# Analysis of Numerical Methods Suitable for Computing Lyapunov Exponents

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**Abstract**

Two standard methods for numerically estimating Lyapunov exponents are reviewed and it is noted that a numerical integration scheme that preserves orthonormality is required. A procedure is introduced for modifying arbitrary $r$th order numerical schemes to preserve orthonormality. Convergence is shown for the particular case when explicit Euler's method is taken as the arbitrary method. This motivates looking at more general systems of ordinary differential equations which conserve orthonormality. An arbitrary convergent numerical method is modified to preserve orthonormality and convergence discussed in this general case. In each case numerical results are presented for the estimation of Lyapunov exponents for the Lorenz equations and results compared with those of other authors.

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# 1 Introduction

Lyapunov exponents are widely used to characterize the asymptotic behaviour of dynamical systems. Introduced by Lyapunov [13], they measure the exponential rate of expansion or contraction of arbitrarily close initial conditions and so generalize the linear stability analysis of steady state solutions to general time dependent solutions. In particular Lyapunov exponents are used to indicate the presence of chaotic attractors for the system. A positive exponent corresponds to exponential growth of perturbations, a negative exponent to exponential decay of perturbations and a zero exponent to perturbations tangent to the solution. For so called "regular" systems (see definition 1.2) the sum of the first $m$ exponents determines the exponential rate of expansion or contraction of an $m$–volume.

For an ergodic system (roughly speaking almost every trajectory visits all the attractor) the dynamics of the attractor may be characterized by the signs of the associated Lyapunov exponents. Further details on Lyapunov exponents and ergodicity may be found in [7, 18].

Lyapunov exponents may be estimated analytically such as in [3, 8, 11], estimated from experimental data as in [7, 15, 19, 25, 27] or estimated numerically such as in [1, 2, 10, 22, 24, 11] and [26]. The most comprehensive analysis of the numerical methods employed for calculating the Lyapunov exponents is by Dieci et al [5] and Dieci and Van Vleck [6].

This paper is organized as follows. We start by recalling the definition of Lyapunov exponents and review two methods for their calculation: the *Standard Method* [1] and *Differential Method* [9]. For the differential method a numerical integration scheme is required which preserves orthonormality of initial vectors.

In section 2 we adapt arbitrary $r$th order convergent methods to obtain schemes for the differential method that preserve orthonormality. Taking explicit Euler's method as the arbitrary convergent method, we prove convergence. This yields a reliable method for computing Lyapunov exponents.

Motivated by this example we consider in section 3 a more general system of ordinary differential equations that preserves orthonormality. We modify arbitrary convergent numerical methods to preserve orthonormality and prove convergence of these schemes to the general system of differential equations.

The paper concludes with some numerical results estimating the Lyapunov exponents for the Lorenz equations using the schemes developed in sections 2 and 3 and our results are compared with those of other authors. In [11] these schemes were used to determine the stability of solutions to a coupled system of partial differential equations and as an indicator of a chaotic attractor.

## 1.1 Short Review of Lyapunov Exponents

Let $\mathbb{R}^p$ be endowed with inner product $\langle \bullet, \bullet \rangle$, induced norm $\| \bullet \|^2 = \langle \bullet, \bullet \rangle$ and consider the linear evolution equation

$$\frac{d}{dt}\xi = A(t)\xi(t) \qquad \text{given } \xi(0) = \xi^0 \in \mathbb{R}^p; \tag{1.1}$$

with solution

$$\xi(t) = L(t)\xi^0. \tag{1.2}$$

We assume that $L : \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{R}^p$ is a compact operator and we let $L(t)^*$ denote the adjoint operator. We are now in a position to define Lyapunov exponents.

**Definition 1.1** Whenever they exists let $\{\xi_i^0\}$ be the set of orthonormal eigenvectors for the operator $(L(t)^*L(t))^{1/2}$ with corresponding eigenvalues $\alpha_i = \alpha_i(L(t))$: so that

$$(L(t)^*L(t))^{1/2}\,\xi_i^0 = \alpha_i \xi_i^0.$$

Then the $i$th *local Lyapunov exponent* $\mu_i$ is defined, whenever the limit exists, by

$$\mu_i = \lim_{t\to\infty} \frac{1}{t} \log \|L(t)\xi_i^0\|. \tag{1.3}$$

We define the $i$th *approximate Lyapunov exponent* $\mu_i(t)$ for $t \in \mathbb{R}^+$ by

$$\mu_i(t) := \frac{1}{t} \log \|L(t)\xi_i^0\|.$$

**Definition 1.2** The linear system (1.1) is said to be *regular* if for some orthonormal basis $\{\xi_i^0\}$ of $\mathbb{R}^p$

$$\sum_i \mu_i = \lim_{t\to\infty} \frac{1}{t} \log \left| \det \left[ \left\langle L(t)\xi_j^0, L(t)\xi_k^0 \right\rangle \right]_{j,k=1,2,3\cdots} \right|.$$

We follow standard convention and order the Lyapunov exponents so that

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots .$$

In Definition 1.1 the existence of the orthonormal set of eigenvectors and existence of the limit is guaranteed for almost all $U^0$ (with respect to some suitable measure) by the *Multiplicative Ergodic Theorem* due to Osceledec [16]. For the more general setting in a Hilbert space we refer the reader to Ruelle [17].

It is shown in Sansone and Conti [20] for (1.1) that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left| \det \left[ \left\langle L(t)\xi_j^0, L(t)\xi_k^0 \right\rangle \right]_{j,k=1,2\cdots} \right| = \limsup_{t\to\ \infty} \frac{1}{t} \int_0^t \mathrm{Tr}\,(A(s))\ ds$$

and so a continuous system in $\mathbb{R}^p$ is regular provided

$$\sum_i \mu_i = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{Tr}\,(A(s))\ ds.$$

We now consider the non–linear evolution equation in $\mathbb{R}^p$ given by

$$U_t = F(U)\ ; \qquad\qquad U(0) = U^0 \text{ given}, \tag{1.4}$$

and the corresponding linear evolution equation

$$\xi_t = DF[U(t)]\xi\ ; \qquad\qquad \xi(0) = \xi^0 \in \mathbb{R}^p \text{ given}, \tag{1.5}$$

where $DF[U(t)]$ denotes the Jacobian of $F$ evaluated at $U(t)$.

We are interested in estimating the set of Lyapunov exponents $\{\mu_1, \mu_2, \cdots, \mu_p\}$ or some subset $\{\mu_1, \mu_2, \cdots, \mu_m\}$, $1 \leq m \leq p$, for the system (1.4) from the linear system (1.5). For the purposes of this paper we assume that the non–linear evolution system (1.4) is solved exactly.

*Example*: Consider the Lorenz equations [12] in the following form

$$\left.\begin{array}{rcl} x_t &=& \sigma(y-x) \\ y_t &=& rx-y-xz \\ z_t &=& xy-\beta z. \end{array}\right\} \qquad (1.6)$$

Then in the notation of (1.4) we have $U := (x, y, z)^T$, and $F$ is given by the right hand side of (1.6). The linear evolution equation (1.5) is given for $\xi^0 \in \mathbb{R}^3$ by

$$\frac{d}{dt}\xi = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z(t) & -1 & -x(t) \\ y(t) & x(t) & -\beta \end{pmatrix} \xi(t).$$

Letting $A(t)$ denote the matrix on the right hand side above we find that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{Tr}(A(s))ds = -\sigma - 1 - \beta \quad \text{and} \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{Tr}(A(s))ds = -\sigma - 1 - \beta.$$

Therefore the Lorenz equations are regular provided $\sum_{i=1}^3 \mu_i = -\sigma - 1 - \beta$.

## 1.2 Numerical Methods

There are two basic methods to estimate numerically the Lyapunov exponents of (1.4). Suppose we are given initial conditions $\{\xi_i^0\}_{i=1}^p$; then to find the Lyapunov exponents we essentially wish to integrate numerically the linear evolution equation (1.5) over a long time interval $[0, \tau]$ and keep the norms $\|\xi_i(\tau)\|$ for $i = 1, \cdots, p$. However in general this strategy will fail when applied directly to (1.5). In the presence of exponential growth in the system (i.e. a positive exponent) the vectors $\{\xi_i\}$ align themselves in the direction associated with the exponential growth and their norms grow exponentially so that numerical overflow rapidly occurs. These problems are well known and are discussed in [1, 22, 9, 26] among others.

The most popular method for computing the Lyapunov exponents $\{\mu_i\}_{i=1}^m$, where $1 \le m \le p$ was suggested by Benettin et al [1] and Shimada and Nagashima [22]. It is known as the *Standard Method* and combines the numerical integration of the linear evolution equation (1.5) with the Gram–Schmidt re–orthonormalization process applied periodically. Suppose we wished to find the approximate exponents $\{\mu_i(t)\}$ at time $t = \tau$, then a renormalization interval $T$ is picked so that $\tau = rT$ with $r \in \mathbb{N}$. The linear evolution equation (1.5) is integrated over $T$ and the Gram–Schmidt process applied. This is repeated a total of $r$ times keeping the norms from the Gram–Schmidt process at each stage and the integration restarted with the orthonormalized vectors.

In implementation the choice of initial vectors $\{\xi_i^0\}$ is normally taken to be the standard basis for $\mathbb{R}^m$ and there are no reports of this approach failing in the literature :- the Gram–Schmidt process ensures that the directions and rates of growth are measured correctly. However there is general agreement that the choice of the re-orthonormalization interval $T$ is a matter of trial and error and that a bad choice of $T$ may cause the algorithm to fail. For further discussion on the standard method we refer the reader to [1, 5, 7, 9, 22] and [27].

Goldhirsch et al [9] propose an alternative method to find the Lyapunov exponents of (1.4): the *Differential Method* in which the re–orthonormalization interval $T$ is made infinitesimal so that the Gram–Schmidt process is being constantly applied.

