ASYMPTOTIC REPRESENTATION OF AN ELASTIC FIELD IN A MULTI–STRUCTURE

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Abstract

A mixed boundary value problem is posed for the Lamé system in an elastic multi–structure which consists of a union of a three–dimensional domain and thin cylinders. A normalized thickness of a rod is characterized by a small parameter ε. We present a rigorous algorithm which yields a multi–scaled asymptotic expansion for the displacement and stress fields in the multi–structure. Justification of the asymptotic algorithm, including the remainder estimate for the asymptotic approximation of the displacement field, is given.

The asymptotic analysis provides the range of applicability of the engineering pile structure model.

It is shown that first three eigenvalues of the spectral problem in the above multi–structure have the order $O(\varepsilon^4)$. The asymptotic formulae for these eigenvalues are obtained.
Keywords: elastic multi–structure, multi–scaled asymptotic expansion, spectral problem.

Contents

1 Introduction

In this section we describe a multi–structure, pose the boundary value problem for the elasticity equations, formulate objectives of the study and present a brief description of the results.

1.1 Background. Objectives of the study. Main results

Whereas engineering theories for thin elastic plates, shells and thin rods are well–established in the literature, the mathematical modelling of elastic fields in multi–structures started its development quite recently. This study includes an asymptotic analysis of solutions to boundary value problems for the Lamé system in regions which depend on a small parameter, say ε, in such a way that, as $\varepsilon \to 0$, the limit set (*skeleton of the multi–structure*) is represented by a union of sets of different dimensions.

Most of results, obtained by 1990 on solutions of the Lamé system for multi–structures are presented or refered to in the book by Ciarlet (1990) [1]. The study of junctions between three–dimensional and two–dimensional linearly elastic structures was performed in the paper of Ciarlet, Le Dret and Nzengwa (1989) [3]. The eigenvalue problems for junctions between elastic structures were studied in the work by Bourquin and Ciarlet (1989) [5], where the authors used a "limit" variational formulation. The modelling of a junction between two rods was developed by Le Dret (1989) [6]. The papers [2], [4] by Sánchez–Palencia (1988, 1990) were devoted to a boundary value problem, where the Neumann boundary conditions are posed in a small region of the surface, and to the analysis of elastic fields in a union of conical and cylindrical domains.

The question of modelling junctions between three–dimensional and one–dimensional elements in a skeleton of a multi–structure was analyzed by Kozlov, Maz'ya and Movchan (1994) [7]. This work deals with a mixed boundary value problem for the Laplacian in a multi–structure consisting of a three–dimensional element and a set of thin rods parallel to each other (see Fig.1). A normalized thickness of a rod is characterized by a small parameter ε.

In the present paper we consider elastic fields in the multi–structure of the same geometry. The material of the multi–structure is assumed to be linearly elastic, isotropic and homogeneous. The traction boundary conditions are prescribed on the surface of the elastic cap and on the lateral surface of the thin cylinders. At the ends of the thin rods displacement components are given. Although we restrict ourselves to a special example of geometry of a multi–structure, the present analysis can be used in a general situation.

The engineering approach to problems of this kind is described in Section 3.7.2, where the three–dimensional element is considered as a rigid body, and the thin rods are replaced by one–dimensional elastic (with certain stiffness coefficients) piles clamped at junction points. In the engineering literature this model is called the pile structure (see, for example, Asplund (1966) [12]). In this model the stress–strain state is usually described by the rigid body displacements of the pile cap and by the reaction forces and moments at junction points (these are called "lock forces" and "lock moments"). The basic relation in the pile structure model is the linear system of equations with respect to components of the rigid body displacements of the pile cap. The 6×6 nonsingular matrix of this system is called the stiffness matrix of the pile structure. Since the stiffness coefficients of elastic piles depend on the normalized thickness of the thin rods, the components of the stiffness matrix include ε. For the case, where the piles are parallel to each other, the principal part of this matrix is singular, and the rigorous analysis was absent in the literature. The question is "Does the pile structure model give correct results in this case?".

Here, we develop an algorithm for constructing the power asymptotic series in ε which is uniformly valid for elastic fields in the entire multi–structure. The leading order term of the displacement vector within the three–dimensional domain Ω is represented by rigid body translations and rotations. In comparison with the engineering model of the pile structure we show, in particular, that the rigorous asymptotic analysis gives the same leading order displacement, provided that the transversal components of the principal force vector and third component of the principal moment do not vanish simultaneously. Otherwise, the formula for the displacement field is different (see Section 6.2). Also, in Section 6.2 the analysis of reaction forces and moments at junction points is given. We show that the classical engineering approach needs modification in order to take into account the elastic interaction between Ω and the rods. The present asymptotic study provides the range of applicability of the model of the pile structure with the degenerate geometry.

One more important feature of elastic fields in Ω should be mentioned. Far from the junction points $\mathbf{a}^{(i)}$, $i = 1, \dots, K$, the stress field is asymptotically equivalent to a solution of the Neumann boundary value problem in Ω with singularities at $\mathbf{a}^{(i)}$. Namely, in order to find the principal term of the asymptotics in Ω outside some neighbourhoods of $\mathbf{a}^{(i)}$, one should set concentrated forces and moments at $\mathbf{a}^{(i)}$. The presence of these singularities does not allow one to apply the variational approach, given by Ciarlet [1], immediately. Here, we use the compound asymptotic expansion technique developed by Maz'ya, Nazarov and Plamenevskii (1991, 1992) [8].

The coefficients of the asymptotic series involve solutions to the following "model problems" in sets independent of ε:

(i) the Neumann problem for the Lamé system in the three–dimensional domain Ω (Fig. 2);

(ii) the "junction layer": the Neumann problem for the Lamé system in the scaled junction region (Fig. 3);

(iii) the "bottom layer": the Dirichlet–Neumann problem in the scaled bottom region of a thin cylinder (Fig. 4);

(iv) the plane Neumann problems for the Lamé and the Laplace equations on the scaled cross–section of a thin rod (Fig. 5a);

(v) the boundary value problems for ordinary differential equations on the axis of a thin rod (Fig. 5b);

(vi) the system of linear equations corresponding to the skeleton of the multi–structure (Fig. 6).

First five model problems are posed for certain elements of the multi–structure. In contrast, the algebraic system from (vi) includes characteristics of all elements of the multi–structure and is closely related to the pile structure model.

In Section 6 we study the spectral problem for the Lamé operator in the multi–structure. We show that first three eigenvalues have the order $O(\varepsilon^4)$, and satisfy the asymptotic formulae

$$\lambda_j = \varepsilon^4 \rho_j + O(\varepsilon^5), \; j = 1, 2, 3,$$

where $\rho_1 \le \rho_2 \le \rho_3$ are eigenvalues of the problem

$$\mathcal{A}\mathbf{q} = \rho \; \mathrm{diag}\{\mathrm{mes}\Omega, \mathrm{mes}\Omega, I_3\} \; \mathbf{q},$$

Here, $\mathcal{A}$ is a 3×3 positive definite matrix defined in Sections 7.3, 3.7.1, and I_3 is the inertia moment of Ω with respect to the axis which is parallel to the rods and contains the inertia centre of Ω. Remaining eigenvalues are greater than Const ε^2.

In *Section* 1 we formulate the boundary value problem and describe the multi–structure. The auxiliary *Section* 2 contains some classical solutions of elasticity equations in a half–space and in cylindrical domains. The model boundary value problems are formulated and analyzed in *Section* 3. *Section* 4 deals with a formal asymptotic expansion of the displacement field. In *Section* 5 we give estimates for the remainder term, and, therefore, justify the asymptotic expansion of the solution. The leading ansatz of the displacement field and the principal parts of the lock forces and moments are analyzed in Section 6, and also the physical interpretation of the results is given. In Section 7 we present asymptotic formulae for first eigenvalues of the spectral problem in the multi–structure. Inequalities of the Korn type, used in the paper, are discussed in Appendix A. The abstract scheme, required for evaluation of the first eigenvalues, is presented in Appendix B.

1.2 Formulation of the problem and description of the multi–structure

Let Ω_ε be a region in R^3 which depends on a small parameter $\varepsilon > 0$ and consists of a bounded domain Ω with the Lipschitz boundary and K parallel thin cylinders $\Pi_\varepsilon^{(k)}$ of the finite length $l^{(k)}$, $k = 1, \dots, K$ (see Fig.1). Saying "thin" we mean that the diameter of the cross–section has the order ε. It is assumed that some open part of the boundary $\partial\Omega$ belongs to the plane $x_3 = 0$ and contains the junction points $\mathbf{a}^{(k)} = (a_1^{(k)}, a_2^{(k)}, 0)$, $k = 1, \dots, K$, which are not located on the same straight line. The thin cylinder at $a^{(k)}$ is given by

$$\Pi_\varepsilon^{(k)} = g_k(\varepsilon) \times [0, l^{(k)}),$$

where

$$g_k(\varepsilon) = \{(x_1, x_2) : \varepsilon^{-1}(x_1 - a_1^{(k)}, x_2 - a_2^{(k)}) \in g\},$$

with g being a two–dimensional bounded Lipschitz domain independent of ε. The part of the boundary, corresponding to the base region of the thin rod $\Pi_\varepsilon^{(k)}$, is denoted by $S_\varepsilon^{(k)}$, and

$$S_\varepsilon = \bigcup_{k=1}^{K} S_\varepsilon^{(k)}.$$

We consider a mixed Dirichlet–Neumann boundary value problem for the Lamé system in Ω_ε. The displacement vector $\mathbf{u}_\varepsilon(\mathbf{x})$ satisfies the system

$$-\mathbf{L}(\partial_{\mathbf{x}})\mathbf{u}_\varepsilon(\mathbf{x}) := -\mu\Delta\mathbf{u}_\varepsilon(\mathbf{x}) - (\lambda + \mu)\nabla\ \nabla \cdot \mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{F}(\mathbf{x}, \varepsilon),\ \mathbf{x} \in \Omega_\varepsilon, \tag{1.2.1}$$

the traction boundary condition

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}_\varepsilon; \mathbf{x}) := \boldsymbol{\sigma}(\mathbf{u}_\varepsilon; \mathbf{x})\mathbf{n} = \mathbf{P}(\mathbf{x}, \varepsilon),\ x \in \partial\Omega_\varepsilon \setminus \overline{S}_\varepsilon, \tag{1.2.2}$$

and the displacement boundary condition

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{\Phi}^{(k)}(\mathbf{x}, \varepsilon),\ \mathbf{x} \in S_\varepsilon^{(k)}.\ k = 1, \dots, K, \tag{1.2.3}$$

where $\mathbf{n}$ is the outward unit normal on $\partial\Omega_\varepsilon$; λ, μ are the Lamé constants; $\boldsymbol{\sigma}$ is the stress tensor with components

$$\sigma_{ij} = \mu(\partial_{x_j} u_i + \partial_{x_i} u_j) + \lambda\delta_{ij}\nabla \cdot \mathbf{u},$$

with δ_{ij} being the Kronecker delta. We suppose that the vector functions $\mathbf{F}$, $\mathbf{P}$ and $\mathbf{\Phi}^{(k)}$ belong to $(L_2(\Omega_\varepsilon))^3$, $(L_2(\partial\Omega \setminus S_\varepsilon))^3$, $(H^{1/2}(S_\varepsilon^{(k)}))^3$, respectively.

We introduce the energy functional

$$\mathcal{E}(\mathbf{u}_\varepsilon;\Omega_\varepsilon) = \frac{1}{2}\sum_{i,j=1}^{3}\int_{\Omega_\varepsilon} \sigma_{ij}(\mathbf{u}_\varepsilon;\mathbf{x})\varepsilon_{ij}(\mathbf{u}_\varepsilon;\mathbf{x})d\mathbf{x}. \tag{1.2.4}$$

Here, ε_{ij} are components of the strain tensor given by

$$\varepsilon_{ij} = \frac{1}{2}\Big(\partial_{x_j}u_i + \partial_{x_i}u_j\Big).$$

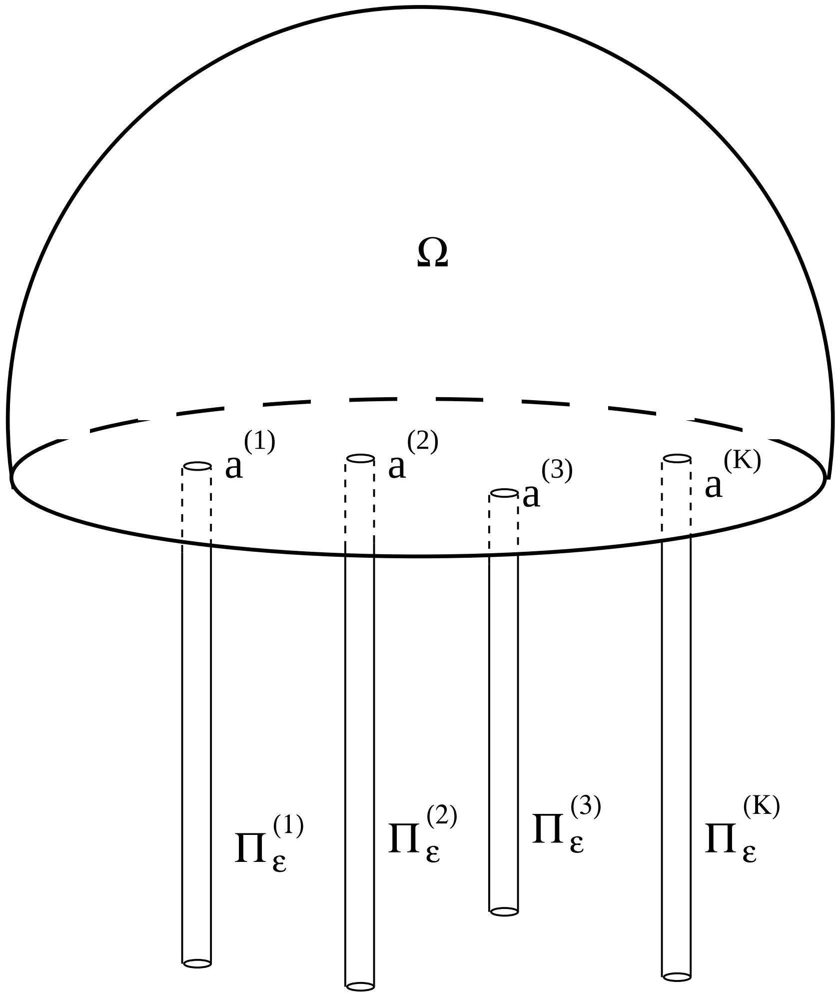

Fig.1 The multi–structure

Since the boundary $\partial\Omega_\varepsilon$ is Lipschitz, the following Korn inequality holds (see, for example, (A.15) in Appendix)

$$\|\mathbf{v}\|^2_{(H^1(\Omega_\varepsilon))^3} < C_\varepsilon\Big\{\mathcal{E}(\mathbf{v};\Omega_\varepsilon) + \|\mathbf{v}\|^2_{(L_2(S_\varepsilon))^3}\Big\}, \tag{1.2.5}$$

for any vector function $\mathbf{v} \in (H^1(\Omega_\varepsilon))^3$, and some C_ε which depends on ε and does not depend on $\mathbf{v}$. It follows from (1.2.5) that the boundary value problem (1.2.1)–(1.2.3) has a unique solution with the finite energy integral (1.2.4).

In Sections 4, 5 we shall construct the asymptotic expansion of the field $\mathbf{u}_\varepsilon$ in the multi–structure Ω_ε.

2 Some Special Solutions in Linear Elasticity

The present section includes well–known solutions of elasticity equations in a half–space and in cylindrical domains.

2.1 Some classical solutions for a half–space

Here we refer to classical results (see, for example, Westergaard (1952) [13]) for the linear elasticity problems in a half–space subjected to a concentrated force applied on the boundary or at an internal point.

2.1.1 Boussinesq–Cerruti's tensor

Let the displacement vector $\mathbf{u}$ satisfy the following boundary value problem

$$\mathbf{L}(\partial_{\mathbf{x}})\mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^3_- = \{(x_1, x_2, x_3) : x_3 < 0\}, \tag{2.1.1}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}; \mathbf{x}) = \mathbf{P}\delta(\mathbf{x}'), \quad x_3 = 0, \ \mathbf{x}' = (x_1, x_2). \tag{2.1.2}$$

where $\mathbf{P} = (P_1, P_2, P_3)^t$ is a constant vector, $\boldsymbol{\sigma}^{(3)} = (\sigma_{13}, \sigma_{23}, \sigma_{33})^t$. We seek a solution which decays at infinity. This field can be represented as a linear combination

$$\mathbf{u} = \sum_{j=1}^{3} P_j \boldsymbol{\mathcal{B}}^{(j)},$$

of the Boussinesq solution $\boldsymbol{\mathcal{B}}^{(3)}$, corresponding to a normal force, and of the Cerruti solutions $\boldsymbol{\mathcal{B}}^{(1)}$, $\boldsymbol{\mathcal{B}}^{(2)}$ for a tangential force. We have no intention to perform the derivation of these solutions. We just present the answer with the reference to the book by Westergaard [13].

The column $\boldsymbol{\mathcal{B}}^{(j)}$, $j = 1, 2, 3$, has components

$$\mathcal{B}_i^{(j)}(\mathbf{x}) = \frac{1}{4\pi\mu R}\left[\frac{x_i x_j}{R^2} + 2(1-\nu)\delta_{ij} - (1-2\nu)p_i^{(j)}(\mathbf{x})\right],$$

where

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

is the Poisson ratio, $R^2 = x_1^2 + x_2^2 + x_3^2$, and

$$\begin{aligned}
p_3^{(3)} &= 0, \\
p_j^{(3)}(\mathbf{x}) &= -p_3^{(j)}(\mathbf{x}) = -\frac{x_j}{R - x_3}, j = 1, 2, \\
p_2^{(1)}(\mathbf{x}) &= p_1^{(2)}(\mathbf{x}) = \frac{x_1 x_2}{(R - x_3)^2}, \\
p_j^{(j)}(\mathbf{x}) &= -\frac{x_3}{R - x_3} + \frac{x_j^2}{(R - x_3)^2}, \ j = 1, 2.
\end{aligned}$$

The matrix $\boldsymbol{\mathcal{B}}$ with the columns $\boldsymbol{\mathcal{B}}^{(i)},\ i=1,2,3$, represents the well–known Boussinesq–Cerutti tensor.

It should be mentioned that the fields

$$\partial_{x_1}\boldsymbol{\mathcal{B}}^{(3)},\ \partial_{x_2}\boldsymbol{\mathcal{B}}^{(3)},\ \frac{1}{2}(\partial_{x_1}\boldsymbol{\mathcal{B}}^{(2)}-\partial_{x_2}\boldsymbol{\mathcal{B}}^{(1)}),$$

correspond to concentrated moments applied at the boundary of the half–space, and the fileds

$$\partial_{x_1}\boldsymbol{\mathcal{B}}^{(1)},\ \partial_{x_2}\boldsymbol{\mathcal{B}}^{(2)},\ \frac{1}{2}(\partial_{x_1}\boldsymbol{\mathcal{B}}^{(2)}+\partial_{x_2}\boldsymbol{\mathcal{B}}^{(1)}),$$

correspond to the concentrated expansion centres and the shear centre respectively.

2.1.2 Mindlin's tensor

Let a concentrated force $\mathbf{P}=(P_1,P_2,P_3)^t$ act at an internal point $\mathbf{x}_0=(0,0,y_3),\ y_3<0$, of the half–space. We are looking for a solution of the boundary value problem

$$\mathbf{L}(\partial_{\mathbf{x}})\mathbf{u}(\mathbf{x})+\mathbf{P}\delta(\mathbf{x}-\mathbf{x}_0)=0,\quad \mathbf{x}\in\mathbb{R}^3_-, \tag{2.1.3}$$

$$\boldsymbol{\sigma}^{(3)}(\mathbf{u};\mathbf{x})=0,\quad x_3=0, \tag{2.1.4}$$

which vanishes at infinity. This problem was solved by Mindlin (1936) [14], and the displacement vector $\mathbf{u}$ admits the representation

$$\mathbf{u}=\sum_{j=1}^{3}P_j\boldsymbol{\mathcal{M}}^{(j)},$$

where $\boldsymbol{\mathcal{M}}^{(j)},\ j=1,2$, correspond to a horisontal force applied at $\mathbf{x}_0$, and $\boldsymbol{\mathcal{M}}^{(3)}$ gives the displacement field for the case of a vertical force. The columns $\boldsymbol{\mathcal{M}}^{(j)},\ j=1,2,3$, are given by

$$\mathcal{M}_i^{(j)}(\mathbf{x}',x_3,y_3)=\frac{1}{16\pi\mu(1-\nu)}\Big[\frac{\hat{x}_i\hat{x}_j}{R_1^3}+(3-4\nu)\frac{\hat{x}_i\hat{x}_j}{R_2^3}+(\frac{1}{R_2}+\frac{3-4\nu}{R_1})\delta_{ij}+t_i^{(j)}(\mathbf{x}',x_3,y_3)\Big],$$

where $\mathbf{x}'=(x_1,x_2)$, $R_1=(x_1^2+x_2^2+(x_3-y_3)^2)^{1/2}$, $R_2=(x_1^2+x_2^2+(x_3+y_3)^2)^{1/2}$, $\hat{x}_i=x_i,\ i=1,2,\ \hat{x}_3=x_3-y_3$, and the terms $t_i^{(j)}$ have the form

$$\begin{aligned}
t_j^{(3)} &= -t_3^{(j)}=\frac{6y_3x_3(x_3+y_3)x_j}{R_2^5}+\frac{4(1-\nu)(1-2\nu)x_j}{R_2(R_2-x_3-y_3)},\ j=1,2,\\
t_3^{(3)} &= \frac{4(1-\nu)(1-2\nu)}{R_2}-\frac{2y_3x_3}{R_2^3}\Big(1-\frac{3(x_3+y_3)^2}{R_2^2}\Big),\\
t_j^{(j)} &= \frac{4(1-\nu)(1-2\nu)}{R_2-x_3-y_3}\Big(1-\frac{x_j^2}{R_2(R_2-x_3-y_3)}\Big)+\frac{2y_3x_3}{R_2^3}\Big(1-\frac{3x_j^2}{R_2^2}\Big),\ j=1,2,\\
t_1^{(2)} &= t_2^{(1)}=-\frac{4(1-\nu)(1-2\nu)x_1x_2}{R_2(R_2-x_3-y_3)^3}-\frac{6y_3x_1x_2x_3}{R_2^5}.
\end{aligned}$$

The matrix function $\boldsymbol{\mathcal{M}}=\boldsymbol{\mathcal{M}}(\mathbf{x}',x_3,y_3)$ with the columns $\boldsymbol{\mathcal{M}}^{(i)},\ i=1,2,3$, represents the Mindlin tensor.

One can see that

$$\boldsymbol{\mathcal{M}}^t(\mathbf{x}'-\mathbf{y}',x_3,y_3)=\boldsymbol{\mathcal{M}}(\mathbf{y}'-\mathbf{x}',y_3,x_3). \tag{2.1.5}$$

The following statement gives a relation between the Boussinesq–Cerutti tensor, the Mindlin tensor and their derivatives, as $y_3\to 0$.

Lemma 2.1 *For any multi–index $\alpha = (\alpha_1, \alpha_2)$ and a non–negative integer k the following formula holds*

$$\partial^\alpha_{\mathbf{y}'} \partial^k_{y_3} \boldsymbol{\mathcal{M}}(\mathbf{x}' - \mathbf{y}', x_3, y_3)|_{\mathbf{y}'=0,\ y_3=0} = \sum_{|\beta|=k} (-\partial_{\mathbf{x}'})^\alpha \partial^\beta_{\mathbf{x}'} \boldsymbol{\mathcal{B}}(\mathbf{x}', x_3) \mathbf{c}_\beta, \tag{2.1.6}$$

where $\mathbf{c}_\beta$ are constant matrices. In particular, $\mathbf{c}_{(0,0)}$ is the identity matrix, and

$$\mathbf{c}_{(1,0)} = \begin{pmatrix} 0 & 0 & \lambda(2\mu+\lambda)^{-1} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathbf{c}_{(0,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda(2\mu+\lambda)^{-1} \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. (i) $k = 0$. Since

$$\boldsymbol{\mathcal{M}}(\mathbf{x}', x_3, 0) = \boldsymbol{\mathcal{B}}(\mathbf{x}', x_3), \tag{2.1.7}$$

we obtain (2.1.6) for $k = 0$.

(ii) $k = 1$. We represent the boundary operator of tractions $\boldsymbol{\sigma}^{(3)}$ in the form

$$\boldsymbol{\sigma}^{(3)}(\mathbf{u}; \mathbf{x}) = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+2\mu \end{pmatrix} (\partial_{x_3} + \mathbb{A}_1 \partial_{x_1} + \mathbb{A}_2 \partial_{x_2}) \mathbf{u}(\mathbf{x}),$$

where

$$\mathbb{A}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \lambda(2\mu+\lambda)^{-1} & 0 & 0 \end{pmatrix}, \ \mathbb{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \lambda(2\mu+\lambda)^{-1} & 0 \end{pmatrix}.$$

Relation (2.1.4) yields

$$(\partial_{y_3} + \mathbb{A}_1 \partial_{y_1} + \mathbb{A}_2 \partial_{y_2}) \boldsymbol{\mathcal{M}}(\mathbf{y}' - \mathbf{x}', y_3, x_3)|_{y_3=0} = 0.$$

Using (2.1.5) we obtain

$$\partial^\alpha_{\mathbf{y}'} \partial_{y_3} \boldsymbol{\mathcal{M}}(\mathbf{x}' - \mathbf{y}', x_3, y_3)|_{y_3=0} = (-\partial_{\mathbf{x}'})^\alpha \Big(\partial_{x_1} \boldsymbol{\mathcal{B}}(\mathbf{x}' - \mathbf{y}', x_3) \mathbb{A}_1^t + \partial_{x_2} \boldsymbol{\mathcal{B}}(\mathbf{x}' - \mathbf{y}', x_3) \mathbb{A}_2^t \Big), \tag{2.1.8}$$

which gives (2.1.6) for $k = 1$.

(iii) $k > 1$. The differential operator $\mathbf{L}$ can be represented in the form

$$\mathbf{L}(\partial_{\mathbf{x}}) = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+2\mu \end{pmatrix} \Big(\partial^2_{x_3} + \sum_{j=1}^2 \mathbb{A}_{j3} \partial_{x_j} \partial_{x_3} + \sum_{j=1}^2 \sum_{k=1}^2 \mathbb{A}_{jk} \partial_{x_j} \partial_{x_k} \Big),$$

where $\mathbb{A}_{j3}$, $\mathbb{A}_{jk}$ are constant matrices. Due to (2.1.5) and (2.1.3),

$$\partial^2_{y_3} \boldsymbol{\mathcal{M}}(\mathbf{x}' - \mathbf{y}', x_3, y_3)$$

$$= \sum_{j=1}^2 \partial_{x_j} \partial_{y_3} \boldsymbol{\mathcal{M}}(\mathbf{x}' - \mathbf{y}', x_3, y_3) \mathbb{A}^t_{j3} - \sum_{j=1}^2 \sum_{k=1}^2 \partial_{x_j} \partial_{x_k} \boldsymbol{\mathcal{M}}(\mathbf{x}' - \mathbf{y}', x_3, y_3) \mathbb{A}^t_{jk}. \tag{2.1.9}$$

Using (2.1.9) m times we deduce

$$\partial^{2m}_{y_3} \boldsymbol{\mathcal{M}}(\mathbf{x}' - \mathbf{y}', x_3, y_3)$$

$$= \sum_{|\beta|=2m-1} \partial^{\beta}_{\mathbf{x}'} \partial_{y_3} \boldsymbol{\mathcal{M}}(\mathbf{x}'-\mathbf{y}', x_3, y_3)\mathbb{A}^t_{\beta 3} - \sum_{|\gamma|=2m} \partial^{\gamma}_{\mathbf{x}'} \boldsymbol{\mathcal{M}}(\mathbf{x}'-\mathbf{y}', x_3, y_3)\mathbb{A}^t_{\gamma}, \tag{2.1.10}$$

where $\mathbb{A}_{\beta 3}$, $\mathbb{A}_{\gamma}$ are constant matrices. Then relations (2.1.10), (2.1.8) and (2.1.7) yield (2.1.6) for even k.

For odd values of k one can differentiate (2.1.10) with respect to y_3 and then use (2.1.9) to expand the term with $\partial^2_{y_3}\boldsymbol{\mathcal{M}}$. As a result we obtain (2.1.6). □

The formula (2.1.6) implies the following relation between Mindlin's and Boussinesq's tensors

$$\frac{1}{2}(\nabla_{\mathbf{y}} \times \boldsymbol{\mathcal{M}}^t(\mathbf{x},\mathbf{y})|_{\mathbf{y}=0})^t$$

$$= \Big[-\partial_{x_2}\boldsymbol{\mathcal{B}}^{(3)}(\mathbf{x}), \partial_{x_1}\boldsymbol{\mathcal{B}}^{(3)}(\mathbf{x}), \frac{1}{2}\Big(\partial_{x_2}\boldsymbol{\mathcal{B}}^{(1)}(\mathbf{x}) - \partial_{x_1}\boldsymbol{\mathcal{B}}^{(2)}(\mathbf{x})\Big)\Big]. \tag{2.1.11}$$

We recall that the right–hand side represents the displacement field corresponding to the concentrated moment applied at the origin.

2.2 Special solutions for a bounded two–dimensional domain

Here we introduce solutions of some boundary value problems for the Laplace operator on the two–dimensional cross–section g of a cylinder which prove to be useful for the description of elastic fields within the cylinder. These solutions play the role of potential functions for the shear stress components, and for the detailed description we refer to the book by Lurie (1970) [17], Chapter 6.

It is assumed that the origin coincides with the inertia centre of g, and the coordinate axes OX_1, OX_2 are parallel to the principal inertia axes, i.e.

$$\int_g X_j d\mathbf{X}' = 0,\ j=1,2;\ \ \int_g X_1 X_2 d\mathbf{X}' = 0. \tag{2.2.1}$$

An important place in the further analysis will be taken by the function φ given as a solution of the Neumann boundary value problem

$$\Delta\varphi(\mathbf{X}') = 0,\ \ \mathbf{X}' \in g, \tag{2.2.2}$$

$$\partial_n\varphi(\mathbf{X}') = X_2 n_1 - X_1 n_2,\ \mathbf{X}' \in \partial g. \tag{2.2.3}$$

Obviously,

$$\int_{\partial g} (X_2 n_1 - X_1 n_2) ds = 0,$$

and, therefore, the problem (2.2.2), (2.2.3) is solvable.

To provide the uniqueness, we assume that

$$\int_g \varphi(\mathbf{X}') d\mathbf{X}' = 0.$$

The function φ is used for the representation of the displacement vector within a cylindrical bar under a torsion load

$$\mathbf{u}(\mathbf{X}) = \beta(-X_2X_3, X_1X_3, \varphi(X_1,X_2)),$$

where β is a twisting per unit length.

It should be mentioned that φ is closely related to the Prandtl potential function Φ for the shear stress components within an elastic rod subjected to a torsion load. Namely,

$$\Phi(\mathbf{X}') = \psi(\mathbf{X}') - \frac{1}{2}(X_1^2 + X_2^2),$$

where ψ is the conjugate function to φ.

Also, in Section 2.3 we shall use the functions χ_1, χ_2 which satisfy the following Neumann boundary value problems

$$\Delta\chi_1(\mathbf{X}') = 0, \quad \mathbf{X}' \in g, \tag{2.2.4}$$

$$\partial_n\chi_1(\mathbf{X}') = X_1^2 n_1 + \frac{\lambda}{\lambda+\mu} X_1 X_2 n_2, \ \mathbf{X}' \in \partial g, \tag{2.2.5}$$

and

$$\Delta\chi_2(\mathbf{X}') = 0, \quad \mathbf{X}' \in g, \tag{2.2.6}$$

$$\partial_n\chi_2(\mathbf{X}') = X_2^2 n_2 + \frac{\lambda}{\lambda+\mu} X_1 X_2 n_1, \ \mathbf{x}' \in \partial g. \tag{2.2.7}$$

One can verify directly that

$$\int_{\partial g} \partial_n \chi_1 ds = \int_{\partial g} \partial_n \chi_2 ds = 0,$$

and, hence, the above formulated Neumann boundary value problems are solvable.

To provide the uniqueness we assume that

$$\int_g \chi_i(\mathbf{X}') d\mathbf{X}' = 0, \ i = 1, 2.$$

In the engineering literature χ_1, χ_2 are used as potential functions for the shear stress components within an elastic rod subjected to a bending load (see Lurie [17], Chapter 6).

Example. Consider the elliptical domain g bounded by

$$\frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = 1,$$

where a and b are non–zero constants. In this case the functions φ and χ_i, $i = 1, 2$, admit the representations

$$\varphi = \frac{b^2 - a^2}{a^2 + b^2} X_1 X_2,$$

$$\chi_1 = \frac{(1-2\nu)a^2}{3(3a^2+b^2)}(X_1^3 - 3X_1X_2^2) + \frac{b^2 + 2a^2(1+\nu)}{3a^2+b^2} a^2 X_1,$$

$$\chi_2 = \frac{(1-2\nu)b^2}{3(3b^2+a^2)}(X_2^3 - 3X_2X_1^2) + \frac{a^2 + 2b^2(1+\nu)}{3b^2+a^2} b^2 X_2.$$

2.3 Some solutions of linear elasticity in an infinite cylinder

Here we describe polynomial (in the axial direction) solutions of the homogeneous elasticity equations in an infinite cylinder with homogeneous traction boundary conditions. We refer to the papers by Arutyunyan, Movchan and Nazarov (1987) [15] and by Shkalikov and Shkred (1991) [16].

