given in columns headed "EOC" in the following tables. The exact solution w of (1.2) is unknown. To check (5.5) we computed (for each case) an approximate solution w^* using $n = 1024$ subintervals and used that as the exact solution. Then $\|w^* - w_h\|_0$ was computed using the two-point Gauss rule on the mesh with 1024 subintervals.

Our first set of results, given in Table 1, are for the case $k = 1$ and two different quadrature rules, the trapezoidal rule and the two point Gauss-Legendre rule, each of which satisfy (4.4). It is clear that the results confirm the prediction (5.5). In Table 2 we present an experiment for the case $k = 2$ using two different quadrature rules. In both cases (5.5) appears to hold even though the one point Gauss rule fails to satisfy (4.4).

As a test of the sensitivity of our numerical method to the choice of grading exponent q we conducted two further experiments. The first was for the case $k = 1$ with the trapezoidal rule and for $q = 1, 2, 3$. The second was for $k = 2$ with the two point Gauss-Legendre rule and for $q = 1, ..., 5$. In both cases we computed with $n = 2^p$, $p = 3, \ldots, 8$. In Figure 1 we plot the base 2 logarithm of the error $\|w^* - w_h\|_0$ against p for some of these experiments. In each case the plain line corresponds to $q = 1$ whereas the lines with $0, \times$ and $+$ correspond to $q = 2, 3$ and 5 respectively. A graph with gradient α corresponds to a rate of convergence of $O(h^\alpha)$.

In the first graph we have plotted only the cases $q = 1, 3$. The case $q = 2$ is graphically identical to $q = 3$ but is slightly less accurate. The choice $q = 3$ is sufficient to satisfy the hyptheses of Theorem 4.1. Both $q = 2$ and $q = 3$ have an optimal empirical convergence rate of $O(h)$. In the second graph we have plotted only the cases $q = 1, 2$ and 5. The cases $q = 3, 4$ are graphically equivalent to $q = 5$, all three cases have an empirical convergence rate of $O(h^2)$ and, of the three, $q = 4$ is marginally most accurate. The choice $q = 5$ is required to satisy the hypotheses of Theorem 4.1.

It appears from these experiments that the results of Theorem 4.1 may remain true for slightly less stringent choices of q than those predicted in Theorem 4.1. However it is not at all obvious how to prove such an extension of Theorem 4.1 since it is fairly easy to show that the regularity result of Theorem 2.2 will not remain true for smaller choices of q than stated there.

Finally we illustrate the use of the method for the computation of solutions to Laplace's equation. If w is the solution to (1.2) then the potential

$$U(\boldsymbol{x}) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \log |\boldsymbol{x} - \boldsymbol{\gamma}(\sigma)| w(\sigma) d\sigma \tag{5.8}$$

satisfies Laplace's equation both in Ω (the region enclosed by Γ) and in Ω' (the complement of $\Omega \cup \Gamma$ in $\mathbb{R}^2$) and coincides with f on Γ. We computed $U_h(\boldsymbol{x})$, the analogue of (5.8), but with w replaced by w_h in the case $k = 2$ using the 2 point Gauss-Legendre rule in (4.12) and also to calculate the potentials (5.8) for various choices of q. Since f is harmonic, we have $U(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega$ and thus errors in U_h can be computed exactly inside Ω. This is not possible in Ω', since f grows exponentially at infinity and an analytic expression for $U(\boldsymbol{x})$ in Ω' is not known. However, unlike the situation inside Ω, the function $U(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega'$ can be expected to have a singularity of the form r^β, where $\beta = (1+\chi)^{-1} \approx 0.57$ and thus presents a reasonable test of the method as a boundary element method. We show in Tables 3-5 the potentials computed at the points

$$\boldsymbol{x}_1 = -0.1\boldsymbol{v}, \quad \boldsymbol{x}_2 = 0.1\boldsymbol{v}, \;\; \boldsymbol{x}_3 = 0.5\boldsymbol{v}, \;\; \boldsymbol{x}_4 = 10\boldsymbol{v}$$

where

$$\boldsymbol{v} = (\cos(1-\chi)(4\pi/9)\ , \ \ \sin(1-\chi)(4\pi/9))^T\ .$$

The points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ lie, respectively, outside and inside Ω but near the singularity at the origin. The point $\boldsymbol{x}_3$ lies near the centre of Ω while $\boldsymbol{x}_4$ lies outside Ω and a long way from Γ. All four potentials converge empirically with $O(h^3)$ when $q = 5$, which is the rate predicted by [4, Corollary 10], i.e. an increase in order of 1 is possible for smooth linear functionals of w. (It is now known that an increase in order of 2 is obtained in the case k odd provided q is appropriately chosen ([6]).) More unexpectedly, the rate $O(h^3)$ is achieved for the potentials at $\boldsymbol{x}_3$ (away from the singular point) even in the uniform mesh case $q = 1$ and for all four potentials in the case $q = 3$. A similar phenomenon was observed in the case $k = 1$ in [4]. However further analysis will be necessary to explain this behaviour.