*The Differential Method*

Let $\{\xi_i^0\}_{i=1}^m$, $1 \leq m \leq p$, be the orthonormal set of initial vectors.

Solve the following system of equations for $t \in [0, \tau]$,

$$\begin{aligned} \frac{d\xi_i}{dt} &= DF[U]\xi_i - \frac{\langle \xi_i, DF[U]\xi_i \rangle}{\langle \xi_i, \xi_i \rangle}\xi_i \\ &\quad - \sum_{j=1}^{i-1} \frac{(\langle \xi_i, DF[U]\xi_j \rangle + \langle \xi_j, DF[U]\xi_i \rangle)}{\langle \xi_j, \xi_j \rangle}\xi_j, \quad \forall i = 1, \cdots, m. \end{aligned} \tag{1.7}$$

Find the $i$th approximate Lyapunov exponent $\mu_i(\tau)$ by evaluating:

$$\mu_i(\tau) = \frac{1}{\tau}\int_0^\tau \langle \xi_i(s), DF[U(s)]\xi_i(s) \rangle \, ds. \tag{1.8}$$

The derivation of this method is presented in [9]. We simply note that the second term on the right hand side of (1.7) corresponds to the re–normalization and the summation term to the re–orthogonalization. From the derivation of these equations in [9] we find the following theorem (which may also be proved by induction).

**Theorem 1.1** *Suppose we are given an orthonormal set of initial vectors* $\{\xi_i^0\}_{i=1}^m$ *for* $1 \leq m \leq p$*, then the system of differential equations* (1.7) *preserves orthonormailty. That is* $\forall i, j = 1, \cdots, m$

$$\left\langle \xi_i^0, \xi_j^0 \right\rangle = \delta_{ij} \implies \langle \xi_i(t), \xi_j(t) \rangle = \delta_{ij} \qquad \forall t > 0.$$

In order to implement the differential method as a numerical scheme the system of differential equations (1.7) have to be integrated numerically over the interval $[0, \tau]$ and the Lyapunov exponents estimated from (1.8) by a quadrature rule.

If we were to try and solve the differential equations (1.7) using an arbitrary convergent numerical integration scheme, then we would probably fail to estimate the Lyapunov exponents accurately since a standard scheme does not conserve the orthonormality of initial vectors. In general all the vectors align themselves in the direction associated with the largest exponent $\mu_1$ and the norm of the vectors would not be conserved.

*Example*: Suppose we approximate (1.7) for $i = 1$ by explicit Euler's method to get

$$\frac{\xi_1^{n+1} - \xi_1^n}{\Delta t} = DF[U]\xi_1^n - \frac{\langle \xi_1^n, DF[U]\xi_1^n \rangle}{\|\xi_1^n\|^2}\xi_1^n.$$

where $DF[U] = DF[U(n\Delta t)]$ and $\Delta t > 0$ denotes our time step.

By considering $\|\xi_1^{n+1}\|$ we see that the norm is not conserved:

$$
\begin{aligned}
\|\xi_1^{n+1}\|^2 &= \left\langle \xi_1^{n+1}, \xi_1^n + \Delta t DF[U]\xi_1^n - \Delta t \left\langle \xi_1^n, DF[U]\xi_1^n \right\rangle \|\xi_1^n\|^{-2}\xi_1^n \right\rangle \\
&= \|\xi_1^n\|^2 + \Delta t^2 \|DF[U]\xi_1^n - \left\langle \xi_1^n, DF[U]\xi_1^n \right\rangle \|\xi_1^n\|^{-2}\xi_1^n\|^2
\end{aligned}
$$

Indeed we have found that $\|\xi_1^{n+1}\| = \|\xi_1^n\| + O(\Delta t)$. This error can have a dramatic effect as illustrated in Figure 1 where the explicit Euler method has been used to solve (1.7) in the case when (1.4) is the Lorenz equations with $r = 45.92$, $\beta = 4$ and $\sigma = 16$. The Lorenz equations were solved using the standard fourth order Runge–Kutta method. The integral (1.8) was estimated by the standard trapezoidal rule. We observe that the norm of the first vector $\xi_1$ appears to grow exponentially. Similarly it is observed numerically, and can be shown analytically, that the explicit Euler method for (1.7) fails to preserve orthogonality.

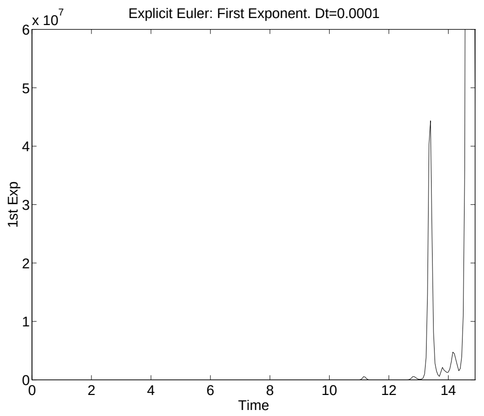


Figure 1: First Lyapunov exponent for Lorenz equation found using explicit Euler's method.

In Dieci et al [5] and Dieci and Van Vleck [6] the differential method is assumed to be implemented using one of the $r$th order unitary (i.e. orthogonality preserving) methods discussed in [4], and a composite trapezoidal rule for the integral (1.8). For this they prove that

$$|\mu_i - \tilde{\mu}_i^c| = O(\Delta t^2) \tag{1.9}$$

where $\mu_i$ is the true Lyapunov exponent from the linear system (1.5) and $\tilde{\mu}_i^c$ is the approximating exponent. They note that by choosing a more accurate integration scheme a higher order of accuracy can be obtained.

For implementation we require an initial set of starting vectors $\{\xi_i^0\}_{i=1}^m$, $1 \leq m \leq p$. If one wishes to calculate all the exponents then the standard basis for $\mathbb{R}^p$ is an adequate choice for $\{\xi_i^0\}_{i=1}^m$. However, if only a subset of the Lyapunov exponents is required, for example just the largest exponent, then it is advisable to pick orthonormal initial vectors $\{\xi_i^0\}_{i=1}^m$, $1 \leq m \leq p$, which are non–zero in every component. This guarantees that all directions of growth are considered.

We present in this paper two collections of schemes suitable for integrating (1.7), i.e. schemes that preserve orthonormality. The first type of scheme, discussed in section 2, is based on integrating the linear evolution equation (1.5) by a standard $r$th order method and applying the Gram–Schmidt to obtain an $r$th order method for the differential equations (1.7). The proof of convergence of the resulting scheme to (1.7) is illustrated by taking explicit Euler as the $r$th order method. Our analysis allows for the set of exponents $\{\mu_1, \cdots, \mu_m\}$, $1 \leq m \leq p$, to be calculated.

The second type of scheme, discussed in section 3, is derived for a more general system of ordinary differential equations of which (1.7) is a specific example. An arbitrary $r$th order method is modified to obtain an $r$th order scheme preserving orthonormality. In this case we prove convergence for the arbitrary $r$th order modified scheme. Our analysis again allows for the set of exponents $\{\mu_1, \cdots, \mu_m\}$, $1 \leq m \leq p$, to be calculated.

We note that for both collections of schemes we can emulate the analysis of Dieci et al [5] and Dieci and Van Vleck [6] and establish error bounds of the form (1.9) - assuming that the non–linear system (1.4) is solved exactly.

We prove convergence of the schemes using the consistency and K– stability results detailed, for example, in [21].

**Definition 1.3 (K-Stability)** Let $h > 0$ be a real parameter, let $U_h \in \mathbb{R}^{p\times N}$ and let $\Phi : \mathbb{R}^{p\times N} \to \mathbb{R}^{p\times N}$ be dependent on $h$. Consider the *discretization*

$$\Phi(U_h) = 0. \tag{1.10}$$

Then (1.10) is said to be *K–stable* if there exists constants $K, h_0 > 0$ and $R \in (0, \infty)$ such that $\forall\, h < h_0$, and for all $V_h, W_h \in B(U_h, R) = \left\{w \in \mathbb{R}^{p\times N} : \|w - U_h\| < R\right\}$ we have that

$$\|V_h - W_h\|_1 \leq K\|\Phi(V_h) - \Phi(W_h)\|_2.$$

The norms $\| \bullet \|_1$ and $\| \bullet \|_2$ are defined for $v = (v^0, \cdots, v^{N-1}) \in \mathbb{R}^{p\times N}$ by

$$\|v\|_1 := \max_n \|v^n\| \quad \text{and} \quad \|v\|_2 := \|v^0\| + h\sum_{n=1}^{N-1} \|v^n\|.$$

**Theorem 1.2** *If a numerical scheme is both consistent and K–stable then it is convergent.*

# 2 Schemes Conserving Orthonormality

The idea behind these schemes is to solve the linear evolution equation (1.5) using a standard method and to project to conserve orthonormality. First we intoduce some notation, we let $\Delta t > 0$ denote our time step and let $t_n = n\Delta t$ for $n \in \mathbb{N}$.

To simplify the equations for the remainder of the paper we use $J = J(t_n)$ to denote the Jacobian $DF[U(t_n)]$, which we assume to be bounded.