Let $\Pi = \{\mathbf{X} \in \mathbb{R}^3 : X_3 \in \mathbb{R}, (X_1, X_2) \in g \subset \mathbb{R}^2\}$, where g is a domain with the Lipschitz boundary. Also, we shall use the notation $\mathbf{X}' = (X_1, X_2)$.

As in the previous section, it is assumed that the coordinate axes OX_1, OX_2 coincide with the principal inertia axes of g.

We seek vector functions, polynomial in X_3, which satisfy the homogeneous boundary value problem

$$\mathbf{L}(\partial_X)\mathbf{\Psi}(\mathbf{X}) = 0, \ \mathbf{X} \in \Pi, \tag{2.3.1}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{\Psi}; \mathbf{X}) = = 0, \ \mathbf{X} \in \partial\Pi, \tag{2.3.2}$$

where $\mathbf{n} = (n_1(\mathbf{X}'), n_2(\mathbf{X}'), 0)$ is the outward unit normal on $\partial\Pi$. We use the following representation for the operators of the problem

$$\mathbf{L}(\partial_X)\mathbf{\Psi} = \mathbf{L}_0(\partial_{X'})\mathbf{\Psi} + \mathbf{L}_1(\partial_{X'})\partial_{X_3}\mathbf{\Psi} + \mathbf{L}_2\partial^2_{X_3}\mathbf{\Psi}, \tag{2.3.3}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{\Psi}; \mathbf{X}) = \mathbf{T}_0(\mathbf{n}, \partial_{X'})\mathbf{\Psi} + \mathbf{T}_1(\mathbf{n})\partial_{X_3}\mathbf{\Psi}. \tag{2.3.4}$$

Here

$$\mathbf{L}_0(\partial_{X'}) = \begin{pmatrix} (2\mu+\lambda)\partial^2_{X_1} & (\lambda+\mu)\partial_{X_1}\partial_{X_2} & 0 \\ (\lambda+\mu)\partial_{X_1}\partial_{X_2} & (2\mu+\lambda)\partial^2_{X_2} + \mu\partial^2_{X_1} & 0 \\ 0 & 0 & \mu(\partial^2_{X_1} + \partial^2_{X_2}) \end{pmatrix}; \tag{2.3.5}$$

$$\mathbf{L}_1(\partial_{X'}) = (\lambda+\mu)\begin{pmatrix} 0 & 0 & \partial_{X_1} \\ 0 & 0 & \partial_{X_2} \\ \partial_{X_1} & \partial_{X_2} & 0 \end{pmatrix}; \ \mathbf{L}_2 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (2\mu+\lambda) \end{pmatrix}; \tag{2.3.6}$$

$$\mathbf{T}_0(\mathbf{n}, \partial_{X'}) = n_1\begin{pmatrix} (2\mu+\lambda)\partial_{X_1} & \lambda\partial_{X_2} & 0 \\ \mu\partial_{X_2} & \mu\partial_{X_1} & 0 \\ 0 & 0 & \mu\partial_{X_1} \end{pmatrix} + n_2\begin{pmatrix} \mu\partial_{X_2} & \mu\partial_{X_1} & 0 \\ \lambda\partial_{X_1} & (2\mu+\lambda)\partial_{X_2} & 0 \\ 0 & 0 & \mu\partial_{X_2} \end{pmatrix}; \tag{2.3.7}$$

$$\mathbf{T}_1(\mathbf{n}) = \begin{pmatrix} 0 & 0 & \lambda n_1 \\ 0 & 0 & \lambda n_2 \\ \mu n_1 & \mu n_2 & 0 \end{pmatrix}. \tag{2.3.8}$$

To obtain required polynomial solutions of (2.3.1), (2.3.2), we consider the spectral problem

$$\mathbf{L}(\partial_{X'}, \Lambda)\boldsymbol{\phi}(X') = 0, \ \mathbf{X}' \in g, \tag{2.3.9}$$

$$\Big(\mathbf{T}_0(\mathbf{n}, \partial_{X'}) + \Lambda\mathbf{T}_1(\mathbf{n})\Big)\boldsymbol{\phi}(X') = 0, \ \mathbf{X}' \in \partial g. \tag{2.3.10}$$

This spectral problem has the eigenvalue $\Lambda = 0$ of the geometric multiplicity 4, and the corresponding eigenvectors are

$$\boldsymbol{\phi}^{(k,0)} = \mathbf{e}^{(k)}, \ k = 1, 2, 3; \tag{2.3.11}$$

$$\boldsymbol{\phi}^{(4,0)} = -X_2\mathbf{e}^{(1)} + X_1\mathbf{e}^{(2)}. \tag{2.3.12}$$

The fields (2.3.11), (2.3.12) are the rigid body translations and the rotation about the OX_3 axis, respectively. The rank of $\boldsymbol{\phi}^{(k,0)}$ is m_k; $m_1 = m_2 = 4,\ m_3 = m_4 = 2$.

The generalized eigenvectors $\boldsymbol{\phi}^{(k,j)},\ k = 1, 2, 3, 4; j = 1, \ldots, m_k - 1$, satisfy the boundary value problems

$$\begin{aligned} \mathbf{L}_0(\partial_{X'})\boldsymbol{\phi}^{(k,j)} &= -(\mathbf{L}_1(\partial_{X'})\boldsymbol{\phi}^{(k,j-1)} + \mathbf{L}_2\boldsymbol{\phi}^{(k,j-2)}), & (2.3.13)\\ \mathbf{T}_0(\mathbf{n}, \partial_{X'})\boldsymbol{\phi}^{(k,j)} &= -\mathbf{T}_1(\mathbf{n})\boldsymbol{\phi}^{(k,j-1)}. & (2.3.14) \end{aligned}$$

Functions with negative indices are assumed to be zero. The vector functions $\boldsymbol{\phi}^{(k,1)},\ k = 3, 4$, have the form

$$\begin{aligned} \boldsymbol{\phi}^{(3,1)} &= -\frac{\lambda}{2(\mu+\lambda)}(X_1\mathbf{e}^{(1)} + X_2\mathbf{e}^{(2)}), & (2.3.15)\\ \boldsymbol{\phi}^{(4,1)} &= \varphi(\mathbf{X}')\mathbf{e}^{(3)}, & (2.3.16) \end{aligned}$$

where φ, introduced in Section 2.2, satisfies the Neumann boundary value problem (2.2.2), (2.2.3).

The generalized eigenvectors $\boldsymbol{\phi}^{(j,k)},\ k = 1, 2, 3$, for $j = 1, 2$, are

$$\begin{aligned} \boldsymbol{\phi}^{(j,1)} &= -X_j\mathbf{e}^{(3)}, & (2.3.17)\\ \boldsymbol{\phi}^{(1,2)} &= \frac{\lambda}{4(\lambda+\mu)}\{(2X_2X_1 + k_1X_1)\mathbf{e}^{(2)} + (X_1^2 - X_2^2 - k_1X_2)\mathbf{e}^{(1)}\}, & (2.3.18)\\ \boldsymbol{\phi}^{(2,2)} &= \frac{\lambda}{4(\lambda+\mu)}\{(2X_2X_1 - k_2X_2)\mathbf{e}^{(1)} + (X_2^2 - X_1^2 + k_2X_1)\mathbf{e}^{(2)}\}, & (2.3.19)\\ \boldsymbol{\phi}^{(j,3)} &= \Big(\frac{4\mu+3\lambda}{12(\lambda+\mu)}X_j^3 + \frac{\lambda}{4(\lambda+\mu)}\frac{X_1^2X_2^2}{X_j} - \chi_j(\mathbf{X}') + k_j\frac{\lambda}{4(\lambda+\mu)}\varphi(\mathbf{X}')\Big)\mathbf{e}^{(3)}, & (2.3.20) \end{aligned}$$

where $k_1,\ k_2$ are constants which will be chosen to satisfy the biorthogonality conditions (2.3.23); the functions $\chi_j,\ j = 1, 2$, are the solutions of (2.2.4)–(2.2.7).

The set $\{\boldsymbol{\phi}^{(s,j)},\ s = 1, 2, 3, 4; j = 1, \ldots, m_s - 1\}$ gives the canonical system of Jordan's chains for any values of the constants $k_1,\ k_2$.

Any X_3–polynomial solution of (2.3.1), (2.3.2) can be represented as a linear combination of the vector functions

$$\sum_{j=0}^{l}\frac{X_3^j}{j!}\boldsymbol{\phi}^{(k,l-j)}(\mathbf{X}'), k = 1, 2, 3, 4;\ l = 0, \ldots, m_k - 1. \quad (2.3.21)$$

Expression (2.3.21) yields 12 linearly independent vector–valued functions

$$\begin{aligned} &\boldsymbol{\psi}^{(i)} = \mathbf{e}^{(i)},\ i = 1, 2, 3;\ \boldsymbol{\psi}^{(4)}(\mathbf{X}) = (-X_2, X_1, 0)^t,\\ &\boldsymbol{\psi}^{(5)}(\mathbf{X}) = (X_3, 0, -X_1)^t,\ \boldsymbol{\psi}^{(6)}(\mathbf{X}) = (0, X_3, -X_2)^t,\\ &\boldsymbol{\xi}^{(i)}(\mathbf{X}) = \frac{1}{6}X_3^3\mathbf{e}^{(i)} + (-\frac{1}{2}X_3^2X_i + \frac{4\mu+3\lambda}{12(\lambda+\mu)}X_i^3\\ &+\frac{\lambda}{4(\lambda+\mu)}\frac{X_1^2X_2^2}{X_i} - \chi_i(\mathbf{X}') + k_i\frac{\lambda}{4(\lambda+\mu)}\varphi(\mathbf{X}'))\mathbf{e}^{(3)} + X_3\boldsymbol{\phi}^{(i,2)}(\mathbf{X}'),\ i = 1, 2,\\ &\boldsymbol{\xi}^{(3)}(\mathbf{X}) = -\lambda[2(\lambda+\mu)]^{-1}(X_1\mathbf{e}^{(1)} + X_2\mathbf{e}^{(2)}) + X_3\mathbf{e}^{(3)}, \end{aligned}$$

$$\boldsymbol{\xi}^{(4)}(\mathbf{X}) = (-X_2X_3, X_1X_3, \varphi(\mathbf{X}'))^t,$$

$$\boldsymbol{\xi}^{(5)}(\mathbf{X}) = (\frac{1}{2}X_3^2, 0, -X_3X_1)^t + \boldsymbol{\phi}^{(1,2)}(\mathbf{X}'),$$

$$\boldsymbol{\xi}^{(6)}(\mathbf{X}) = (0, \frac{1}{2}X_3^2, -X_3X_2)^t + \boldsymbol{\phi}^{(2,2)}(\mathbf{X}').$$

It can be also mentioned that any X_3–polynomial solution of (2.3.1), (2.3.2) can be given as a linear combination of the vector–valued functions $\boldsymbol{\xi}^{(i)}$, $i = 1, 2, 3, 4$, and their derivatives with respect to X_3.

The vectors $\boldsymbol{\psi}^{(i)}$, $i = 1, 2, \ldots, 6$, are the rigid body displacements, whereas the vector polynomials $\boldsymbol{\xi}^{(i)}$, $i = 1, 2, \ldots, 6$ have infinite elastic energy.

Let the constants k_1, k_2 from (2.3.18), (2.3.19) satisfy

$$\frac{\lambda}{4(\lambda+\mu)} k_j \int_g (\|\mathbf{X}'\|^2 - \|\nabla\varphi\|^2) d\mathbf{X}' = \int_g \Big\{ X_1 \partial_{X_2}\chi_j - X_2 \partial_{X_1}\chi_j \pm \frac{\mu}{\lambda+\mu} X_j X_1 X_2 \Big\} d\mathbf{X}', \ j = 1, 2, \tag{2.3.22}$$

where the sign "+" corresponds to $j = 1$, and "-" is related to the case $j = 2$.

The coefficient near k_j in (2.3.22) is positive. It follows from

$$\int_g \nabla\varphi \cdot (X_2\mathbf{e}^{(1)} - X_1\mathbf{e}^{(2)}) d\mathbf{X}' = \int_g \|\nabla\varphi\|^2 d\mathbf{X}',$$

and, therefore,

$$\int_g (\|\mathbf{X}'\|^2 - \|\nabla\varphi\|^2) d\mathbf{X}' = \int_g \|\nabla\varphi - X_2\mathbf{e}^{(1)} + X_1\mathbf{e}^{(2)}\|^2 d\mathbf{X}' > 0.$$

Lemma 2.2 *If k_1, k_2 satisfy (2.3.22) then the following biorthogonality conditions hold*

$$\Big(\int_g \boldsymbol{\sigma}^{(3)}(\boldsymbol{\xi}^{(k)}; \mathbf{X}) \cdot \boldsymbol{\psi}^{(i)}(\mathbf{X}) d\mathbf{X}'\Big)_{k,i=1}^6 = diag\{-D_1, -D_2, D_3, D_4, D_1, D_2\}, \tag{2.3.23}$$

where $\boldsymbol{\sigma}^{(3)} = (\sigma_{13}, \sigma_{23}, \sigma_{33})^t$, the integrals in (2.3.23) do not depend on X_3, and the elements of the diagonal matrix are

$$D_k = \mu \frac{2\mu + 3\lambda}{\lambda + \mu} \int_g X_k^2 d\mathbf{X}', \ k = 1, 2,$$

$$D_3 = (\lambda+\mu)^{-1}\mu(2\mu + 3\lambda) mes_2 g,$$

$$D_4 = \mu \int_g \|\nabla\varphi - X_2\mathbf{e}^{(1)} + X_1\mathbf{e}^{(2)}\|^2 d\mathbf{X}'.$$

Proof. First, applying the Betti formula to the vector functions $\boldsymbol{\xi}^{(k)}$, $\boldsymbol{\psi}^{(i)}$, $i, k = 1, \ldots, 6$, in the region $\{\mathbf{x} \in \Pi, a < X_3 < b\}$ for some finite a, b, we obtain that the integral in the left–hand side (2.3.23) does not depend on X_3.

Since

$$\partial_{X_3}\boldsymbol{\xi}^{(1)} = \boldsymbol{\xi}^{(5)} \text{ and } \partial_{X_3}\boldsymbol{\xi}^{(2)} = \boldsymbol{\xi}^{(6)},$$

the relations (2.3.23) for $k = 5, 6$, follow from those obtained for $k = 1, 2$.

Now, we prove (2.3.23) for k=1. We have

$$\sigma_{13}(\boldsymbol{\xi}^{(1)};\mathbf{X}) = \mu\Big\{X_1^2 - \partial_{X_1}\chi_1 + \frac{\lambda k_1}{4(\lambda+\mu)}(\partial_{X_1}\varphi - X_2)\Big\},$$

$$\sigma_{23}(\boldsymbol{\xi}^{(1)};\mathbf{X}) = \mu\Big\{\frac{\lambda}{\lambda+\mu}X_1X_2 - \partial_{X_2}\chi_1 + \frac{\lambda k_1}{4(\lambda+\mu}(\partial_{X_2}\varphi + X_1)\Big\},$$

$$\sigma_{33}(\boldsymbol{\xi}^{(1)};\mathbf{X}) = -\mu\frac{2\mu+3\lambda}{\lambda+\mu}X_1X_3.$$

Since χ_1 satisfies the problem (2.2.4), (2.2.5),

$$\int_g \partial_{X_1}\chi_1 d\mathbf{X}' = \int_{\partial g}(\partial_n\chi_1)X_1 ds = \frac{4\mu+3\lambda}{\lambda+\mu}\int_g X_1^2 d\mathbf{X}',$$

$$\int_g \partial_{X_2}\chi_1 d\mathbf{X}' = \int_{\partial g}(\partial_n\chi_1)X_2 ds = 0.$$

Similarly,

$$\int_g \partial_{X_i}\varphi d\mathbf{X}' = 0, \ \ i=1,2.$$

Then, direct calculations give (2.3.23) for $k=1, \ i=1,2,3,5,6$. For $i=4$ the relation (2.3.23) is valid due to the choice of the coefficient k_1 (see (2.3.22)). The proof of (2.3.23) for $k=2$ follows the same pattern. The verification of (2.3.23) for k=3,4 is trivial. □

The quantities $D_j, \ j=1,2,3,4$, occur in the theory of elastic rods and can be interpreted as stiffness coefficients , where $D_1, \ D_2$ are related to transversal displacements in the directions of the principal inertia axes of the cross–section, and $D_3, \ D_4$ correspond to the longitudinal displacement and axial rotation, respectively.

For example, for an elastic cylinder with a circular cross–section of the radius a we have

$$D_1 = D_2 = \frac{\mu\pi}{4}\frac{2\mu+3\lambda}{\lambda+\mu}a^4, \quad D_3 = \frac{\mu(2\mu+3\lambda)}{\lambda+\mu}\pi a^2, \quad D_4 = \frac{\mu\pi a^4}{2}.$$

Further, we shall need the following consequence of Lemma 2.2.

Lemma 2.3 *The biorthogonality relations hold*

$$\int_g(\mathbf{L}_1\boldsymbol{\phi}^{(k,m_k-1)} + \mathbf{L}_2\boldsymbol{\phi}^{(k,m_k-2)})\cdot\boldsymbol{\phi}^{(j,0)}d\mathbf{X}'$$
$$-\int_{\partial g}\mathbf{T}_1\boldsymbol{\phi}^{(k,m_k-1)}\cdot\boldsymbol{\phi}^{(j,0)}dl = \mp\delta_{kj}\mathcal{D}_k, \ j=1,2,3,4. \tag{2.3.24}$$

where "-" corresponds to $k=1,2$, and "+" to $k=3,4$.

Proof. Let $\eta=\eta(X_3)$ be a smooth function which is equal to 1 for large X_3 and 0 for $X_3<0$. First, we show that

$$\Big\{\int_\Pi \mathbf{L}(\partial_\mathbf{X})(\eta\boldsymbol{\xi}^{(j)})\cdot\boldsymbol{\psi}^{(k)}d\mathbf{X} - \int_{\partial\Pi}\boldsymbol{\sigma}^{(n)}(\eta\boldsymbol{\xi}^{(j)};\mathbf{X})\cdot\boldsymbol{\psi}^{(k)}ds\Big\}_{j,k=1}^6$$
$$= \mathrm{diag}\{-D_1,-D_2,D_3,D_4,D_1,D_2\}. \tag{2.3.25}$$

Indeed, applying the Betti formula to $\eta\boldsymbol{\xi}^{(j)}$ and $\boldsymbol{\psi}^{(k)}$ in the domain $\{\mathbf{X} \in \Pi : 0 < X_3 < c\}$, where c is a large positive number, we obtain that the left–hand side (2.3.25) is equal to

$$\Big\{ \int_g \boldsymbol{\sigma}^{(3)}(\boldsymbol{\xi}^{(j)}; \mathbf{X}) \cdot \boldsymbol{\psi}^{(k)} d\mathbf{X}'|_{X_3=c} \Big\}_{j,k=1}^{6}.$$

Hence, (2.3.25) follows from (2.3.23).

Using the representations (2.3.3) and (2.3.4) one has

$$\int_\Pi \mathbf{L}(\partial_\mathbf{X})(\eta\boldsymbol{\xi}^{(j)}) \cdot \boldsymbol{\psi}^{(k)} d\mathbf{X} - \int_{\partial\Pi} \boldsymbol{\sigma}^{(n)}(\eta\boldsymbol{\xi}^{(j)}; \mathbf{X}) \cdot \boldsymbol{\psi}^{(k)} ds$$

$$= \int_\Pi \eta'(\mathbf{L}_1\boldsymbol{\xi}^{(j)} + \mathbf{L}_2\partial_{X_3}\boldsymbol{\xi}^{(j)}) \cdot \boldsymbol{\psi}^{(k)} d\mathbf{X} - \int_{\partial\Pi} \eta'(\mathbf{T}_1\boldsymbol{\xi}^{(j)}) \cdot \boldsymbol{\psi}^{(k)} ds, \tag{2.3.26}$$

for $k = 1, 2, 3, 4$. The left–hand side of (2.3.26) does not depend on η. Hence, we can replace η by η_s such that $\eta'_s \to \delta(0)$ as $s \to \infty$. Taking the limit $s \to \infty$, one arrives at

$$\int_g (\mathbf{L}_1\boldsymbol{\xi}^{(j)} + \mathbf{L}_2\partial_{X_3}\boldsymbol{\xi}^{(j)}) \cdot \boldsymbol{\psi}^{(k)} d\mathbf{X}'|_{X_3=0} - \int_{\partial g} (\mathbf{T}_1\boldsymbol{\xi}^{(j)}) \cdot \boldsymbol{\psi}^{(k)} dl|_{X_3=0} = \mp\delta_{kj} D_k, \tag{2.3.27}$$

where "$-$" corresponds to $k = 1, 2$, and "$+$" to $k = 3, 4$. The left–hand sides (2.3.27) and (2.3.24) do coincide. Thus, the proof is complete. $\square$

Remark 2.1. *Similarly to (2.3.25), one can show that*

$$\int_\Pi \mathbf{L}(\partial_\mathbf{X})(\eta\boldsymbol{\psi}^{(j)}) \cdot \boldsymbol{\psi}^{(k)} d\mathbf{X} - \int_{\partial\Pi} \boldsymbol{\sigma}^{(n)}(\eta\boldsymbol{\psi}^{(j)}; \mathbf{X}) \cdot \boldsymbol{\psi}^{(k)} ds = 0, \tag{2.3.28}$$

for $j, k = 1, \dots, 6$.

3 Model problems

The asymptotic description of elastic fields in the multi–structure Ω_ε requires six types of boundary value problems in domains, independent of ε. Here we formulate basic results related to solvability and the asymptotic representation of solutions to these model problems.

3.1 Limit domains

We assign to the multi–structure Ω_ε the following set of domains, independent of ε.

(i) The bounded domain Ω with the Lipschitz boundary (see Fig. 2) corresponds to "the pile cap" of the multi–structure. It is supposed that $\partial\Omega$ includes a plane part containing the junction points $\mathbf{a}^{(k)} = (a_1^{(k)}, a_2^{(k)}, a_3^{(k)})$, $k = 1, \dots, K$.

(ii) The domain G is a union of the half–space $\mathbb{R}^3_- = \{\mathbf{X} : X_3 < 0\}$ and the semi–infinite cylinder Π_+ located in $\mathbb{R}^3_+$

$$\Pi_+ = \{\mathbf{X} : X_3 \geq 0, (X_1, X_2) \in g\},$$

where g is a two–dimensional bounded domain with the Lipschitz boundary (see Fig. 3). This limit domain describes a scaled junction region of the multi–structure, where the junction layer problem will be posed.

(iii) The domain Π_- is the semi–infinite cylinder $\{\mathbf{Y} : Y_3 < 0, (Y_1, Y_2) \in g\}$ which is shown on Fig. 4. The bottom layer will be characterized by a solution of the mixed boundary value problem with the Dirichlet data on the base region S of the boundary $\partial\Pi_-$.

(iv) Asymptotic analysis of the elastic fields in a thin rod will require auxiliary boundary value problems specified on the scaled cross–section g shown on Fig. 5(a).

(v) Certain coefficients of the asymptotic expansions of components of the displacement vector in a thin cylinder satisfy ordinary differential equations on the finite interval (see Fig. 5 (b)).

(vi) Finally, we should use the "skeleton" of the multi–structure, represented by a union of the three–dimensional region Ω and one–dimensional segments connected with Ω at junction points (see Fig. 6). Here we introduce the pile structure model and the corresponding matrix operators.

3.2 The model problem for the three–dimensional body Ω

In this section we consider the boundary value problem for the Lamé system in Ω with traction boundary conditions. We admit that the tractions may be singular at the points $\mathbf{a}^{(i)}, i = 1, \dots, K$ (see Fig. 2).

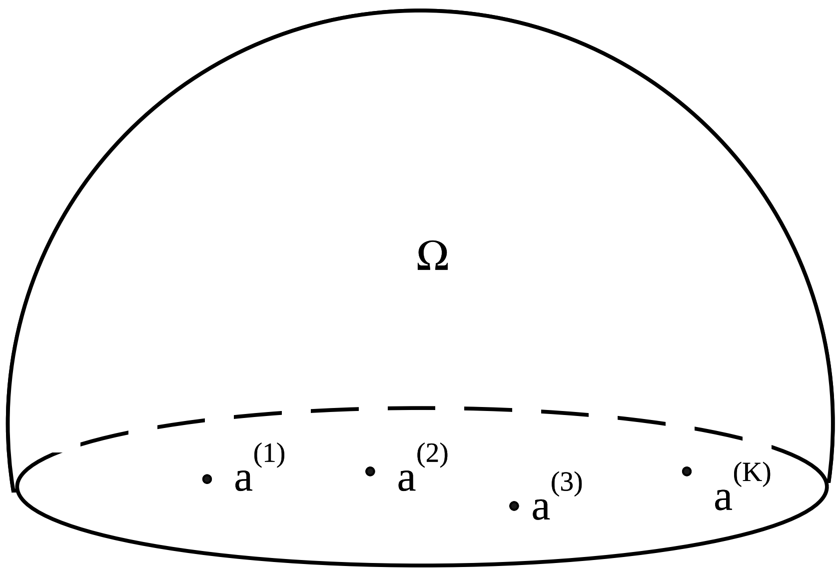

Fig.2 Three–dimensional "cap" of the multi–structure

We seek a vector valued function $\mathbf{u}$ subjected to the Lamé system

$$-\mathbf{L}(\partial_{\mathbf{x}})\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \ \mathbf{x} \in \Omega, \tag{3.2.1}$$

and the traction boundary condition

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u};\mathbf{x}) = \mathbf{p}(\mathbf{x}), \ \mathbf{x} \in \partial\Omega \setminus \cup_i \mathbf{a}_i. \tag{3.2.2}$$

We do not assume that $\mathbf{f}$ and $\mathbf{p}$ have zero principal force and moment vectors, and admit that the displacement $\mathbf{u}$ may have singularities at the points $\mathbf{x} = \mathbf{a}^{(i)}, \ i = 1, \dots, K$.

We seek a solution of (3.2.1), (3.2.2) in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_r(\mathbf{x}) + \sum_{i=1}^{K} \left\{ \boldsymbol{\mathcal{N}}(\mathbf{x},\ \mathbf{a}^{(i)})\mathfrak{F}^{(i)} + \frac{1}{2}(\nabla_y \times \boldsymbol{\mathcal{N}}^t(\mathbf{x},\ \mathbf{y})|_{\mathbf{y}=\mathbf{a}^{(i)}})^t \mathfrak{M}^{(i)} \right\}, \tag{3.2.3}$$

where $\mathbf{u}_r$ is a vector function from the Sobolev space $(H^1(\Omega))^3$; $\mathfrak{F}^{(i)}$, $\mathfrak{M}^{(i)}$ are constant vectors; $\boldsymbol{\mathcal{N}}(\mathbf{x},\mathbf{y})$ is the Green tensor for the Neumann boundary value problem for the Lamé operator in Ω. The representation (3.2.3) corresponds to the setting of the concentrated forces $\mathfrak{F}^{(i)}$ and the concentrated moments $\mathfrak{M}^{(i)}$ at the junction points $\mathbf{a}^{(i)}$, $i = 1, \dots, K$.

Let us recall the definition of the Green tensor $\boldsymbol{\mathcal{N}}(\mathbf{x},\mathbf{y})$ which can be written nicely in the system of coordinates where the axes coincide with the principal inertia axes of the body Ω. The unit basis vectors are denoted by $\mathbf{e}^{(j)}$, $j = 1, 2, 3$. Then columns $\boldsymbol{\mathcal{N}}^{(j)}$ of the tensor $\boldsymbol{\mathcal{N}}$ satisfy the relations

$$-\mathbf{L}(\partial_{\mathbf{x}})\boldsymbol{\mathcal{N}}^{(j)}(\mathbf{x},\mathbf{y}) = (\delta(\mathbf{x}-\mathbf{y}) - \frac{1}{\mathrm{mes}\Omega})\mathbf{e}^{(j)} - J_j^{-1}\mathbf{x} \times (\mathbf{e}^{(j)} \times \mathbf{y}),\ \mathbf{x} \in \Omega, \tag{3.2.4}$$

$$\sigma^{(n)}(\boldsymbol{\mathcal{N}}^{(j)}(\cdot,\mathbf{y});\mathbf{x}) = \mathbf{0},\ \mathbf{x} \in \partial\Omega,\ \ \mathbf{y} \in \Omega, \tag{3.2.5}$$

where J_j, $j = 1, 2, 3$, are the principal inertia moments

$$J_j = \int_\Omega \left(||\mathbf{x}||^2 - x_j^2 \right) d\mathbf{x}. \tag{3.2.6}$$

Problem (3.2.4), (3.2.5) is solvable due to the orthogonality of the right–hand sides to the rigid body displacements. To provide the uniqueness of $\boldsymbol{\mathcal{N}}^{(j)}$, we set the orthogonality conditions

$$\int_\Omega \boldsymbol{\mathcal{N}}^{(j)}(\mathbf{x},\mathbf{y})d\mathbf{x} = \mathbf{0},\ \ \int_\Omega \mathbf{x} \times \boldsymbol{\mathcal{N}}^{(j)}(\mathbf{x},\mathbf{y})d\mathbf{x} = \mathbf{0}.$$

The matrix function $\boldsymbol{\mathcal{N}}$ satisfies the relation

$$\boldsymbol{\mathcal{N}}(\mathbf{x},\mathbf{y}) = \boldsymbol{\mathcal{N}}^t(\mathbf{y},\mathbf{x}).$$

Now, we formulate the result on solvability of the problem (3.2.1)–(3.2.3).

We shall use the following notations for the principal force and moment vectors

$$\mathbb{F}(\mathbf{f},\mathbf{p},\Omega) = \int_\Omega \mathbf{f}(\mathbf{x})d\mathbf{x} + \int_{\partial\Omega} \mathbf{p}(\mathbf{x})ds, \tag{3.2.7}$$

$$\mathbb{M}(\mathbf{f},\mathbf{p},\Omega) = \int_\Omega \mathbf{x} \times \mathbf{f}(\mathbf{x})d\mathbf{x} + \int_{\partial\Omega} \mathbf{x} \times \mathbf{p}(\mathbf{x})ds. \tag{3.2.8}$$

Theorem 3.1 *Let* $\mathbf{f} \in (L_2(\Omega))^3$, $\mathbf{p} \in (L_2(\partial\Omega))^3$, *and* $\mathfrak{F}^{(i)}$, $\mathfrak{M}^{(i)}$, $i = 1, 2, \dots, K$, *be the constant vectors such that*

$$\mathbb{F}(\mathbf{f},\mathbf{p},\Omega) + \sum_{i=1}^{K} \mathfrak{F}^{(i)} = \mathbf{0}, \tag{3.2.9}$$

$$\mathbb{M}(\mathbf{f},\mathbf{p},\Omega) + \sum_{i=1}^{K} \left\{ \mathfrak{M}^{(i)} + \mathbf{a}^{(i)} \times \mathfrak{F}^{(i)} \right\} = \mathbf{0}. \tag{3.2.10}$$

Then, there exists a solution $\mathbf{u}$ *of the problem (3.2.1), (3.2.2) which admits the representation (3.2.3). The solution* $\mathbf{u}$ *is unique provided the vector function* $\mathbf{u}_r$ *from (3.2.3) satisfies the orthogonality conditions*

$$\int_{\Omega} \mathbf{u}_r(\mathbf{x})d\mathbf{x} = \mathbf{0}, \ \int_{\Omega} \mathbf{x} \times \mathbf{u}_r(\mathbf{x})d\mathbf{x} = \mathbf{0}. \tag{3.2.11}$$

Proof. Since the statement of the theorem is invariant with respect to translations and rotations of the coordinate system, we shall use coordinates where the axes coincide with the principal inertia axes of Ω. Substituting (3.2.3) in (3.2.1), (3.2.2) and using the definition of the Green tensor, we find that the vector $\mathbf{u}_r$ satisfies the boundary value problem

$$\mathbf{L}(\partial_{\mathbf{x}})\mathbf{u}_r(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathfrak{C} + \mathfrak{A} \times \mathbf{x}, \ \mathbf{x} \in \Omega, \tag{3.2.12}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}_r; \mathbf{x}) = \mathbf{p}(\mathbf{x}), \ \mathbf{x} \in \partial\Omega, \tag{3.2.13}$$

where

$$\mathfrak{C} + \mathfrak{A} \times \mathbf{x} = \sum_{i=1}^{K} \left\{ J_i^{-1}(\mathbf{a}^{(i)} \times \mathfrak{F}^{(i)}) \times \mathbf{x} + (\mathrm{mes}\Omega)^{-1}\mathfrak{F}^{(i)} + J_i^{-1}\mathfrak{M}^{(i)} \times \mathbf{x} \right\}.$$

Hence, applying (3.2.9), (3.2.10) one can see that the right–hand sides (3.2.12), (3.2.13) are orthogonal to the rigid–body translations and rotations. Thus, the problem (3.2.12), (3.2.13) is solvable in $(H^1(\Omega))^3$. The orthogonality conditions (3.2.11) provide the uniqueness of $\mathbf{u}_r$. $\square$

Now, we turn back and consider the Green tensor $\boldsymbol{\mathcal{N}}$ in the coordinate system such that the plane part of $\partial\Omega$ belongs to the plane $x_3 = 0$, and the Ox_3 axis is directed downwards. The matrix valued function $\mathbf{y} \to \boldsymbol{\mathcal{N}}(\mathbf{x}, \mathbf{y}), \ \ \mathbf{y} \in \Omega$, can be extended by continuity to $\mathbf{a}^{(i)}$ for any $\mathbf{x} \in \Omega$, and the following asymptotic representation holds in a neighbourhood of $\mathbf{a}^{(i)}$

$$\boldsymbol{\mathcal{N}}(\mathbf{x}, \ \mathbf{a}^{(i)}) = \boldsymbol{\mathcal{B}}(\mathbf{x}, \mathbf{a}^{(i)}) + \mathbf{N}^{(i)}(\mathbf{x}), \tag{3.2.14}$$

$$\begin{aligned} \frac{1}{2}(\nabla_{\mathbf{y}} \times \boldsymbol{\mathcal{N}}^t(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{a}^{(i)}})^t = \Big[&- \partial_{x_2}\boldsymbol{\mathcal{B}}^{(3)}(\mathbf{x} - \mathbf{a}^{(i)}), \partial_{x_1}\boldsymbol{\mathcal{B}}^{(3)}(\mathbf{x} - \mathbf{a}^{(i)}), \\ &\frac{1}{2}\Big(\partial_{x_2}\boldsymbol{\mathcal{B}}^{(1)}(\mathbf{x} - \mathbf{a}^{(i)}) - \partial_{x_1}\boldsymbol{\mathcal{B}}^{(2)}(\mathbf{x} - \mathbf{a}^{(i)})\Big)\Big] + \mathbf{D}^{(i)}(\mathbf{x}), \end{aligned} \tag{3.2.15}$$

where $\boldsymbol{\mathcal{B}}$ is the Boussinesq tensor for the half–space $\mathbb{R}^3_-$, and the matrices $\mathbf{N}^{(i)}$ and $\mathbf{D}^{(i)}$ are smooth.