	Trapezoidal rule		*Two point Gauss-Legendre rule*	
n	$\Vert w^* - w_h \Vert_2$	EOC	$\Vert w^* - w_h \Vert_0$	EOC
8	6.23 (-1)		6.61 (-1)	
		0.5		0.6
16	4.29 (-1)		4.35 (-1)	
		1.0		1.0
32	2.17 (-1)		2.18 (-1)	
		1.0		1.0
64	1.08 (-1)		1.09 (-1)	
		1.0		1.0
128	5.39 (-2)		5.39 (-2)	
		1.0		1.0
256	2.63 (-2)		2.63 (-2)	

Table 1: $\chi = 0.76$, $k = 1$, $q = 3$, various quadrature rules

	One point Gauss-Legendre rule		*Two point Gauss-Legendre rule*	
n	$\Vert w^* - w_h \Vert_2$	EOC	$\Vert w^* - w_h \Vert_0$	EOC
8	7.91 (-1)		7.46 (-1)	
		2.8		2.8
16	1.17 (-1)		1.06 (-1)	
		2.3		2.3
32	2.31 (-2)		2.10 (-2)	
		2.3		2.1
64	5.37 (-3)		4.96 (-3)	
		2.0		2.0
128	1.37 (-3)		1.28 (-3)	
		1.8		1.8
256	3.95 (-4)		3.74 (-4)	