Given $\{\xi_i^0\}_{i=1}^m$, $1 \leq m \leq p$, let $S_i(t) : \mathbb{R}^p \to \mathbb{R}^p$ be the semi–group associated with the linear evolution equation (1.5) so that,

$$\xi_i(t) = S_i(t)\xi_i^0 \qquad \forall t \geq 0.$$

Furthermore let $S_i^1 : \mathbb{R}^p \to \mathbb{R}^p$ be the semi–group of an arbitrary convergent numerical method for (1.5), arising, for example, from a standard Runge–Kutta or linear multi–step method. For convenience we assume in this section that each of the schemes $\{S_i^1\}_{i=1}^m$ is of the same order $r$. Thus there exists $C_i > 0$ such that for $V \in \mathbb{R}^p$

$$\|S_i(\Delta t)V - S_i^1 V\| \leq C_i \Delta t^{r+1}, \quad \forall i = 1, \cdots, m.$$

We derive the modified scheme as follows. Given an orthonormal set $\{\xi_i^0\}_{i=1}^m$ define $\xi_i^*$ by

$$\xi_i^* := S_i^1 \xi_i^n - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 \xi_i^n, \xi_j^{n+1} \right\rangle}{\|\xi_j^{n+1}\|^2} \xi_j^{n+1}, \tag{2.1}$$

and define $\xi_i^{n+1}$ by

$$\xi_i^{n+1} := \xi_i^* \frac{\|\xi_i^n\|}{\|\xi_i^*\|}. \tag{2.2}$$

Then the mappings $\tilde{S}_i^1 : \mathbb{R}^p \to \mathbb{R}^p$ defined by $\tilde{S}_i^1 \xi_i^n = \xi_i^{n+1}$, $\forall n \in \mathbb{N}$, preserve orthonormality.

**Theorem 2.1** *Suppose we are given orthonormal initial vectors* $\{\xi_i^0\}_{i=1}^m$, $1 \leq m \leq p$. *Then the scheme (2.1,2.2) conserves orthonormality. That is* $\forall i, k = 1, \cdots, m$

$$\left\langle \xi_i^0, \xi_k^0 \right\rangle = \delta_{ik} \implies \langle \xi_i^n, \xi_k^n \rangle = \delta_{ik}, \qquad \forall n \geq 0.$$

**Proof** This is proved by induction on $n$. We present below the inductive step.

First consider the case $i = k$. Suppose $\xi_i^n$ is known and satisfies $\|\xi_i^n\| = \|\xi_i^0\|$ for all $i = 1, \cdots, m$. Then

$$\|\xi_i^{n+1}\|^2 = \left\langle \xi_i^{n+1}, \xi_i^{n+1} \right\rangle = \langle \xi_i^*, \xi_i^* \rangle \frac{\|\xi_i^n\|^2}{\|\xi_i^*\|^2} = \|\xi_i^n\|^2 = \|\xi_i^0\|^2.$$

For the case $i \neq k$ assume without loss of generality that $k < i$. Then we wish to prove that

$$\left\langle \xi_i^{n+1}, \xi_k^{n+1} \right\rangle = 0$$

which we do by induction on $i$ and $k$.

**Induction I** : we prove that $\left\langle \xi_i^{n+1}, \xi_1^{n+1} \right\rangle = 0 \ \forall i \geq 2$. The case $i = 2$ is straightforward to show. Assuming **I** true for $i-1$ we prove **I** true for $i$ by noting that

$$\left\langle \xi_i^{n+1}, \xi_1^{n+1} \right\rangle = \left\{ \left\langle S_i^1 \xi_i^n, \xi_1^{n+1} \right\rangle - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 \xi_i^n, \xi_j^{n+1} \right\rangle}{\|\xi_j^{n+1}\|^2} \left\langle \xi_j^{n+1}, \xi_1^{n+1} \right\rangle \right\} \frac{\|\xi_i^n\|}{\|\xi_i^*\|}.$$

By the inductive hypothesis of **I** we get that

$$\begin{aligned}\left\langle \xi_i^{n+1}, \xi_1^{n+1} \right\rangle &= \left\{ \left\langle S_i^1 \xi_i^n, \xi_1^{n+1} \right\rangle - \left\langle S_i^1 \xi_i^n, \xi_1^{n+1} \right\rangle \right\} \frac{\|\xi_i^n\|}{\|\xi_i^*\|}. \\ &= 0,\end{aligned}$$

and the induction is complete.

**Induction II**: we prove that $\left\langle \xi_i^{n+1}, \xi_k^{n+1} \right\rangle = 0 \;\; \forall i \geq 1, \;\forall k < i$. First note this is true for $k = 1$ by **Induction I**. Assuming **II** true for $k \leq i-2$. Then for $k = i-1$ we have

$$\left\langle \xi_i^{n+1}, \xi_{i-1}^{n+1} \right\rangle = \left\{ \left\langle S_i^1 \xi_i^n, \xi_{i-1}^{n+1} \right\rangle - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 \xi_i^n, \xi_j^{n+1} \right\rangle}{\|\xi_j^{n+1}\|^2} \left\langle \xi_j^{n+1}, \xi_{i-1}^{n+1} \right\rangle \right\} \frac{\|\xi_i^n\|}{\|\xi_i^*\|}$$

which by the inductive hypothesis **II** becomes

$$\begin{aligned}\left\langle \xi_i^{n+1}, \xi_{i-1}^{n+1} \right\rangle &= \left\{ \left\langle S_i^1 \xi_i^n, \xi_{i-1}^{n+1} \right\rangle - \left\langle S_i^1 \xi_i^n, \xi_{i-1}^{n+1} \right\rangle \right\} \frac{\|\xi_i^n\|}{\|\xi_i^*\|} \\ &= 0.\end{aligned}$$

Thus our proof of **II** is complete.

This concludes the induction on $n$ and the theorem is proved. □

It would be possible to prove that the scheme (2.1, 2.2) converges to (1.7) as $\Delta t \to 0$ by proving that the projection (or Gram–Schmidt process) was of order $\Delta t^r$. Once we have proved convergence we shall have a reliable method for calculating Lyapunov exponents. To illustrate the convergence proof we take as the arbitrary $r$th order method the explicit Euler method (other cases may be treated similarly). Thus we have that

$$S_i^1 \xi_i^n = \xi_i^n + \Delta t J \xi_i^n. \tag{2.3}$$

In which case equations (2.1) and (2.2) respectively become

$$\xi_i^* = \xi_i^n + \Delta t J \xi_i^n - \sum_{j=1}^{i-1} \frac{\left\langle \xi_i^n + \Delta t J \xi_i^n, \xi_j^{n+1} \right\rangle}{\left\langle \xi_j^{n+1}, \xi_j^{n+1} \right\rangle} \xi_j^{n+1}; \tag{2.4}$$

and

$$\xi_i^{n+1} = \xi_i^* \frac{\|\xi_i^n\|}{\|\xi_i^*\|}, \quad \forall i = 1, \cdots, m; \; 1 \leq m \leq p. \tag{2.5}$$

We re–write (2.4,2.5) in a more familiar form: one that looks like a discretized ordinary differential equation.

$$\frac{\xi_i^{n+1} - \xi_i^n}{\Delta t} = f_i(\xi_i^n) \tag{2.6}$$

where $i = 1, \cdots, m$, and $f_i : \mathbb{R}^{p\times i} \to \mathbb{R}^p$ is given for $w_i \in \mathbb{R}^p$ and $\left\{w_j^{n+1}\right\}_{j=1}^{i-1}$, $w_j \in \mathbb{R}^p$, by

$$\begin{aligned} f_i(w_i) &:= f(w_i; w_{i-1}^{n+1}, w_{i-2}^{n+1}, \cdots, w_1^{n+1}) \\ &= \{(\|w_i^n\| - \|w_i^*\|)\, w_i^n + \Delta t \|w_i^n\| J w_i^n \\ &\quad -\|w_i^n\| \sum_{j=1}^{i-1} \frac{\left\langle w_i^n + \Delta t J w_i^n, w_j^{n+1}\right\rangle}{\left\langle w_j^{n+1}, w_j^{n+1}\right\rangle} w_j^{n+1} \Bigg\} \times \frac{1}{\Delta t \|w_i^*\|} \quad . \end{aligned} \tag{2.7}$$

Let $\alpha \in \mathbb{R}$ and let $\xi \in \mathbb{R}^{p\times N}$, $\xi = (\xi_i^0, \xi_i^1, \ldots, \xi_i^{N-1})$. Then we define the discretization of the $i$th equation by $\Phi_i = 0$, where $\Phi_i : \mathbb{R}^{p\times N} \to \mathbb{R}^{p\times N}$, and

$$\Phi_i(\xi) = \begin{pmatrix} \xi_i^0 & - & \alpha \\ \frac{\xi_i^1 - \xi_i^0}{\Delta t} & - & f_i(\xi_i^1) \\ & \vdots & \\ \frac{\xi_i^{N-1} - \xi_i^{N-2}}{\Delta t} & - & f_i(\xi_i^{N-2}) \end{pmatrix} = \begin{pmatrix} \Phi_i(\xi)^0 \\ \Phi_i(\xi)^1 \\ \vdots \\ \Phi_i(\xi)^{N-1} \end{pmatrix}. \tag{2.8}$$

Finally, for the rest of the paper we let $C_i = C_i(J\xi_i^n, \xi_{i-1}^n, J\xi_{i-1}^n, \cdots, J\xi_1^n, \xi_1^n) \in \mathbb{R}$ denote an arbitrary constant.

We start by working towards proving consistency of the scheme, for which we require a preliminary lemma.