3.3 The junction layer

Consider the domain G which coincides with the union of the half–space $\mathbb{R}^3_- = \{\mathbf{X} : X_3 < 0\}$ and the semi–infinite cylinder

$$\Pi_+ = \{\mathbf{X} : X_3 \geq 0, (X_1, X_2) \in g\},$$

located in $\mathbb{R}^3_+$, where g is a two–dimensional bounded domain with the Lipschitz boundary (see Fig. 3).

A vector function $\boldsymbol{\mathcal{W}}$ is said to belong to the space $\mathbf{E}(G)$ if $\boldsymbol{\mathcal{W}} \in (\mathbf{H}^1(G \cap B_r))^3$ for all positive r and

$$\mathcal{E}(\boldsymbol{\mathcal{W}}; G) + \int_{R^3_- \setminus B_1} \|\boldsymbol{\mathcal{W}}\|^2 \frac{d\mathbf{X}}{\|\mathbf{X}\|^2} < \infty,$$

where $B_r = \{\mathbf{x} : \|x\| < r\}$.

In accordance with Lemma A1 from Appendix and with the Hardy inequality we obtain

$$\int_{R^3_- \setminus B_1} \|\boldsymbol{\mathcal{W}}\|^2 \frac{d\mathbf{X}}{\|\mathbf{X}\|^2} \leq \mathrm{Const}\ \mathcal{E}(\boldsymbol{\mathcal{W}}; G).$$

Thus, the following norm may be defined in the space $\mathbf{E}(G)$

$$\|\boldsymbol{\mathcal{W}}\|_{\mathbf{E}(G)} = (\mathcal{E}(\boldsymbol{\mathcal{W}}; G))^{1/2}.$$

The Korn inequality for vector–valued functions , defined on a bounded Lipschitz domain (see Gobert (1962) [18], Fichera (1984) [11]), yields the completeness of the space $\mathbf{E}(G)$. Thus, $\mathbf{E}(G)$ is the Hilbert space.

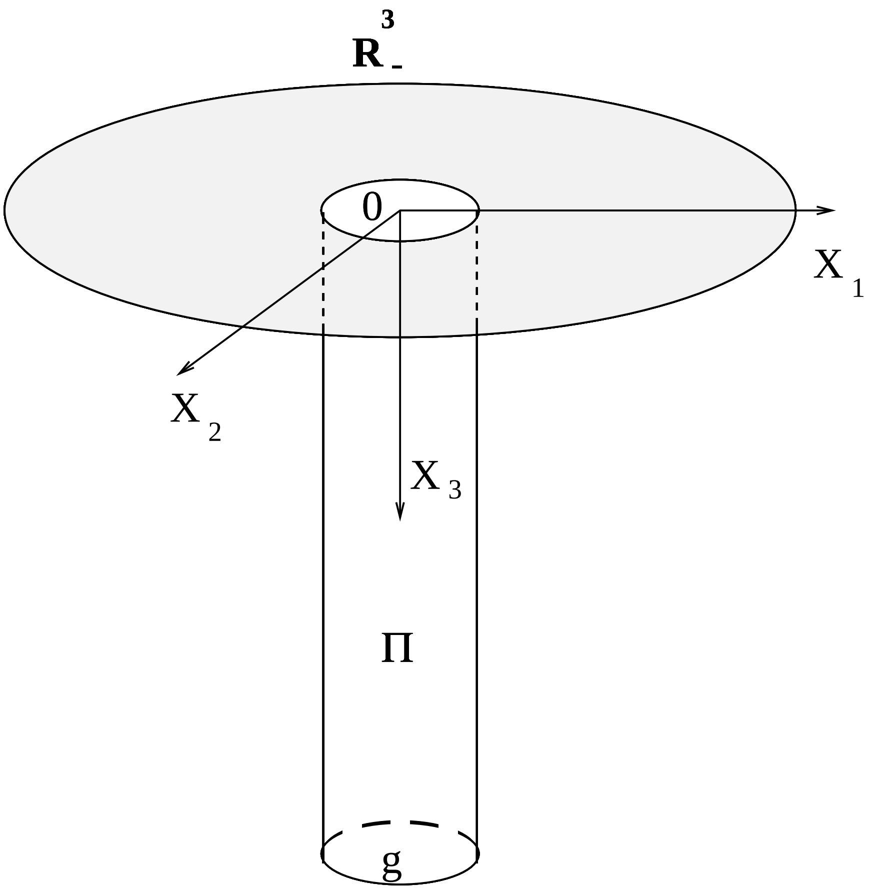

Fig.3 The junction layer region

Consider the boundary value problem in G with prescribed tractions on ∂G

$$-\mathbf{L}(\partial_{\mathbf{X}})\boldsymbol{\mathcal{W}}(\mathbf{X}) = \mathbf{F}(\mathbf{X}),\ \mathbf{X} \in G, \tag{3.3.1}$$

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{\mathcal{W}}; \mathbf{X}) = \mathbf{P}(\mathbf{X}),\ \mathbf{X} \in \partial G. \tag{3.3.2}$$

where $\mathbf{F} \in (L_2(G))^3$, $\mathbf{P} \in (L_2(\partial G))^3$ have compact supports. We seek a weak solution of the problem (3.3.1), (3.3.2) from $\mathbf{E}(G)$ which satisfies the relation

$$\sum_{i,j=1}^{3} \int_G \sigma_{ij}(\boldsymbol{\mathcal{W}}; \mathbf{X}) \varepsilon_{ij}(\boldsymbol{\mathcal{V}}; \mathbf{X}) d\mathbf{X} = h(\boldsymbol{\mathcal{V}}), \tag{3.3.3}$$

for any $\mathcal{V} \in \mathbf{E}(G)$, where

$$h(\mathcal{V}) = \int_G \mathbf{F}(\mathbf{X}) \cdot \mathcal{V}(\mathbf{X}) d\mathbf{X} + \int_{\partial G} \mathbf{P}(\mathbf{X}) \cdot \mathcal{V}(\mathbf{X}) dS_X \tag{3.3.4}$$

is the linear continuous functional on $\mathbf{E}(G)$. The following statement deals with the solvability of the problem (3.3.1)–(3.3.2) and the asymptotic representation of its solution.

Theorem 3.2 *Let $\mathbf{F}$ and $\mathbf{P}$ be vector–valued functions from $(\mathbf{L}_2(G))^3$ and $(\mathbf{L}_2(\partial G))^3$ with compact supports.*

i)The problem (3.3.1)–(3.3.2) is uniquely solvable in $\mathbf{E}(G)$.

ii) The solution $\mathcal{W}$ admits the asymptotic representation in the half–space $\mathbb{R}^3_-$

$$\begin{aligned}\mathcal{W}(\mathbf{X}) \sim\ & \boldsymbol{\mathcal{B}}(\mathbf{X})\mathfrak{F} + (\mathfrak{M}_1\partial_{X_1} - \mathfrak{M}_2\partial_{X_2})\boldsymbol{\mathcal{B}}^{(3)}(\mathbf{X}) \\ & + \frac{1}{2}\mathfrak{M}_3(-\partial_{X_1}\boldsymbol{\mathcal{B}}^{(2)}(\mathbf{X}) + \partial_{X_2}\boldsymbol{\mathcal{B}}^{(1)}(\mathbf{X})) + \sum_{i=1}^{2} b_i \partial_{X_i}\boldsymbol{\mathcal{B}}^{(i)}(\mathbf{X}) \\ & + \frac{1}{2}b_3(\partial_{X_1}\boldsymbol{\mathcal{B}}^{(2)}(\mathbf{X}) + \partial_{X_2}\boldsymbol{\mathcal{B}}^{(1)}(\mathbf{X})) + \sum_{|\alpha|\ge 2} \partial^{\alpha}_{\mathbf{X}'}\boldsymbol{\mathcal{B}}(\mathbf{X})\mathbf{B}_\alpha, \\ & \mathbf{X} \in \mathbb{R}^3_-,\ \|\mathbf{X}\| \to \infty,\end{aligned} \tag{3.3.5}$$

where $\boldsymbol{\mathcal{B}}$ is the Boussinesq tensor (see Section 2.1), b_j are constants, $\mathbf{B}_\alpha$ are constant vectors, and $\mathfrak{F}$ and $\mathfrak{M}$ are given by

$$\mathfrak{F} = \mathbb{F}(\mathbf{F}, \mathbf{P}, G),$$

$$\mathfrak{M} = \mathbb{M}(\mathbf{F}, \mathbf{P}, G),$$

where the functionals $\mathbb{F}$, $\mathbb{M}$ were defined in (3.2.7), (3.2.8). The asymptotic expansion (3.3.5) should be understood in the following sense: the difference of $\mathcal{W}(\mathbf{X})$ and the terms, homogeneous of order m, $m \ge -M$, from the right hand side has the norm $O(\|\mathbf{X}\|^{-M-1})$ in $\mathbb{C}^3$, and the above asymptotic expansion can be differentiated.

iii) In the cylindrical extension Π_+ the vector $\mathcal{W}$ has the asymptotics

$$\mathcal{W}(\mathbf{X}) = \mathbf{a} + \mathbf{b} \times \mathbf{X} + \mathbf{R}(\mathbf{X}) \tag{3.3.6}$$

where $\mathbf{a}$, $\mathbf{b}$ are constant vectors, and

$$\|\partial^j_{X_3}\mathbf{R}(\cdot, X_3)\|_{(H^1(g))^3} \le C_j exp(-\gamma X_3),\ \gamma > 0, \tag{3.3.7}$$

for large positive X_3 and some constant C_j, $j = 0, 1, \ldots$.

Proof. i) Due to our assumption, the formula (3.3.4) defines a linear continuous functional on $\mathbf{E}(G)$. From the Riesz theorem on the representation of a linear continuous functional in a Hilbert space we obtain the existence and the uniqueness for the variational problem (3.3.3).

ii) Let r be a positive constant such that the ball B_r contains the supports of $\mathbf{F}$ and $\mathbf{P}$. We introduce a smooth function $\theta = \theta(\mathbf{Y})$ in such a way that $\theta(\mathbf{Y}) = 0$ in $B_r \cup \Pi_+$ and $\theta(\mathbf{Y}) = 1$ in $\mathbb{R}^3_- \setminus B_{2r}$. Since the supports of the vector functions $\mathbf{L}\theta\mathcal{W}$ and $\boldsymbol{\sigma}^{(n)}(\theta\mathcal{W}; \mathbf{X})$ are compact and are contained in the closure of $\mathbb{R}^3_-$, one can represent $\mathcal{W}$ as

$$\mathcal{W}(\mathbf{X}) = -\int_{\mathbb{R}^3_-} \boldsymbol{\mathcal{M}}(\mathbf{X}, \mathbf{Y})\mathbf{L}(\partial_{\mathbf{Y}})(\theta(\mathbf{Y})\mathcal{W}(\mathbf{Y}))d\mathbf{Y}$$

$$+\int_{X_3=0} \boldsymbol{\mathcal{M}}(\mathbf{X},\mathbf{Y})\boldsymbol{\sigma}^{(3)}(\theta\boldsymbol{\mathcal{W}};\mathbf{Y})dS,$$

for $\mathbf{X} \in \overline{\mathbb{R}}^3_- \setminus B_{2r}$. Here, $\boldsymbol{\mathcal{M}}$ is the Mindlin tensor.

Using the Taylor expansion of $\boldsymbol{\mathcal{M}}$ in a neighbourhood of $\mathbf{Y}$ and Lemma 2.1, we arrive at (3.3.5).

iii) By Lemma A2, the vector function $X \to e^{-\varepsilon X_3}\boldsymbol{\mathcal{W}}(\mathbf{X})$ belongs to $(H^1(\Pi_+))^3$ for all positive ε. Taking into account that the right–hand sides (3.3.1), (3.3.2) have compact supports and applying the results by Kondrat'ev (1967) [9] we obtain

$$\boldsymbol{\mathcal{W}}(\mathbf{X}) = \mathbf{a} + \mathbf{b} \times \mathbf{X} + \sum_{k=1}^{6} c_k \boldsymbol{\xi}^{(k)}(\mathbf{X}) + \mathbf{R}(X),$$

where $\boldsymbol{\xi}^{(k)}$ are the same as in Section 2.3, and the function $\mathbf{X} \to e^{\gamma X_3}\mathbf{R}(\mathbf{X})$ and its derivatives with respect to X_3 belong to $(H^1(\Pi_+))^3$ for some positive γ. Since the field $\boldsymbol{\mathcal{W}}$ has the finite elastic energy, the coefficients c_k, $k = 1, \ldots, 6$, vanish. □

Theorem 3.2 gives the solution $\boldsymbol{\mathcal{W}}$ which does not decay in the cylindrical extension Π_+ and vanishes like $O(\|\mathbf{X}\|^{-1})$ at infinity in the half–space $\mathbb{R}^3_-$. We change the boundary value problem introducing 12 constants in its formulation in such a way that the solution decays in both directions.

Let η be a smooth cut–off function such that $\eta(z) = 0$ for $z < 1$ and $\eta(z) = 1$ for $z > 2$. Consider the boundary value problem

$$\begin{aligned} -\mathbf{L}(\partial_{\mathbf{X}})\boldsymbol{\mathcal{W}}(\mathbf{X}) &= \mathbf{F}(\mathbf{X}) + \sum_{j=1}^{6} \Big\{ d_j^{(I)} \mathbf{L}(\partial_{\mathbf{X}})\eta(X_3)\boldsymbol{\xi}^{(j)}(\mathbf{X}) \\ &\quad + d_j^{(II)} \mathbf{L}(\frac{\partial}{\partial \mathbf{X}})\eta(X_3)\boldsymbol{\psi}^{(j)}(\mathbf{X}) \Big\}, \ \mathbf{X} \in G, \end{aligned} \tag{3.3.8}$$

$$\begin{aligned} \boldsymbol{\sigma}^{(n)}(\boldsymbol{\mathcal{W}};\mathbf{X}) &= \mathbf{P}(\mathbf{X}) - \sum_{j=1}^{6} \Big\{ d_j^{(I)} \boldsymbol{\sigma}^{(n)}(\eta\boldsymbol{\xi}^{(j)};\mathbf{X}) \\ &\quad + d_j^{(II)} \boldsymbol{\sigma}^{(n)}(\eta\boldsymbol{\psi}^{(j)};\mathbf{X}) \Big\}, \mathbf{X} \in \partial G, \end{aligned} \tag{3.3.9}$$

where $d_j^{(I)}$, $d_j^{(II)}$, $j = 1, 2, \ldots, 6$ are constant.

Theorem 3.3 *Let* $\mathbf{F} \in (\mathbf{L}_2(G))^3$, $\mathbf{P} \in (\mathbf{L}_2(\partial G))^3$ *have compact supports. There exists a unique set*

$$\{\boldsymbol{\mathcal{W}}; d_1^{(I)}, \ldots, d_6^{(I)}; d_1^{(II)}, \ldots, d_6^{(II)}\}, \tag{3.3.10}$$

which satisfies the boundary value problem (3.3.8), (3.3.9), where $\boldsymbol{\mathcal{W}} \in E(G)$ *admits the asymptotic expansion (3.3.5) in* $\mathbb{R}^3_-$ *with* $\mathfrak{F} = 0$, $\mathfrak{M} = 0$, *and decays exponentially in* Π_+. *The constants* $d_j^{(I)}$ *are given by*

$$d_q^{(I)} = D_q^{-1}\mathbb{F}_q(\mathbf{F},\mathbf{P},G), \ q = 1, 2, \tag{3.3.11}$$

$$d_3^{(I)} = -D_3^{-1}\mathbb{F}_3(\mathbf{F},\mathbf{P},G), \tag{3.3.12}$$

$$d_4^{(I)} = -D_4^{-1}\mathbb{M}_3(\mathbf{F},\mathbf{P},G), \tag{3.3.13}$$

$$d_5^{(I)} = -D_1^{-1}\mathbb{M}_2(\mathbf{F},\mathbf{P},G), \tag{3.3.14}$$

$$d_6^{(I)} = D_2^{-1}\mathbb{M}_1(\mathbf{F},\mathbf{P},G), \tag{3.3.15}$$

Proof. First, we prove the existence of the solution (3.3.10) of the problem (3.3.8), (3.3.9). Let $d_j^{(I)}$, $j = 1, \dots, 6$, be given by (3.3.11)–(3.3.15). Due to the biorthogonality conditions (2.3.25), (2.3.28), the principal force and moment vectors for the right–hand sides of (3.3.8), (3.3.9) are equal to zero. Let $d_j^{(II)} = 0, j = 1, \dots, 6$. Then, by Theorem 3.2 the problem (3.3.8), (3.3.9) is solvable, and the asymptotics (3.3.5) has $\mathfrak{F} = \mathfrak{M} = 0$ whereas the asymptotic formula (3.3.6) holds in a cylindrical extension Π_+ with some constant vectors $\mathbf{a}$, $\mathbf{b}$. Subtracting from this field the rigid body displacement $\mathbf{a}+\mathbf{b}\times\mathbf{x}$ multiplied by the cut–off function η and taking

$$d_j^{(II)} = a_j,\ j = 1, 2, 3;\ d_4^{(II)} = b_3,\ d_4^{(II)} = b_2,\ d_6^{(II)} = -b_1, \tag{3.3.16}$$

we arrive at the required solution of the problem (3.3.8), (3.3.9).

Now we prove the uniqueness.

One can introduce the linear finite dimensional operator

$$\{d_j^{(I)},\ d_j^{(II)}\}_{j=1}^{6} \to \{\mathfrak{F}, \mathfrak{M}, \mathbf{a}, \mathbf{b}\}, \tag{3.3.17}$$

where the right–hand side includes vectors from the asymptotic formulae (3.3.5), (3.3.6) for the solution of the problem (3.3.8), (3.3.9), which is given in Theorem 3.2, with fixed coefficients $d_j^{(I)}$, $d_j^{(II)}$. The operator (3.3.17) acts from $\mathbb{R}^{12}$ into $\mathbb{R}^{12}$. Let us prove that it represents a surjective operator. Let $\{\mathfrak{F}, \mathfrak{M}, \mathbf{a}, \mathbf{b}\}$ be an element of $\mathbb{R}^{12}$. We introduce a smooth cut–off function χ such that $\chi = 1$ for large $\|\mathbf{X}\|$ in the half–space and $\chi = 0$ in the semi–cylinder. Let

$$\boldsymbol{\mathcal{U}}(\mathbf{X}) = \chi(\mathbf{X})\Big\{\boldsymbol{\mathcal{B}}(\mathbf{X})\mathfrak{F} + (\mathfrak{M}_1\partial_{X_1} - \mathfrak{M}_2\partial_{X_2})\boldsymbol{\mathcal{B}}^{(3)}(\mathbf{X}) + \frac{1}{2}\mathfrak{M}_3(-\partial_{X_1}\boldsymbol{\mathcal{B}}^{(2)}(\mathbf{X}) + \partial_{X_2}\boldsymbol{\mathcal{B}}^{(1)}(\mathbf{X}))$$

$$+\sum_{i=1}^{2} b_i\partial_{X_i}\boldsymbol{\mathcal{B}}^{(i)}(\mathbf{X}) + \frac{1}{2}b_3(\partial_{X_1}\boldsymbol{\mathcal{B}}^{(2)}(\mathbf{X}) + \partial_{X_2}\boldsymbol{\mathcal{B}}^{(1)}(\mathbf{X}))\Big\} + \eta(X_3)\Big\{\mathbf{a} + \mathbf{b}\times\mathbf{X}\Big\}.$$

Then the vector functions

$$\mathbf{F}(\mathbf{X}) = -\mathbf{L}(\partial_{\mathbf{X}})\boldsymbol{\mathcal{U}}(\mathbf{X}),\ \mathbf{P}(\mathbf{X}) = \boldsymbol{\sigma}^{(n)}(\boldsymbol{\mathcal{U}}; \mathbf{X})$$

have compact supports, and, as proved above, the problem (3.3.8), (3.3.9) with just specified right–hand sides has the solution set (3.3.10). This implies that the operator (3.3.17) is surjective. Since it is finite–dimensional, its kernel is trivial.

In order to complete the proof of the uniqueness, we assume that $\boldsymbol{\mathcal{W}}$ satisfies (3.3.8), (3.3.9) with $\mathbf{F}$, $\mathbf{P}$ and the coefficients $\mathfrak{M}$, $\mathfrak{F}$ in (3.3.5) and $\mathbf{a}$, $\mathbf{b}$ in (3.3.6) being equal to zero. Since the operator (3.3.17) is injective we obtain

$$d_j^{(I)} = d_j^{(II)} = 0,\ j = 1, \dots, 6.$$

Applying Theorem 3.2 we obtain that $\boldsymbol{\mathcal{W}}$ is zero. $\square$

3.4 The model problem for the bottom layer

Let Π_- be the semi–cylinder $\{\mathbf{Y} : Y_3 < 0, (Y_1, Y_2) \in g\}$ (see Fig. 4), with g being the same as in the previous section . We denote by S the base region of the boundary $\partial\Pi_-$.

A vector–valued function $\boldsymbol{\mathcal{U}}$ is said to belong to the space $\mathbf{E}(\Pi_-)$ if $\boldsymbol{\mathcal{U}} \in (H^1(\Pi_- \cap B_r))^3$ for any positive r, and $\mathcal{E}(\boldsymbol{\mathcal{U}}; \Pi_-) < \infty$.

In the space $\mathbf{E}(\Pi_-)$ we define the norm

$$\|\mathcal{U}\|_{\mathbf{E}(\Pi_-)} = \left(\mathcal{E}(\mathcal{U}, \Pi_-) + \int_{B_1 \cap \Pi_-} \|\mathcal{U}(\mathbf{Y})\|^2 d\mathbf{Y} \right)^{1/2}.$$

Due to the Korn inequality for vector functions in a bounded Lipschitz domain [18], the space $\mathbf{E}(\Pi_-)$ is Hilbert. Clearly, $\mathbf{E}(\Pi_-)$ includes the subspace $\mathcal{R}$ which consists of the rigid body displacements.

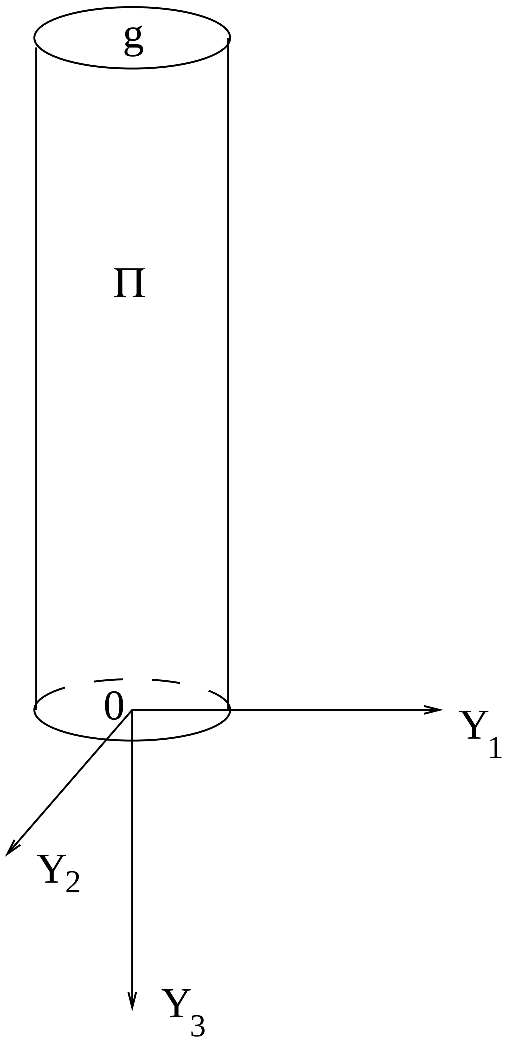

Fig.4 The bottom layer region

We introduce the subspace

$$\mathbf{E}^0(\Pi_-; S) = \{\mathcal{V} \in \mathbf{E}(\Pi_-) : \mathcal{V} = 0 \text{ on } S\}.$$

of the space $\mathbf{E}(\Pi_-)$. Due to Lemma A2, $(\mathcal{E}(\mathbf{V}; \Pi_-))^{1/2}$ is an equivalent norm in $\mathbf{E}^0(\Pi_-; S)$.

Consider a variational formulation in Π_-. Let h be a linear bounded functional on the subspace $\mathbf{E}^0(\Pi_-;S)$. We seek a vector–valued function $\mathbf{V} \in \mathbf{E}(\Pi_-)$ satisfying the equation

$$\sum_{i,j=1}^{3} \int_{\Pi_-} \sigma_{ij}(\mathbf{V};\mathbf{Y})\varepsilon_{ij}(\boldsymbol{\mathcal{V}};\mathbf{Y})d\mathbf{Y} = h(\boldsymbol{\mathcal{V}}), \tag{3.4.1}$$

for all $\boldsymbol{\mathcal{V}} \in \mathbf{E}^0(\Pi_-;S)$ and the boundary condition

$$\mathbf{V}(\mathbf{Y}) = \boldsymbol{\psi}(\mathbf{Y}),\ \mathbf{Y} \in S, \tag{3.4.2}$$

where $\boldsymbol{\psi} \in (H^{1/2}(S))^3$.

The above variational formulation corresponds to the following mixed boundary value problem: the displacement vector $\mathbf{V}$ satisfies the Lamé system

$$-\mathbf{L}(\partial_{\mathbf{Y}})\mathbf{V}(\mathbf{Y}) = \mathbf{G}(\mathbf{Y}),\ \ \mathbf{Y} \in \Pi_-, \tag{3.4.3}$$

the traction boundary condition

$$\boldsymbol{\sigma}^{(n)}(\mathbf{V};\mathbf{Y}) = \mathbf{H}(\mathbf{Y}),\ \ \mathbf{Y} \in \partial\Pi_- \setminus \bar{S}, \tag{3.4.4}$$

and the displacement boundary condition (3.4.2).

The right–hand side h in the variational formulation admits the form (3.3.4) with $\mathbf{F}$, $\mathbf{P}$ and G being replaced by $\mathbf{G}$, $\mathbf{H}$ and Π_-.

Theorem 3.4 *The problem (3.4.2)–(3.4.4), where $\mathbf{G} \in (L_2(\Pi_-))^3$; $\mathbf{H} \in (L_2(\partial\Pi_- \setminus \overline{S}))^3$ have compact supports, is uniquely solvable in the class $\mathbf{E}(\Pi_-)$ and its solution admits the asymptotic representation*

$$\mathbf{V}(\mathbf{Y}) = \mathbf{a} + \mathbf{b} \times \mathbf{Y} + \mathbf{R}(\mathbf{Y}), \tag{3.4.5}$$

where $\mathbf{a}$, $\mathbf{b}$ are constant vectors, and

$$\|\partial^j_{Y_3}\mathbf{R}(\cdot,Y_3)\|_{(H^1(g))^3} \le C_j\, exp(\gamma Y_3),\ \gamma > 0, \tag{3.4.6}$$

for large $|Y_3|$ and some constant C_j, $j = 0,1,\dots$.

Proof. Let $\mathbf{V}_0$ be a vector function from $\mathbf{E}(\Pi_-)$ with a compact support such that $\mathbf{V}_0 = \boldsymbol{\psi}$ on S. We seek a solution in the form $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1$. Then, the vector function $\mathbf{V}_1$ satisfies the variational problem (3.4.1) with h given by

$$h(\boldsymbol{\mathcal{V}}) = -2\mathcal{E}(\mathbf{V}_0,\boldsymbol{\mathcal{V}}) + \Big\{\int_{\Pi_-} \mathbf{G}\cdot\boldsymbol{\mathcal{V}}d\mathbf{Y} + \int_{\partial\Pi_-} \mathbf{H}\cdot\boldsymbol{\mathcal{V}}dS\Big\},$$

and $\mathbf{V}_1 = 0$ on S. It is clear that h is a linear bounded functional on $\mathbf{E}^0(\Pi_-;S)$. Using the Riesz theorem we obtain the existence and the uniqueness of $\mathbf{V}_1$. The proof of the asymptotic representation (3.4.5) is similar to that of Theorem 3.2. $\Box$

Similar to the junction layer problem (3.3.8)–(3.3.9) we introduce 6 constants and change the boundary conditions at S in such a way that the solution of the new problem will decay at infinity.

Consider the system (3.4.3) and the traction boundary condition (3.4.4). On the base region S we specify the following displacement

$$\mathbf{V}(\mathbf{Y}) = \boldsymbol{\psi}(\mathbf{Y}) + \sum_{j=1}^{6} c_j\boldsymbol{\psi}^{(j)}(Y_1,Y_2,0),\ \mathbf{Y} \in S, \tag{3.4.7}$$

where $\boldsymbol{\psi}^{(j)}$, $j = 1,2,\dots,6$, are rigid body displacements introduced in Section 2.2, c_j are constant.

Theorem 3.5 *Let* $\mathbf{G} \in (\mathbf{L}_2(\Pi_-))^3$, $\mathbf{H} \in (\mathbf{L}_2(\partial\Pi_- \setminus \overline{S}))^3$ *have compact supports and* $\boldsymbol{\psi} \in (H^{1/2}(S))^3$. *There exists a unique set* $\{\mathbf{V}, c_1, \ldots, c_6\}$ *which satisfies the boundary value problem (3.4.3), (3.4.4), (3.4.7), where* $\mathbf{V}$ *decays exponentially at infinity.*

With some simplifications the proof follows Theorem 3.3.

Remark 3.1. *One can obtain the following explicit relations for the coefficients in (3.4.7) and (3.4.5)*

$$c_i = -a_i, \ i = 1,2,3; \ c_4 = -b_3, \ c_5 = -b_2, \ c_6 = b_1.$$

3.5 The model problems for a bounded two–dimensional domain

Here we introduce model boundary value problems for the anti–plane shear and the plane strain on the scaled cross–section of a thin elastic rod.

3.5.1 Plane strain

Consider a bounded two–dimensional domain g with the Lipschitz boundary, and assume that the displacement vector $\mathbf{u} = (u_1, \ u_2)^t$ satisfies the following boundary value problem for the two–dimensional Lamé operator

$$-\mathbf{L}_0^{(1)}(\frac{\partial}{\partial \mathbf{X}'})\mathbf{u}(\mathbf{X}') = \mathbf{f}(\mathbf{X}'), \ \mathbf{X}' = (X_1, \ X_2) \in g, \tag{3.5.1}$$

$$\mathbf{T}_0^{(1)}(\mathbf{n}, \partial'_{\mathbf{X}})\mathbf{u}(\mathbf{X}') = \mathbf{p}(\mathbf{X}'), \ \mathbf{X}' \in \partial g. \tag{3.5.2}$$

Here $\mathbf{L}_0^{(1)}$, $\mathbf{T}_0^{(1)}$ are the first 2×2 diagonal blocks of the matrix differential operators $\mathbf{L}_0$, $\mathbf{T}_0$ specified in (2.3.5) , (2.3.7); vector–valued functions $\mathbf{p} \in (L_2(\partial g))^2$ and $\mathbf{f} \in (L_2(g))^2$ represent the traction vector and the body force density, respectively.

Problem (3.5.1)–(3.5.2) is solvable in $(H^1(g))^2$ if and only if the right–hand sides satisfy the following orthogonality conditions

$$\int_g f_i(\mathbf{X}')d\mathbf{X}' + \int_{\partial g} p_i(\mathbf{X}')ds = 0, \ i = 1,2, \tag{3.5.3}$$

$$\int_g (f_1(\mathbf{X}')X_2 - f_2(\mathbf{X}')X_1)d\mathbf{X}' + \int_{\partial g} (p_1(\mathbf{X}')X_2 - p_2(\mathbf{X}')X_1)ds = 0. \tag{3.5.4}$$

The physical sense of the above conditions is clear: these are the balance relations for the components of the principal force vector and the principal moment.

To provide the uniqueness, we assume that the displacement vector is subjected to

$$\int_g u_i(\mathbf{X}')d\mathbf{X}' = 0, \ i = 1,2, \tag{3.5.5}$$

and

$$\int_g (u_1(\mathbf{X}')X_2 - u_2(\mathbf{X}')X_1)d\mathbf{X}' = 0. \tag{3.5.6}$$

Otherwise one would be allowed to add any rigid body translation and rotation in the OX_1X_2 plane.

The vector function $\mathbf{u}$ gives first two components of the displacement vector due to the plane–strain load.