Table 2: $\chi = 0.76$, $k = 2$, $q = 5$, various quadrature rules

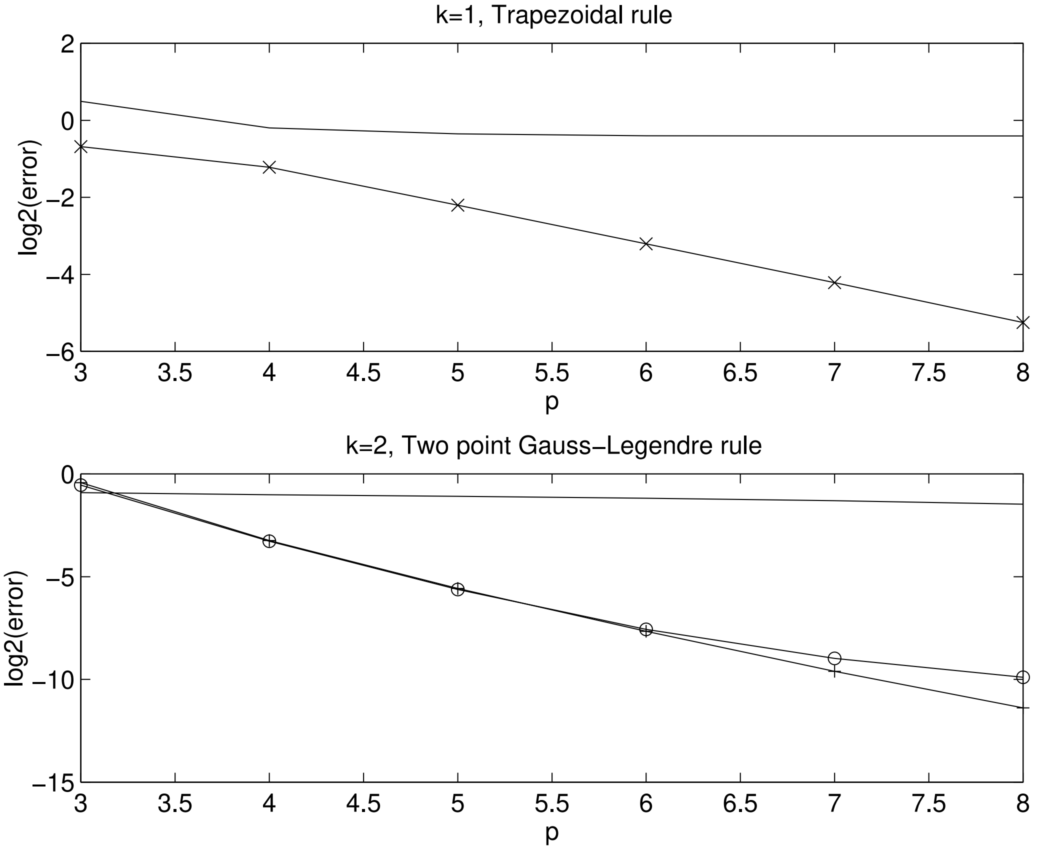

Figure 1: Performance of methods as q varies

	$q=1$		$q=3$		$q=5$	
n	$U_h(\boldsymbol{x}_1)$	EOC	$U_h(\boldsymbol{x}_1)$	EOC	$U_h(\boldsymbol{x}_1)$	EOC
8	0.60027		0.55839		0.55867	
16	0.58612		0.55852		0.55857	
		0.2		0.8		1.3
32	0.57416		0.55859		0.55861	
		0.7		3.5		2.4
64	0.56655		0.55860		0.55860	
		0.9		5.1		1.7
128	0.56243		0.55860		0.55860	
		1.0		3.3		2.8
256	0.56039		0.55860		0.55860	

Table 3: $k=2$, 2 point Gauss-Legendre: Potentials at $\boldsymbol{x}_1$

	$q=1$		$q=3$		$q=5$	
n	$\lvert U_h(\boldsymbol{x}_2)-U(\boldsymbol{x}_2)\rvert$	EOC	$\lvert U_h(\boldsymbol{x}_2)-U(\boldsymbol{x}_2)\rvert$	EOC	$\lvert U_h(\boldsymbol{x}_2)-U(\boldsymbol{x}_2)\rvert$	EOC
8	2.20(-1)		2.17(-2)		1.71(-2)	
		2.5		3.5		1.6
16	3.86(-2)		1.88(-3)		5.69(-3)	
		2.5		6.2		3.9
32	6.86(-3)		2.57(-5)		3.90(-4)	
		4.0		6.2		5.2
64	4.38(-4)		3.44(-7)		1.08(-5)	
		7.1		0.4		4.6
128	3.09(-6)		2.64(-7)		4.55(-7)	
		1.3		3.0		3.0
256	1.27(-6)		3.29(-8)		5.64(-8)	

Table 4: $k=2$, 2 point Gauss-Legendre: Errors in potentials at $\boldsymbol{x}_2$

	$q=1$		$q=3$		$q=5$	
n	$\lvert U_h(\boldsymbol{x}_3) - U(\boldsymbol{x}_3)\rvert$	EOC	$\lvert U_h(\boldsymbol{x}_3) - U(\boldsymbol{x}_3)\rvert$	EOC	$\lvert U_h(\boldsymbol{x}_3) - U(\boldsymbol{x}_3)\rvert$	EOC
8	9.61(-3)		4.31(-3)		1.88(-2)	
		6.0		0.8		4.7
16	1.49(-4)		2.56(-3)		7.07 (-4)	
		2.6		4.2		6.0
32	2.43(-5)		1.44(-4)		1.13(-5)	
		2.7		3.2		*
64	3.68(-6)		1.58(-5)		1.80(-5)	
		3.0		3.0		3.0
128	4.74(-7)		2.00(-6)		2.24(-6)	
		3.0		3.0		3.0
256	5.98(-8)		2.50(-7)		2.81(-7)	

Table 5: $k = 2$, 2 point Gauss-Legendre: Errors in potentials at $\boldsymbol{x}_3$

	$q=1$		$q=3$		$q=5$	
n	$U_h(\boldsymbol{x}_4)$	EOC	$U_h(\boldsymbol{x}_4)$	EOC	$U_h(\boldsymbol{x}_4)$	EOC
8	-3.18405		-3.12481		-3.12376	
16	-3.15376		-3.12970		-3.12957	
		1.2		3.8		3.6
32	-3.14066		-3.13006		-3.13006	
		1.2		3.3		3.8
64	-3.13486		-3.13010		-3.13010	
		1.2		3.1		3.4
128	-3.13226		-3.13010		-3.13010	
		1.1		3.0		3.0
256	-3.13108		-3.13010		- 3.13010	

Table 6: $k = 2$, 2 point Gauss-Legendre: Potentials at $\boldsymbol{x}_4$

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