**Lemma 2.1** *Given orthonormal initial vectors $\{\xi_i^0\}_{i=1}^m$, $1 \le m \le p$, suppose that $\{\xi_i^n\}_{i=1}^m$ is found from (2.6). Then, for all $i = 1, \cdots, m$, there exists a constant $C_i$ such that*

$$\|\xi_i^*\|^2 = \|\xi_i^n\|^2 + 2\Delta t \langle \xi_i^n, J\xi_i^n\rangle + C_i \Delta t^2. \tag{2.9}$$

**Proof** Take the inner product of $\xi_i^*$ and $\xi_i^*$ and expand to get

$$\begin{aligned} \langle \xi_i^*, \xi_i^*\rangle &= \langle \xi_i^n, \xi_i^n\rangle + 2\Delta t \langle \xi_i^n, J\xi_i^n\rangle - 2\sum_{j=1}^{i-1} \frac{\left\langle \xi_i^n + \Delta t J\xi_i^n, \xi_j^{n+1}\right\rangle}{\left\langle \xi_j^{n+1}, \xi_j^{n+1}\right\rangle} \left\langle \xi_i^n, \xi_j^{n+1}\right\rangle \\ &\quad + \Delta t^2 \langle J\xi_i^n, J\xi_i^n\rangle - 2\Delta t \sum_{j=1}^{i-1} \frac{\left\langle \xi_i^n + \Delta t J\xi_i^n, \xi_j^{n+1}\right\rangle}{\left\langle \xi_j^{n+1}, \xi_j^{n+1}\right\rangle} \left\langle J\xi_i^n, \xi_j^{n+1}\right\rangle \\ &\quad + \sum_{j=1}^{i-1} \frac{\left\langle \xi_i^n + \Delta t J\xi_i^n, \xi_j^{n+1}\right\rangle^2}{\left\langle \xi_j^{n+1}, \xi_j^{n+1}\right\rangle^2} \left\langle \xi_j^{n+1}, \xi_j^{n+1}\right\rangle \\ &\quad + \sum_{r,s\ r\neq s}^{i-1} \frac{\langle \xi_i^n + \Delta t J\xi_i^n, \xi_r^{n+1}\rangle}{\langle \xi_r^{n+1}, \xi_r^{n+1}\rangle} \frac{\langle \xi_i^n + \Delta t J\xi_i^n, \xi_s^{n+1}\rangle}{\langle \xi_s^{n+1}, \xi_s^{n+1}\rangle} \left\langle \xi_r^{n+1}, \xi_s^{n+1}\right\rangle. \end{aligned} \tag{2.10}$$

By inspection of (2.10), using Theorem 2.1, our proof will be complete if we can show that for all $j < i$

$$\left\langle \xi_i^n, \xi_j^{n+1}\right\rangle = \Delta t \langle \xi_i^n, J\xi_i^n\rangle + C_i \Delta t^2. \tag{2.11}$$

**Proof** of (2.11) : We prove equation (2.11) holds by induction and present here the inductive step. Assume that (2.11) holds for $1 \leq j \leq i-2$. Then for $j = i-1$ we have

$$\begin{aligned}
&\left\langle \xi_i^n, \xi_{i-1}^{n+1} \right\rangle \\
&= \left\{ \left\langle \xi_i^n, \xi_{i-1}^n \right\rangle + \Delta t \left\langle \xi_i^n, J\xi_{i-1}^n \right\rangle - \sum_{k=1}^{i-2} \frac{\left\langle \xi_{i-1}^n + \Delta t J \xi_{i-1}^n, \xi_k^{n+1} \right\rangle}{\left\langle \xi_k^{n+1}, \xi_k^{n+1} \right\rangle} \left\langle \xi_k^n, \xi_{i-1}^{n+1} \right\rangle \right\} \frac{\|\xi_{i-1}^n\|}{\|\xi_{i-1}^*\|}.
\end{aligned} \tag{2.12}$$

Let us look at each term on the right hand side of (2.12) separately. By Theorem 2.1 and the inductive hypothesis the term in large curly brackets is of order $\Delta t$, so that

$$\begin{aligned}
\left\langle \xi_i^n, \xi_{i-1}^n \right\rangle + \Delta t \left\langle \xi_i^n, J\xi_{i-1}^n \right\rangle - \sum_{k=1}^{i-2} \frac{\left\langle \xi_{i-1}^n + \Delta t J \xi_{i-1}^n, \xi_k^{n+1} \right\rangle}{\left\langle \xi_k^{n+1}, \xi_k^{n+1} \right\rangle} \left\langle \xi_k^n, \xi_{i-1}^{n+1} \right\rangle \\
= \Delta t \left\langle \xi_i^n, J\xi_{i-1}^n \right\rangle + C_{i-2} \Delta t^2.
\end{aligned}$$

For the last term of (2.12) we note that (2.9) holds by the inductive hypothesis for $1 \leq j \leq i-1$. If we apply Taylor's theorem we find

$$\begin{aligned}
\frac{\|\xi_{i-1}^n\|}{\|\xi_{i-1}^*\|} &= \left\{ \frac{\left\langle \xi_{i-1}^n, \xi_{i-1}^n \right\rangle}{\langle \xi_{i-1}^n, \xi_{i-1}^n \rangle + 2\Delta t \langle \xi_{i-1}^n, J\xi_{i-1}^n \rangle + C_{i-1}\Delta t^2} \right\}^{\frac{1}{2}} \\
&= \frac{\left\langle \xi_{i-1}^n, \xi_{i-1}^n \right\rangle^{1/2}}{\langle \xi_{i-1}^n, \xi_{i-1}^n \rangle^{1/2}} \left\{ 1 - \Delta t \frac{\left\langle \xi_{i-1}^n, J\xi_{i-1}^n \right\rangle}{\langle \xi_{i-1}^n, \xi_{i-1}^n \rangle} + C_{i-1}\Delta t^2 \right\} \\
&= 1 - \Delta t \frac{\left\langle \xi_{i-1}^n, J\xi_{i-1}^n \right\rangle}{\langle \xi_{i-1}^n, \xi_{i-1}^n \rangle} + C_{i-1}\Delta t^2.
\end{aligned}$$

Returning to (2.12) we have

$$\left\langle \xi_i^n, \xi_{i-1}^{n+1} \right\rangle = \Delta t \langle \xi_i^n, J\xi_i^n \rangle + C_{i-1}\Delta t^2, \tag{2.13}$$

and so (2.11) is proved.

All that remains is to apply (2.11) to (2.10) to get (2.9).□

By two straightforward applications of Taylor's theorem we deduce the following corollary.

**Corollary 2.1** *For all $i = 1, \cdots, m$, $1 \leq m \leq p$, there exists constants $C_i$ such that we can expand $\|\xi_i^*\|$ and $\|\xi_i^*\|^{-1}$ as*

$$\|\xi_i^*\| = \|\xi_i^n\| \left\{ 1 + \Delta t \frac{\langle \xi_i^n, J\xi_i^n \rangle}{\|\xi_i^n\|^2} + C_i \Delta t^2 \right\}, \tag{2.14}$$

*and*

$$\|\xi_i^*\|^{-1} = \|\xi_i^n\| \left\{ 1 - \Delta t \frac{\langle \xi_i^n, J\xi_i^n \rangle}{\|\xi_i^n\|^2} + C_i \Delta t^2 \right\}. \tag{2.15}$$

**Theorem 2.2 (Consistency)** *Let $\chi_i^n$ denote the solution of the $i^{th}$ differential equation of* (1.7) *at time $t = n\Delta t$. Then the local truncation error, $\eta_i^n$, satisfies:*

$$\eta_i^n := \frac{\chi_i^{n+1} - \chi_i^n}{\Delta t} - f_i(\chi_i^n) = O(\Delta t),$$

*and hence the scheme (2.4,2.5) is consistent of order* 1.

**Proof** Start by applying (2.11) and (2.3) to the summation term in (2.6,2.7 to get

$$\begin{aligned}
&\frac{\chi_i^{n+1} - \chi_i^n}{\Delta t} \\
&= f_i(\chi_i^n) + \eta_i^n \\
&= \left\{ (\|\chi_i^n\| - \|\chi_i^*\|)\chi_i^n + \Delta t \|\chi_i^n\| J\chi_i^n - \Delta t \|\chi_i^n\| \sum_{j=1}^{i-1} \frac{\left\langle \chi_i^n, J\chi_j^n \right\rangle + \left\langle J\chi_i^n, \chi_j^n \right\rangle}{\|\chi_j^{n+1}\|^2} \chi_j^{n+1} \right\} \\
&\quad \times \frac{1}{\Delta t \|\chi_i^*\|} + \eta_i^n + C_i \Delta t.
\end{aligned}$$

Now use the expansion (2.14) for $\|\chi_i^*\|$ inside the large brackets:

$$\begin{aligned}
\frac{\chi_i^{n+1} - \chi_i^n}{\Delta t} &= \left\{ \|\chi_i^n\| J\chi_i - \frac{\langle \chi_i^n, J\chi_i^n \rangle}{\|\chi_i^n\|} \chi_i^n \right. \\
&\quad \left. - \|\chi_i^n\| \sum_{j=1}^{i-1} \frac{\left\langle \chi_i^n, J\chi_j^n \right\rangle + \left\langle J\chi_i^n, \chi_j^n \right\rangle}{\left\langle \chi_j^{n+1}, \chi_j^{n+1} \right\rangle} \chi_j^{n+1} \right\} \|\chi_i^*\|^{-1} + \eta_i^n + C_i \Delta t.
\end{aligned}$$

Finally use the expansion of $\|\chi_i^*\|^{-1}$ given by (2.15) to get

$$\begin{aligned}
\eta_i^n &= \frac{\chi_i^{n+1} - \chi_i^n}{\Delta t} \\
&\quad - \left\{ J\chi_i^n - \frac{\langle \chi_i^n, J\chi_i^n \rangle}{\langle \chi_i^n, \chi_i^n \rangle} \chi_i^n - \sum_{j=1}^{i-1} \frac{\left\langle \chi_i, J\chi_j^n \right\rangle + \left\langle \chi_j^n, J\chi_i^n \right\rangle}{\left\langle \chi_j^{n+1}, \chi_j^{n+1} \right\rangle} \chi_j^{n+1} \right\} + C_i \Delta t,
\end{aligned}$$

and the result follows. □

Before proving that the scheme is K–stable, we show that the functions $f_i$, $i = 1, \cdots, m$, defined in (2.7) are locally Lipschitz continuous.