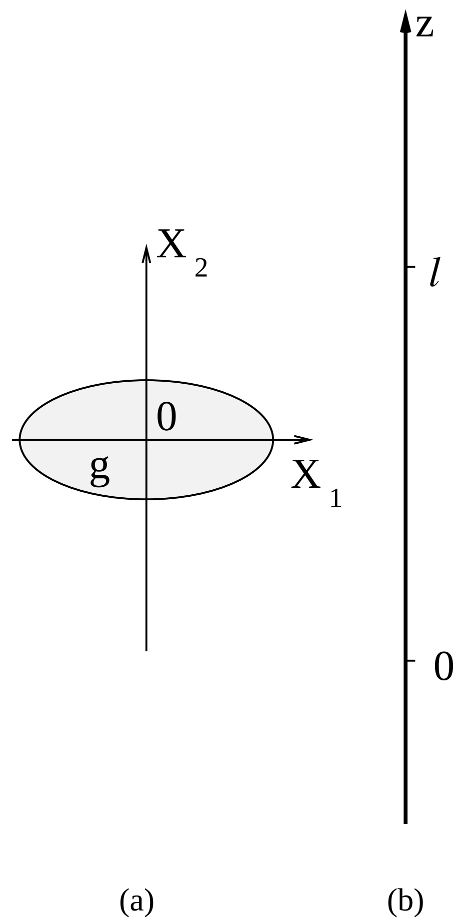

Fig.5 Model regions for an elastic rod

3.5.2 Anti–plane shear

In the same domain g consider the Neumann boundary value problem for the Laplace operator

$$-\mu\Delta u(\mathbf{X}') = f(\mathbf{X}'),\ \mathbf{X}' \in g, \tag{3.5.7}$$

$$\mu\frac{\partial u}{\partial n}(\mathbf{X}') = p(\mathbf{X}'),\ \mathbf{X}' \in \partial g. \tag{3.5.8}$$

where $\mathbf{n}$ is the outward unit normal on the boundary ∂g; the right–hand sides f, g belong to $L_2(g)$, $L_2(\partial g)$, respectively, and satisfy the orthogonality condition

$$\int_g f(\mathbf{X}')d\mathbf{X}' + \int_{\partial g} p(\mathbf{X}')ds = 0. \tag{3.5.9}$$

The problem (3.5.7)–(3.5.8) has a solution from $(H^1(g))^2$ specified up to an arbitrary additive constant. To provide the uniqueness we set the following constraint

$$\int_g u(\mathbf{X}')d\mathbf{X}' = 0. \tag{3.5.10}$$

The scalar function u represents the third component of the displacement vector due to an anti–plane shear of the elastic domain g.

3.6 The model problems on the axis of an elastic rod

It will be shown further that the leading order terms of the longitudinal, transversal and torsional displacements within a one–dimensional elastic rod (see Fig.5 (b)) satisfy the ordinary differential equations of the form

$$\frac{d^4v}{dz^4}(z) = R(z),\ z \in (0,l), \tag{3.6.1}$$

$$\frac{d^2w}{dz^2}(z) = r(z),\ z \in (0,l), \tag{3.6.2}$$

and the boundary conditions at the ends of the interval $(0,l)$:

$$v(\frac{1}{2}(l \pm l)) = B_0^{\pm}, \tag{3.6.3}$$

$$v'(\frac{1}{2}(l \pm l)) = B_1^{\pm}, \tag{3.6.4}$$

$$w(\frac{1}{2}(l \pm l)) = b^{\pm}. \tag{3.6.5}$$

Here, we include the elastic constants characterizing the rod into the right–hand sides R and r. The solutions of these problems can be represented as

$$\begin{aligned} v(z) = \frac{1}{2}\Big\{ &\int_0^z \Big(lt^2\phi_{II}(\frac{z}{l}) - \frac{t^3}{3}\phi_I(\frac{z}{l})\Big) R(t)dt \\ -&\int_z^l \Big(\frac{1}{3}(l-t)^3\psi_I(\frac{z}{l}) + l(l-t)^2\psi_{II}(\frac{z}{l})\Big) R(t)dt\Big\} \\ + \phi_I(\frac{z}{l})B_0^- &+ l\phi_{II}(\frac{z}{l})B_1^- + \psi_I(\frac{z}{l})B_0^+ + l\psi_{II}(\frac{z}{l})B_1^+, \end{aligned} \tag{3.6.6}$$

where

$$\phi_I(z) = (z-1)^2(2z+1),\ \phi_{II}(z) = (z-1)^2 z,$$
$$\psi_I(z) = z^2(3-2z),\ \psi_{II}(z) = z^2(z-1),$$

and

$$w(z) = (\frac{z}{l} - 1)\int_0^z tr(t)dt + \frac{z}{l}\int_z^l (t-l)r(t)dt$$

$$+ (1 - \frac{z}{l}) b^- + \frac{z}{l} b^+. \tag{3.6.7}$$

Also, in further sections we shall need the following linear functionals on the right–hand sides of (3.6.1)–(3.6.5):

$$\mathbb{Q}^{(I)}(r, b^+, b^-) := \frac{dw}{dz}|_{z=0} = \int_0^l (\frac{t}{l} - 1) r(t) dt + \frac{1}{l}(b^+ - b^-), \tag{3.6.8}$$

$$\mathbb{Q}^{(II)}(R, B_0^+, B_0^-, B_1^+, B_1^-) := \frac{d^2 v}{dz^2}|_{z=0}$$

$$= \int_0^l t(1 - \frac{t}{l})^2 R(t) dt + \frac{6}{l^2}(B_0^+ - B_0^-) - \frac{2}{l}(2B_1^- + B_1^+), \tag{3.6.9}$$

$$\mathbb{Q}^{(III)}(R, B_0^+, B_0^-, B_1^+, B_1^-) := \frac{d^3 v}{dz^3}|_{z=0}$$

$$= - \int_0^l (1 - \frac{t}{l})^2 (1 + \frac{2t}{l}) R(t) dt + \frac{12}{l^3}(B_0^- - B_0^+) + \frac{6}{l^2}(B_1^- + B_1^+). \tag{3.6.10}$$

3.7 Model matrices and the pile structure

In the presnt section we introduce a set of matrices independent of ε which play an important role for the asymptotic evaluation of the solution of the problem (1.2.1)–(1.2.3). The components of these matrices include information about all elements of the multi–structure. The matrices mentioned can be interpreted as the normalized stiffness and coupling matrices for the pile structure related to the skeleton of the multi–structure.

3.7.1 Model matrices

The further asymptotic algorithm requires certain model matrices:

$$\mathcal{A} = \sum_{k=1}^{K} \frac{12}{l_k^3} \begin{pmatrix} D_1 & 0 & -D_1 a_2^{(k)} \\ 0 & D_2 & D_2 a_1^{(k)} \\ -D_1 a_2^{(k)} & D_2 a_1^{(k)} & \begin{matrix} (a_1^{(k)})^2 D_2 + (a_2^{(k)})^2 D_1 \\ +l_k^2 (D_4/12) \end{matrix} \end{pmatrix}, \tag{3.7.1}$$

$$\mathcal{B} = \sum_{k=1}^{K} \frac{D_3}{l_k} \begin{pmatrix} 1 & a_2^{(k)} & -a_1^{(k)} \\ a_2^{(k)} & (a_2^{(k)})^2 & -a_1^{(k)} a_2^{(k)} \\ -a_1^{(k)} & -a_1^{(k)} a_2^{(k)} & (a_1^{(k)})^2 \end{pmatrix}, \tag{3.7.2}$$

$$\mathcal{D} = \sum_{k=1}^{K} \frac{6}{l_k^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -D_2 & -D_2 a_1^{(k)} \\ D_1 & 0 & -D_1 a_2^{(k)} \end{pmatrix}, \tag{3.7.3}$$

and

$$\mathcal{E} = \sum_{k=1} \frac{4}{l_k} \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_1 \end{pmatrix}, \tag{3.7.4}$$

where $D_i,\ i = 1, \ldots, 4$, are the same as in Section 2.3. The quantities $a_j^{(k)}$ are coordinates of the junction points calculated in the coordinate system such that the plane Ox_1x_2 contains the junction points, the Ox_3 axis is directed downwards, and the Ox_1, Ox_2 axes are parallel to the principal inertia axes of the cross–sections of thin rods.

Since for any nonzero constant vector $X \in \mathbb{R}^3$

$$X^t \mathcal{A} X = 12 \sum_k \frac{1}{l_k^3} \Big\{ D_1 (X_1 - X_3 a_2^{(k)})^2 + D_2 (X_2 + X_3 a_1^{(k)})^2 + X_3^2 l_k^2 D_4/12 \Big\} > 0, \tag{3.7.5}$$

the matrix $\mathcal{A}$ is nonsingular. Similarly,

$$X^t \mathcal{B} X = D_3 \sum_k \frac{1}{l_k} (X_1 + a_2^{(k)} X_2 - a_1^{(k)} X_3)^2 > 0, \tag{3.7.6}$$

provided the junction points $\mathbf{a}^{(k)}$, $k = 1, \dots, K$, are not located on the same straight line.

Lemma 3.1 *The following inequality holds*

$$X^t \mathcal{A} X + 2Y^t \mathcal{D} X + Y^t \mathcal{E} Y \geq \frac{1}{4} X^t \mathcal{A} X, \tag{3.7.7}$$

for any $X, Y \in \mathbb{R}^3$.

Proof. We denote by $\mathcal{A}_k$, $\mathcal{D}_k$ and $\mathcal{E}_k$ the terms with the number k from the sums (3.7.1), (3.7.3), (3.7.4), respectively. Let

$$\mathcal{E}_k^{(1)} = \frac{l_k}{4} \mathrm{diag}\{0, D_2^{-1}, D_1^{-1}\}.$$

Then

$$|Y^t \mathcal{D}_k X| = |Y^t \mathcal{E}_k \mathcal{E}_k^{(1)} \mathcal{D}_k X| \leq \Big\{ (Y^t \mathcal{E}_k Y)(X^t \mathcal{D}_k^t \mathcal{E}_k^{(1)} \mathcal{D}_k X) \Big\}^{1/2}.$$

One can verify directly that

$$\mathcal{D}_k^t \mathcal{E}_k^{(1)} \mathcal{D}_k = \frac{3}{4} \Big\{ \mathcal{A}_k - \mathrm{diag}\{0, 0, \frac{D_4}{l_k}\} \Big\}.$$

Therefore,

$$|Y^t \mathcal{D}_k X| \leq \Big\{ \frac{3}{4} (Y^t \mathcal{E}_k Y)(X^t A_k X) \Big\}^{1/2} \leq \frac{1}{2} (Y^t \mathcal{E}_k Y + \frac{3}{4} X^t \mathcal{A}_k X)).$$

Taking the summation with respect to k we arrive at (3.7.7). □

The matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ occur naturaly in the pile structure model described in the next section.

3.7.2 The pile structure

The classical pile structure model deals with the set of elastic piles of different orientations (i.e. the vectors directed along the piles span $\mathbb{R}^3$) encased at their heads in the rigid pile cap. The main relation is the 6×6 linear system which provides the correspondence between the rigid body displacements of the pile cap and the principal forces and moments of the external load (see Asplund [12], pp. 291–297). All nonzero elements of the matrix of this system have the same order of magnitude.

In the present section we relate the pile structure model to the skeleton of the multi–structure represented by a set of one–dimensional rods r_k, $k = 1, \dots, K$ and the rigid cap Ω. This case can be regarded as a degenerate pile structure, and the matrix of the system for the rigid body displacements includes components of different orders of magnitude.

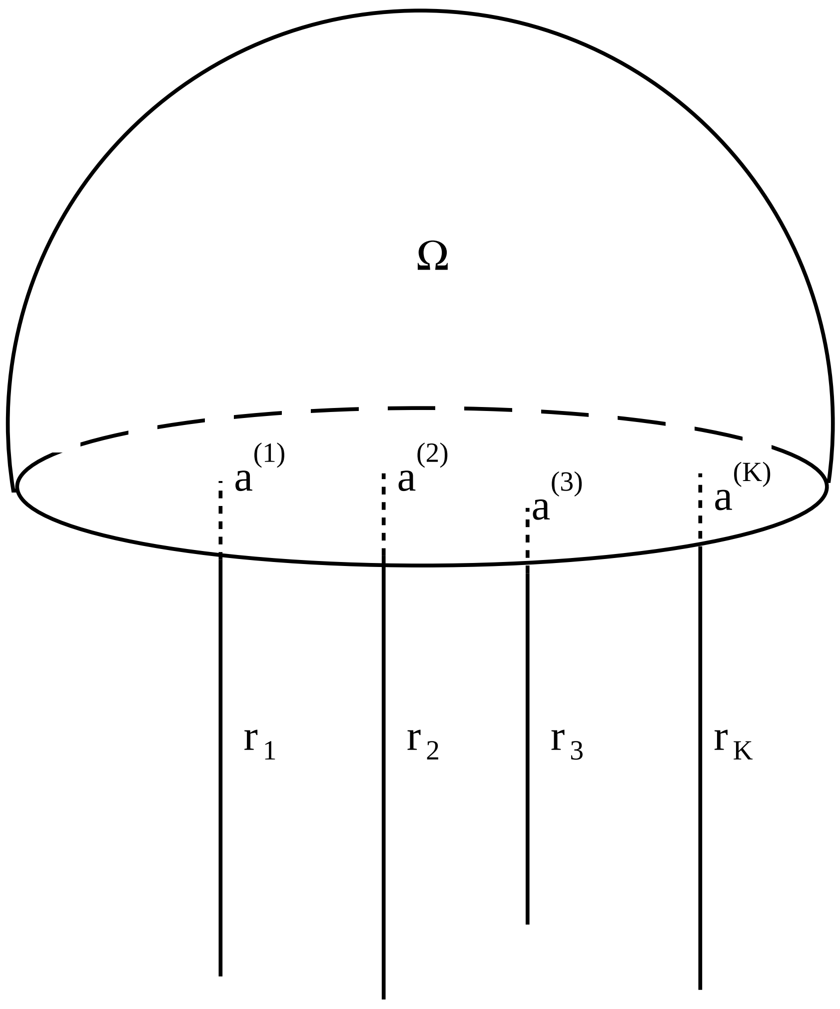

Fig.6 Skeleton of the multi–structure

We fix the same Cartesian coordinate system as in the previous section. The elastic rod r_k, corresponding to the junction point $\mathbf{x} = \mathbf{a}^{(k)}$ is assumed to have the length l_k, and it is characterized by the stiffness coefficients S_1, S_2, S_3, S_4 which are supposed to be positive. The first two coefficients S_1 and S_2 are related to transversal displacements in the direction of the Ox_1 and Ox_2 axes. The coefficients S_3 and S_4 correspond to the longitudinal displacement and the axial rotation, respectively. Our multi–structure is related to the pile structure with

$$S_i = \varepsilon^4 D_i, \ i = 1, 2, 4; \ S_3 = \varepsilon^2 D_3, \tag{3.7.8}$$

where $D_k,\ k = 1, \ldots, 4$, are the same constants as in Section 2.3, and ε is the non–dimensional small parameter (see Section 1.2). The region Ω is assumed to be rigid, and the displacements of the ends of rods at the junction points and the rotations are equal to the displacements and the rotations of Ω at the same points. The dead ends of elastic rods are assumed to be clamped. The interaction between the rigid body Ω and elastic rods is characterized by lock forces $\mathfrak{F}^{(k)}$ and lock moments $\mathfrak{M}^{(k)}$ which are applied at the junction points. Assume that the external force $\mathfrak{F}$ and the moment $\mathfrak{M}$ are applied to Ω.

The equilibrium equations have the form

$$\sum_{k=1}^{K} \mathfrak{F}^{(k)} + \mathfrak{F} = 0, \tag{3.7.9}$$

$$\sum_{k=1}^{K} (\mathfrak{M}^{(k)} + \mathbf{a}^{(k)} \times \mathfrak{F}^{(k)}) + \mathfrak{M} = 0. \tag{3.7.10}$$

The objective is to derive a system of linear equations with respect to components of rigid body translations and rotations of Ω and a representation for components of lock forces and moments at the junction points. The displacements of points of the one–dimensional elastic rod r_k are characterized by functions $v_1^{(k)},\ v_2^{(k)},\ v_3^{(k)},\ v_4^{(k)}$ which correspond to the transversal displacements, the longitudinal displacement and the axial rotation, respectively.

Now, we describe the mathematical model. The rigid body displacements of Ω are specified by

$$\boldsymbol{\alpha} + \boldsymbol{\beta} \times \mathbf{x}, \tag{3.7.11}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant vectors.

The functions $v_i^{(k)},\ i = 1, 2, 3, 4$, satisfy the ordinary differential equations

$$\frac{d^4 v_i^{(k)}}{dx_3^4}(x_3) = 0, \quad x_3 \in (0, l_k),\ i = 1, 2, \tag{3.7.12}$$

$$\frac{d^2 v_i^{(k)}}{dx_3^2}(x_3) = 0, \quad x_3 \in (0, l_k),\ i = 3, 4. \tag{3.7.13}$$

The clamping conditions at the dead ends of the rods r_k are

$$v_i^{(k)}(l_k) = 0,\ i = 1, 2, 3, 4, \tag{3.7.14}$$

$$\frac{dv_i^{(k)}}{dx_3}(l_k) = 0,\ i = 1, 2. \tag{3.7.15}$$

The continuity conditions of rotations and translations at the junction points $\mathbf{a}^{(k)},\ k = 1, \ldots, K$, can be written as

$$v_i^{(k)}(0) = \alpha_i + (\boldsymbol{\beta} \times \mathbf{a}^{(k)})_i,\ i = 1, 2, 3, \tag{3.7.16}$$

$$v_4^{(k)}(0) = \beta_3,\ \frac{dv_1^{(k)}}{dx_3}(0) = \beta_2,\ \frac{dv_2^{(k)}}{dx_3}(0) = -\beta_1. \tag{3.7.17}$$

Forces and moments at junction points, exerted by elastic rods on the body Ω, are specified by

$$\mathfrak{F}_i^{(k)} = -S_i \frac{d^3 v_i^{(k)}}{dx_3^3}(0),\ i = 1, 2, \tag{3.7.18}$$

$$\mathfrak{F}_3^{(k)} = S_3 \frac{dv_3^{(k)}}{dx_3}(0), \tag{3.7.19}$$

$$\mathfrak{M}_{3-i}^{(k)} = (-1)^{i+1} S_i \frac{d^2 v_i^{(k)}}{dx_3^2}(0), \; i = 1, 2, \tag{3.7.20}$$

$$\mathfrak{M}_3^{(k)} = S_4 \frac{dv_4^{(k)}}{dx_3}(0). \tag{3.7.21}$$

By formulae (3.6.6), (3.6.7), the solutions of (3.7.12)–(3.7.17) have the form

$$v_1^{(k)} = (\alpha_1 - a_2^{(k)}\beta_3)\phi_I(\frac{x_3}{l_k}) + \beta_2 l_k \phi_{II}(\frac{x_3}{l_k}), \tag{3.7.22}$$

$$v_2^{(k)} = (\alpha_2 + a_1^{(k)}\beta_3)\phi_I(\frac{x_3}{l_k}) - \beta_1 l_k \phi_{II}(\frac{x_3}{l_k}), \tag{3.7.23}$$

$$v_3^{(k)} = (\alpha_3 + \beta_1 a_2^{(k)} - a_1^{(k)}\beta_2)(1 - \frac{x_3}{l_k}), \tag{3.7.24}$$

$$v_4^{(k)} = \beta_3(1 - \frac{x_3}{l_k}), \tag{3.7.25}$$

where ϕ_I, ϕ_{II} are the same as in Section 3.6.

Evaluating the right–hand sides of (3.7.18)–(3.7.21) we obtain

$$\mathfrak{F}_1^{(k)} = -\frac{6S_1}{l_k^3}\Big\{2(\alpha_1 - \beta_3 a_2^{(k)}) + l_k\beta_2\Big\}, \tag{3.7.26}$$

$$\mathfrak{F}_2^{(k)} = -\frac{6S_2}{l_k^3}\Big\{2(\alpha_2 + \beta_3 a_2^{(k)}) - l_k\beta_1\Big\}, \tag{3.7.27}$$

$$\mathfrak{F}_3^{(k)} = -\frac{S_3}{l_k}\Big\{\alpha_3 + \beta_1 a_2^{(k)} - a_1^{(k)}\beta_2\Big\}, \tag{3.7.28}$$

$$\mathfrak{M}_1^{(k)} = \frac{2S_2}{l_k^2}\Big\{3(\alpha_2 + \beta_3 a_1^{(k)}) - 2l_k\beta_1\Big\}, \tag{3.7.29}$$

$$\mathfrak{M}_2^{(k)} = -\frac{2S_1}{l_k^2}\Big\{3(\alpha_1 - \beta_3 a_2^{(k)}) + 2l_k\beta_2\Big\}, \tag{3.7.30}$$

$$\mathfrak{M}_3^{(k)} = -\frac{S_4\beta_3}{l_k}. \tag{3.7.31}$$

Substituting these expressions into the balance relations (3.7.9), (3.7.10) we arrive at the following system of linear equations

$$\begin{pmatrix} \varepsilon^2\mathcal{A} & \varepsilon^2\mathcal{D}^t \\ \varepsilon^2\mathcal{D} & \mathcal{B} + \varepsilon^2\mathcal{E} \end{pmatrix} (\alpha_1, \alpha_2, \beta_3, \alpha_3, \beta_1, \beta_2)^t = \varepsilon^{-2}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{M}_3, \mathfrak{F}_3, \mathfrak{M}_1, \mathfrak{M}_2)^t, \tag{3.7.32}$$

where the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ are given by (3.7.1)–(3.7.4)

Due to (3.7.5)–(3.7.7), the matrix of the system (3.7.32) is positive definite, and, hence, there exists a unique set of coefficients α_j, β_j, $j = 1, 2, 3$, solving this system. The components of the lock forces and moments are evaluated by (3.7.26)–(3.7.31).

The system (3.7.32) is the basic relation in the described model. The diagonal 3×3 blocks $\mathcal{A}$ and $\mathcal{B} + \mathcal{E}$ can be interpreted as the normalized stiffness matrices for transversal and longitudinal displacements. The matrix $\mathcal{D}$ is the coupling matrix.

The matrix of the system (3.7.32) degenerates for $\varepsilon = 0$. This causes for displacement components of the body Ω to have different orders of magnitude for transversal and longitudinal types of the load.

(i) First, suppose that $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{M}_3$ in (3.7.32) do not vanish simultaneously. Using (3.7.32) we obtain

$$\mathcal{A}(\alpha_1, \alpha_2, \beta_3)^t = \varepsilon^{-4}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{M}_3)^t + O(\varepsilon^{-2}).$$

and

$$\mathcal{B}(\alpha_3, \beta_1, \beta_2)^t + \varepsilon^2 \mathcal{D}(\alpha_1, \alpha_2, \beta_3)^t = \varepsilon^{-2}(\mathfrak{F}_3, \mathfrak{M}_1, \mathfrak{M}_2)^t + O(1).$$

Although the quantities α_3, β_1, β_2 are small in comparison with α_1, α_2, β_3, we need them for the evaluation of the longitudinal components of lock forces (see (3.7.28)). The components of lock forces and moments are determined by (3.7.26)–(3.7.31). The last terms in curly brackets (3.7.26), (3.7.27), (3.7.29)–(3.7.31) are much smaller than the remaining terms (their ratio is of order $O(\varepsilon^2)$).

The above analysis shows that one can use a simplified model of the pile structure where the local rotations with respect to Ox_1, Ox_2 are neglected, i.e. the right–hand sides of two last relations (3.7.17) are zero. Consequently, the stiffness matrix for this model has the block–diagonal structure.

(ii) Second, consider the case where $\mathfrak{F}_1 = \mathfrak{F}_2 = \mathfrak{M}_3 = 0$. Again, neglecting terms of order ε^2 one has

$$\mathcal{B}(\alpha_3, \beta_1, \beta_2)^t = \varepsilon^{-2}(\mathfrak{F}_3, \mathfrak{M}_1, \mathfrak{M}_2)^t + O(1), \tag{3.7.33}$$

$$\mathcal{A}(\alpha_1, \alpha_2, \beta_3)^t = -\mathcal{D}^t(\alpha_3, \beta_1, \beta_2)^t. \tag{3.7.34}$$

It follows from (3.7.26)–(3.7.31) that all components of lock forces and moments, except $\mathfrak{F}_3^{(k)}$, are $O(\varepsilon^2)$, and

$$\mathfrak{F}_3^{(k)} = -\frac{\varepsilon^2 D_3}{l_k}\{\alpha_3 + \beta_1 a_2^{(k)} - a_1^{(k)}\beta_2\}.$$

Here, we also obtain the block–diagonal structure of the stiffness matrix. However, the components of the rigid–body translations and rotations have the same order of magnitude.

4 Asymptotic expansion of the solution

In the present section we construct an asymptotic series for a solution of a mixed boundary value problem in the multi–structure which consists of the union of a three–dimensional cap Ω and a set of cylindrical extensions $\Pi_\varepsilon^{(i)}$, $i = 1, \ldots, K$.

Section 4.1 deals with the leading ansatz of the asymptotic expansion of the displacement field in a thin elastic cylinder. Local coordinates are defined in Section 4.2. A multi–scaled asymptotic representation of the right–hand sides is specified in Section 4.3. In Section 4.4 we describe the asymptotic expansion of the displacement field in Ω_ε. The general ansatz of this asymptotic series is constructed in Sections 4.5–4.9. In Section 4.10 we present the recurrent sequence for construction of all terms of the asymptotic series.

4.1 Special solutions of the Lamé system in a thin elastic rod

Consider a thin cylinder $\Pi_\varepsilon = \{\mathbf{x} : 0 < x_3 < l, \varepsilon^{-1}(x_1, x_2) \in g \subset \mathbb{R}^2\}$. Here $\varepsilon > 0$ is a non–dimensional small parameter, g is a bounded domain with the Lipschitz boundary. In this section we introduce $\zeta = \varepsilon^{-1}(x_1, x_2)$ and $z = x_3$. The notations used follow Section 2.3.

Theorem 4.1 *Let $\boldsymbol{\mathcal{F}} \in C^\infty([0,l];(L_2(g))^3)$ and $\boldsymbol{\mathcal{P}} \in C^\infty([0,l];(L_2(\partial g))^3)$. Then the vector–valued function*

$$\mathfrak{U}(\zeta,z,\varepsilon) = \varepsilon^{-4}\sum_{k=1}^{2}\sum_{j=0}^{3}\varepsilon^j\frac{d^j}{dz^j}v_k(z)\boldsymbol{\phi}^{(k,j)}(\zeta) + \varepsilon^{-2}\sum_{k=3}^{4}\sum_{j=0}^{1}\varepsilon^j\frac{d^j}{dz^j}v_k(z)\boldsymbol{\phi}^{(k,j)}(\zeta) + \mathbf{U}(\zeta,z), \quad (4.1.1)$$

satisfies the equations

$$-\mathbf{L}(\partial_{\mathbf{x}})\mathfrak{U} - \varepsilon^{-2}\boldsymbol{\mathcal{F}}(\zeta,z) = \varepsilon^{-1}\boldsymbol{\mathcal{F}}^{(0)}(\zeta,z) + \boldsymbol{\mathcal{F}}^{(1)}(\zeta,z),\ \mathbf{x}\in\Pi_\varepsilon, \quad (4.1.2)$$

$$\boldsymbol{\sigma}^{(n)}(\mathfrak{U};\mathbf{x}) - \varepsilon^{-1}\boldsymbol{\mathcal{P}}(\zeta,z) = \boldsymbol{\mathcal{P}}^{(0)}(\zeta,z),\ (\zeta,z)\in\partial g\times[0,l], \quad (4.1.3)$$

with $\boldsymbol{\mathcal{F}}^{(j)} \in C^\infty([0,l];(L_2(g))^3)$, $j=0,1$, and $\boldsymbol{\mathcal{P}}^{(0)} \in C^\infty([0,l];(L_2(\partial g))^3)$, provided

$$\frac{d^4}{dz^4}v_k(z) = -D_k^{-1}\{\int_g \mathcal{F}_k(\zeta,z)d\zeta + \int_{\partial g}\mathcal{P}_k(\zeta,z)dl_\zeta\},\ k=1,2, \quad (4.1.4)$$

$$\frac{d^2}{dz^2}v_3(z) = D_3^{-1}\{\int_g \mathcal{F}_3(\zeta,z)d\zeta + \int_{\partial g}\mathcal{P}_3(\zeta,z)dl_\zeta\}, \quad (4.1.5)$$

$$\frac{d^2}{dz^2}v_4(z) = D_4^{-1}\{\int_g (\zeta_1\mathcal{F}_2(\zeta,z) - \zeta_2\mathcal{F}_1(\zeta,z))d\zeta + \int_{\partial g}(\zeta_1\mathcal{P}_2(\zeta,z) - \zeta_2\mathcal{P}_1(\zeta,z))dl_\zeta\}, \quad (4.1.6)$$

and the vector $\mathbf{U} \in C^\infty([0,l];(H^1(g))^3)$ satisfies the two–dimensional boundary value problem

$$\begin{aligned}-\mathbf{L}_0\mathbf{U} &= \boldsymbol{\mathcal{F}}(\zeta,z) + \mathbf{L}_1\Big(\frac{d^4v_1}{dz^4}\boldsymbol{\phi}^{(1,3)} + \frac{d^4v_2}{dz^4}\boldsymbol{\phi}^{(2,3)} + \frac{d^2v_4}{dz^2}\boldsymbol{\phi}^{(4,1)} + \frac{d^2v_3}{dz^2}\boldsymbol{\phi}^{(3,1)}\Big)\\ &+\mathbf{L}_2\Big(\frac{d^4v_1}{dz^4}\boldsymbol{\phi}^{(1,2)} + \frac{d^4v_2}{dz^4}\boldsymbol{\phi}^{(2,2)} + \frac{d^2v_3}{dz^2}\boldsymbol{\phi}^{(3,0)} + \frac{d^2v_4}{dz^2}\boldsymbol{\phi}^{(4,0)}\Big), \zeta\in g, \qquad (4.1.7)\\ \mathbf{T}_0\mathbf{U} &= \boldsymbol{\mathcal{P}}(\zeta,z) - \mathbf{T}_1(\frac{d^4v_1}{dz^4}\boldsymbol{\phi}^{(1,3)} + \frac{d^4v_2}{dz^4}\boldsymbol{\phi}^{(2,3)}\\ &+\frac{d^2v_3}{dz^2}\boldsymbol{\phi}^{(3,1)} + \frac{d^2v_4}{dz^2}\boldsymbol{\phi}^{(4,1)}), \zeta\in\partial g, \qquad (4.1.8)\end{aligned}$$

which is solvable for every $z\in[0,l]$. Here $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{T}_1$ are matrix operators defined by (2.3.6) and (2.3.8).

Proof. Applying the operators

$$\mathbf{L}(\partial_{\mathbf{x}}) = \varepsilon^{-2}\mathbf{L}_0(\partial_\zeta) + \varepsilon^{-1}\mathbf{L}_1(\partial_\zeta)\frac{d}{dz} + \mathbf{L}_2\frac{d^2}{dz^2}, \quad (4.1.9)$$

$$\boldsymbol{\sigma}^{(n)}(\cdot;\mathbf{x}) = \varepsilon^{-1}\mathbf{T}_0(\mathbf{n},\partial_\zeta) + \mathbf{T}_1(\mathbf{n})\frac{d}{dz}, \quad (4.1.10)$$

(compare with (2.3.3), (2.3.4)) to the vector $\mathfrak{U}$ and using the equalities (2.3.13), (2.3.14) we derive

$$\begin{aligned}\mathbf{L}(\partial_{\mathbf{x}})\mathfrak{U} = \varepsilon^{-2}\sum_{k=1}^{2}\Big\{\frac{d^4v_k(z)}{dz^4}(\mathbf{L}_2\boldsymbol{\phi}^{(k,2)} + \mathbf{L}_1(\partial_\zeta)\boldsymbol{\phi}^{(k,3)})\\ + \frac{d^2v_{2+k}(z)}{dz^2}(\mathbf{L}_2\boldsymbol{\phi}^{(2+k,0)} + \mathbf{L}_1(\partial_\zeta)\boldsymbol{\phi}^{(2+k,1)})\Big\} + \varepsilon^{-2}\mathbf{L}_0\mathbf{U} + \varepsilon^{-1}\boldsymbol{\mathcal{F}}^{(0)}(\zeta,z) + \boldsymbol{\mathcal{F}}^{(1)}(\zeta,z), \quad (4.1.11)\end{aligned}$$

where

$$\boldsymbol{\mathcal{F}}^{(0)} = \sum_{k=1}^{2} \left\{ \frac{d^5 v_k(z)}{dz^5} \mathbf{L}_2 \boldsymbol{\phi}^{(k,3)} + \frac{d^3 v_{k+2}(z)}{dz^3} \mathbf{L}_2 \boldsymbol{\phi}^{(k+2,1)} \right\} + \mathbf{L}_1 \mathbf{U},$$

and

$$\boldsymbol{\mathcal{F}}^{(1)} = \mathbf{L}_2 \mathbf{U}.$$

Similarly,

$$\boldsymbol{\sigma}^{(n)}(\mathfrak{U}; \mathbf{x}) = \varepsilon^{-1} \mathbf{T}_1(\mathbf{n}) \sum_{k=1}^{2} \left\{ \frac{d^4 v_k(z)}{dz^4} \boldsymbol{\phi}^{(k,3)} + \frac{d^2 v_{k+2}(z)}{dz^2} \boldsymbol{\phi}^{(k+2,1)} \right\}$$
$$+ \varepsilon^{-1} \mathbf{T}_0(\mathbf{n}, \partial_\zeta) \mathbf{U} + \boldsymbol{\mathcal{P}}^{(0)}, \tag{4.1.12}$$

where

$$\boldsymbol{\mathcal{P}}^{(0)} = \mathbf{T}_1(\mathbf{n}) \mathbf{U}.$$

Substituting (4.1.11), (4.1.12) into the system (4.1.2) and the traction boundary condition (4.1.3) and equating the coefficients near ε^{-2}, ε^{-1}, respectively, we arrive at the two–dimensional boundary value problem (4.1.7), (4.1.8) which must be satisfied for every $z \in [0, l]$. This problem is solvable if and only if the right–hand sides are orthogonal to $\boldsymbol{\psi}^{(j)}$, $j = 1, 2, 3, 4$, for every $z \in [0, l]$. Due to the biorthogonality condition (2.3.24), this is equivalent to (4.1.4)–(4.1.6). □

In order to provide the uniqueness of $\mathbf{U}$, we assume

$$\int_g \mathbf{U}(\boldsymbol{\zeta}, z) d\boldsymbol{\zeta} = 0, \quad \int_g \Big(\zeta_1 U_2^{(0)}(\boldsymbol{\zeta}, z) - \zeta_2 U_1^{(0)}(\boldsymbol{\zeta}, z) \Big) d\boldsymbol{\zeta} = 0. \tag{4.1.13}$$

We would like to emphasize that the operators $\mathbf{L}_0$, $\mathbf{T}_0$ have the block–diagonal structure which shows that (4.1.7), (4.1.8) can be split into the plane strain and the anti–plane shear boundary value problems discussed in Section 3.5.