**Theorem 2.3** *For all $\Delta t$ sufficiently small the function $f_i : I\!R^{p \times i} \to I\!R^p$, $i = 1, \cdots, m$, ($1 \le m \le p$) defined by* (2.7) *is locally Lipschitz continuous and so for $w_i, v_i \in B(\rho)$, a ball of radius $\rho$ in $I\!R^p$, there exists a constant $M = M(\rho) > 0$ such that*

$$\|f_i(w_i) - f_i(v_i)\| \le M \|w_i - v_i\|, \quad \forall i = 1, \cdots, m.$$

**Proof** By the same analysis as Theorem 2.2 we find, using Lemma 2.1, that for all $w_i \in \mathbb{R}^p$ there exists a constant $C_i$ such that

$$f_i(w_i) = F_i(w_i) + C_i \Delta t,$$

where

$$F_i(w_i) := Jw_i - \frac{\langle w_i, Jw_i \rangle}{\langle w_i, w_i \rangle} w_i - \sum_{j=1}^{i-1} \frac{\left\langle w_i, Jw_j^n \right\rangle + \left\langle w_j^n, Jw_i \right\rangle}{\left\langle w_j^{n+1}, w_j^{n+1} \right\rangle} w_j^{n+1}.$$

Choose $\Delta t$ sufficiently small so that $\Delta t \leq \|w_i - v_i\|$. Then consider $F_i(w_i) - F_i(v_i)$ and take norms to get

$$\begin{aligned} \|F_i(w_i) - F_i(v_i)\| \leq\ & \|Jw_i - Jv_i\| + \left\| \frac{\langle w_i, Jw_i \rangle}{\|w_i\|^2} w_i - \frac{\langle v_i, Jv_i \rangle}{\|v_i\|^2} v_i \right\| \\ & + \sum_{j=1}^{i-1} \|w_j^{n+1}\|^{-2} \left\| \left\langle w_i, Jw_j^n \right\rangle w_j^{n+1} - \left\langle v_i, Jw_j^n \right\rangle w_j^{n+1} \right\| \\ & + \sum_{j=1}^{i-1} \|w_j^{n+1}\|^{-2} \left\| \left\langle w_j^n, Jw_i \right\rangle w_j^{n+1} - \left\langle w_j^n, Jv_i \right\rangle w_j^{n+1} \right\|. \end{aligned} \tag{2.16}$$

We examine (2.16) term by term, making use of $J$ being a bounded linear operator. For the first term we simply note that

$$\|Jw_i - Jv_i\| \leq \|J\| \|w_i - v_i\|. \tag{2.17}$$

The second term is dealt with as follows:

$$\begin{aligned} \left\| \frac{\langle w_i, Jw_i \rangle}{\|w_i\|^2} w_i - \frac{\langle v_i, Jv_i \rangle}{\|v_i\|^2} v_i \right\| \leq\ & \frac{1}{\|w_i\|^2} \left\| \langle w_i, Jw_i \rangle w_i - \langle w_i, Jw_i \rangle v_i \right\| \\ & + \left\| \frac{\langle w_i, Jw_i \rangle}{\|w_i\|^2} v_i - \frac{\langle v_i, Jv_i \rangle}{\|v_i\|^2} v_i \right\|. \end{aligned}$$

Assume, without loss of generality, that $\|w_i\| \geq \|v_i\|$ to get

$$\begin{aligned} & \left\| \frac{\langle w_i, Jw_i \rangle}{\|w_i\|^2} w_i - \frac{\langle v_i, Jv_i \rangle}{\|v_i\|^2} v_i \right\| \\ \leq\ & \|J\| \|w_i - v_i\| + \|v_i\|^{-1} \left\| \langle w_i - v_i, Jw_i \rangle + \langle v_i, J(w_i - v_i) \rangle \right\| \\ \leq\ & \|J\| \|w_i - v_i\| + \|v_i\|^{-1} \|J\| \|w_i\| \|w_i - v_i\| + \|J\| \|w_i - v_i\| \\ \leq\ & 3\|J\| \|w_i - v_i\|. \end{aligned} \tag{2.18}$$

The final two terms are dealt with by noting that

$$\left( \left\langle w_i, Jw_j^n \right\rangle - \left\langle v_i, Jw_j^n \right\rangle \right) w_j^{n+1} = \left\langle w_i - v_i, Jw_j^n \right\rangle w_j^{n+1},$$

and

$$\left(\left\langle w_j^n, Jw_i\right\rangle - \left\langle w_j^n, Jv_i\right\rangle\right) w_j^{n+1} = \left\langle w_j^n, J(w_i - v_i)\right\rangle w_j^{n+1};$$

so that

$$\left\|\left(\left\langle w_i, Jw_j^n\right\rangle - \left\langle v_i, Jv_j^n\right\rangle\right) w_j^{n+1}\right\| \le \|w_j^{n+1}\|\|w_j^n\|\|J\|\|w_i - v_i\|, \tag{2.19}$$

and

$$\left\|\left(\left\langle w_j^n, Jw_i\right\rangle - \left\langle w_j^n, Jv_i\right\rangle\right) w_j^{n+1}\right\| \le \|w_j^{n+1}\|\|w_j^n\|\|J\|\|w_i - v_i\|. \tag{2.20}$$

Combining equations (2.17–2.20) into (2.16) yields the desired result. □

We now proceed to prove the scheme is K–stable.

**Theorem 2.4 (stability)** *Let $\chi_i^n$ be the true solution of the $i^{th}$ differential equation evaluated at $t_n$ and define $\chi_i := (\chi_i^0, \cdots, \chi_i^{N-1})$. Let $B(\chi_i, r)$ be the open ball centered at $\chi_i$ of radius $r$ where $0 < r \le \infty$. Suppose we are also given $\boldsymbol{\xi}$, $\mathbf{a} \in I\!R^{p\times N}$, $\boldsymbol{\xi} = (\xi_i^0, \cdots, \xi_i^{N-1})$ and $\mathbf{a} = (a_i^0, \cdots, a_i^{N-1})$ defined in $B(\chi_i, r)$ both arising from the discretization (2.8).*

*Then there exists a constant $M = M(B(\chi_i, r))$, independent of $\Delta t$ such that*

$$\|\boldsymbol{\xi} - \mathbf{a}\|_1 \le e^{N\Delta tM}\|\Phi_i(\boldsymbol{\xi}) - \Phi_i(\mathbf{a})\|_2.$$

**Proof** First we bound $\|\xi_i^{n+1} - a_i^{n+1}\|$ in terms of $\|\Phi_i(\boldsymbol{\xi})^{n+1} - \Phi_i(\mathbf{a})^{n+1}\|$ for $n = 1, \cdots, N-1$ . Note that

$$\Phi_i(\boldsymbol{\xi})^0 = \xi_i^0 - \alpha, \qquad \Phi_i(\boldsymbol{\xi})^{n+1} = \frac{\xi_i^{n+1} - \xi_i^n}{\Delta t} - f_i(\xi_i^n)$$
$$\Phi_i(\mathbf{a})^0 = a_i^0 - \alpha, \qquad \Phi_i(\mathbf{a})^{n+1} = \frac{a_i^{n+1} - a_i^n}{\Delta t} - f_i(a_i^n).$$

Subtract to get :

$$\Phi_i(\boldsymbol{\xi})^0 - \Phi_i(\mathbf{a})^0 = \xi_i^0 - a_i^0$$

and

$$\xi_i^{n+1} - a_i^{n+1} = (\xi_i^n - a_i^n) + \Delta t(f_i(\xi_i^n) - f_i(a_i^n)) + \Delta t(\Phi_i(\boldsymbol{\xi})^{n+1} - \Phi_i(\mathbf{a})^{n+1}) \tag{2.21}$$

Taking norms in equation (2.21) and using the local Lipschitz continuity of $f_i$ shown in Theorem 2.3 we find

$$\|\xi_i^{n+1} - a_i^{n+1}\| \le \|\xi_i^n - a_i^n\|(1 + \Delta tM) + \Delta t\|\Phi_i(\boldsymbol{\xi})^{n+1} - \Phi_i(\mathbf{a})^{n+1}\|.$$

Now using a standard induction argument and taking norms we get

$$\begin{aligned}
\|\boldsymbol{\xi} - \mathbf{a}\|_1 &= \max_n \|\xi_i^n - a_i^n\| \\
&\le e^{N\Delta tM}\left(\|\Phi_i(\boldsymbol{\xi})^0 - \Phi_i(\mathbf{a})^0\| + \sum_{n=1}^{N} \Delta t\|\Phi_i(\boldsymbol{\xi})^n - \Phi_i(\mathbf{a})^n\|\right) \\
&\le e^{N\Delta tM}\|\Phi_i(\boldsymbol{\xi}) - \Phi_i(\mathbf{a})\|_2.
\end{aligned}$$

Hence the scheme is K-stable.□

**Theorem 2.5 (Convergence)** *The scheme* (2.6) *is convergent of* $O(\Delta t)$ *to the differential equations* (1.7) *for* $i = 1, \cdots, m$; $1 \leq m \leq p$.

**Proof** By Theorem 2.2 the scheme (2.6) is consistent with order $\Delta t$ to (1.7) and by Theorem 2.4 is K–stable. Hence by Theorem 1.2 the scheme is convergent. □

# 3 A More General Problem

Consider the following system of ordinary differential equations in $\mathbb{R}^p$

$$\frac{dV_i}{dt} = G_i(V_i, V_{i-1}, \cdots, V_1) \quad i = 1, \cdots, m; \quad 1 \leq m \leq p, \tag{3.1}$$

$$V(0) = V^0 \ \in \mathbb{R}^p$$

subject to

$$\langle V_i(t), G_j(V_j(t), \cdots, V_1(t)) \rangle + \langle G_i(V_i(t), \cdots, V_1(t)), V_j(t) \rangle \ = 0 \tag{3.2}$$

for all $t \geq 0$ and for all $i, j = 1, \cdots, m$.

Henceforth, to simplify the notation, we omit the dependence of the functions $G_i$ on $V_j$ for $j < i$.

*Assumptions A*

A1 We assume there exists a continuous semigroup $S_i(t) : \mathbb{R}^p \to \mathbb{R}^p$ such that for all $i = 1, \cdots, m$
$$V_i(t) = S_i(t)V_i^0 \ \ \forall \ t \geq 0.$$

A2 We assume that $G_i$ is locally Lipschitz continuous with Lipschitz constant $L$ for all $i = 1, \cdots, p$. That is, for $V$, $W \ \in B(\rho)$, an open ball of radius $\rho > 0$ in $\mathbb{R}^p$, there exists $L > 0$ such that
$$\|G_i(V) - G_i(W)\| \leq L\|V - W\|, \ \forall i = 1, \cdots, m.$$

Then, as the following theorem shows, the system (3.1–3.2) preserves the orthonormality of initial vectors.

**Theorem 3.1** *The system of ordinary differential equations* (3.1) *preserves the orthonormality of the initial vectors. That is, for all* $t \geq 0$

$$\left\langle V_i^0, V_j^0 \right\rangle = \delta_{ij} \implies \left\langle S_i(t)V_i^0, S_j(t)V_j^0 \right\rangle \ = \delta_{ij} \quad i, j = 1, \cdots, m. \tag{3.3}$$

**Proof** Note that $\forall i, j = 1, \cdots, m$ we have from equation (3.2) that

$$\langle V_i(t), G_j(V_j(t)) \rangle + \langle G_j(V_j(t)), V_i(t) \rangle = \frac{1}{2}\frac{d}{dt} \langle V_i(t), V_j(t) \rangle = 0.$$

Thus, for all $i, j = 1, \cdots, m$, we have

$$\langle V_i(t), V_j(t)\rangle = \left\langle V_i^0, V_j^0\right\rangle \quad \forall t \geq 0. \quad \Box \tag{3.4}$$

Let $S_i^1 : \mathbb{R}^p \to \mathbb{R}^p$ be the semi–group associated with an arbitrary convergent numerical method of order $r_i$ for the $i$th of differential equation of (3.1–3.2), arising for example from a standard Runge–Kutta or linear multistep approximation. Since $S_i^1$ is a convergent scheme of order $r_i$, and hence consistent, $\exists\ C_i \in \mathbb{R}$ such that for any $V \in \mathbb{R}^p$

$$\|S_i(\Delta t)V - S_i^1 V\| \leq C_i \Delta t^{r_i+1}. \tag{3.5}$$

Note that we are no longer assuming that each $\{S_i^1\}_{i=1}^m$ is of the same order. In general this arbitrary scheme will not preserve orthonormality as the following lemma illustrates.

**Lemma 3.1** *Suppose we are given an arbitrary $r_i$th order convergent numerical method for the $i$th equation of (3.1–3.2) with semi–group $S_i^1 : \mathbb{R}^p \to \mathbb{R}^p$, for all $i = 1, \cdots, m$. Then there exists constants $C_i = C_i(v_i^n, \cdots, v_1^n) \in \mathbb{R}$ such that $\forall i, j = 1, \cdots, m$*

$$\|S_i^1 v_i^n\|^2 = \|v_i^n\|^2 + C_i \Delta t^{r_i+1}, \quad \forall n \in \mathbb{N}$$

*and, for $j \leq i$,*

$$\left\langle S_i^1 v_i^n, S_j^1 v_j^n\right\rangle = \left\langle v_i^0, v_j^0\right\rangle + C_i \Delta t^{r_{ij}+1}, \quad \forall n \in \mathbb{N}$$

*where $r_{ij} := \min\{r_i, r_j\}$.*

**Proof** Both parts are proved by induction on $n$ and we give here the inductive step. Assume without loss of generality that $i \geq j$. First note that by the consistency of the scheme $S_i^1$ we have that

$$S_i^1 v_i^n = S_i(\Delta t) v_i^n + \Delta t^{r_i+1} \eta_i^n + O(\Delta t^{r_i+2}), \tag{3.6}$$

where $\eta_i^n$ is the truncation error at the $n$th iteration.

The first equality is established by noting that by Theorem 3.1 and (3.6)

$$\begin{aligned}
\|S_i^1 v_i^n\|^2 &= \|S_i(\Delta t) v_i^n\|^2 + \|S_i^1 v_i^n - S_i(\Delta t) v_i^n\|^2 \\
&= \|v_i^n\|^2 + C_i \Delta t^{r_i+1}.
\end{aligned}$$

Similarly the second equality is found by equation (3.6) and Theorem 3.1 and the inductive hypothesis,

$$\begin{aligned}
\left\langle S_i^1 v_i^n, S_j^1 v_j^n\right\rangle &= \left\langle S_i(\Delta t) v_i^n, S_j(\Delta t) v_j^n\right\rangle + \Delta t^{r_{ij}+1}\left(\left\langle S_i(\Delta t) v_i^n, \eta_j^n\right\rangle + \left\langle S_j(\Delta t) v_j^n, \eta_i^n\right\rangle\right) \\
&\quad + O(\Delta t_{ij}^{r+2}) \\
&= \left\langle S_i(\Delta t) v_i^n, S_j(\Delta t) v_j^n\right\rangle + C_i \Delta t^{r_{ij}+1} \\
&= \left\langle v_i^n, v_j^n\right\rangle + C_i \Delta t^{r_{ij}+1} \\
&= \left\langle v_i^0, v_j^0\right\rangle + C_i \Delta t^{r_{ij}+1}. \quad \Box
\end{aligned}$$

We seek numerical methods to solve (3.1) which, unlike the general scheme given by $\{S_i^1\}_{i=1}^m$, preserve orthonormality of initial vectors. We propose schemes which are based on arbitrary convergent numerical methods with an added projection. We prove convergence by proving consistency and K–stability.

## 3.1 Schemes Conserving Orthonormality

Let $\{S_i(\Delta t)\}_{i=1}^m$ be the the set of arbitrary convergent numerical schemes, where $S_i : \mathbb{R}^p \to \mathbb{R}^p$ is the $r_i$ order scheme used to approximate the $i$th differential equation of (3.1) subject to (3.2). For convenience of notation we redefine $r_i$ as

$$r_i := \min_{1\leq k\leq i} r_k.$$

We now modify these schemes to preserve orthonormality.

Let $v_i^*$ be defined by

$$v_i^* := S_i^1 v_i^n - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i^n, v_j^{n+1}\right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1}\right\rangle} v_j^{n+1} \quad \forall\ i = 1, \cdots, m \tag{3.7}$$

and define our approximation to (3.1) by

$$v_i^{n+1} = v_i^* \frac{\|v_i^n\|}{\|v_i^*\|}, \quad \forall\ i = 1, \cdots, m. \tag{3.8}$$

Let the corresponding semi–group $\widetilde{S}_i^1 : \mathbb{R}^p \to \mathbb{R}^p$ be given by $\widetilde{S}_i^1 v_i^n := v_i^{n+1},\ \forall\ i = 1, \cdots, m$. The summation term in (3.7) corresponds to a re–orthogonalization and the choice of weighting in (3.8) corresponds to the re–normalization. The scheme given by (3.7–3.8) may be re–written to resemble a discretized ordinary differential equation :

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = f_i(v_i^n), \quad \forall i = 1, \cdots, m,$$

where $f_i : \mathbb{R}^{p\times i} \to \mathbb{R}^p$ is given for $w_i \in \mathbb{R}^p$ and for $\left\{w_j^{n+1}\right\}_{j=1}^{i-1}$, $w_j^{n+1} \in \mathbb{R}^p$, by

$$\begin{aligned} f_i(w_i) &:= f_i(w_i; w_{i-1}^{n+1}, \cdots, w_1^{n+1}) \\ &= \left\{ S_i^1 w_i \|w_i\| - \|w_i^*\| w_i - \|w_i\| \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 w_i, v_j^{n+1}\right\rangle}{\left\langle w_j^{n+1}, w_j^{n+1}\right\rangle} w_j^{n+1} \right\} / \Delta t \|w_i^*\| \end{aligned} \tag{3.9}$$

Finally we let $C_i = C_i(v_i^n, \cdots, v_1^n)$ denote an arbitrary constant.

**Theorem 3.2** *Given an orthonormal set $\{v_i^0\}_{i=1}^m$ the scheme given by (3.7-3.8) conserves othonormality of initial vectors. That is, for all $i, k = 1, \cdots, m$*

$$\left\langle v_i^0, v_k^0\right\rangle = \delta_{ik} \implies \left\langle v_i^n, v_k^n\right\rangle = \delta_{ik} \qquad \forall n \in I\!N,$$

**Proof** The proof is by induction on $n$ exactly like the proof of Theorem 2.1 and is omitted here. $\Box$

We prove convergence of the scheme given by (3.7–3.8) to the system (3.1–3.2) using the consistency and K–stability result stated in Theorem 1.2. Our first Lemma is a step towards consistency.