Remark 4.1. *It is appropriate to mention that the right–hand sides (4.1.2), (4.1.3) and (4.1.7), (4.1.8) are specified uniquely by $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$.*

4.2 Local coordinates

We recall that the global system of coordinates $\mathbf{x} = (x_1, x_2, x_3)$ is chosen in such a way that the junction points $\mathbf{a}^{(i)}$, $i = 1, \dots, K$, are located on the plane $x_3 = 0$, and all cylinders are parallel to the Ox_3 axis directed downwards (see Fig.1). We suppose that the axes Ox_1, Ox_2 are parallel to the principal inertia axes of the cross–section of a thin cylinder, and the first two coordinates $a_i^{(j)}$, $i = 1, 2$, coincide with the inertia centre of the cross–section of the cylinder $\Pi_\varepsilon^{(j)}$.

In the vicinity of the junction points we shall use the coordinates

$$\mathbf{X}^{(j)} = \varepsilon^{-1}(\mathbf{x} - \mathbf{a}^{(j)}), \ j = 1, \dots, K.$$

The coordinate system

$$(\zeta_1^{(j)}, \zeta_2^{(j)}, z) = (X_1^{(j)}, X_2^{(j)}, x_3),$$

is introduced in $\Pi_\varepsilon^{(i)}$.

Scaled variables

$$\mathbf{Y}^{(j)} = (X_1^{(j)}, X_2^{(j)}, \varepsilon^{-1}(x_3 - l^{(j)}))$$

are used in the vicinity of the end of $\Pi_\varepsilon^{(j)}$.

We define the cut–off function for Ω

$$\mathfrak{X}(\mathbf{x},\varepsilon)=\prod_{j=1}^{K}\chi(\mathbf{X}^{(j)}),\ \mathbf{x}\in\Omega,$$

where χ is smooth, $\chi(\mathbf{X})=0$ for $\|\mathbf{X}\|<1$ and $\chi(\mathbf{X})=1$ for $\|\mathbf{X}\|>2$; $\mathfrak{X}=0$ in the cylinders $\Pi_\varepsilon^{(j)}$.

Also, in thin cylinders we use the smooth cut–off function $\eta(Z)$ which equals 0 for $Z<1$, and $\eta(Z)=1$ for $Z>2$.

The third smooth cut–off function Ξ for the junction layer is defined in $\mathbb{R}^2$ in such a way that $\Xi(\mathbf{x})=0$ for $(x_1^2+x_2^2)^{1/2}>d/2$, and $\Xi(\mathbf{x})=1$ as $(x_1^2+x_2^2)^{1/2}<d/4$; $d=\min_{i,j}\operatorname{dist}(\mathbf{a}^{(i)},\mathbf{a}^{(j)})$.

4.3 The asymptotic representation of the right–hand sides

The right–hand sides of the boundary value problem (1.2.1)–(1.2.3) are represented as asymptotic sums of four types of terms which correspond to the region Ω, the junction regions, thin rods and the bottom region. Namely, $\mathbf{F},\mathbf{P}$ and $\Phi^{(j)}$, $j=1,\dots,K$, in (1.2.1)–(1.2.3) admit the asymptotic expansions

$$\begin{aligned}\mathbf{F}(\mathbf{x},\varepsilon)\sim\sum_{k=0}^{\infty}\varepsilon^k\Big\{&\mathfrak{X}(\mathbf{x},\varepsilon)\mathbf{f}^{(k)}(\mathbf{x})+\sum_{j=1}^{K}\Big[\varepsilon^{-4}\mathbf{F}^{(k,j)}(\mathbf{X}^{(j)})\\&+\varepsilon^{-2}\boldsymbol{\mathcal{F}}^{(k,j)}(\zeta^{(j)},z)\eta(\frac{z}{\varepsilon})+\varepsilon^{-4}\mathbf{G}^{(k,j)}(\mathbf{Y}^{(j)})\Big]\Big\},\ \mathbf{x}\in\Omega_\varepsilon,\end{aligned}\tag{4.3.1}$$

$$\begin{aligned}\mathbf{P}(\mathbf{x},\varepsilon)\sim\sum_{k=0}^{\infty}\varepsilon^k\Big\{&\mathfrak{X}(\mathbf{x},\varepsilon)\mathbf{p}^{(k)}(\mathbf{x})+\sum_{j=1}^{K}\Big[\varepsilon^{-3}\mathbf{P}^{(k,j)}(\mathbf{X}^{(j)})\\&+\varepsilon^{-1}\boldsymbol{\mathcal{P}}^{(k,j)}(\zeta^{(j)},z)\eta(\frac{z}{\varepsilon})+\varepsilon^{-3}\mathbf{H}^{(k,j)}(\mathbf{Y}^{(j)})\Big]\Big\},\ \mathbf{x}\in\partial\Omega_\varepsilon\setminus\bar{S}_\varepsilon,\end{aligned}\tag{4.3.2}$$

$$\Phi^{(j)}(\mathbf{x},\varepsilon)\sim\sum_{k=0}^{\infty}\varepsilon^{k-2}\Phi^{(k,j)}(Y_1^{(j)},Y_2^{(j)}),\ \mathbf{x}\in S_\varepsilon.\tag{4.3.3}$$

It is assumed that $\mathbf{f}^{(k)}\in(L_2(\Omega))^3$, $\mathbf{p}^{(k)}\in(L_2(\partial\Omega))^3$ are smooth in the vicinity of the junction points. The vector–valued functions $\mathbf{F}^{(k,j)}$, $\mathbf{P}^{(k,j)}$ belong to $(L_2(G))^3$, $(L_2(\partial G))^3$ and have compact supports; $\boldsymbol{\mathcal{F}}^{(k,j)}\in C^\infty([0,l_j];(L_2(g))^3)$, $\boldsymbol{\mathcal{P}}^{(k,j)}\in C^\infty([0,l_j];(L_2(\partial g))^3)$. We make the following technical assumption which does not cause any loss of generality

$$\int_G\mathbf{F}^{(0,j)}d\mathbf{X}+\int_{\partial G}\mathbf{P}^{(0,j)}ds=0.\tag{4.3.4}$$

The vector functions $\mathbf{G}^{(k,j)}\in L_2(\Pi_-)$, $\mathbf{H}^{(k,j)}\in L_2(\partial\Pi_-\setminus\bar{S})$ have compact supports, and $\Phi^{(k,j)}\in H^{1/2}(g)$.

By $\sim$ in expansions (4.3.1), (4.3.2) we mean that the L_2-norm of the difference between the left–hand sides and the first n terms in the asymptotic expansion is estimated by Const ε^{n-3}. A similar statement holds for (4.3.3) with $H^{1/2}(g)$-norm taken instead of L_2-norm.

4.4 Description of the asymptotic series for displacement in Ω_ε

The displacement field $\mathbf{u}(\mathbf{x},\varepsilon)$ will be sought in the form of the asymptotic series

$$\mathbf{u}(\mathbf{x},\varepsilon) \sim \sum_{k=-1}^{\infty} \varepsilon^k \mathbf{u}^{(k)}(\mathbf{x},\varepsilon), \tag{4.4.1}$$

with

$$\begin{aligned}\mathbf{u}^{(k)}(\mathbf{x},\varepsilon) &= \varepsilon^{-4}\{\sum_{i=1}^{2}\alpha_i^{(k)}\mathbf{e}^{(i)} + \varepsilon^2\alpha_3^{(k)}\mathbf{e}^{(3)} + (\varepsilon^2\sum_{i=1}^{2}\beta_i^{(k)}\mathbf{e}^{(i)} + \beta_3^{(k)}\mathbf{e}^{(3)})\times\mathbf{x}\}\\ &+\mathfrak{X}(\mathbf{x},\varepsilon)\mathbf{u}_\Omega^{(k)}(\mathbf{x}) + \sum_{j=1}^{K}\Big\{\eta(\frac{z}{\varepsilon})\mathfrak{U}^{(k,j)}(\zeta,z,\varepsilon)\\ &+\varepsilon^{-2}\Xi(\mathbf{x}-\mathbf{a}^{(j)})\boldsymbol{\mathcal{W}}^{(k,j)}(\mathbf{X}^{(j)}) + \varepsilon^{-2}\eta(\frac{z}{\varepsilon})\mathbf{V}^{(k,j)}(\mathbf{Y}^{(j)})\Big\}.\end{aligned} \tag{4.4.2}$$

Here $\alpha_i^{(k)}$, $\beta_i^{(k)}$, $i=1,2,3$, are constant, and the terms in the first curly brackets in (4.4.2) represent rigid body displacements; these coefficients are defined by the system of linear equations of the form (3.7.32). The vector valued function $\mathbf{u}_\Omega^{(k)}$ solves the problem of the type (3.2.1), (3.2.2), and can be represented as

$$\mathbf{u}_\Omega^{(k)}(\mathbf{x}) = \mathbf{u}_r^{(k)}(\mathbf{x}) + \mathbf{u}_s^{(k)}(\mathbf{x}). \tag{4.4.3}$$

The field $\mathbf{u}_r^{(k)} \in (H^1(\Omega))^3$ is a solution of the boundary value problem (3.2.12), (3.2.13) with self–balanced right–hand sides from L_2 which are smooth in the vicinity of the junction points; hence it is also smooth in neighbourhoods of the junction points. The vector function $\mathbf{u}_s^{(k)}$ is defined by

$$\mathbf{u}_s^{(k)}(\mathbf{x}) = \sum_{j=1}^{K}\Big\{\boldsymbol{\mathcal{N}}(\mathbf{x},\mathbf{a}^{(i)})\mathfrak{F}^{(k,j)} + \frac{1}{2}(\nabla_{\mathbf{x}^*}\times\boldsymbol{\mathcal{N}}^t(\mathbf{x},\mathbf{x}^*)|_{\mathbf{x}^*=\mathbf{a}^{(i)}})^t\mathfrak{M}^{(k,j)}\Big\}, \tag{4.4.4}$$

where $\boldsymbol{\mathcal{N}}$ is the Green tensor from Section 3.2; $\mathfrak{F}^{(k,j)}$, $\mathfrak{M}^{(k,i)}$ are constant vectors, which satisfy the following balance relations

$$\mathbb{F}(\mathbf{f}^{(k)},\mathbf{p}^{(k)},\Omega) + \textstyle\sum_{j=1}^{K}\mathfrak{F}^{(k,j)} = 0, \tag{4.4.5}$$

$$\mathbb{M}(\mathbf{f}^{(k)},\mathbf{p}^{(k)},\Omega) + \textstyle\sum_{j=1}^{K}\Big\{\mathfrak{M}^{(k,j)} + \mathbf{a}^{(j)}\times\mathfrak{F}^{(k,j)}\Big\} = 0, \tag{4.4.6}$$

with $\mathbb{F}$ and $\mathbb{M}$ being introduced in (3.2.7), (3.2.8).

The product

$$\mathfrak{X}(\mathbf{x},\varepsilon)\mathbf{u}_\Omega^{(k)}$$

is extended as zero into thin cylinders.

The ansatz $\mathfrak{U}^{(k,j)}$ corresponds to the displacement in a thin cylinder $\Pi_\varepsilon^{(j)}$, and it is given by

$$\begin{aligned}\mathfrak{U}^{(k,j)}(\zeta,z,\varepsilon) &= \varepsilon^{-4}\sum_{i=1}^{2}\sum_{q=0}^{3}\varepsilon^q\frac{d^q}{dz^q}v_i^{(k,j)}(z)\boldsymbol{\phi}^{(i,q)}(\zeta)\\ &+\varepsilon^{-2}\sum_{i=3}^{4}\sum_{q=0}^{1}\varepsilon^q\frac{d^q}{dz^q}v_i^{(k,j)}(z)\boldsymbol{\phi}^{(i,q)}(\zeta) + \mathbf{U}^{(k,j)}(\zeta,z),\end{aligned} \tag{4.4.7}$$

where the functions $v_i^{(k,j)}$ depend on the longitudinal variable and are subjected to the ordinary differential equations of the type (4.1.4)–(4.1.6); the vector function $\mathbf{U}^{(k,j)}$ belongs to $\mathbb{C}^\infty([0,l_j];(H^1(g))^3)$ and for every $z \in [0,l_j]$ it satisfies a boundary value problem on the cross–section g of the form (4.1.7), (4.1.8), and the orthogonality conditions (4.1.13).

The junction layer $\boldsymbol{\mathcal{W}}^{(k,j)}$ and constants $d_i^{(I,k,j)}$, $d_i^{(II,k,j)}$, $i=1,2,\dots,6$, give a solution of the boundary value problem of the type (3.3.8), (3.3.9) in G, and $\boldsymbol{\mathcal{W}}^{(k,j)}$ decays at infinity (see Theorem 3.3).

The bottom layer $\mathbf{V}^{(k,j)}$ and constants $c_q^{(k,j)}$, $q=1,2,\dots,6$, are specified as a solution of (3.4.3)–(3.4.7), where $\mathbf{V}^{(k,j)}$ decays at infinity (see Theorem 3.5). The constants $d_i^{(I,k,j)}$, $d_i^{(II,k,j)}$, $c_i^{(k,j)}$ will be included in the boundary conditions for the functions $v_i^{(k,j)}$.

In the following five sections we describe the algorithm for the calculation of the term $\mathbf{u}^{(m)}$ of the expansion (4.4.1). Looking at the expansion (4.4.2) one can deduce that the first group of terms in curly brackets leaves a discrepancy in the Dirichlet boundary condition on S_ε; the term $\mathfrak{X}\mathbf{u}_\Omega^{(k)}$ will produce an error in the vicinity of the junction points; the term $\eta\mathfrak{U}^{(k,j)}$ yields a discrepancy in the vicinity of the junction point as well as at S_ε. Since $\boldsymbol{\mathcal{W}}^{(k,j)}$ decays exponentially in the cylindrical extension, we have to take into account just an error in the three–dimensional region Ω. The last term does not give any discrepancy in our asymptotic expansion. In the asymptotic procedure the right–hand sides absorb discrepancy terms from the previous steps. The modified right–hand sides will be denoted by the same letters with "$\sim$"; the structure of the modified right–hand sides will be described in each case.

To simplify formulae, we shall use the notations $\mathbf{X},\mathbf{Y},\boldsymbol{\zeta}$ without an upper index, and we hope that it will not cause any confusion.

4.5 Expansions for displacement in a thin rod

The field $\mathfrak{U}^{(m,j)}$ is a special solution of the elasticity problem in a thin cylinder described in Theorem 4.1; the vector functions $\boldsymbol{\mathcal{F}}$, $\boldsymbol{\mathcal{P}}$ in (4.1.2) and (4.1.3) should be replaced by $\tilde{\boldsymbol{\mathcal{F}}}^{(m,j)}$, $\tilde{\boldsymbol{\mathcal{P}}}^{(m,j)}$, absorbing $\boldsymbol{\mathcal{F}}^{(m,j)}$, $\boldsymbol{\mathcal{P}}^{(m,j)}$ and discrepancy terms in the right–hand sides (4.1.2), (4.1.3) which occur on the previous steps of the asymptotic algorithm. Therefore, due to Remark 4.1, $\tilde{\boldsymbol{\mathcal{F}}}^{(m,j)}$, $\tilde{\boldsymbol{\mathcal{P}}}^{(m,j)}$ depend on $\boldsymbol{\mathcal{F}}^{(k,j)}$, $\boldsymbol{\mathcal{P}}^{(k,j)}$, $k \le m$, only.

The functions $v_k^{(m,j)}$ in the expansion (4.1.1) for the field $\mathfrak{U}^{(m,j)}$ satisfy the ordinary differential equations (4.1.4)–(4.1.6) on $(0,l_j)$, and the vector function $\mathbf{U}^{(k,j)}$ can be found as a solution of the boundary value problem (4.1.7), (4.1.8). In all these equations the vector functions $\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}}$ should be replaced by $\tilde{\boldsymbol{\mathcal{F}}}^{(m,j)}$, $\tilde{\boldsymbol{\mathcal{P}}}^{(m,j)}$.

The boundary conditions for $v_k^{(m,j)}$ will be derived in the sub–sections related to the analysis of the junction and the bottom layers. To this end we shall need the following expansion of the ansatz $\mathfrak{U}^{(m,j)}$ in the vicinity of the end point $z=0$

$$\mathfrak{U}^{(m,j)}(\zeta,z,\varepsilon) = \sum_{k=1}^{4}\left\{\varepsilon^{-m_k} v_k^{(m,j)}(0)\boldsymbol{\psi}^{(k)}(\mathbf{X}) + \varepsilon^{-1}\frac{d^{m_k-1}v_k^{(m,j)}}{dz^{m_k-1}}(0)\boldsymbol{\xi}^{(k)}(\mathbf{X})\right\}$$

$$+\sum_{k=1}^{2}\left\{\varepsilon^{-3}\frac{dv_k^{(m,j)}}{dz}(0)\boldsymbol{\psi}^{(4+k)}(\mathbf{X}) + \varepsilon^{-3}\frac{d^2v_k^{(m,j)}}{dz}(0)\boldsymbol{\xi}^{(4+k)}(\mathbf{X})\right\} + \tilde{\mathfrak{U}}^{(m,j)}(\zeta,X_3,\varepsilon), \tag{4.5.1}$$

which follows from (4.4.7). Here $\boldsymbol{\psi}^{(k)}$, $\boldsymbol{\xi}^{(k)}$ and m_k are the same as in Section 2.3; $\tilde{\mathfrak{U}}^{(m,j)}(\zeta,X_3,\varepsilon)$ is smooth with respect to X_3, ε, and belongs to $(H^1(g))^3$ as a function of ζ.

Near the end point $z = l_j$ of a thin rod $\Pi_\varepsilon^{(j)}$ one can use the expansion of $\mathfrak{U}^{(m,j)}$ in scaled coordinates $\mathbf{Y}$. This expansion is similar to (4.5.1) and has the form

$$\mathfrak{U}^{(m,j)}(\zeta, z, \varepsilon) = \sum_{k=1}^{4} \Big\{ \varepsilon^{-m_k} v_k^{(m,j)}(l_j) \boldsymbol{\psi}^{(k)}(\mathbf{Y}) + \varepsilon^{-1} \frac{d^{m_k-1} v_k^{(m,j)}}{dz^{m_k-1}}(l_j) \boldsymbol{\xi}^{(k)}(\mathbf{Y}) \Big\}$$

$$+ \sum_{k=1}^{2} \Big\{ \varepsilon^{-3} \frac{dv_k^{(m,j)}}{dz}(l_j) \boldsymbol{\psi}^{(4+k)}(\mathbf{Y}) + \varepsilon^{-3} \frac{d^2 v_k^{(m,j)}}{dz}(l_j) \boldsymbol{\xi}^{(4+k)}(\mathbf{Y}) \Big\} + \tilde{\mathfrak{V}}^{(m,j)}(\zeta, Y_3, \varepsilon), \qquad (4.5.2)$$

where $\tilde{\mathfrak{V}}^{(m,j)}$ is smooth with respect to Y_3, ε, and belongs to $(H^1(g))^3$ as a function of ζ. The vector functions $\tilde{\mathfrak{U}}^{(m,j)}$, $\tilde{\mathfrak{V}}^{(m,j)}$ include high order derivatives of the functions $v_k^{(m,j)}$, $k = 1, 2, 3, 4$, $\mathbf{U}^{(m,j)}$, and, therefore, depend on the $\boldsymbol{\mathcal{F}}^{(k,j)}$, $\boldsymbol{\mathcal{P}}^{(k,j)}$, $k \le m$ (see Remark 4.1).

4.6 The junction layer

Let us introduce the notation

$$\boldsymbol{\sigma}^{(n)}(\mathbf{x}, \partial_{\mathbf{x}}) = \boldsymbol{\sigma}^{(n)}(\cdot; \mathbf{x}),$$

and let $[A, B] = AB - BA$ denote the commutator of the operators A and B.

The junction layer $\boldsymbol{\mathcal{W}}^{(m,j)}$ should compensate the right–hand sides $\mathbf{F}^{(m,j)}$, $\mathbf{P}^{(m,j)}$ and the discrepancy terms caused by the fields in Ω and in thin cylinders.

First, we compensate the error terms brought by the vector functions $\tilde{\mathfrak{U}}^{(k,j)}$, $k \le m$, in (4.5.1), by $\mathbf{u}_s^{(m)}$ and by the bounded (near the junction points) part of $\mathbf{u}_\Omega^{(k)}$ with $k < m$ (see (4.4.4), (4.4.3)). It can be done with the solution of the junction layer problem

$$\begin{pmatrix} -L(\partial_{\mathbf{X}}) \\ \boldsymbol{\sigma}^{(n)}(\mathbf{X}, \partial_{\mathbf{X}}) \end{pmatrix} \boldsymbol{\mathcal{W}}^{(m,j)}(\mathbf{X}) = \begin{pmatrix} \tilde{\mathbf{F}}^{(m,j)}(\mathbf{X}) \\ \tilde{\mathbf{P}}^{(m,j)}(\mathbf{X}) \end{pmatrix}$$

$$+ \begin{pmatrix} [L(\partial_{\mathbf{X}}), \chi(\mathbf{X})] \\ -[\boldsymbol{\sigma}^{(n)}(\mathbf{X}, \partial_{\mathbf{X}}), \chi(\mathbf{X})] \end{pmatrix} \Big\{ \boldsymbol{\mathcal{M}}(\mathbf{X}, 0) \mathfrak{F}^{(m-1,j)} + \frac{1}{2} (\nabla_{\mathbf{X}^*} \times \boldsymbol{\mathcal{M}}^t(\mathbf{X}, \mathbf{X}^*)|_{\mathbf{X}^*=0})^t \mathfrak{M}^{(m,j)} \Big\}$$

$$+ \begin{pmatrix} [\mathbf{L}(\partial_{\mathbf{X}}), \eta(X_3)] \\ -[\boldsymbol{\sigma}^{(n)}(\mathbf{X}, \partial_{\mathbf{X}}), \eta(X_3)] \end{pmatrix} \sum_{k=1}^{6} \Big\{ d_k^{(I,m,j)} \boldsymbol{\xi}^{(k)}(\mathbf{X}) + d_k^{(II,m,j)} \boldsymbol{\psi}^{(k)}(\mathbf{X}) \Big\}, \ \text{ on } G \times \partial G. \qquad (4.6.1)$$

The vector–valued functions $\tilde{\mathbf{F}}^{(m,j)}$, $\tilde{\mathbf{P}}^{(m,j)}$ absorb $\mathbf{F}^{(m,j)}$, $\mathbf{P}^{(m,j)}$ and two kinds of discrepancy terms: first, terms that come from the expansion (4.5.1) and contain $\tilde{\mathfrak{U}}^{(k,j)}$, $k \le m - 2$, and their derivatives, and, consequently, $\boldsymbol{\mathcal{F}}^{(k,j)}, \boldsymbol{\mathcal{P}}^{(k,j)}$ for $k \le m - 2$; second , terms that come from the expansion of the bounded part of $\mathbf{u}_\Omega^{(k)}$, $k \le m - 2$, at junction points.

Using Theorem 3.3 we obtain that there exists a unique set

$$\{\boldsymbol{\mathcal{W}}^{(m,j)}; d_1^{(I,m,j)}, \dots, d_6^{(I,m,j)}; d_1^{(II,m,j)}, \dots, d_6^{(II,m,j)}\}, \qquad (4.6.2)$$

such that the vector function $\boldsymbol{\mathcal{W}}^{(m,j)}$ satisfies the conditions of decay from the Theorem 3.3.

The last term in (4.6.1) should be compensated by the discrepancy that comes from first two sums of the expansion (4.5.1) multiplied by the cut–off function η. Thus, we relate the values $v_k^{(s,j)}$ and their derivatives to the coefficients $d_k^{(i,m,j)}$, $k = 1, \dots, 6; i = I, II$. Using (3.3.11)–(3.3.15) we deduce

$$v_q^{(m+m_q-2,j)}(0) = d_q^{(II,m,j)}, \ q = 1, 2, 3, 4; \qquad (4.6.3)$$

$$\frac{dv_q^{(m+1,j)}}{dz}(0) = d_{4+q}^{(II,m,j)}, \ q = 1, 2; \qquad (4.6.4)$$

$$\frac{d^3 v_q^{(m-1,j)}}{dz^3}(0) = d_q^{(I,m,j)} = D_q^{-1}\Big\{ - \mathfrak{F}_q^{(m-1,j)} + \mathbb{F}_q(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}, G)\Big\},\ q=1,2, \quad (4.6.5)$$

$$\frac{dv_3^{(m-1,j)}}{dz}(0) = d_3^{(I,m,j)} = -D_3^{-1}\Big\{ - \mathfrak{F}_3^{(m-1,j)} + \mathbb{F}_3(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}, G)\Big\}, \quad (4.6.6)$$

$$\frac{dv_4^{(m-1,j)}}{dz}(0) = d_4^{(I,m,j)} = -D_4^{-1}\Big\{ - \mathfrak{M}_3^{(m,j)} + \mathbb{M}_3(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}, G)\Big\}, \quad (4.6.7)$$

$$\frac{d^2 v_q^{(m,j)}}{dz^2}(0) = d_{4+q}^{(I,m,j)} = (-1)^q D_q^{-1}\Big\{ - \mathfrak{M}_{3-q}^{(m,j)} + \mathbb{M}_{3-q}(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}, G)\Big\},$$
$$q=1,2, \quad (4.6.8)$$

where the functionals $\mathbb{F}$, $\mathbb{M}$ were defined in (3.2.7) and (3.2.8).

4.7 Displacement in Ω

The vector function $\mathbf{u}_\Omega^{(m)}$ from (4.4.3) has to compensate the right–hand sides $\mathbf{f}^{(m)}$, $\mathbf{p}^{(m)}$ and the discrepancies caused by the junction layer fields. It satisfies the boundary value problem

$$\begin{pmatrix} -\mathbf{L}(\partial_{\mathbf{x}}) \\ \boldsymbol{\sigma}^{(n)}(\mathbf{x}, \partial_{\mathbf{x}}) \end{pmatrix} \mathbf{u}_\Omega^{(m)}(\mathbf{x}) = \begin{pmatrix} \tilde{\mathbf{f}}^{(m)}(\mathbf{x}) \\ \tilde{\mathbf{p}}^{(m)}(\mathbf{x}) \end{pmatrix} \quad (4.7.1)$$

$$+\sum_{j=1}^{K} \begin{pmatrix} \Big[\mathbf{L}(\partial_{\mathbf{x}}), \Xi(\mathbf{x}-\mathbf{a}^{(j)})\Big] \\ -\Big[\boldsymbol{\sigma}^{(n)}(\mathbf{x}, \partial_{\mathbf{x}}), \Xi(\mathbf{x}-\mathbf{a}^{(j)})\Big] \end{pmatrix} \Big\{ \sum_{q=1}^{2} b_q^{(m,j)} \partial_{x_q} \boldsymbol{\mathcal{B}}^{(q)}(\mathbf{x}-\mathbf{a}^{(j)})$$

$$+\frac{1}{2} b_3^{(m,j)} (\partial_{x_1} \boldsymbol{\mathcal{B}}^{(2)}(\mathbf{x}-\mathbf{a}^{(j)}) + \partial_{x_2} \boldsymbol{\mathcal{B}}^{(1)}(\mathbf{x}-\mathbf{a}^{(j)}))\Big\} \text{ in } \begin{pmatrix} \Omega \\ \partial\Omega \setminus \cup_j \mathbf{a}^{(j)} \end{pmatrix},$$

where $b_q^{(m,j)}$ are the coefficients from the expansion (3.3.5) for the function $\boldsymbol{\mathcal{W}}^{(m,j)}$; the principal force and moment vectors for the last sum in (4.7.1) are equal to zero. The vectors $\tilde{\mathbf{f}}^{(m)}$, $\tilde{\mathbf{p}}^{(m)}$ are given by the sum of $\mathbf{f}^{(m)}$, $\mathbf{p}^{(m)}$ and the self–balanced junction layer discrepancies from the previous steps of the asymptotic algorithm. These error terms are brought by the commutators of the cut–off functions and differential operators of the Lamé system and traction boundary conditions applied to corresponding terms of order $O(\|\mathbf{X}\|^{-k})$, $k \geq 3$, of the asymptotic expansion of the junction layer $\boldsymbol{\mathcal{W}}^{(k,j)}$, $k < m$, in a half–space at infinity. Therefore, the principal force and moment vectors for the right hand side of (4.7.1) are the same as for $\mathbf{f}^{(m)}$, $\mathbf{p}^{(m)}$.

By Theorem 3.1, for the constant vectors $\mathfrak{F}^{(m,j)}$, $\mathfrak{M}^{(m,j)}$, satisfying the balance relations (4.4.5), (4.4.6), there exists a unique solution of the problem (4.7.1) which admits the representation (4.4.3) where the regular part is orthogonal to the rigid–body rotations and translations in $(L_2(\Omega))^3$. Since the right hand side (4.7.1) is smooth in the vicinity of the junction points, the regular part $\mathbf{u}_r^{(m)}$ is also smooth there (see (3.2.12), (3.2.13)).

4.8 The bottom layer

The special solution $\mathfrak{U}^{(m,j)}$ of the elasticity problem in a thin cylinder and the rigid body translations and rotations from the the expansion (4.4.2) give the error terms in the Dirichlet boundary condition on $S_\varepsilon^{(j)}$. Hence, using (4.5.2) we obtain that the boundary value problem for the bottom layer has the form

$$-\mathbf{L}(\partial_{\mathbf{Y}})\mathbf{V}^{(m,j)}(\mathbf{Y}) = \mathbf{G}^{(m,j)}(\mathbf{Y}),\ \mathbf{Y} \in \Pi_-, \tag{4.8.1}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{V}^{(m,j)};\mathbf{Y}) = \mathbf{H}^{(m,j)}(\mathbf{Y})\ ,\mathbf{Y} \in \partial\Pi_- \setminus S, \tag{4.8.2}$$

$$\begin{aligned}\mathbf{V}^{(m,j)}(\mathbf{Y}) = \tilde{\mathbf{\Phi}}^{(m,j)}(\mathbf{Y}) - \sum_{k=1}^{2} v_k^{(m+2,j)}(l_j)\boldsymbol{\psi}^{(k)} - v_3^{(m,j)}(l_j)\boldsymbol{\psi}^{(3)} \\ -v_4^{(m,j)}(l_j)\boldsymbol{\psi}^{(4)}(\mathbf{Y}) - \sum_{k=1}^{2}\{\frac{dv_k^{(m+1,j)}}{dz}(l_j)\boldsymbol{\psi}^{(4+k)}(\mathbf{Y}) + \alpha_k^{(m+2)}\mathbf{e}^{(k)}\} \\ -\alpha_3^{(m)}\mathbf{e}^{(3)} - (\sum_{k=1}^{2}\beta_k^{(m)}\mathbf{e}^{(k)} + \beta_3^{(m+2)}\mathbf{e}^{(3)}) \times (l_j\mathbf{e}^{(3)} + a_1^{(j)}\mathbf{e}^{(1)} + a_2^{(j)}\mathbf{e}^{(2)}) \\ - (\sum_{k=1}^{2}\beta_k^{(m-1)}\mathbf{e}^{(k)} + \beta_3^{(m+1)}\mathbf{e}^{(3)}) \times \mathbf{Y},\ \mathbf{Y} \in S,\end{aligned} \tag{4.8.3}$$

where $\tilde{\mathbf{\Phi}}^{(m,j)}$ involves $\mathbf{\Phi}^{(m,j)}$ and traces on S_ε of the functions $v_1^{(k,j)},\ v_2^{(k,j)},\ k \le m,\ \ v_3^{(k,j)}, v_4^{(k,j)}$, $k \le m-1$, and the vector functions $\tilde{\mathfrak{V}}^{(k,j)},\ k \le m-2$, and their derivatives which correspond to the terms in (4.5.2), different from the rigid body displacements (multipliers of $\boldsymbol{\psi}^{(j)},\ j = 1,\dots,6$).

Due to Theorem 3.5, there exists a unique set

$$\{\mathbf{V}^{(m,j)};\ v_k^{(m+2,j)}(l_j), \frac{dv_k^{(m+1,j)}}{dz}(l_j), v_{k+2}^{(m,j)}(l_j),\ k = 1,2\}, \tag{4.8.4}$$

which satisfies the problem (4.8.1)–(4.8.3), and $\mathbf{V}^{(m,j)}$ decays exponentially at infinity.