**Lemma 3.2** *For all $i = 1, \cdots, m$ there exists $C_i \in \mathbb{R}$ such that*

$$\|v_i^*\|^2 = \|v_i^n\|^2 + C_i \Delta t^{2(r_i+1)}.$$

*Hence there exists $\widetilde{C}_i \in \mathbb{R}$ such that*

$$\frac{\|v_i^n\|}{\|v_i^*\|} = 1 + \widetilde{C}_i \Delta t^{r_i+1}.$$

**Proof** Consider the expansion of $\langle v_i^*, v_i^* \rangle$:

$$\begin{aligned}
\langle v_i^*, v_i^* \rangle &= \left\langle S_i^1 v_i^n - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i^n, v_j^{n+1}\right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1}\right\rangle} v_j^{n+1}, S_i^1 v_i^n - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i^n, v_j^{n+1}\right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1}\right\rangle} v_j^{n+1} \right\rangle \\
&= \left\langle S_i^1 v_i^n, S_i^1 v_i^n \right\rangle - 2 \left\langle S_i^1 v_i^n, \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i^n, v_j^{n+1}\right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1}\right\rangle} v_j^{n+1} \right\rangle \\
&\quad + \left\langle \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i^n, v_j^{n+1}\right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1}\right\rangle} v_j^{n+1}, \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i^n, v_j^{n+1}\right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1}\right\rangle} v_j^{n+1} \right\rangle . \qquad (3.10)
\end{aligned}$$

By Lemma 3.1 and inspection the proof will be completed by proving that there exists $C_i \in \mathbb{R}$ such that

$$\left\langle S_i^1 v_i^n, v_j^{n+1} \right\rangle = C_i \Delta t^{r_i+1} \quad \forall j < i. \qquad (3.11)$$

**Proof** of (3.11) : We prove (3.11) by induction. The proof for $j = 1$ is straightforward. Supposing true for $j = 2, \cdots, i-2$ we prove true for $j = i-1$. By Lemma 3.1 and the inductive hypothesis we have that

$$\begin{aligned}
\left\langle S_i^1 v_i^n, v_{i-1}^{n+1} \right\rangle &= \left\langle S_i^1 v_i^n, S_i^1 v_{i-1}^n \right\rangle - \sum_{k=1}^{i-2} \frac{\left\langle S_i^1 v_i^n, v_k^{n+1}\right\rangle}{\left\langle v_k^{n+1}, v_k^{n+1}\right\rangle} \left\langle v_k^{n+1}, S_i^1 v_i^n \right\rangle + \widetilde{C}_{i-1} \Delta t^{r+1} \\
&= C_i \Delta t^{r_i+1}
\end{aligned}$$

and so (3.11) is established.

The first equality follows by applying (3.11) in (3.10) and the second is found by an application of Taylor's theorem. $\Box$

**Theorem 3.3 (Consistency)** *Let $V_i(t_n)$ denote the solution of the ith differential equation of the system (3.1–3.2) at time $t = t_n$. Then the local truncation error $\widetilde{\eta}_i^n$ satisfies*

$$\widetilde{\eta}_i^n := S_i(\Delta t) V_i^n - \widetilde{S}_i^1 V_i^n = \eta_i^n + O(\Delta t^{r_i+1})$$

*where $\eta_i^n$ is the truncation error for the numerical method given by $S_i^1$. Thus the scheme (3.7–3.8) is consistent of order $r := \min_{1 \le k \le m} r_k$.*

**Proof** The proof of consistency follows from Lemma 3.2,

$$\begin{aligned} S_i(\Delta t)V_i^n - \tilde{S}_i^1 V_i^n &= S_i(\Delta t)V_i^n - \left\{ S_i^1 V_i^n - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 V_i^n, V_j^{n+1} \right\rangle}{\left\langle V_j^{n+1}, V_j^{n+1} \right\rangle} V_j^{n+1} \right\} \frac{\|V_i^n\|}{\|V_i^*\|} \\ &= S_i(\Delta t)V_i^n - S_i^1 V_i^n + \tilde{C}_i \Delta t^{r_i+1} + O(\Delta t^{r_i+2}) \\ &= S_i(\Delta t)V_i^n - S_i^1 V_i^n + O(\Delta t^{r_i+1}) \\ &= \eta_i^n + O(\Delta t^{r_i+1}). \quad \square \end{aligned}$$

We now apply Lemma 3.2 to prove that the function $f_i$, $i = 1, \cdots, m$, defined in (3.9) is locally Lipschitz continuous.

**Theorem 3.4** *There exists $\Delta t_0 > 0$ such that $\forall \Delta t < \Delta t_0$ the function $f_i : I\!R^{p \times i} \to I\!R^p$, $i = 1, \cdots, m$, defined by (3.9) is locally Lipschitz continuous. Thus for all $v_i, w_i \in B(\rho)$, a ball of radius $\rho > 0$ in $I\!R^p$, $\exists M = M(\rho) > 0$ such that*

$$\|f_i(v_i) - f_i(w_i)\| \leq M \|v_i - w_i\| \quad \forall\ i = 1, \cdots, p.$$

**Proof** First consider the differential equations Tr

$$\frac{dv_i}{dt} = G_i(v_i) \ \& \ \frac{dw_i}{dt} = G_i(w_i).$$

Subtracting, taking the inner product with $v_i - w_i$ and using assumption A2 we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_i(t) - w_i(t)\|^2 &\leq \langle G_i(v_i(t)) - G_i(w_i(t)), v_i(t) - w_i(t) \rangle \\ &\leq L\|v_i(t) - w_i(t)\|^2. \end{aligned}$$

Thus, $\forall i = 1, \cdots, m$ we find

$$\|v_i(t) - w_i(t)\|^2 \leq e^{2Lt}\|v_i - w_i\|^2, \quad \forall t \geq 0. \tag{3.12}$$

Now consider the function $f_i$, $i = 1, \cdots, m$ defined in (3.9). By substituting into (3.9) from Lemma 3.2 for $\|v_i^*\|$ and $\|v_i^*\|^{-1}$, we find that

$$\begin{aligned} f_i(v_i) &= \left\{ S_i^1 v_i \|v_i\| - (1 - C_i \Delta t) \|v_i\| v_i - \|v_i\| \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i, v_j^{n+1} \right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1} \right\rangle} v_j^{n+1} \right\} \\ &\quad \times \frac{1}{\Delta t} \|v_i\|^{-1} \left(1 + C_i \Delta t^{r_i+1}\right) \\ &= \left\{ S_i^1 v_i - v_i - \sum_{j=1}^{i-1} \frac{\left\langle S_i^1 v_i, v_j^{n+1} \right\rangle}{\left\langle v_j^{n+1}, v_j^{n+1} \right\rangle} v_j^{n+1} \right\} \frac{1}{\Delta t} + C_i \Delta t^{r_i}. \end{aligned}$$

By Lemma 3.1 we have that the summation term above is of order $O\left(\Delta t^{r_i+1}\right)$. Hence there exists $C_i \in \mathbb{R}$ such that

$$f_i(v_i) = \frac{S_i^1 v_i - v_i}{\Delta t} + C_i \Delta t_i^r.$$

Further, by the consistency of the scheme given by $S_i^1$, we have from equation (3.6) that

$$f_i(v_i) = \frac{S_i(\Delta t)v_i - v_i}{\Delta t} + C_i \Delta t^{r_i},$$

where $C_i$ is used to denote a generic constant. Let $\Delta t_0 > 0$ be chosen so that

$$\Delta t_0 \leq \|v_i - w_i\|^{1/r};$$

then

$$\begin{aligned} f_i(v_i) - f_i(w_i) &= \frac{S_i(\Delta t)v_i - v_i}{\Delta t} - \frac{S_i(\Delta t)w_i - w_i}{\Delta t} + C_i \Delta t^{r_i} \\ &= F_i(v_i) - F_i(w_i) + C_i \Delta t^{r_i} \end{aligned}$$

where

$$F_i(u) := \frac{S_i(\Delta t)u - u}{\Delta t}.$$

We have that

$$\begin{aligned} F_i(v_i) - F_i(w_i) &= \frac{1}{\Delta t} \int_0^{\Delta t_0} \left( \frac{dv_i}{dt} - \frac{dw_i}{dt} \right) dt \\ &= \frac{1}{\Delta t} \int_0^{\Delta t_0} \left( G_i(v_i(t)) - G_i(w_i(t)) \right) dt. \end{aligned}$$

Hence,

$$\|F_i(v_i) - F_i(w_i)\| \leq \frac{1}{\Delta t} \int_0^{\Delta t_0} \|G_i(v_i(t)) - G_i(w_i(t))\| \, dt,$$

and so by the local Lipschitz property of $G_i$ (assumption A2) and equation (3.12) we get

$$\begin{aligned} \|F_i(v_i) - F_i(w_i)\| &\leq \frac{1}{\Delta t} \int_0^{\Delta t} L \|v_i(t) - w_i(t)\| dt \\ &\leq \|v_i - w_i\| \frac{1}{\Delta t} \int_0^{\Delta t} L e^{Lt} dt \\ &\leq \|v_i - w_i\| L^2 \frac{e^{L\Delta t} - 1}{\Delta t}. \end{aligned}$$

Now noting that

$$\begin{aligned} \frac{e^{L\Delta t} - 1}{\Delta t} &= \frac{1}{\Delta t} \left( \sum_{k=0}^{\infty} \frac{(L\Delta t)^k}{k!} - 1 \right) \quad = \quad L \sum_{k=1}^{\infty} \frac{(L\Delta t)^{k-1}}{k!} \\ &= L \sum_{k=0}^{\infty} \frac{(L\Delta t)^k}{(k+1)!} \quad \leq \quad L \sum_{k=0}^{\infty} \frac{(L\Delta t)^k}{k!} \\ &= L e^{L\Delta t} \quad \leq \quad L e^{L\Delta t_0}, \end{aligned}$$

we see that the function $F_i$, $i = 1, \cdots, m$, is locally Lipschitz continuous. Hence so is the function $f_i$, $i = 1, \cdots, m$. $\Box$

We now state our convergence theorem for the scheme given by (3.7–3.8).

**Theorem 3.5 (Convergence)** *The scheme (3.7–3.8) is convergent of of order* $r := \min_{1\leq k\leq m} r_k$ *to the differential equations* (3.1) *for all* $i = 1, \cdots, m$.