One can represent the constant components from (4.8.4) in the form

$$v_1^{(m+2,j)}(l_j) = -(\alpha_1^{(m+2)} + l_j\beta_2^{(m)} - \beta_3^{(m+2)}a_2^{(j)}) - c_1^{(m,j)}, \tag{4.8.5}$$

$$v_2^{(m+2,j)}(l_j) = -(\alpha_2^{(m+2)} + \beta_3^{(m+2)}a_1^{(j)} - \beta_1^{(m)}l_j) - c_2^{(m,j)}, \tag{4.8.6}$$

$$v_3^{(m,j)}(l_j) = -(\alpha_3^{(m)} + \beta_1^{(m)}a_2^{(j)} - \beta_2^{(m)}a_1^{(j)}) - c_3^{(m,j)}, \tag{4.8.7}$$

$$v_4^{(m,j)}(l_j) = -\beta_3^{(m+1)} - c_4^{(m,j)}, \tag{4.8.8}$$

$$\frac{dv_1^{(m+1,j)}}{dz}(l_j) = -\beta_2^{(m-1)} - c_5^{(m,j)}, \tag{4.8.9}$$

$$\frac{dv_2^{(m+1,j)}}{dz}(l_j) = \beta_1^{(m-1)} - c_6^{(m,j)}. \tag{4.8.10}$$

Then the set

$$\{\mathbf{V}^{(m,j)}; c_q^{(m,j)},\ q = 1,2,...,6\} \tag{4.8.11}$$

is the unique solution of the system (4.8.1) with the traction boundary condition (4.8.2) and the following displacement boundary condition

$$\mathbf{V}^{(m,j)}(\mathbf{Y}) = \tilde{\mathbf{\Phi}}^{(m,j)}(Y_1,Y_2) + \sum_{q=1}^{6} c_q^{(m,j)}\boldsymbol{\psi}^{(q)}(\mathbf{Y}),\ \mathbf{Y} \in S. \tag{4.8.12}$$

Therefore, the set (4.8.11) depends on $\mathbf{G}^{(m,j)}$, $\mathbf{H}^{(m,j)}$, $\mathbf{\Phi}^{(m,j)}$, $\boldsymbol{\mathcal{F}}^{(k,j)}$, $\boldsymbol{\mathcal{P}}^{(k,j)}$ with $k \le m-2$, the functions $v_1^{(k,j)},\ v_2^{(k,j)},\ k \le m$ and $v_3^{(k,j)},\ v_4^{(k,j)},\ k \le m-1$.

4.9 Calculation of the coefficients $\mathfrak{F}^{(k)}$, $\mathfrak{M}^{(k)}$

Let $Q_i^{(m,j)}$ denote the right–hand sides of the problems (4.1.4)–(4.1.6) corresponding to $v_i^{(m,j)}$, with $\mathcal{F}_i$, $\mathcal{P}_i$ replaced by $\tilde{\mathcal{F}}_i^{(m,j)}$, $\tilde{\mathcal{P}}_i^{(m,j)}$. Equations (4.1.4) and the boundary conditions (4.6.3), (4.6.4), for $z=0$ and (4.8.5)–(4.8.10) for $z=l_j$ enable us to find $v_i^{(m,j)}$, $i=1,2$, in the form

$$\begin{aligned} v_1^{(m,j)}(z) &= -(\alpha_1^{(m)}+l_j\beta_2^{(m-2)}-a_2^{(j)}\beta_3^{(m)})\psi_I(\frac{z}{l_j})-\beta_2^{(m-2)}l_j\psi_{II}(\frac{z}{l_j})+\mathcal{V}_1^{(m,j)}(z), && (4.9.1)\\ v_2^{(m,j)}(z) &= -(\alpha_2^{(m)}+a_1^{(j)}\beta_3^{(m)}-\beta_1^{(m-2)}l_j)\psi_I(\frac{z}{l_j})+\beta_1^{(m-2)}l_j\psi_{II}(\frac{z}{l_j})+\mathcal{V}_2^{(m,j)}(z), && (4.9.2)\end{aligned}$$

with ψ_I, ψ_{II} being the same as in Section 3.6; the functions $\mathcal{V}_i^{(m,j)}$, $i=1,2$, are given as the solutions of the boundary value problems

$$\begin{aligned} \frac{d^4\mathcal{V}_i^{(m,j)}}{dz^4}(z) &= Q_i^{(m,j)}(z),\ z\in(0,l_j), && (4.9.3)\\ \mathcal{V}_i^{(m,j)}(0) &= d_i^{(II,m-2,j)}, && (4.9.4)\\ \frac{d\mathcal{V}_i^{(m,j)}}{dz}(0) &= d_{i+4}^{(II,m-1,j)}, && (4.9.5)\\ \mathcal{V}_i^{(m,j)}(l_j) &= -c_i^{(m-2,j)}, && (4.9.6)\\ \frac{d\mathcal{V}_i^{(m,j)}}{dz}(l_j) &= -c_{4+i}^{(m-1,j)},\ i=1,2. && (4.9.7)\end{aligned}$$

Equations (4.1.5), (4.1.6) and the boundary conditions (4.6.3), (4.8.7), (4.8.8) lead to the representations

$$\begin{aligned} v_3^{(m,j)}(z) &= -(\alpha_3^{(m)}+\beta_1^{(m)}a_2^{(j)}-\beta_2^{(m)}a_1^{(j)})\frac{z}{l_j}+\mathcal{V}_3^{(m,j)}, && (4.9.8)\\ v_4^{(m,j)}(z) &= -\beta_3^{(m+1)}\frac{z}{l_j}+\mathcal{V}_4^{(m,j)}, && (4.9.9)\end{aligned}$$

where $\mathcal{V}_i$, $i=3,4$, satisfy the boundary value problems

$$\begin{aligned} \frac{d^2\mathcal{V}_i^{(m,j)}}{dz^2}(z) &= Q_i^{(m)}(z),\ z\in(0,l_j), && (4.9.10)\\ \mathcal{V}_i^{(m,j)}(0) &= d_i^{(II,m,j)}, && (4.9.11)\\ \mathcal{V}_i^{(m,j)}(l_j) &= -c_i^{(m,j)}, i=3,4. && (4.9.12)\end{aligned}$$

Formulae (4.6.5)–(4.6.8) yield the relations between the components of forces and moments and the constants $\alpha_j^{(k)}, \beta_j^{(k)}$

$$\begin{aligned} \mathfrak{F}_1^{(m,j)} &= \mathbb{F}_1(\tilde{\mathbf{F}}^{(m+1,j)},\tilde{\mathbf{P}}^{(m+1,j)};G)-D_1\Big\{\frac{12}{l_j^3}(\alpha_1^{(m)}-a_2^{(j)}\beta_3^{(m)})+\frac{6\beta_2^{(m-2)}}{l_j^2}\\ &\quad+\mathbb{Q}^{(III)}(Q_1^{(m,j)},d_1^{(II,m-2,j)},-c_1^{(m-2,j)},d_5^{(II,m-1,j)},-c_5^{(m-1,j)})\Big\}, && (4.9.13)\\ \mathfrak{F}_2^{(m,j)} &= \mathbb{F}_2(\tilde{\mathbf{F}}^{(m+1,j)},\tilde{\mathbf{P}}^{(m+1,j)};G)-D_2\Big\{\frac{12}{l_j^3}(\alpha_2^{(m)}+a_1^{(j)}\beta_3^{(m)})-\frac{6\beta_1^{(m-2)}}{l_j^2}\\ &\quad+\mathbb{Q}^{(III)}(Q_2^{(m,j)},d_2^{(II,m-2,j)},-c_2^{(m-2,j)},d_6^{(II,m-1,j)},-c_6^{(m-1,j)})\Big\}, && (4.9.14)\end{aligned}$$

$$\mathfrak{M}_3^{(m,j)} = \mathbb{M}_3(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}; G) + D_4\Big\{ -\frac{\beta_3^{(m)}}{l_j} + \mathbb{Q}^{(I)}(Q_4^{(m-1,j)}, d_4^{(II,m-1,j)}, -c_4^{(m-1,j)})\Big\}, \tag{4.9.15}$$

$$\mathfrak{F}_3^{(m,j)} = \mathbb{F}_3(\tilde{\mathbf{F}}^{(m+1,j)}, \tilde{\mathbf{P}}^{(m+1,j)}; G) + D_3\Big\{ -(\alpha_3^{(m)} + a_2^{(j)}\beta_1^{(m)} - a_1^{(j)}\beta_2^{(m)})/l_j + \mathbb{Q}^{(I)}(Q_3^{(m,j)}, d_3^{(II,m,j)}, -c_3^{(m,j)})\Big\}, \tag{4.9.16}$$

$$\mathfrak{M}_1^{(m,j)} = \mathbb{M}_1(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}; G) - D_2\Big\{ -\frac{6}{l_j^2}(\alpha_2^{(m)} + a_1^{(j)}\beta_3^{(m)}) + \frac{4\beta_1^{(m-2)}}{l_j} + \mathbb{Q}^{(II)}(Q_2^{(m,j)}, d_2^{(II,m-2,j)}, -c_2^{(m-2,j)}, d_6^{(II,m-1,j)}, -c_6^{(m-1,j)})\Big\}, \tag{4.9.17}$$

$$\mathfrak{M}_2^{(m,j)} = \mathbb{M}_2(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}; G) + D_1\Big\{ -\frac{6}{l_j^2}(\alpha_1^{(m)} - a_2^{(j)}\beta_3^{(m)}) - \frac{4\beta_2^{(m-2)}}{l_j} + \mathbb{Q}^{(II)}(Q_1^{(m,j)}, d_1^{(II,m-2,j)}, -c_1^{(m-2,j)}, d_5^{(II,m-1,j)}, -c_5^{(m-1,j)})\Big\}, \tag{4.9.18}$$

the functionals $\mathbb{Q}^{(I)}$, $\mathbb{Q}^{(II)}$, $\mathbb{Q}^{(III)}$ were defined in Section 3.6 where the length l of the interval on the Oz axis should be replaced by l_j (we hope that this notation will not cause any confusion).

Using the balance equations (4.4.5), (4.4.6) we arrive at the system of recurrent relations with respect to components of $\boldsymbol{\alpha}^{(m)}$, $\boldsymbol{\beta}^{(m)}$

$$\mathcal{A}\begin{pmatrix}\alpha_1^{(m)}\\ \alpha_2^{(m)}\\ \beta_3^{(m)}\end{pmatrix} + \mathcal{D}^t\begin{pmatrix}\alpha_3^{(m-2)}\\ \beta_1^{(m-2)}\\ \beta_2^{(m-2)}\end{pmatrix} = \begin{pmatrix}h_1^{(m)}\\ h_2^{(m)}\\ h_3^{(m)}\end{pmatrix}, \tag{4.9.19}$$

$$\mathcal{D}\begin{pmatrix}\alpha_1^{(m)}\\ \alpha_2^{(m)}\\ \beta_3^{(m)}\end{pmatrix} + \mathcal{B}\begin{pmatrix}\alpha_3^{(m)}\\ \beta_1^{(m)}\\ \beta_2^{(m)}\end{pmatrix} + \mathcal{E}\begin{pmatrix}\alpha_3^{(m-2)}\\ \beta_1^{(m-2)}\\ \beta_2^{(m-2)}\end{pmatrix} = \begin{pmatrix}h_4^{(m)}\\ h_5^{(m)}\\ h_6^{(m)}\end{pmatrix}, \tag{4.9.20}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ are matrices from (3.7.1)–(3.7.4). The quantities $h_i^{(m)}$, $i = 1, \dots, 6$, are

$$h_q^{(m)} = \mathbb{F}_q(\mathbf{f}^{(m)}, \mathbf{p}^{(m)}; \Omega) + \sum_j \Big\{\mathbb{F}_q(\tilde{\mathbf{F}}^{(m+1,j)}, \tilde{\mathbf{P}}^{(m+1,j)}, G) - D_q\mathbb{Q}^{(III)}(Q_q^{(m,j)}, d_q^{(II,m-2,j)}, -c_q^{(m-2,j)}, d_{q+4}^{(II,m-1,j)}, -c_{q+4}^{(m-1,j)})\Big\},\ q = 1, 2, \tag{4.9.21}$$

$$\begin{aligned} h_3^{(m)} = {} & \mathbb{M}_3(\mathbf{f}^{(m)}, \mathbf{p}^{(m)}; \Omega) + \sum_j \Big\{\mathbb{M}_3(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}; G) \\ & + a_1^{(j)}\mathbb{F}_2(\tilde{\mathbf{F}}^{(m+1,j)}, \tilde{\mathbf{P}}^{(m+1,j)}, G) - a_2^{(j)}\mathbb{F}_1(\tilde{\mathbf{F}}^{(m+1,j)}, \tilde{\mathbf{P}}^{(m+1,j)}, G) \\ & + D_4\mathbb{Q}^{(I)}(Q_4^{(m-1,j)}, d_4^{(II,m-1,j)}, -c_4^{(m-1,j)}) \\ & - a_1^{(j)}D_2\mathbb{Q}^{(III)}(Q_2^{(m,j)}, d_2^{(II,m-2,j)}, -c_2^{(m-2,j)}, d_6^{(II,m-1,j)}, -c_6^{(m-1,j)}) \\ & + a_2^{(j)}D_1\mathbb{Q}^{(III)}(Q_1^{(m,j)}, d_1^{(II,m-2,j)}, -c_1^{(m-2,j)}, d_5^{(II,m-1,j)}, -c_5^{(m-1,j)})\Big\}, \end{aligned} \tag{4.9.22}$$

$$h_4^{(m)} = \mathbb{F}_3(\mathbf{f}^{(m)}, \mathbf{p}^{(m)}, \Omega) + \sum_j \Big\{\mathbb{F}_3(\tilde{\mathbf{F}}^{(m+1,j)}, \tilde{\mathbf{P}}^{(m+1,j)}, G) + D_3\mathbb{Q}^{(I)}(Q_3^{(m,j)}, d_3^{(II,m,j)}, -c_3^{(m,j)})\Big\}, \tag{4.9.23}$$

$$h_5^{(m)} = \mathbb{M}_1(\mathbf{f}^{(m)}, \mathbf{p}^{(m)}; \Omega) + \sum_j \Big\{\mathbb{M}_1(\tilde{\mathbf{F}}^{(m,j)}, \tilde{\mathbf{P}}^{(m,j)}; G)$$

$$
\begin{aligned}
&+a_2^{(j)}\Big[\mathbb{F}_3(\tilde{\mathbf{F}}^{(m+1,j)},\tilde{\mathbf{P}}^{(m+1,j)},G)+D_3\mathbb{Q}^{(I)}(Q_3^{(m,j)},d_3^{(II,m,j)},-c_3^{(m,j)})\Big]\\
&-D_2\mathbb{Q}^{(II)}(Q_2^{(m,j)},d_2^{(II,m-2,j)},-c_2^{(m-2,j)},d_6^{(II,m-1,j)},-c_6^{(m-1,j)})\Big\}, \qquad (4.9.24)
\end{aligned}
$$

$$
\begin{aligned}
h_6^{(m)} &= \mathbb{M}_2(\mathbf{f}^{(m)},\mathbf{p}^{(m)};\Omega)+\sum_j\Big\{\mathbb{M}_2(\tilde{\mathbf{F}}^{(m,j)},\tilde{\mathbf{P}}^{(m,j)};G)\\
&-a_1^{(j)}\Big[D_3\mathbb{F}_3(\tilde{\mathbf{F}}^{(m+1,j)},\tilde{\mathbf{P}}^{(m+1,j)},G)+\mathbb{Q}^{(I)}(Q_3^{(m,j)},d_3^{(II,m,j)},-c_3^{(m,j)})\Big]\\
&+D_1\mathbb{Q}^{(II)}(Q_1^{(m,j)},d_1^{(II,m-2,j)},-c_1^{(m-2,j)},d_5^{(II,m-1,j)},-c_5^{(m-1,j)})\Big\}, \qquad (4.9.25)
\end{aligned}
$$

We remind the reader that, by assumption, $a_3^{(j)}=0,\ j=1,\dots,K$.

The coefficients $\boldsymbol{\alpha}^{(k)},\ \boldsymbol{\beta}^{(k)}$ with negative indices are assumed to equal to zero. The recurrent relations (4.9.19), (4.9.20) together with the equalities (4.9.21)–(4.9.25) for $h_q^{(m)}$ yield the values of the components of $\boldsymbol{\alpha}^{(m)},\ \boldsymbol{\beta}^{(m)}$ and the components of the lock forces $\mathfrak{F}^{(m,j)}$ and moments $\mathfrak{M}^{(m,j)},\ j=1,2,\dots,K$.

4.10 The recurrent procedure for the asymptotic expansion

To conclude, we shall describe the order of the steps of the asymptotic algorithm.

First, we solve the boundary value problem (4.9.3)–(4.9.7) for $\mathcal{V}_k^{(m,j)},\ k=1,2$. The knowledge of these functions enables us to find $h_i^{(m)},\ i=1,2,3$, from the right–hand side of (4.9.19) and, therefore, obtain the coefficients $\alpha_1^{(m)},\ \alpha_2^{(m)},\ \beta_3^{(m)}$ near the rigid body displacements of Ω.

Using the formulae (4.9.1), (4.9.2) we obtain $v_1^{(m,j)},\ v_2^{(m,j)}$.

Formulae (4.9.13)–(4.9.14) and (4.9.17), (4.9.18) give the transversal components $\mathfrak{F}_1^{(m,j)},\mathfrak{F}_2^{(m,j)}$ of forces and components $\mathfrak{M}_k^{(m,j)},\ k=1,2,3$, of moments at junction points.

Next, we obtain the solution

$$\Big\{\boldsymbol{\mathcal{W}}^{(m,j)},d_1^{(I,m,j)},\dots,d_6^{(I,m,j)};d_1^{(II,m,j)},\dots,d_6^{(II,m,j)}\Big\}$$

of the junction layer problem (4.6.1).

The bottom layer problem (4.8.1), (4.8.2), (4.8.12) gives the set

$$\Big\{\mathbf{V}^{(m,j)};c_q^{(m,j)},q=1,2,\dots,6\Big\}.$$

Then, the boundary value problems (4.9.10)–(4.9.12) yield the functions $\mathcal{V}_3^{(m,j)},\ \mathcal{V}_4^{(m,j)}$.

Consequently, one can calculate $h_4^{(m)},\ h_5^{(m)},\ h_6^{(m)}$ from the right–hand side of (4.9.20) and determine the remaining coefficients $\alpha_3^{(m)},\beta_1^{(m)},\ \beta_2^{(m)}$ near the rigid body displacements of Ω.

The longitudinal components $\mathfrak{F}_3^{(m,j)}$ of forces at junction points are calculated in accordance with (4.9.16). Knowing components of forces and moments at junction points one has the singular part $\mathbf{u}_s^{(m)}$ of the displacement in Ω (see (4.4.4), $k=m$).

In accordance with Section 4.7, we obtain the regular term of the field $\mathbf{u}_\Omega^{(m)}$ in Ω, and, consequently, complete the evaluation of $\mathbf{u}_\Omega^{(m)}$.

Following the formulae (4.9.8), (4.9.9) we determine the functions $v_3^{(m,j)}$ completely, and $v_4^{(m,j)}$ up to a linear term $\beta_3^{(m,j)}z/l_j$. The boundary value problem (4.2.7), (4.2.8) with $\mathbf{U},\ v_k,\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{P}}$ being replaced by $\mathbf{U}^{(m,j)},v_k^{(m,j)},\tilde{\boldsymbol{\mathcal{F}}}^{(m,j)},\tilde{\boldsymbol{\mathcal{P}}}^{(m,j)}$, respectively, gives the vector function $\mathbf{U}^{(m,j)}$. Thus, the displacement field $\mathfrak{U}^{(m,j)}$ in a thin cylinder has been determined up to the constant $\beta_3^{(m+1)}$. Repeating the first two steps of the iterative procedure with m replaced by $m+1$ we obtain the constants $\alpha_1^{(m+1)}, \alpha_2^{(m+1)}, \ \beta_3^{(m+1)}$ and complete the construction of the coefficient $\mathbf{u}^{(m)}$ in the asymptotic expansion (4.4.1).

Remark 4.2. Substitution of

$$s_m = \sum_{j=-1}^{m} \varepsilon^j \mathbf{u}^{(j)}(\mathbf{x}, \varepsilon)$$

into the equations (1.2.1)–(1.2.3) gives the right–hand sides

$$\mathbf{F} - \mathbf{F}_m, \ \ \mathbf{P} - \mathbf{P}_m, \ \ \mathbf{\Phi}^{(k)} - \mathbf{\Phi}_m^{(k)},$$

where the terms $\mathbf{F}_m, \ \mathbf{P}_m, \ \mathbf{\Phi}_m^{(k)}$ admit the asymptotic representation of the form (4.3.1)–(4.3.3) with $k \geq m+1$.

5 The remainder estimate

Here, we present the justification of the asymptotic expansion.

Theorem 5.1 *The solution $\mathbf{u}(\mathbf{x}, \varepsilon)$ of the problem (1.2.1)–(1.2.3) with the right–hand sides of the form (4.3.1)–(4.3.3) has the asymptotic representation (4.4.1) in the following sense*

$$\|\mathbf{u} - \sum_{j=-1}^{k} \varepsilon^j \mathbf{u}^{(j)}(\mathbf{x}, \varepsilon)\|_{(H^1(\Omega_\varepsilon))^3} \leq Const \ \varepsilon^{k-3}, \tag{5.1}$$

where the constant does not depend on ε.

The proof of Theorem will be preceded by the statement

Lemma 5.1 *For the solution of the problem (1.1)-(1.3) with the right hand sides of the form (4.1)-(4.3) the following inequality holds*

$$\|\mathbf{u}\|_{(H^1(\Omega_\varepsilon))^3} \leq Const \, \varepsilon^{-\delta} \{ \|\mathbf{Lu}\|_{(L_2(\Omega_\varepsilon))^3}$$

$$+ \|\boldsymbol{\sigma}^{(n)}(\mathbf{u})\|_{(L_2(\partial\Omega_\varepsilon \setminus S_\varepsilon))^3} + \sum_i \|\mathbf{u}\|_{(H^{1/2}(S_\varepsilon))^3} \}, \tag{5.2}$$

where the constant does not depend on ε, and δ is a certain positive exponent (we are not interested in its exact value).

Proof. By $\dot{H}^{1/2}(S_\varepsilon)$ we denote the closure of $C_0^\infty(S_\varepsilon)$ in $H^{1/2}(S_\varepsilon)$, and $H^{-1/2}(S_\varepsilon)$ denotes the adjoint space. Using Betti's formula and Hölder's inequality we obtain

$$\mathcal{E}(\mathbf{u}; \Omega_\varepsilon) \leq \text{Const} \ \{ \|\mathbf{Lu}\|_{(L_2(\Omega_\varepsilon))^3} \|\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3}$$

$$+ \|\boldsymbol{\sigma}^{(n)}(\mathbf{u})\|_{(L_2(\partial\Omega_\varepsilon \setminus S_\varepsilon))^3} \|\mathbf{u}\|_{(L_2(\partial\Omega_\varepsilon \setminus S_\varepsilon))^3} + \|\mathbf{u}\|_{(H^{1/2}(S_\varepsilon))^3} \|\boldsymbol{\sigma}^{(n)}(\mathbf{u})\|_{(H^{-1/2}(S_\varepsilon))^3} \}.$$

First, we estimate $\|\boldsymbol{\sigma}^{(n)}(\mathbf{u})\|_{(H^{-1/2}(S_\varepsilon))^3}$. Let $\boldsymbol{\varphi} \in (\dot{H}^{1/2}(S_\varepsilon))^3$. Then there exists an extension $\mathbf{v} \in (H^1(\Omega_\varepsilon))^3$ such that $\mathbf{v} = 0$ on $\partial\Omega_\varepsilon \setminus S_\varepsilon$, and the following estimate holds

$$\|\mathbf{v}\|_{(H^1(\Omega_\varepsilon))^3} \leq \text{Const} \ \varepsilon^{-1/2} \|\boldsymbol{\varphi}\|_{(H^{1/2}(S_\varepsilon))^3}. \tag{5.3}$$

It follows from the Betti formula that

$$\int_{S_\varepsilon} \boldsymbol{\sigma}^{(n)}(\mathbf{u};\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) ds = \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \sigma_{ij}(\mathbf{u};\mathbf{x}) \varepsilon_{ij}(\mathbf{v};\mathbf{x}) d\mathbf{x} + \int_{\Omega_\varepsilon} \mathbf{v}(\mathbf{x}) \cdot (\mathbf{L}\mathbf{u})(\mathbf{x}) d\mathbf{x}. \tag{5.4}$$

Using the estimate (5.3) and the formula (5.4) we obtain

$$|\int_{S_\varepsilon} \boldsymbol{\sigma}^{(n)}(\mathbf{u};\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) ds| \leq \text{Const } \varepsilon^{-1/2} \|\boldsymbol{\varphi}\|_{(H^{1/2}(S_\varepsilon))^3} ((\mathcal{E}(\mathbf{u};\Omega_\varepsilon))^{1/2} + \|\mathbf{L}\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3}).$$

Therefore,

$$\|\boldsymbol{\sigma}^{(n)}(\mathbf{u};\mathbf{x})\|_{(H^{-1/2}(\Omega_\varepsilon))^3} \leq \text{Const } \varepsilon^{-1/2} \Big(\|\mathbf{L}\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3} + (\mathcal{E}(\mathbf{u};\Omega_\varepsilon))^{1/2} \Big).$$

Using this estimate together with (A.15) we obtain that

$$\mathcal{E}(\mathbf{u};\Omega_\varepsilon) \leq \text{Const } \varepsilon^{-\delta} \Big\{ \|\mathbf{L}\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3} + \|\boldsymbol{\sigma}^{(n)}(\mathbf{u})\|_{(L_2(\partial\Omega_\varepsilon \setminus S_\varepsilon))^3}$$

$$+ \|\mathbf{u}\|_{(H^{1/2}(S_\varepsilon))^3} \Big\} \Big((\mathcal{E}(\mathbf{u};\Omega_\varepsilon))^{1/2} + \|\mathbf{L}\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3} + \|\mathbf{u}\|_{(L_2(S_\varepsilon))^3} \Big). \tag{5.5}$$

Since $\|\mathbf{u}\|^2_{(L_2(S_\varepsilon))^3}$ and $\|\mathbf{L}\mathbf{u}\|^2_{(L_2(S_\varepsilon))^3}$ are estimated by the right–hand side (5.5), we derive from (5.5)

$$(\mathcal{E}(\mathbf{u},\Omega_\varepsilon) + \|\mathbf{u}\|^2_{(L_2(S_\varepsilon))^3})^{1/2} \leq \text{Const } \varepsilon^{-\delta} \Big\{ \|\mathbf{L}\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3}$$

$$+ \|\boldsymbol{\sigma}^{(n)}(\mathbf{u})\|_{(L_2(\partial\Omega_\varepsilon \setminus S_\varepsilon))^3} + \|\mathbf{u}\|_{(H^{1/2}(S_\varepsilon))^3} \Big\}.$$

This together with Korn's inequality (see Corollary in Appendix A) give the statement of Lemma. □

Proof of Theorem 5.1. For large n we write the inequality

$$\|\mathbf{u} - \mathbf{s}_k\|^2_{(H^1(\Omega_\varepsilon))^3} \leq 2(\|\mathbf{u} - \mathbf{s}_n\|^2_{(H^1(\Omega_\varepsilon))^3} + \|\mathbf{s}_n - \mathbf{s}_k\|^2_{(H^1(\Omega_\varepsilon))^3}), \tag{5.6}$$

where s_k denotes the sum from (5.2).

Let us denote by $\mathbf{F}_n, \mathbf{P}_n, \boldsymbol{\Phi}_n^{(j)}$ the right–hand sides of the problem (1.2.1)–(1.2.3) for $\mathbf{u} - \mathbf{s}_n$. Then, by Remark 4.2, the vector functions $\mathbf{F}_n, \mathbf{P}_n$ and $\boldsymbol{\Phi}_n^{(j)}$ have the asymptotic form (4.3.1)–(4.3.3) with $k \geq n+1$. Therefore,

$$\|\mathbf{F}_n\|_{(L_2(\Omega_\varepsilon))^3} + \|\mathbf{P}_n\|_{(L_2(\partial\Omega_\varepsilon \setminus S_\varepsilon))^3} + \sum_k \|\boldsymbol{\Phi}_n^{(k)}\|_{(H^{1/2}(S_\varepsilon^{(k)}))^3} \leq \text{Const } \varepsilon^{n-1}.$$

Using Lemma 5.1, we obtain

$$\|\mathbf{u} - \mathbf{s}_n\|_{(H^1(\Omega_\varepsilon))^3} \leq \text{Const } \varepsilon^{n-1-\delta}.$$

The last term from (5.6) satisfies the estimate

$$\|\mathbf{s}_n - \mathbf{s}_k\|^2_{(H^1(\Omega_\varepsilon))^3} \leq \text{Const } \varepsilon^{k-3}.$$

Choosing n such that $n \geq k - 2 + \delta$ we complete the proof. □

6 The leading ansatz and physical interpretation

The present part is related to the analysis of the leading ansatz of the asymptotic expansion of the displacement field in the multi–structure. Physical interpretation is given, and results of comparison of the asymptotic approach with the pile structure analysis are presented.

6.1 The leading ansatz of the asymptotic expansion

We construct two terms of orders ε^{-1} and ε^{0} of the asymptotic expansion (4.4.1) under the assumption that the principal force vector, corresponding to $\mathbf{F}^{(0,j)}$, $\mathbf{P}^{(0,j)}$, is equal to zero (see (4.3.4)). First, consider the term with $k=-1$ in the expansion (4.4.1). Since $\mathcal{V}_1^{(-1,j)} = \mathcal{V}_2^{(-1,j)} = 0$, and, therefore, the right–hand sides of (4.9.19) and (4.9.20), $m=-1$, are equal to zero, one has

$$\alpha_i^{(-1)} = \beta_i^{(-1)} = 0, \ i = 1,2,3. \tag{6.1.1}$$

It follows from (4.9.1)–(4.9.2) that $v_1^{(-1,j)} = v_2^{(-1,j)} = 0$. Thus, due to (4.9.13)–(4.9.18) and (4.3.4), the components $\mathfrak{F}_1^{(-1,j)}, \mathfrak{F}_2^{(-1,j)}$ and $\mathfrak{M}_k^{(-1,j)}$, $k=1,2,3$, are also zero. From (4.6.1), m=-1, we obtain that the set

$$\left\{\boldsymbol{\mathcal{W}}^{(-1,j)}, d_1^{(I,-1,j)}, \ldots, d_6^{(I,-1,j)}, d_1^{(II,-1,j)}, \ldots, d_6^{(II,-1,j)}\right\}$$

is trivial.

The right–hand sides of (4.8.1), (4.8.2), (4.8.12) with $m=-1$ equal zero, and, consequently,

$$\mathbf{V}^{(-1,j)} = 0, \ c_q^{(-1,j)} = 0, \ q = 1,2,\ldots,6.$$

The homogeneous boundary value problems (4.9.10)–(4.9.12), $m=-1$, give $\mathcal{V}_3^{(-1,j)} = \mathcal{V}_4^{(-1,j)} = 0$.

From (4.9.23)–(4.9.25) we derive that $h_4^{(-1)} = h_5^{(-1)} = h_6^{(-1)} = 0$, and, therefore, $\alpha_3^{(-1)} = \beta_1^{(-1)} = \beta_2^{(-1)} = 0$.

In accordance with (4.9.16), (4.3.4) the longitudinal components $\mathfrak{F}_3^{(-1,j)}$ of forces at junction points are zero. Thus, $\mathbf{u}_s^{(-1)} = 0$. Since $\mathbf{f}^{(-1)} = 0$, $\mathbf{p}^{(-1)} = 0$, the regular term $\mathbf{u}_r^{(-1)}$ from (4.4.3) also vanishes, and, therefore, $\mathbf{u}_\Omega^{(-1)} = 0$.

Formulae (4.9.8)–(4.9.9) yield $v_3^{(-1,j)} = 0$, and $v_4^{(-1,j)}(z) = \beta_3^{(0)} z/l_j$, and, consequently (see (4.1.7), (4.1.8), (4.1.13)), the vector function $\mathbf{U}^{(-1,j)}$ is zero.

Thus, we have shown that

$$\boldsymbol{\mathcal{U}}^{(-1,j)}(\zeta,z,\varepsilon) = \varepsilon^{-2} v_4^{(-1,j)}(z)\boldsymbol{\phi}^{(4,0)}(\zeta) + \varepsilon^{-1}\frac{d}{dz} v_4^{(-1,j)}(z)\boldsymbol{\phi}^{(4,1)}(\zeta),$$

where $\boldsymbol{\phi}^{(4,0)}$, $\boldsymbol{\phi}^{(4,1)}$ were defined by (2.3.12), (2.3.16), and, consequently, we obtain

$$\mathbf{u}^{(-1)}(\mathbf{x},\varepsilon) = \varepsilon^{-1}\beta_3^{(0)}\sum_{j=1}^{K}\eta(z/\varepsilon)\frac{1}{l_j}\Big(z(-(x_2-a_2^{(j)})\mathbf{e}^{(1)} + (x_1-a_1^{(j)})\mathbf{e}^{(2)})$$

$$+\varphi(\frac{x_1-a_1^{(j)}}{\varepsilon}, \frac{x_2-a_2^{(j)}}{\varepsilon})\mathbf{e}^{(3)}\Big).$$

with φ being defined in Section 2.2. The constant $\beta_3^{(0)}$ will be found during the calculations of $\mathbf{u}^{(0)}(\mathbf{x},\varepsilon)$.

Now, consider the term $\mathbf{u}^{(0)}$. According to the general representation (4.4.2), we have

$$\begin{aligned}\mathbf{u}^{(0)}(\mathbf{x},\varepsilon) = \varepsilon^{-4}\{\sum_{i=1}^{2}\alpha_i^{(0)}\mathbf{e}^{(i)} + \varepsilon^2\alpha_3^{(0)}\mathbf{e}^{(3)} + (\varepsilon^2\sum_{i=1}^{2}\beta_i^{(0)}\mathbf{e}^{(i)} + \beta_3^{(0)}\mathbf{e}^{(3)})\times\mathbf{x}\}\\ +\mathfrak{X}(\mathbf{x},\varepsilon)\mathbf{u}_\Omega^{(0)}(\mathbf{x}) + \sum_{j=1}^{K}\Big\{\eta(\frac{z}{\varepsilon})\mathfrak{U}^{(0,j)}(\zeta,z,\varepsilon)\\ +\varepsilon^{-2}\mathcal{W}^{(0,j)}(\mathbf{X}^{(j)})\Xi(\mathbf{x}-\mathbf{a}^{(j)}) + \varepsilon^{-2}\mathbf{V}^{(0,j)}(\mathbf{Y}^{(j)})\Big\},\end{aligned} \quad (6.1.2)$$

Next, in accordance with the recurrent procedure, described in Section 4.10, we find the functions $\mathcal{V}_i^{(0,j)}$, $i = 1, 2$, which satisfy the boundary value problems (4.9.3)–(4.9.7) with homogeneous boundary conditions and

$$Q_i^{(0,j)} = -D_i^{-1}\{\int_g \mathcal{F}_i^{(0,j)}(\zeta,z)d\zeta + \int_{\partial g}\mathcal{P}_i^{(0,j)}(\zeta,z)ds\},\ i = 1,2.$$

The equation (4.9.19) takes the form

$$\mathcal{A}\begin{pmatrix}\alpha_1^{(0)}\\ \alpha_2^{(0)}\\ \beta_3^{(0)}\end{pmatrix} = \begin{pmatrix}h_1^{(0)}\\ h_2^{(0)}\\ h_3^{(0)}\end{pmatrix}, \quad (6.1.3)$$

where

$$\begin{aligned}h_i^{(0)} = \mathbb{F}_i(\mathbf{f}^{(0)},\mathbf{p}^{(0)};\Omega) + \sum_{j=1}^{K}\Big\{\mathbb{F}_i(\mathbf{F}^{(1,j)},\mathbf{P}^{(1,j)};G)\\ -D_i\mathbb{Q}^{(III)}(Q_i^{(0,j)},0,0,0,0)\Big\},\quad i=1,2.\end{aligned}$$

For $h_3^{(0)}$ we have

$$\begin{aligned}h_3^{(0)} = \mathbb{M}_3(\mathbf{f}^{(0)},\mathbf{p}^{(0)},\Omega) + \sum_{j=1}^{K}\Big\{\mathbb{M}_3(\mathbf{F}^{(0,j)},\mathbf{P}^{(0,j)},G)\\ -a_1^{(j)}D_2\mathbb{Q}^{(III)}(Q_2^{(0,j)},0,0,0,0) + a_2^{(j)}D_1\mathbb{Q}^{(III)}(Q_1^{(0,j)},0,0,0,0)\\ +\int_G(\mathbf{a}^{(j)}\times\mathbf{F}^{(1,j)})_3 d\mathbf{X} + \int_{\partial G}(\mathbf{a}^{(j)}\times\mathbf{P}^{(1,j)})_3 ds\Big\}.\end{aligned}$$

Formulae (4.9.1), (4.9.2) give expressions for $v_1^{(0,j)}$, $v_2^{(0,j)}$, $j = 1,\ldots,K$. Now, one can calculate transversal components of forces and components of moments at junction points. Formulae (4.9.13)–(4.9.15), (4.9.17), (4.9.18) give

$$\begin{aligned}\mathfrak{F}_1^{(0,j)} &= -D_1\Big\{\frac{12}{l_j^3}(\alpha_1^{(0)} - a_2^{(j)}\beta_3^{(0)}) + \mathbb{Q}^{(III)}(Q_1^{(0,j)},0,0,0,0)\Big\}\\ &\quad+\mathbb{F}_1(\mathbf{F}^{(1,j)},\mathbf{P}^{(1,j)},G), &(6.1.4)\\ \mathfrak{F}_2^{(0,j)} &= -D_2\Big\{\frac{12}{l_j^3}(\alpha_2^{(0)} + a_1^{(j)}\beta_3^{(0)}) + \mathbb{Q}^{(III)}(Q_2^{(0,j)},0,0,0,0)\Big\}\\ &\quad+\mathbb{F}_2(\mathbf{F}^{(1,j)},\mathbf{P}^{(1,j)},G), &(6.1.5)\\ \mathfrak{M}_1^{(0,j)} &= -D_2\Big\{-\frac{6}{l_j^2}(\alpha_2^{(0)} + a_1^{(j)}\beta_3^{(0)}) + \mathbb{Q}^{(II)}(Q_2^{(0,j)},0,0,0,0)\Big\}\end{aligned}$$

$$+\mathbb{M}_1\,(\mathbf{F}^{(0,j)},\mathbf{P}^{(0,j)},G), \tag{6.1.6}$$

$$\mathfrak{M}_2^{(0,j)} = D_1\Big\{-\frac{6}{l_j^2}(\alpha_1^{(0)}-a_2^{(j)}\beta_3^{(0)})+\mathbb{Q}^{(II)}(Q_1^{(0,j)},0,0,0,0)\Big\}$$

$$+\mathbb{M}_2\,(\mathbf{F}^{(0,j)},\mathbf{P}^{(0,j)},G), \tag{6.1.7}$$

$$\mathfrak{M}_3^{(0,j)} = -D_4\frac{\beta_3^{(0)}}{l_j}+\mathbb{M}_3\,(\mathbf{F}^{(0,j)},\mathbf{P}^{(0,j)},G). \tag{6.1.8}$$

Next, we determine the solution

$$\Big\{\boldsymbol{\mathcal{W}}^{(0,j)};d_1^{(I,0,j)},\dots,d_6^{(I,0,j)};d_1^{(II,0,j)},\dots,d_6^{(II,0,j)}\Big\}, \tag{6.1.9}$$

of the junction layer problem (4.6.1) with $\tilde{\mathbf{F}}^{(m,j)}$, $\tilde{\mathbf{P}}^{(m,j)}$, $\mathfrak{M}^{(m,j)}$ replaced by $\mathbf{F}^{(0,j)}$, $\mathbf{P}^{(0,j)}$, $\mathfrak{M}^{(0,j)}$ and $\mathfrak{F}^{(-1,j)}$ being zero. Note that, due to (4.3.4),

$$d_1^{(I,0,j)}=d_2^{(I,0,j)}=d_3^{(I,0,j)}=0.$$

Due to Theorem 3.3, the vector–function $\boldsymbol{\mathcal{W}}^{(0,j)}$ decays exponentially in Π_+ and has the following asymptotics in the half–space

$$\boldsymbol{\mathcal{W}}^{(0,j)}(\mathbf{X}) \sim \sum_{i=1}^{2} b_i^{(0,j)}\partial_{X_i}\boldsymbol{\mathcal{B}}^{(i)}(\mathbf{X})+\frac{1}{2}b_3^{(0,j)}\Big(\partial_{X_1}\boldsymbol{\mathcal{B}}^{(2)}(\mathbf{X})+\partial_{X_2}\boldsymbol{\mathcal{B}}^{(1)}(\mathbf{X})\Big)$$

$$+O(\|\mathbf{X}\|^{-3}) \ \text{ as } \ \|\mathbf{X}\|\to\infty,\mathbf{X}\in\mathbb{R}^3_-. \tag{6.1.10}$$

Consider the base regions of the thin cylinders. The set

$$\Big\{\mathbf{V}^{(0,j)},\ c_1^{(0,j)},\dots,c_6^{(0,j)}\Big\} \tag{6.1.11}$$

solves the problem (4.8.1), (4.8.2), (4.8.12) with $m=0$ and

$$\tilde{\boldsymbol{\Phi}}^{(0,j)}(\mathbf{Y}')=\boldsymbol{\Phi}^{(0,j)}(\mathbf{Y}')-\frac{dv_4^{(-1,j)}}{dz}(l_j)\boldsymbol{\xi}^{(4)}(\mathbf{Y}',0)-\sum_{k=1}^{2}\frac{d^2v_k^{(0,j)}}{dz^2}(l_j)\boldsymbol{\xi}^{(4+k)}(\mathbf{Y}',0).$$

Now, we determine the functions $\mathcal{V}_i^{(0,j)}, i=3,4$, which satisfy the boundary value problems (4.9.10)–(4.9.12) with

$$Q_3^{(0,j)}(z)=D_3^{-1}\Big\{\int_g\mathcal{F}_3^{(0,j)}(\zeta,z)d\zeta+\int_{\partial g}\mathcal{P}_3^{(0,j)}(\zeta,z)ds\Big\},\ z\in(0,l_j),$$

$$Q_4^{(0,j)}(z)=D_4^{-1}\Big\{\int_g\Big(\zeta_1\mathcal{F}_2^{(0,j)}(\zeta,z)-\zeta_2\mathcal{F}_1^{(0,j)}(\zeta,z)\Big)d\zeta$$

$$+\int_{\partial g}\Big(\zeta_1\mathcal{P}_2^{(0,j)}(\zeta,z)-\zeta_2\mathcal{P}_1^{(0,j)}(\zeta,z)\Big)ds\Big\}.$$

To complete the calculation of the rigid body displacements of the three–dimensional element Ω, we have to obtain the coefficients $\alpha_3^{(0)}$, $\beta_1^{(0)}$, $\beta_2^{(0)}$ in the ansatz (6.1.2). Due to (4.9.20), one has

$$\begin{pmatrix}\alpha_3^{(0)}\\ \beta_1^{(0)}\\ \beta_2^{(0)}\end{pmatrix}=\mathcal{B}^{-1}\Big\{\begin{pmatrix}h_4^{(0)}\\ h_5^{(0)}\\ h_6^{(0)}\end{pmatrix}-\mathcal{D}\begin{pmatrix}\alpha_1^{(0)}\\ \alpha_2^{(0)}\\ \beta_3^{(0)}\end{pmatrix}\Big\}, \tag{6.1.12}$$

where

$$h_4^{(0)} = \mathbb{F}_3(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega) + \sum_{j=1}^{K} \Big\{ D_3 \mathbb{Q}^{(I)}(Q_3^{(0,j)}, d_3^{(II,0,j)}, -c_3^{(0,j)}) + \mathbb{F}_3(\mathbf{F}^{(1,j)}, \mathbf{P}^{(1,j)}, G) \Big\},$$

$$h_5^{(0)} = \mathbb{M}_1(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega) + \sum_{j=1}^{K} \Big\{ - D_2 \mathbb{Q}^{(II)}(Q_2^{(0,j)}, 0, 0, 0, 0) + \mathbb{M}_1(\mathbf{F}^{(0,j)}, \mathbf{P}^{(0,j)}, G)$$
$$+ a_2^{(j)} \Big[D_3 \mathbb{Q}^{(I)}(Q_3^{(0,j)}, d_3^{(II,0,j)}, -c_3^{(0,j)}) + \mathbb{F}_3(\mathbf{F}^{(1,j)}, \mathbf{P}^{(1,j)}, G) \Big] \Big\}$$

$$h_6^{(0)} = \mathbb{M}_2(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega) + \sum_{j=1}^{K} \Big\{ D_1 \mathbb{Q}^{(II)}(Q_1^{(0,j)}, 0, 0, 0, 0) + \mathbb{M}_2(\mathbf{F}^{(0,j)}, \mathbf{P}^{(0,j)}, G)$$
$$- a_1^{(j)} \Big[D_3 \mathbb{Q}^{(I)}(Q_3^{(0,j)}, d_3^{(II,0,j)}, -c_3^{(0,j)}) + \mathbb{F}_3(\mathbf{F}^{(1,j)}, \mathbf{P}^{(1,j)}, G) \Big] \Big\}$$

Due to (4.9.16), the longitudinal force at junction points $\mathbf{a}^{(j)}$, $j = 1, \dots, K$, is given by

$$\mathfrak{F}_3^{(0,j)} = D_3 \Big\{ - (\alpha_3^{(0)} + \beta_1^{(0)} a_2^{(j)} - \beta_2^{(0)} a_1^{(j)}) \frac{1}{l_j} + \mathbb{Q}^{(I)}(Q_3^{(0,j)}, d_3^{(II,0,j)}, -c_3^{(0,j)}) \Big\} \qquad (6.1.13)$$
$$+ \mathbb{F}_3(\mathbf{F}^{(1,j)}, \mathbf{P}^{(1,j)}, G).$$

Following (4.4.4) with $k = 0$, we can obtain the singular part $\mathbf{u}_s^{(0)}$ of the displacement $\mathbf{u}_\Omega^{(0)}$ in Ω.

At this stage, we can find the regular part $\mathbf{u}_r^{(0)}$ of the term $\mathbf{u}_\Omega^{(0)}$ from the ansatz (6.1.2). According to Sections 4.7, 4.6 and the boundary value problem (4.7.1), the vector $\mathbf{u}_r^{(0)}$ satisfies (3.2.12), (3.2.13) with $\mathbf{f}$, $\mathbf{p}$ being replaced by $\mathbf{f}^{(0)}$, $\mathbf{p}^{(0)}$ plus discrepancy terms brought by the commutators of the cut–off function Θ and the differential operators of the Lamé system and the traction boundary conditions applied to the leading part of the asymptotic expansion (6.1.10). It should be noted that the principal force and moment of these discrepancy terms are equal to zero.

Using the formulae (4.9.8), (4.9.9) we determine the functions $v_3^{(0,j)}$ completely, and $v_4^{(0,j)}$ up to a linear term $\beta_3^{(1)} z / l_j$. The boundary value problem (4.1.7), (4.1.8) with $\mathbf{U}$, v_k, $\boldsymbol{\mathcal{F}}$, $\boldsymbol{\mathcal{P}}$ replaced by $\mathbf{U}^{(0,j)}$, $v_k^{(0,j)}$, $\boldsymbol{\mathcal{F}}^{(0,j)}$, $\boldsymbol{\mathcal{P}}^{(0,j)}$, respectively, provides the vector function $\mathbf{U}^{(0,j)}$. The constant $\beta_3^{(1)}$ is to be found on the next iteration.

6.2 Physical interpretation of the results

In the physical model of the pile structure important characteristics are the components of the rigid–body displacements of the pile cap and the reaction forces and moments (these are called "lock forces" and "lock moments") at junction points. In the present section we analyze these quantities on the basis of the asymptotic algorithm developed for the region Ω_ε, which enables us to compare the pile structure model with the model of elastic multi–structure.

Let us assume that the load is applied to the region Ω only. Consider two cases.

(a) Suppose that the first two components of the principal force vector and the third component of the principal moment do not vanish simultaneously. In this case the leading order terms in the representation of rigid body displacements of the pile cap Ω are given by

$$\varepsilon^{-4}(\alpha_1^{(0)} \mathbf{e}^{(1)} + \alpha_2^{(0)} \mathbf{e}^{(2)} + \beta_3^{(0)} \mathbf{e}^{(3)} \times \mathbf{x}), \qquad (6.2.1)$$

where, in accordance with (6.1.3)

$$\begin{pmatrix} \alpha_1^{(0)} \\ \alpha_2^{(0)} \\ \beta_3^{(0)} \end{pmatrix} = \mathcal{A}^{-1} \begin{pmatrix} \mathbb{F}_1(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}; \Omega) \\ \mathbb{F}_2(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}; \Omega) \\ \mathbb{M}_3(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}; \Omega) \end{pmatrix}. \tag{6.2.2}$$

The leading order terms of the components of the reaction forces and moments at junction points of the elastic multi–structure Ω_ε are specified by (see formulae (6.1.4)–(6.1.8), (6.1.13))

$$\begin{aligned}
\mathfrak{F}_1^{(0,j)} &= -D_1 \frac{12}{l_j^3}(\alpha_1^{(0)} - a_2^{(j)}\beta_3^{(0)}), \\
\mathfrak{F}_2^{(0,j)} &= -D_2 \frac{12}{l_j^3}(\alpha_2^{(0)} + a_1^{(j)}\beta_3^{(0)}), \\
\mathfrak{M}_3^{(0,j)} &= -D_4 \frac{\beta_3^{(0)}}{l_j}, \\
\mathfrak{M}_1^{(0,j)} &= \frac{6D_2}{l_j^2}(\alpha_2^{(0)} + a_1^{(j)}\beta_3^{(0)}), \\
\mathfrak{M}_2^{(0,j)} &= -\frac{6D_1}{l_j^2}(\alpha_1^{(0)} - a_2^{(j)}\beta_3^{(0)}), \\
\mathfrak{F}_3^{(0,j)} &= -\frac{D_3}{l_j}(\alpha_3^{(0)} + a_2^{(j)}\beta_1^{(0)} - a_1^{(j)}\beta_2^{(0)} - d_3^{(II,0,j)} - c_3^{(0,j)}),
\end{aligned}$$

where, due to (6.1.12), the coefficients $\alpha_3^{(0)}$, $\beta_1^{(0)}$, $\beta_2^{(0)}$ admit the representation

$$(\alpha_3^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})^t = \mathcal{B}^{-1}\Big\{(h_4^{(0)}, h_5^{(0)}, h_6^{(0)})^t - \mathcal{D}(\alpha_1^{(0)}, \alpha_2^{(0)}, \beta_3^{(0)})^t\Big\}.$$

Here

$$\begin{aligned}
h_4^{(0)} &= \mathbb{F}_3(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega) + D_3 \sum_{j=1}^{K} \frac{d_3^{(II,0,j)} + c_3^{(0,j)}}{l_j}, \\
h_5^{(0)} &= \mathbb{M}_1(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega) + D_3 \sum_{j=1}^{K} a_2^{(j)} \frac{d_3^{(II,0,j)} + c_3^{(0,j)}}{l_j}, \\
h_6^{(0)} &= \mathbb{M}_2(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega) - D_3 \sum_{j=1}^{K} a_1^{(j)} \frac{d_3^{(II,0,j)} + c_3^{(0,j)}}{l_j}.
\end{aligned}$$

The constants $d_3^{(II,0,j)}$, $c_3^{(0,j)}$ come from the solutions of the junction layer and of the bottom layer problems (see (6.1.9), (6.1.11)).

Comparing the asymptotic approach with the pile structure model (see Section 3.7.2), we make the following comments:

(i) the coefficients $\alpha_1^{(0)}$, $\alpha_2^{(0)}$, $\beta_3^{(0)}$ in the representation of the leading order rigid body displacements of Ω are the same for the elastic multi–structure Ω_ε and for the pile structure described in Section 3.7.2;

(ii) the components $\mathfrak{F}_1^{(0,k)}$, $\mathfrak{F}_2^{(0,k)}$, $\mathfrak{M}_1^{(0,k)}$, $\mathfrak{M}_2^{(0,k)}$, $\mathfrak{M}_3^{(0,k)}$ of the transversal force and of the moment at junction points $\mathbf{a}^{(k)}$, $k = 1, \dots, K$, of the multi–structure Ω_ε are determined by the same formulae (see (3.7.18), (3.7.20), (3.7.21) and (3.7.22)–(3.7.25)) as used for calculation of the leading part of lock forces and moments in the pile structure;

(iii) the longitudinal components $\mathfrak{F}_3^{(0,k)}$ of the forces at junction points will differ from those calculated on the basis of the model of the pile structure (see Section 3.7.2); the group of coefficients $\alpha_3^{(0)}$, $\beta_1^{(0)}$, $\beta_2^{(0)}$ for the rigid body displacements can be calculated by means of the same formulae as (3.7.32). However, for the case of elastic multi–structure Ω_ε correction terms, involving $d_3^{(II,0,j)}$, $c_3^{(0,j)}$, occur in the right–hand side of the system of equations (see (6.1.12)).

Formula (6.2.2) shows that the leading order terms of the displacement field (see coefficients $\alpha_i^{(0)}$, $i = 1, 2$, and $\beta_3^{(0)}$) in the pile cap outside neighbourhoods of the junction points follow the classical theory of pile structures where the pile cap is treated as a rigid body. However, high order corrections are needed to evaluate the components $\alpha_3^{(0)}$, $\beta_i^{(0)}$, $i = 1, 2$, and the third component $\mathfrak{F}_3^{(j)}$ of the lock forces. To get these values, we ought to perform the asymptotic study of the elastic field in the vicinity of the junction points and the thin rods. This procedure requires the junction layer and the bottom layer type solutions.

(b) Now, assume that the first two components of the principal force vector and the third component of the principal moment do vanish. In this case one can deduce that the coefficients $\alpha_1^{(j)}$, $\alpha_2^{(j)}$, $\beta_3^{(j)}$, $j = 0, 1$, are equal to zero. Therefore, the leading order term of the displacement in Ω is specified by the rigid body translations and rotations of the form

$$\varepsilon^{-2}\Big\{\sum_{i=1}^{2}\alpha_i^{(2)}\mathbf{e}^{(i)} + \beta_3^{(2)}\mathbf{e}^{(3)} \times \mathbf{x} + \alpha_3^{(0)}\mathbf{e}^{(3)} + \sum_{i=1}^{2}\beta_i^{(0)}\mathbf{e}^{(i)} \times \mathbf{x}\Big\}, \tag{6.2.3}$$

where the coefficients are given by

$$\begin{pmatrix}\alpha_3^{(0)}\\ \beta_1^{(0)}\\ \beta_2^{(0)}\end{pmatrix} = \mathcal{B}^{-1}\begin{pmatrix}\mathbb{F}_3(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}; \Omega)\\ \mathbb{M}_1(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}; \Omega)\\ \mathbb{M}_2(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}; \Omega)\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1^{(2)}\\ \alpha_2^{(2)}\\ \beta_3^{(2)}\end{pmatrix} = \mathcal{A}^{-1}\begin{pmatrix}h_1^{(2)}\\ h_2^{(2)}\\ h_3^{(2)}\end{pmatrix} - \mathcal{A}^{-1}\mathcal{D}^t\begin{pmatrix}\alpha_3^{(0)}\\ \beta_1^{(0)}\\ \beta_2^{(0)}\end{pmatrix},$$

where the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are the same as in (4.9.19), (4.9.20), and the components $h_i^{(2)}$, $i = 1, 2, 3$, have the form

$$h_q^{(2)} = -\sum_j D_q\mathbb{Q}^{(III)}(0, 0, 0, d_{q+4}^{(II,1,j)}, -c_{q+4}^{(1,j)}),\ q = 1, 2,$$

$$h_3^{(2)} = \sum_j\Big\{D_4\mathbb{Q}^{(I)}(0, d_4^{(II,1,j)}, -c_4^{(1,j)}) - a_1^{(j)}D_2\mathbb{Q}^{(III)}(0, 0, 0, d_6^{(II,1,j)}, -c_6^{(1,j)})$$
$$+ a_2^{(j)}D_1\mathbb{Q}^{(III)}(0, 0, 0, d_5^{(II,1,j)}, -c_5^{(1,j)})\Big\},$$

with the functionals $\mathbb{Q}^{(I)}$, $\mathbb{Q}^{(III)}$ being the same as in Section 3.6, and the constants $d_i^{(II,1,j)}$, $c_i^{(1,j)}$, $i = 5, 6$, brought by the solutions of the junction layer and the bottom layer problems. The set (4.6.2) with $m = 1$ solves the boundary value problem (4.6.1) with

$$\tilde{\mathbf{F}}^{(1,j)} = 0,\ \tilde{\mathbf{P}}^{(1,j)} = 0,\ \mathfrak{M}^{(1,j)} = 0,\ \mathfrak{F}_1^{(0,j)} = \mathfrak{F}_2^{(0,j)} = 0,$$

and $\mathfrak{F}_3^{(0,j)}$ being defined by

$$\mathfrak{F}_3^{(0,j)} = -\frac{D_3}{l_j}(\alpha_3^{(0)} + \beta_1^{(0)}a_2^{(j)} - \beta_2^{(0)}a_1^{(j)}). \tag{6.2.4}$$

The set (4.8.11) solves the boundary value problem (4.8.1), (4.8.2), (4.8.12) where

$$\mathbf{G}^{(1,j)} = 0, \ \mathbf{H}^{(1,j)} = 0,$$

and

$$\tilde{\mathbf{\Phi}}^{(1,j)} = \frac{1}{l_j}\Big(\alpha_3^{(0)} + \beta_1^{(0)} a_2^{(j)} - \beta_2^{(0)} a_1^{(j)}\Big)\boldsymbol{\xi}^{(3)}(\mathbf{Y}', 0).$$

In this case all components of the lock forces and moments for $m = 0$, except the longitudinal force, are equal to zero. The longitudinal force at a junction point $\mathbf{a}^{(j)}$ is given by (6.2.4).

Comparing the two models, we should mention that the quantities $\alpha_3^{(0)}$, $\beta_1^{(0)}$, $\beta_2^{(0)}$, and $\mathfrak{F}_3^{(0,j)}$ are the same both for the elastic multi–structure Ω_ε and for the pile structure (see (3.7.32) and (3.7.19), (3.7.24)). The formulae for $\alpha_1^{(2)}$, $\alpha_2^{(2)}$, $\beta_3^{(2)}$ are similar for both models (compare with (3.7.34)). However, in the case of the elastic multi–structure Ω_ε the right–hand side of the system includes correction terms $h_i^{(2)}$, $1, 2, 3$, that come from solutions of the boundary layer and of the junction layer problems.

For the general case of geometry of the cross–section g the quantities $d_{q+4}^{(II,1,j)}$, $c_{q+4}^{(1,j)}$, $q = 1, 2$, may differ from zero. However, if g is centrally symmetric, then the above coefficients are identically zero, and, therefore,

$$h_q^{(2)} = 0, \ q = 1, 2; \ \ h_3^{(2)} = \sum_j D_4 \mathbb{Q}^{(I)}(0, d_4^{(II,1,j)}, -c_4^{(1,j)}).$$

Let us consider an example where the engineering approach works perfectly well. We assume that the cross–section g is centrally symmetric, and in addition suppose that the tractions and body forces are applied in such a way that the rigid body rotation is absent, i.e. $\beta_1^{(0)} = \beta_2^{(0)} = \beta_3^{(2)} = 0$, and that the first two components of the principal force vector are zero. Then,

$$\alpha_1^{(2)} = \alpha_2^{(2)} = 0, \ \alpha_3^{(0)} = \frac{(\lambda+\mu)\mathbb{F}_3(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega)}{\mu(3\lambda + 2\mu)\mathrm{mes}_2 g \sum_{i=1}^K 1/l_i},$$

and the leading part of the *longitudinal components of the lock forces* is

$$\mathfrak{F}_3^{(0,j)} = -\frac{\mathbb{F}_3(\mathbf{f}^{(0)}, \mathbf{p}^{(0)}, \Omega)}{l_j \sum_{k=1}^K 1/l_k}.$$

7 Asymptotics of first eigenvalues of the Lamé operator in the multi–structure

7.1 The spectral problem

Here we derive the asymptotic approximation for the first three eigenvalues of the spectral problem

$$-\mathbf{L}(\partial_{\mathbf{x}})\mathbf{u} = \lambda \mathbf{u}, \text{ in } \Omega_\varepsilon, \tag{7.1.1}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}; \mathbf{x}) = 0, \text{ on } \partial\Omega \setminus S_\varepsilon, \tag{7.1.2}$$

$$\mathbf{u}(\mathbf{x}) = 0, \text{ on } S_\varepsilon. \tag{7.1.3}$$

We shall use Theorem B1, presented in Appendix B. In our case the notations, used in Appendix B, have the form

$$X = (L_2(\Omega_\varepsilon))^3, \ \boldsymbol{\mathcal{H}} = (H^1(\Omega_\varepsilon))^3.$$

The semi–linear form $a(\cdot,\cdot)$ is given by

$$a(u,v) = \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \sigma_{ij}(\mathbf{u};\mathbf{x})\overline{\varepsilon_{ij}(\mathbf{v};\mathbf{x})}d\mathbf{x}.$$

It is clear that the properties 1), 2) from the Appendix B hold. Note that

$$a(\mathbf{u},\mathbf{u}) = 2\mathcal{E}(\mathbf{u};\Omega_\varepsilon),$$

where the functional $\mathcal{E}$ was defined by (1.2.4).

The notation $\mathbf{H}$ is used for a subspace of functions from $(H^1(\Omega_\varepsilon))^2$ vanishing on S_ε.

It remains to specify the subspace X_0 and obtain the estimate (B.1).

7.2 Korn's type inequalities

One can assume that the system of coordinates (x_1,x_2,x_3) has been chosen in such a way that the centre of mass of the body Ω is located on the Ox_3 axis, i.e.

$$\int_\Omega x_i d\mathbf{x} = 0,\ i=1,2.$$

Introduce the following vector functions

$$\begin{gathered}
\mathbf{\Psi}^{(1)}(\mathbf{x}) = \mathbf{e}^{(1)} - (I_2I_1 - Q_3^2)^{-1}I_1\kappa\mathbf{e}^{(1)}\times\mathbf{x} - (I_2I_1-Q_3^2)^{-1}Q_3\kappa\mathbf{e}^{(2)}\times\mathbf{x},\\
\mathbf{\Psi}^{(2)}(\mathbf{x}) = \mathbf{e}^{(2)} + (I_2I_1 - Q_3^2)^{-1}I_2\kappa\mathbf{e}^{(1)}\times\mathbf{x} + (I_2I_1-Q_3^2)^{-1}Q_3\kappa\mathbf{e}^{(2)}\times\mathbf{x},\\
\mathbf{\Psi}^{(3)}(\mathbf{x}) = \mathbf{e}^{(3)}\times\mathbf{x} + (I_2I_1-Q_3^2)^{-1}(I_1Q_1+Q_2Q_3)\mathbf{e}^{(1)}\times\mathbf{x}\\
+(I_2I_1-Q_3^2)^{-1}(I_2Q_2+Q_1Q_2)\mathbf{e}^{(2)}\times\mathbf{x},
\end{gathered}$$

where

$$I_i = \int_\Omega (\|\mathbf{x}\|^2 - x_i^2)d\mathbf{x},\ i=1,2,3, \tag{7.2.1}$$

are the moments of inertia, and

$$Q_j = \int_\Omega x_j^{-1}x_1x_2x_3 d\mathbf{x},\ \kappa = \int_\Omega x_3 d\mathbf{x}.$$

One can verify by direct calculations that

$$\int_\Omega \mathbf{e}^{(3)}\cdot\mathbf{\Psi}^{(j)}d\mathbf{x} = \int_\Omega \left(\mathbf{e}^{(1)}\times\mathbf{x}\right)\cdot\mathbf{\Psi}^{(j)}d\mathbf{x} = \int_\Omega \left(\mathbf{e}^{(2)}\times\mathbf{x}\right)\cdot\mathbf{\Psi}^{(j)}d\mathbf{x} = 0, \tag{7.2.2}$$

for $j=1,2,3$. Moreover, the vector functions $\mathbf{\Psi}^{(1)}, \mathbf{\Psi}^{(2)}, \mathbf{\Psi}^{(3)}$ are linearly independent.

We start with the following statement

Lemma 7.1 *If $\mathbf{u}\in\mathbf{H}$ satisfies*

$$\int_\Omega \mathbf{u}\cdot\mathbf{\Psi}^{(j)}\mathbf{x} = 0,\ j=1,2,3, \tag{7.2.3}$$

then

$$\int_{\Omega_\varepsilon} \|\mathbf{u}\|^2 d\mathbf{x} \le C\varepsilon^{-2}a(\mathbf{u},\mathbf{u}), \tag{7.2.4}$$

with a positive constant C independent of ε.

Proof. Denote by Z the set of functions $\mathbf{v} \in (H^1(\Omega))^3$ subjected to

$$\int_\Omega \mathbf{v}(\mathbf{x}) \cdot (\mathbf{a} + \mathbf{b} \times \mathbf{x}) d\mathbf{x} = 0,$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$. By C we shall denote a positive constant independent of ε, which may be different in different estimates. The following version of Korn's inequality holds: if $\mathbf{v} \in Z$ then

$$\int_\Omega \|\mathbf{v}(\mathbf{x})\|^2 d\mathbf{x} + \int_{\partial\Omega} \|\mathbf{v}(\mathbf{x})\|^2 ds \le C\mathcal{E}(\mathbf{v}; \Omega). \tag{7.2.5}$$

Due to (7.2.2), in the region Ω the vector function $\mathbf{u}$ admits the representation

$$\mathbf{u} = \mathbf{v} + \mathbf{w},$$

where $\mathbf{v} \in Z$,

$$\mathbf{w}(\mathbf{x}) = \alpha \mathbf{e}^{(3)} + \beta \mathbf{e}^{(1)} \times \mathbf{x} + \gamma \mathbf{e}^{(2)} \times \mathbf{x}, \tag{7.2.6}$$

with constant α, β, γ; the vector functions $\mathbf{v}$, $\mathbf{w}$ satisfy the relation

$$\int_\Omega \mathbf{v} \cdot \mathbf{w} d\mathbf{x} = 0. \tag{7.2.7}$$

Using (7.2.5) and the orthogonality condition (7.2.7) we deduce

$$\int_\Omega \|\mathbf{u}(\mathbf{x})\|^2 d\mathbf{x} \le C\mathcal{E}(\mathbf{v}; \Omega) + \int_{\partial\Omega} \|\mathbf{w}\|^2 d\mathbf{x}. \tag{7.2.8}$$

Let Γ_ε denote the junction region of the boundary $\partial\Omega$ i.e.

$$\Gamma_\varepsilon = \Big\{\mathbf{x} \in \partial\Omega : x_3 = 0, \varepsilon^{-1}(x_1 - a_1^{(j)}, x_2 - a_2^{(j)}) \in g,\ j = 1, \ldots, K\Big\}.$$

Introduce

$$d(\varepsilon) = \sup \int_\Omega \|\mathbf{w}\|^2 d\mathbf{x} \Big(\int_{\Gamma_\varepsilon} |w_3|^2 dx_1 dx_2 \Big)^{-1}$$

where the supremum is taken over all $\mathbf{w}$ of the form (7.2.6) with $|\alpha| + |\beta| + |\gamma| > 0$. Since the points $\mathbf{a}^{(j)}, j = 1, \ldots, K$, do not lie on the same straight line we obtain

$$d(\varepsilon) \le C\varepsilon^{-2},$$

with C being independent of ε. Using the definition of $d(\varepsilon)$, (7.2.5), and the equality $\varepsilon_{ij}(\mathbf{w}) = 0$ we derive

$$\int_\Omega \|\mathbf{u}\|^2 d\mathbf{x} \le c\mathcal{E}(\mathbf{u}; \Omega) + c_0 \varepsilon^{-2} \int_{\Gamma_\varepsilon} |w_3|^2 dx_1 dx_2. \tag{7.2.9}$$

By (7.2.5)

$$\int_{\Gamma_\varepsilon} |w_3|^2 d\mathbf{x} \le 2 \int_{\Gamma_\varepsilon} (|u_3|^2 + |v_3|^2) d\mathbf{x} \le 2 \int_{\Gamma_\varepsilon} |u_3|^2 + c\mathcal{E}(\mathbf{u}; \Omega). \tag{7.2.10}$$

Combining (7.2.9) and (7.2.10) we arrive at

$$\int_\Omega \|\mathbf{u}\|^2 d\mathbf{x} \le c\varepsilon^{-2} \Big(\mathcal{E}(\mathbf{u}; \Omega) + \int_{\Gamma_\varepsilon} |u_3|^2 dx_1 dx_2 \Big).$$

Since

$$\int_{\Gamma_\varepsilon} |u_3|^2 dx_1 dx_2 \le C \sum_{j=1}^{N} \int_{\Pi_\varepsilon^{(j)}} |\partial_3 u_3|^2 d\mathbf{x}$$

with C independent of ε, we get

$$\int_\Omega \|\mathbf{u}\|^2 d\mathbf{x} \le C\varepsilon^{-2}\mathcal{E}(\mathbf{u};\Omega_\varepsilon). \tag{7.2.11}$$

Inequality (A.9) and the relation $\mathbf{u} = 0$ on S_ε provide the estimate

$$\sum_{j=1}^{K} \int_{\Pi_\varepsilon^{(j)}} \|\mathbf{u}\|^2 d\mathbf{x} \le C\varepsilon^{-2}\mathcal{E}(\mathbf{u};\bigcup_j \Pi_\varepsilon^{(j)}). \tag{7.2.12}$$

Using (7.2.11), (7.2.12) we arrive at (7.2.4). □

We show that the inequality (7.2.4) is still valid if in the orthogonality condition (7.2.3) the domain Ω is replaced by Ω_ε.