**Proof** The proof of K-stability is analogous to that given in Theorem 2.4 using the Lipschitz continuity of $f_i$, $i = 1, \cdots, m$, proved in Theorem 3.4; and hence is omitted. By Theorem 3.3 the scheme is consistent and so convergence follows from Theorem 1.2. □

# 4 Numerical Results

To illustrate the schemes of section 2 we estimated the Lyapunov exponents of the Lorenz equation (1.6) by the differential method (1.7, 1.8). Explicit Euler's method was taken as the arbitrary convergent scheme in (2.1,2.2), just as it was taken in (2.4,2.5) to illustrate the convergence proof. The integral (1.8) was estimated using the standard trapezoidal rule. The resulting method for finding the exponents is denoted below by OE, standing for "Orthogonal Euler".

To illustrate the schemes (3.7–3.8) of section 3 we noted that (1.7) is a specific example of a system of the form (3.1) subject to (3.2). We used (1.7) in conjuiction with (1.8) to find the Lyapunov exponents for the Lorenz equations (1.6). For a direct comparaison with the results from section 2 we chose in each case as our arbitrary methods $\{S_i^1\}_{i=1}^3$ in (3.7–3.8) explicit Euler's method, and so each scheme $S_i^1$, $i = 1, \cdots 3$ is of the same order. The integral (1.8) was estimated by the standard trapezoidal rule. The resulting method for computing the Lyapunov exponents is denoted by GOE below, standing for "Generalized Orthogonal Euler". By the analysis of section 3 we could pick each schemes $S_i$ of different order, for example $r_1 > r_2 \geq r_3$, and hence (in theory) find a more acurate approximation to the first exponent than the second exponent or third. This, however would not normally be required.

Recall that OE and GOE are distinct schemes. The scheme OE is found by approximating the linear evolution equation (1.5) by an arbitrary scheme (explicit Euler's method) and modifying this to approximate (1.7) and preserve orthonormality. The scheme GOE is found by approximating (1.7) directly by an arbitrary scheme (explicit Euler's method) and then modifying to preserve orthonormality.

In practice the choice of explicit Euler's method for both OE and GOE is not so restrictive since, in general, a high order numerical integration scheme is not normally required to solve (1.7). Finding Lyapunov exponents is computationally expensive: first the size of the system to be solved increases with the number of Lyapunov exponents, and doubles if all exponents are required; secondly long run times are required since Lyapunov exponents are a time asymptotic quantity. The accuracy of the exponent is limited by the accuracy of the solution to the nonlinear system (1.4), by the accuracy of the solution of (1.7) and by the accuracy of the quadrature rule for (1.8). Clearly there is no point solving (1.7) more accurately than the nonlinear system (1.4). In applications it is effectively the quadrature rule that determines the accuracy of the exponents since to keep computational costs low a low order quadrature rule is commonly used.

For both OE and GOE the Lorenz equations (1.6) were integrated by the standard 4th order Runge–Kutta method. We present results for two values of the parameter $r$, $r = 40$ and $r = 45.92$. In each case we took $\sigma = 16$ and $\beta = 4$. These are parameter regimes for which it is commonly believed that there exists a "chaotic attractor". For further details on the Lorenz equations see [23] and for chaos in the Lorenz equations, see [14].

In Table 1 we compare results from OE, GOE and those found by other authors. We compare OE and GOE to a differential method (DffMthd) and the standard method (StdMthd) :

DffMthd [5]: The Lyapunov exponents were found from the differential method using a fourth order unitary integrator (ie orthogonality preserving) to integrate the equations (1.7) and a composite trapezoidal rule to estimate the integral (1.8). Results were taken from [5].

StdMthd : This is the standard method (see section 1.2) for finding the Lyapunov exponents. Results were taken from [5] and [27].

In each case the total integration time was taken to be $T = 1000$. For the schemes OE and GOE we took a step size of $\Delta t = 0.0001$.

| Method | r | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_1 + \mu_2 + \mu_3$ |
|---|---|---|---|---|---|
| StdMthd [5] | 40.0 | 1.33961 | -0.01055 | -22.30930 | -20.98024 |
| DffMthd [5] | 40.0 | 1.36006 | 0.00570 | -22.36576 | -21.00 |
| OE | 40.0 | 1.353239 | -0.006204 | -22.347250 | -21.000215 |
| StdMthd [27] | 45.92 | 1.497 | 0.0 | -22.458 | -20.961 |
| StdMthd [5] | 45.92 | 1.47829 | -0.01086 | -22.44645 | -20.97902 |
| DffMthd [5] | 45.92 | 1.48804 | 0.00452 | -22.52772 | -21.03516 |
| GOE | 45.92 | 1.476634 | -0.005988 | -22.470644 | -20.999998 |

Table 1: Comparison of different methods.

We note that our results agree very well with each other and the other computed values. Recall for these parameter values of the Lorenz equations we have

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t (DF(U(s)))ds = -\sigma - \beta - 1 = -21.$$

Hence the numerical results suggest that the Lorenz equations are regular since the sum of global Lyapunov exponents is $-21$. Indeed if the system were non–regular it is exceedingly unlikely that any reasonable numerical results would have been found [5].

In Table 2 we compare OE and GOE for different step sizes, integrating for a fixed time of $T = 1000$. We know the second exponent should be zero since the chaotic trajectory is stable and a zero exponent corresponds to perturbations along the trajectory. It is clear from the table that as the mesh size is refined we find a better approximation, and that the methods OE and GOE yield the same results to order $10^{-3}$.

It is of interest that the sum of the exponents is estimated far more accurately than the individual exponents ( see for example $\Delta t = 0.01$). This illustrates that the sum of exponents should not be used as a stopping criteria in codes.

It should be remembered that there are two limiting processes at work: there is the limit as $\Delta t \to 0$ and the limit as $T \to \infty$. Figures 2 illustrate the two limiting processes. The $x$ co–ordinate of each plot is $T$ and the $y$ co–ordinate of each plot is $\log(\Delta t)$. If we fix a point on the $\log(\Delta t)$ axis and look along the $T$ axis we are looking at convergence for a fixed $\Delta t$ in time. If we fix $T$ and look along the $\log(\Delta t)$ axis we are looking at convergence as $\Delta t \to 0$. On these plots the point $(50, -10, \bullet)$ is the most accurate approximation. We clearly see from Figure 2 that the sum of exponents converges faster than the individual exponents as both $\Delta t \to 0$ and $T \to \infty$. Suppose an exponent was required to a certain fixed accuracy, then Figure 2 suggests this may be achieved either by fixing a small step size and integrating over a certain time interval, or fixing a larger step size and integrating over a longer time interval.

| Method | $r$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_1+\mu_2+\mu_3$ | $\Delta t$ |
|---|---|---|---|---|---|---|
| OE | 40.0 | 0.945100 | -1.096546 | -20.848659 | -21.000105 | 0.01 |
| OE | 40.0 | 1.308091 | -0.102461 | -22.205609 | -20.999979 | 0.001 |
| OE | 40.0 | 1.353239 | -0.006204 | -22.347250 | -21.0000215 | 0.0001 |
| GOE | 40.0 | 0.431585 | -0.671833 | -20.759647 | -20.999895 | 0.01 |
| GOE | 40.0 | 1.265091 | -0.059433 | -22.205648 | -20.99999 | 0.001 |
| GOE | 40.0 | 1.352325 | -0.005025 | -22.347299 | -20.9999999 | 0.0001 |
| OE | 45.92 | 1.206253 | -1.277307 | -20.928842 | -20.996896 | 0.01 |
| OE | 45.92 | 1.429261 | -0.119475 | -22.309775 | -20.999989 | 0.001 |
| OE | 45.92 | 1.478099 | -0.007642 | -22.470456 | -20.999999 | 0.0001 |
| GOE | 45.92 | 0.511754 | -0.773511 | -20.738033 | -20.99979 | 0.01 |
| GOE | 45.92 | 1.386088 | -0.074614 | -22.311453 | -20.999979 | 0.001 |
| GOE | 45.92 | 1.476634 | -0.005988 | -22.470644 | -20.999998 | 0.0001 |

Table 2: Comparaison of the methods OE and GOE.

In conclusion we have examined numerical schemes for integrating the system (1.7) used for finding the first $m$ Lyapunov exponents of a dynamical system. An arbitrary convergent numerical scheme was modified to preserve orthonormality and convergence proved with explicit Euler's method as the arbitrary scheme. This led to examining a more general system of which (1.7) is an example. Arbitrary convergent schemes were modified to preserve orthonormality and convergence proved in the general case. Some numerical results were then presented.

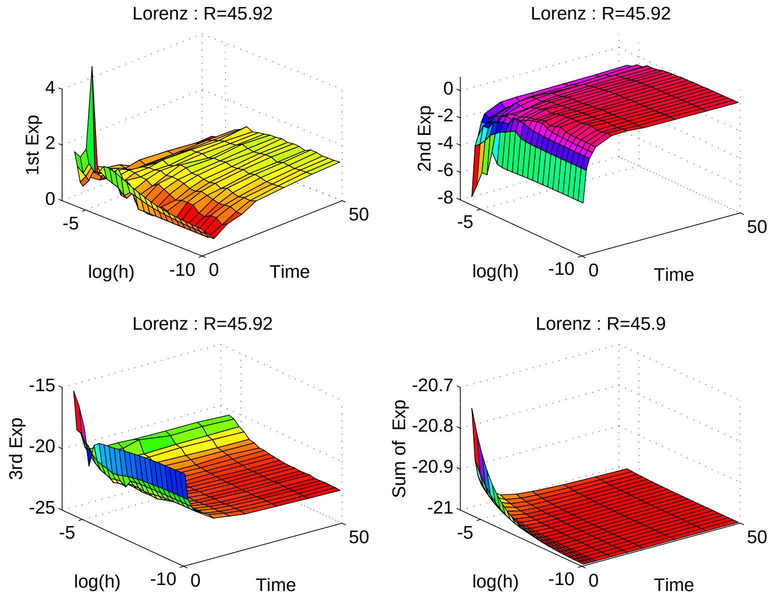


Figure 2: Exponents and sum of exponents for Lorenz equations ($r = 45.92$).


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