Lemma 7.2 *If* $\mathbf{u} \in \mathbf{H}$ *satisfies*

$$\int_{\Omega_\varepsilon} \mathbf{u}\cdot \mathbf{\Psi}^{(j)} d\mathbf{x} = 0,\ j = 1,2,3, \tag{7.2.13}$$

then the inequality (7.2.4) holds, where the constant C is independent of ε.

Proof. Suppose that the equalities (7.2.13) hold. Let us find constants $c_k = c_k(\mathbf{u})$, $k = 1,2,3$, such that

$$\int_\Omega (\mathbf{u} - \sum_{k=1}^{3} c_k \mathbf{\Psi}^{(k)}) \cdot \mathbf{\Psi}^{(j)} d\mathbf{x} = 0, j = 1,2,3,$$

or, equivalently,

$$\sum_{k=1}^{3} c_k \int_\Omega \mathbf{\Psi}^{(k)} \cdot \mathbf{\Psi}^{(j)} d\mathbf{x} = \int_\Omega \mathbf{u}\cdot \mathbf{\Psi}^{(j)},\ j = 1,2,3.$$

Since the matrix

$$\Big\{ \int_\Omega \mathbf{\Psi}^{(k)} \cdot \mathbf{\Psi}^{(j)} d\mathbf{x} \Big\}_{k,j=1}^{3}$$

is nonsingular and due to the orthogonality condition (7.2.13), we have that there exists a unique set $\{c_1, c_2, c_3\}$, and

$$\sum_{k=1}^{3} |c_k| \le C\varepsilon \|\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3}, \tag{7.2.14}$$

where C does not depend on ε and $\mathbf{u}$.

Let $\eta = \eta(z)$ be a smooth function defined for $z \ge 0$ in such a way that $\eta(0) = 1$ and $\eta(z) = 0$ for $z > \delta$, where δ is a positive number which does not exceed $\min_j l_j$. We represent the vector function $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{v} + \mathbf{w},$$

where

$$\mathbf{v}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \sum_{k=1}^{3} c_k \mathbf{\Psi}^{(k)}(\mathbf{x}) \ \text{ in } \ \Omega,$$

and

$$\mathbf{v}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \sum_{k=1}^{3} c_k \Big(\mathbf{\Psi}^{(k)}(\mathbf{x}) - \eta(l_j - x_3)\mathbf{\Psi}^{(k)}(l_j) \Big) \ \text{ in } \ \Pi_\varepsilon^{(j)}.$$

Clearly, $\mathbf{v} \in \mathbf{H}$ and

$$\int_\Omega \mathbf{v} \cdot \mathbf{\Psi}^{(j)} d\mathbf{x} = 0 \ \text{ for } \ j = 1, 2, 3.$$

Therefore, by Lemma 7.1,

$$\|\mathbf{u}\|^2_{(L_2(\Omega_\varepsilon))^3} \le C \Big(\varepsilon^{-2} a(\mathbf{v}, \mathbf{v}) + \|\mathbf{w}\|^2_{(L_2(\Omega_\varepsilon))^3} \Big). \tag{7.2.15}$$

Using (7.2.14) one has

$$\|\mathbf{w}\|^2_{(L_2(\Omega_\varepsilon))^3} \le C\varepsilon^2 \|\mathbf{u}\|_{(L_2(\Omega_\varepsilon))^3}. \tag{7.2.16}$$

The definition of $\mathbf{v}$ provides the estimate

$$a(\mathbf{v}, \mathbf{v}) \le 2 \Big(a(\mathbf{u}, \mathbf{u}) + 2 \sum_{j=1}^{K} \sum_{k=1}^{3} |c_k|^2 \mathcal{E}(\mathbf{w}^{(j,k)}; \Pi_\varepsilon^{(j)}) \Big),$$

where

$$\mathbf{w}^{(j,k)} = \eta(l_j - x_3)\mathbf{\Psi}^{(k)}(l_j).$$

Due to the inequality

$$\mathcal{E}(\mathbf{w}^{(j,k)}; \Pi_\varepsilon^{(j)}) \le C\varepsilon^2$$

and (7.2.14), we arrive at

$$a(\mathbf{v}, \mathbf{v}) \le 2a(\mathbf{u}, \mathbf{u}) + C\varepsilon^4 \|\mathbf{u}\|^2_{(L_2(\Omega_\varepsilon))^3}. \tag{7.2.17}$$

The estimate (7.2.15) together with (7.2.16), (7.2.17) lead to (7.2.4). □

Thus, the space X_0, specified in Appendix B, will be chosen as a linear hull of the vector functions $\mathbf{\Psi}^{(1)}$, $\mathbf{\Psi}^{(2)}$, $\mathbf{\Psi}^{(3)}$, and the inequality (7.2.4), proved in Lemma 7.2, yields the estimate (B.1) with $q = \text{Const}\ \varepsilon^{-2}$.

7.3 Asymptotic expansion for eigenvalues

Let $\rho_1 \le \rho_2 \le \rho_3$ be eigenvalues of the problem

$$\mathcal{A}\mathbf{q} = \rho \mathbf{J} \mathbf{q}, \tag{7.3.1}$$

where $\mathbf{q} \in \mathbb{C}^3$, and

$$\mathbf{J} = \text{diag}\{\text{mes}\Omega, \text{mes}\Omega, I_3\}. \tag{7.3.2}$$

The matrix $\mathcal{A}$ is given by (3.7.1), where the coordinates of the junction points are regarded to the Cartesian system of coordinates such that Ox_1, Ox_2 are parallel to the principal inertia axes of the region g, all junction points are located in the plane $x_3 = 0$, and the inertia centre of the body Ω is placed on the Ox_3 axis.

Here I_3 is the inertia moment of Ω with respect to Ox_3 axis specified by (7.2.1). Since $\mathcal{A}$ is positive definite (see (3.7.5)), the eigenvalues ρ_i are positive.

The following statement gives the asymptotics for first three eigenvalues of the spectral problem (7.1.1)–(7.1.3).

Theorem 7.1 *First three eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$ of the spectral problem (7.1.1)–(7.1.3) admit the asymptotic representation*

$$\lambda_j = \varepsilon^4 \rho_j + O(\varepsilon^5). \tag{7.3.3}$$

Remaining eigenvalues are greater than $C\varepsilon^2$ with some positive constant C independent of ε.

Proof. Here, we use the notations $X, X_0, \boldsymbol{\mathcal{H}}$ and $\mathbf{H}$ introduced in the previous subsection.

Let $\mathfrak{N}(\boldsymbol{\Psi})$ be the solution of the boundary value problem (1.2.1)–(1.2.3) with $\mathbf{F} = \boldsymbol{\Psi}$, $\mathbf{P} = 0, \boldsymbol{\Phi}^{(k)} = 0$, $k = 1, \ldots, K$. Then, the space $\mathbf{H}_0$, used in Appendix B, is $\mathfrak{N}(X_0)$. The ratio (B.4) of quadratic forms can be written as

$$\frac{2\mathcal{E}(\mathfrak{N}(\boldsymbol{\Psi}); \Omega_\varepsilon)}{\|\mathfrak{N}(\boldsymbol{\Psi})\|^2_{(L_2(\Omega_\varepsilon))^3}}, \ \boldsymbol{\Psi} \in X_0,$$

or, equivalently,

$$\frac{\int_{\Omega_\varepsilon} \mathfrak{N}(\boldsymbol{\Psi}) \cdot \overline{\boldsymbol{\Psi}} d\mathbf{x}}{\|\mathfrak{N}(\boldsymbol{\Psi})\|^2_{(L_2(\Omega_\varepsilon))^3}}, \boldsymbol{\Psi} \in X_0. \tag{7.3.4}$$

Let $\nu_1 \le \nu_2 \le \nu_3$ denote the eigenvalues of the spectral problem induced by (7.3.4). According to (B.3) and Theorem B1 (we recall that in our case $q = \text{Const}\ \varepsilon^{-2}$)

$$\lambda_4 \ge \text{Const}\ \varepsilon^2,$$

and

$$\frac{\nu_j}{1 + C\varepsilon^{-2}\nu_j} \le \lambda_j \le \nu_j, \ j = 1, 2, 3. \tag{7.3.5}$$

Now, we need the asymptotic representation for ν_j. First, represent the spectral problem, corresponding to (7.3.4), in the coordinate form.

Let

$$\boldsymbol{\Psi}(\mathbf{x}) = \sum_{j=1}^{3} b_j \boldsymbol{\Psi}_j(\mathbf{x}). \tag{7.3.6}$$

Due to Theorem 5.1 and formulae for $\mathbf{u}^{(-1)}$, $\mathbf{u}^{(0)}$ from Section 6.1, the vector function $\mathfrak{N}(\boldsymbol{\Psi})$ admits the asymptotic representation

$$\mathfrak{N}(\boldsymbol{\Psi}) = \varepsilon^{-4}(\alpha \mathbf{e}^{(1)} + \beta \mathbf{e}^{(2)} + \gamma \mathbf{e}^{(3)} \times \mathbf{x}) + \mathbf{w}, \tag{7.3.7}$$

where the remainder satisfies the estimate

$$\|\mathbf{w}\|_{(L_2(\Omega_\varepsilon))^3} \le C\varepsilon^{-3} \|\boldsymbol{\Psi}\|_{(L_2(\Omega_\varepsilon))^3}. \tag{7.3.8}$$

The constants α, β, γ solve the linear system

$$\mathcal{A} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},$$

with

$$h_j = \int_\Omega \mathbf{e}^{(j)} \cdot \boldsymbol{\Psi} d\mathbf{x}, \ j = 1, 2, \tag{7.3.9}$$

$$h_3 = \int_\Omega (\mathbf{e}^{(3)} \times \mathbf{x}) \cdot \mathbf{\Psi} d\mathbf{x} = \int_\Omega (\mathbf{x} \times \mathbf{\Psi})_3 d\mathbf{x}. \tag{7.3.10}$$

Let us introduce the matrix $\mathbf{N} = \{N_{jk}\}_{j,k=1}^3$ with elements

$$N_{jk} = \int_\Omega \mathbf{e}^{(j)} \cdot \mathbf{\Psi}^{(k)} d\mathbf{x} \ \text{ for } j = 1, 2, \ k = 1, 2, 3, \tag{7.3.11}$$

$$N_{3k} = \int_\Omega (\mathbf{e}^{(3)} \times \mathbf{x}) \cdot \mathbf{\Psi}^{(k)} d\mathbf{x} \ \text{ for } k = 1, 2, 3. \tag{7.3.12}$$

Using the definition of $\mathbf{\Psi}^{(j)}$, $j = 1, 2, 3$, one can verify by direct calculations that the matrix $\mathbf{N}$ is symmetric. It follows from linear independence of $\mathbf{\Psi}^{(1)}, \mathbf{\Psi}^{(2)}, \mathbf{\Psi}^{(3)}$ and the orthogonality conditions (7.2.2) that the matrix $\mathbf{N}$ is nonsingular. Now, the equalities (7.3.9), (7.3.10) can be written in the form

$$h_j = \sum_{k=1}^3 N_{jk} b_k, \ j = 1, 2, 3,$$

where b_k are constant coefficients from (7.3.6).

Therefore,

$$(\alpha, \beta, \gamma)^t = \mathcal{A}^{-1} \mathbf{N} \mathbf{b},$$

where $\mathbf{b} = (b_1, b_2, b_3)^t$. Hence,

$$\int_{\Omega_\varepsilon} \mathfrak{N}(\mathbf{\Psi}) \cdot \mathbf{\Psi} d\mathbf{x} = (\mathcal{A}^{-1} \mathbf{N} \mathbf{b}, \mathbf{N} \mathbf{b})_{\mathbb{C}^3} \varepsilon^{-4} + O(\varepsilon^{-3}).$$

One can verify that

$$\|\mathfrak{N}(\mathbf{\Psi})\|^2_{(L_2(\Omega_\varepsilon))^3} = \varepsilon^{-8} \Big(\mathbf{J} (\alpha, \beta, \gamma)^t, (\alpha, \beta, \gamma)^t \Big)_{\mathbb{C}^3} + O(\varepsilon^{-7}),$$

where $\mathbf{J}$ was defined in (7.3.2). Therefore,

$$\|\mathfrak{N}(\mathbf{\Psi})\|^2_{(L_2(\Omega_\varepsilon))^3} = \varepsilon^{-8} (\mathbf{J} \mathcal{A}^{-1} \mathbf{N} \mathbf{b}, \mathcal{A}^{-1} \mathbf{N} \mathbf{b})_{\mathbb{C}^3} + O(\varepsilon^{-7}).$$

Now, the coordinate form of the spectral problem, corresponding to (7.3.4), is

$$\Big(\mathbf{N} \mathcal{A}^{-1} \mathbf{N} + O(\varepsilon) \Big) \mathbf{b} = \nu \varepsilon^{-4} \Big(\mathbf{N} \mathcal{A}^{-1} \mathbf{J} \mathcal{A}^{-1} \mathbf{N} + O(\varepsilon) \Big) \mathbf{b}, \tag{7.3.13}$$

where the perturbation operators are symmetric.

Let

$$\mathbf{q} = \mathcal{A}^{-1} \mathbf{N} \mathbf{b}.$$

The spectral problem (7.3.13) can be rewritten as

$$\Big(\mathcal{A} + O(\varepsilon) \Big) \mathbf{q} = \nu \varepsilon^{-4} \Big(\mathbf{J} + O(\varepsilon) \Big) \mathbf{q},$$

with the perturbation operators being symmetric. Hence, the eigenvalues ν_1, ν_2, ν_3 admit the asymptotic approximation

$$\nu_j = \varepsilon^4 \rho_j + O(\varepsilon^5).$$

This equality together with (7.3.5) yield the formula (7.3.3). □

8 Appendix A. Korn's Inequalities

Here we present several versions of the Korn inequality which were used to justify the formal asymptotic expansion (4.4.1), (4.4.2).

The following Korn's inequality for vector functions in a half–space is well-known and can be proved by the Fourier transform in x_1, x_2.

Lemma A1. *Assume that* $\mathbf{u} = (u_1, u_2, u_3) \in (H^1(\mathbb{R}^3_-))^3$. *Then the following inequality holds*

$$\|\nabla \mathbf{u}\|^2_{(L_2(\mathbb{R}^3_-))^3} \le C\mathcal{E}(\mathbf{u}; \mathbb{R}^3_-). \tag{A.1}$$

Korn's inequality, established in the next Lemma, was stated by Nazarov (1991) [10] with a sketch of the proof. We give a detailed proof for reader's convenience.

Lemma A2. *Let* Π *denote a semi–cylinder* $\{\mathbf{x} : (x_1, x_2) \in g, x_3 > 1\}$, *where* g *is a bounded Lipschitz domain in* $\mathbb{R}^2$. *Consider* $\mathbf{u} \in (H^1_{loc}(\overline{\Pi}))^3$. *Then the inequality is valid*

$$\int_\Pi (|\nabla' u_3|^2 + |u_3|^2 + \|\partial_{x_3}\mathbf{u}'\|^2)\frac{dx}{x_3^2} + \int_\Pi (\|\nabla'\mathbf{u}'\|^2 + \|\mathbf{u}'\|^2)\frac{dx}{x_3^4} \le$$

$$\le C(\mathcal{E}(\mathbf{u}, \Pi) + \int_g \|\mathbf{u}(x', 1)\|^2 dx'), \tag{A.2}$$

where $\mathbf{u}' = (u_1, u_2), x' = (x_1, x_2), \nabla' = (\partial_{x_1}, \partial_{x_2})$.

Proof. Consider a bounded cylinder $\Pi_l = \{\mathbf{x} : x' \in g, a < x_3 < a + l\},\ l \ge 1$. We shall prove the inequality

$$\frac{C_0}{l^2}\int_{\Pi_l} (\|\nabla' u_3\|^2 + |u_3|^2)dx + \frac{1}{2l}\int_g |u_3(x', a+l)|^2 dx' \le$$

$$\le C_1 \sum_{i,j=1}^3 \int_{\Pi_l} |\varepsilon_{ij}(\mathbf{u})|^2 dx + \frac{1}{l}\int_g |u_3(x', a)|^2 dx', \tag{A.3}$$

where C_0 and C_1 do not depend of a, l and u. Introducing new variables

$$x_3 = ly_3 + a,\ x' = y',\ U_3 = lu_3,\ \mathbf{U}' = \mathbf{u}',$$

we reduce the proof of the inequality to the case $a = 0,\ l = 1$. The direct consequence of the Newton–Leibnitz formula is

$$\int_g |U_3(y', 1)|^2 dy' \le C\int_{\Pi_1} |\varepsilon_{33}(\mathbf{U})|^2 dy + \frac{3}{2}\int_g |U_3(y', 0)|^2 dy'. \tag{A.4}$$

We use the well–known (see, for example [18]) Korn's inequality

$$\|\mathbf{v}\|_{(H^1(\Pi_1))^3} \le C\left(\sum_{i,j=1}^3 \int_{\Pi_1} |\varepsilon_{ij}(\mathbf{v})|^2 dy\right)^{1/2}, \tag{A.5}$$

which is valid for all $\mathbf{v}$ from a closed subspace of $(H^1(\Pi_1))^3$ excluding rigid body displacements. This yields that for all $\mathbf{U} \in (H^1(\Pi_1))^3$

$$\|\mathbf{U}\|^2_{(H^1(\Pi_1))^3} \le C\mathcal{E}(\mathbf{U}; \Pi_1) + \sum_{j=1}^6 |F_j(\mathbf{U})|^2,$$

where F_j are linear bounded functionals on $(H^1(\Pi_1))^3$ which do not vanish simultaneously on the rigid body displacements. Clearly, the following choice can be made

$$F_j(\mathbf{U}) = \int_g U_3(\mathbf{y}', 0) y_j d\mathbf{y}', \ j = 1, 2,$$

$$F_{6-j}(\mathbf{U}) = \int_g U_j(\mathbf{y}', 0) d\mathbf{y}', \ j = 1, 2, 3,$$

$$F_6(\mathbf{U}) = \int_g (U_1(\mathbf{y}', 0) y_2 - U_2(\mathbf{y}', 0) y_1) d\mathbf{y}'.$$

Thus,

$$C_0 \int_{\Pi_1} (\|\nabla' U_3\|^2 + |U_3|^2) d\mathbf{y} \leq C\mathcal{E}(\mathbf{U}; \Pi_1) + \frac{1}{4} \int_g |U_3(\mathbf{y}', 0)|^2 d\mathbf{y}' + \sum_{j=4}^{6} |F_j(\mathbf{U})|^2. \tag{A.6}$$

Instead of $\mathbf{U}$ one can substitute the vector

$$\mathbf{U} + C_1 \mathbf{e}^{(1)} + C_2 \mathbf{e}^{(2)} + C_3 (\mathbf{y} \times \mathbf{e}^{(3)}).$$

In the last inequality the left–hand side and the first two terms on the right do not change. However the last term in the right–hand side will be zero provided the constants C_j satisfy

$$F_j(\mathbf{U}) + C_1 F_j \mathbf{e}^{(1)} + C_2 F_j \mathbf{e}^{(2)} + C_3 F_j (\mathbf{y} \times \mathbf{e}^{(3)}) = 0,$$

for $j = 4, 5, 6$. Since the matrix of this system is nonsingular, the constants C_k are defined uniquely. Thus, the last term in the right–hand side (A.6) may be omitted. Next, using (A.4) we obtain (A.3).

Let $a = l = 2^k, \ k = 0, 1, \ldots$. Then

$$c_0 2^{-2k} \int_{\Pi^{(k)}} (\|\nabla' u_3\|^2 + |u_3|^2) d\mathbf{x} + 2^{-k-1} \int_g |u_3(\mathbf{x}', 2^{k+1})|^2 d\mathbf{x}' \leq$$

$$\leq c_1 \mathcal{E}(\mathbf{u}; \Pi^{(k)}) + 2^{-k} \int_g |u_3(\mathbf{x}', 2^k)|^2 d\mathbf{x}', \tag{A.7}$$

where $\Pi^{(k)} = \{\mathbf{x} : \mathbf{x}' \in g, 2^k < x_3 < 2^{k+1}\}$. Taking a sum of (A.7) with respect to k we derive

$$\int_\Pi (\|\nabla' u_3\|^2 + |u_3|^2) \frac{d\mathbf{x}}{x_3^2} \leq C \left(\mathcal{E}(\mathbf{u}; \Pi) + \int_g |u_3(\mathbf{x}', 1)|^2 d\mathbf{x}' \right). \tag{A.8}$$

Hence, by $\partial_{x_3} u_i(\mathbf{x}) = 2\varepsilon_{i3}(\mathbf{u}; \mathbf{x}) - \partial_{x_i} u_3(\mathbf{x}), \ i = 1, 2$, we obtain

$$\int_\Pi \|\partial_{x_3} \mathbf{u}'\|^2 \frac{d\mathbf{x}}{x_3^2} \leq C(\mathcal{E}(\mathbf{u}; \Pi) + \int_g |u_3(\mathbf{x}', 1)|^2 d\mathbf{x}'). \tag{A.9}$$

Applying the Hardy inequality one has

$$\int_\Pi \|\mathbf{u}'\|^2 \frac{d\mathbf{x}}{x_3^4} \leq C(\int_\Pi \|\partial_{x_3} \mathbf{u}'\|^2 \frac{d\mathbf{x}}{x_3^2} + \int_g |\mathbf{u}'(\mathbf{x}', 1)|^2 d\mathbf{x}'). \tag{A.10}$$

The inequalities (A.9) and (A.10) yield the estimate of the integral in the left hand side (A.10) by the expression in the right hand side (A.2).

Using the two–dimensional Korn inequality

$$\int_g \|\nabla' \mathbf{u}'(\mathbf{x}', x_3)\|^2 d\mathbf{x}' \le C(\sum_{i,j=1}^{2} \int_g |\varepsilon_{ij}(\mathbf{u}; \mathbf{x}', x_3)|^2 d\mathbf{x}' + \int_g \|\mathbf{u}'(\mathbf{x}', x_3)\|^2 d\mathbf{x}'),$$

together with (A.10), we estimate the term, including $\nabla' \mathbf{u}'$ in (A.2), and complete the proof. □

Remark 1. During the proof of the Lemma we also obtained the following Korn's inequality for vector–valued functions in the cylinder $\Pi_l = \{\mathbf{x} : \mathbf{x}' \in g,\ 0 < x_3 < l\}$, $l \ge 1$,

$$l^{-2} \int_{\Pi} (\|\nabla' u_3\|^2 + |u_3|^2 + \|\partial_{x_3} \mathbf{u}'\|^2) d\mathbf{x} + l^{-4} \int_{\Pi} (\|\nabla' \mathbf{u}'\|^2 + \|\mathbf{u}'\|^2) d\mathbf{x} \le$$

$$\le C(\mathcal{E}(\mathbf{u}; \Pi_l) + \int_g \|\mathbf{u}(\mathbf{x}', 0)\|^2 d\mathbf{x}'), \tag{A.11}$$

where C does not depend of l and $\mathbf{u}$.

Taking l to be ε^{-1} and substituting $\varepsilon^{-1}\mathbf{x}$ instead of $\mathbf{x}$ we derive Korn's inequality for the thin cylinder $\pi_\varepsilon = \{\mathbf{x} : \varepsilon^{-1} x' \in g,\ 0 < x_3 < 1\}$

$$\int_{\pi_\varepsilon} (|u_3|^2 + \varepsilon^2(\|\nabla' u_3\|^2 + \|\partial_{x_3} \mathbf{u}'\|^2 + \|\mathbf{u}'\|^2) + \varepsilon^4 \|\nabla' \mathbf{u}'\|^2) d\mathbf{x} \le$$

$$\le C(\mathcal{E}(\mathbf{u}; \pi_\varepsilon) + \varepsilon^{-1} \int_{\varepsilon g} \|\mathbf{u}(\mathbf{x}', 0\|^2 d\mathbf{x}'), \tag{A.12}$$

where C does not depend on ε and $\mathbf{u}$.

Lemma A3. *Let Ω be a bounded domain with the Lipschitz boundary. Also, let ω be a non–empty subdomain of Ω. The following inequality is valid for all $\mathbf{u} \in (H^1(\Omega))^3$*

$$\int_\Omega \|\mathbf{u}(\mathbf{x})\|^2 d\mathbf{x} \le C\alpha_\omega(\mathcal{E}(\mathbf{u}; \Omega) + \int_\omega \|\mathbf{u}\|^2 d\mathbf{x}), \tag{A.13}$$

with C independent of ω and

$$\alpha_\omega = \sup_{\mathbf{a},\mathbf{b}} \frac{\int_\Omega \|\mathbf{a} + \mathbf{b} \times \mathbf{x}\|^2 d\mathbf{x}}{\int_\omega \|\mathbf{a} + \mathbf{b} \times \mathbf{x}\|^2 d\mathbf{x}}.$$

Here, the supremum is taken over all non–zero constant vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

Proof. Consider $X_0 = \{\mathbf{a} + \mathbf{b} \times \mathbf{x} : \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}$ and represent $(H^1(\Omega))^3$ as the direct sum $X_0 + X_1$ where

$$X_1 = \{\mathbf{v} \in (H^1(\Omega))^3 : \int_\Omega \mathbf{v}(\mathbf{x}) \cdot (\mathbf{a} + \mathbf{b} \times \mathbf{x}) d\mathbf{x} = 0, \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}.$$

Let $\mathbf{u} \in (H^1(\Omega))^3$ and $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in X_1$, $\mathbf{w} \in X_0$. The vector–valued function $\mathbf{v}$ satisfies the Korn inequality

$$\|\mathbf{v}\|_{(H^1(\Omega))^3} \le C\mathcal{E}(\mathbf{v}; \Omega). \tag{A.14}$$

Hence

$$\int_\Omega \|\mathbf{u}\|^2 dx \le 2 \int_\Omega (\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2) d\mathbf{x} \le 2C\mathcal{E}(\mathbf{u}; \Omega) + 2\alpha_\omega \int_\omega \|\mathbf{w}\|^2 d\mathbf{x}.$$

To complete the proof we use (A.14) and write the inequality

$$\int_{\omega}\|\mathbf{w}\|^2 dx \leq 2\int_{\omega}(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2)d\mathbf{x} \leq$$

$$\leq 2\int_{\omega}\|\mathbf{u}\|^2 d\mathbf{x} + 2C\mathcal{E}(\mathbf{u};\Omega).$$

□

Remark 2. In a similar way we obtain the estimate

$$\int_{\Omega}\|\nabla\mathbf{u}\|^2 d\mathbf{x} \leq C\beta_{\omega}(\mathcal{E}(\mathbf{u};\Omega)+\int_{\omega}\|\mathbf{u}\|^2 d\mathbf{x}),$$

where

$$\beta_{\omega}=\sup_{\mathbf{a},\mathbf{b}}\frac{\|\mathbf{b}\|^2}{\int_{\omega}\|\mathbf{a}+\mathbf{b}\times\mathbf{x}\|^2 d\mathbf{x}}.$$

Corollary. *Let Ω_{ε} be the multi–structure described in Introduction. An arbitrary vector–valued function $\mathbf{u}\in(H^1(\Omega_{\varepsilon}))^3$ satisfies Korn's inequality*

$$\|\mathbf{u}\|_{(H^1(\Omega_{\varepsilon}))^3} \leq C\varepsilon^{-4}\{\mathcal{E}(\mathbf{u},\Omega_{\varepsilon})+\varepsilon^{-1}\int_{S_{\varepsilon}}\|\mathbf{u}\|^2 d\mathbf{x}'\}^{1/2}. \tag{A.15}$$

Proof. Let Π be a thin cylindrical element of the multi–structure. This cylinder can be extended into the three-dimensional region by a distance δ; let Σ denote this extension. Assume that Σ belongs to $\bar{\Omega}$. By Remark 1

$$\|\mathbf{u}\|_{(H^1(\Pi\cup\Sigma))^3} \leq C\varepsilon^{-2}(\mathcal{E}(\mathbf{u};\Pi\cup\Sigma)+\varepsilon^{-1}\int_{S}\|\mathbf{u}\|^2 d\mathbf{x}')^{1/2}, \tag{A.16}$$

where S is the base part of the cylinder surface $\partial(\Pi\cup\Sigma)$. Lemma A3 and Remark 2 yield

$$\|\mathbf{u}\|_{(H^1(\Omega))^3} \leq C\varepsilon^{-2}(\mathcal{E}(\mathbf{u};\Omega)+\int_{\Sigma}\|\mathbf{u}\|^2 d\mathbf{x})^{1/2}. \tag{A.17}$$

Combining (A.16) and (A.17) we obtain inequality (A.15). □

9 Appendix B. Abstract scheme of estimating the eigenvalues

Here we present an abstract scheme of spectral analysis for first eigenvalues of a boundary value problem in the multi–structure. This is a slight modification of the scheme presented in Appendix of our paper [7].

Let X and $\mathcal{H}$ be Hilbert spaces with the scalar products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_{\mathcal{H}}$ and with the norms

$$\| x \|_X=(x,x)_X^{1/2}, \| h \|_{\mathcal{H}}=(h,h)_{\mathcal{H}}^{1/2}.$$

We suppose that the space $\mathcal{H}$ is densely and compactly embedded in the space X.

Let us consider on $\mathcal{H}$ a semi–linear form $a(\cdot,\cdot)$ such that

1) $a(h,h)\geq 0,\ \ \forall h\in\mathcal{H}$
2) $a(h,h)\geq c_0 \| h \|_{\mathcal{H}}^2 - c_1 \| h \|_X^2,\ \ \forall h\in\mathcal{H}$

where c_0, c_1 are positive constants.

Due to these properties, the form can degenerate only on a finite–dimensional subspace. We fix a subspace X_0 of the above finite dimensional subspace and denote by $P : X \to X_0$ the corresponding orthogonal projector. Let $n = \dim X_0$.

Also, let $\mathbf{H}$ be a closed subspace in $\boldsymbol{\mathcal{H}}$, dense in X. We suppose that

$$\|h - Ph\|_X^2 \leq qa(h,h), \text{ for all } h \in \mathbf{H}, \tag{B.1}$$

where q is a positive constant.

We are interested in eigenvalues of the spectral problem induced by the ratio of quadratic forms

$$\frac{a(h,h)}{\| h \|_X^2}, \quad h \in \mathbf{H}. \tag{B.2}$$

We anumerate the eigenvalues (taking into account their multiplicities) in the non–decreasing order

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

Our objective is to obtain the estimates of $\lambda_i,\ i = 1, \dots, n$, in terms of eigenvalues of some finite–dimensional spectral problem.

Let us formulate this problem. Suppose that

$$\mathbf{H}_1 = \{h \in \mathbf{H} : Ph = 0\}.$$

Thus, $\mathbf{H}_1$ is a closed subspace in $\mathbf{H}$ with the codimension n. Let

$$\mathbf{H}_0 = \{h \in \mathbf{H} :\ a(h,g) = 0,\ \forall g \in \mathbf{H}_1\}.$$

One can see that $\dim \mathbf{H}_0 = n$ and $\mathbf{H}$ is the direct sum of $\mathbf{H}_0$ and $\mathbf{H}_1$.

Since the ratio in (B.2) is bounded below by $1/q$ for all $0 \neq h \in \mathbf{H}_1$, then

$$\lambda_{n+1} \geq 1/q. \tag{B.3}$$

We introduce the auxiliary finite–dimensional spectral problem induced by the quotient

$$\frac{a(h,h)}{\| h \|_X^2}, \quad h \in \mathbf{H}_0, \tag{B.4}$$

and denote by $0 \leq \nu_1 \leq \dots \leq \nu_n$ the eigenvalues.

Theorem B1. *The following inequalities hold*

$$\frac{\nu_j}{1 + q\nu_j} \leq \lambda_j \leq \nu_j, \tag{B.5}$$

for $j = 1, \dots, n$.

The proof is literally the same as in the Theorem A2 of Appendix of the paper [7].

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