DYNAMIC WEIGHT FUNCTIONS FOR A MOVING CRACK. I. MODE I LOADING

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ABSTRACT

Dynamic weight functions are discussed, for arbitrary time-dependent loading of a plane semi-infinite crack extending at constant speed in an infinite isotropic elastic body. Then, the weight function appropriate to the case of general normal (or Mode I) loading is constructed explicitly, employing Fourier transforms to develop and solve a Wiener-Hopf problem. Transforms are inverted by a variant of Cagniard's technique. The weight function is then employed to develop a relationship, in the framework of first-order perturbation theory, between the Mode I stress intensity factor and a small but otherwise arbitrary time-varying deviation from straightness of the edge of the crack.

1. INTRODUCTION

Weight functions were introduced into elastostatics by Bueckner (1970). They provide weights for the loads applied to the crack surfaces, such that their weighted integrals over the crack surfaces provide the stress intensity factors at a chosen point. Thus, they are related to Green's functions for the crack: they are, in fact, the stress intensity factors associated with concentrated point loading of the crack surfaces but they can also be constructed as solutions of the equations of equilibrium (or of motion) with zero tractions on the crack surfaces but with an unphysical singularity at the crack edge. For plane cracks in two-dimensional elastostatics, weight functions are relatively easy to find: it is necessary only to have available one physical solution to a crack problem, for arbitrary crack length a, and then to differentiate this with respect to a (Bueckner 1970, Rice 1972). A two-dimensional weight function was also constructed by Burridge (1976) for a semi-infinite crack, extending non-uniformly in an infinite isotropic body. Three-dimensional weight functions are scarce, corresponding to the fact that three-dimensional Green's functions are hard to find. Even for statics, one of the few solutions that is known is for a semi-infinite crack in an infinite body (Kassir and Sih, 1973). The main object of this article is to produce a weight function for a corresponding dynamic problem.

The motivation for the work derived from a recent study of Rice, Ben-Zion and Kim (1994), of the perturbation from straightness of the edge of a propagating semi-infinite crack and the associated perturbation of the stress intensity factor. Rice *et al.* solved a model problem, for the scalar wave equation; the present work provides the solution for the full equations of elastodynamics. The method employed here differs greatly from that of Rice *et al.* and, when applied to their scalar problem, reproduces their basic representation very directly.

The plan of the work is as follows. First, since it is convenient to employ coordinates that move with the crack, the equations of motion appear in non-self-adjoint form and a "reciprocal theorem" is derived for such equations. Then, the properties that are desired for three-dimensional dynamic weight functions are introduced. For such weight functions, the integrals that appear in the "reciprocal theorem" can be recognized as convolutions, and this opens the way to their evaluation via Fourier transforms. The Fourier transform of the weight function for Mode I loading is next found, by use of the Wiener-Hopf technique, and the transform is formally inverted. The main application, however, is to relate a small deviation from straightness of the crack front to the associated perturbation of the Mode I stress intensity factor; this is done most efficiently by remaining in the transform domain until the very last stage.

The main results of the paper are the explicit construction of the Mode I weight function and its use in a first-order perturbation analysis of the deviation from straightness of the crack edge. The case of coupled Mode II/Mode III loading will be addressed in a sequel, designated Part II. Detailed implications of the solutions will be explored elsewhere.

2. PROBLEM SPECIFICATION

A crack extends dynamically, so that, at time t, it occupies the region

$$x_1 - Vt < 0, \quad -\infty < x_2 < \infty, \quad x_3 = 0. \tag{2.1}$$

The medium through which it propagates is linearly elastic and isotropic, with Lamé moduli λ and μ, and mass density ρ. By employing superposition as necessary, the medium is taken to be unstressed at infinity, while the crack faces are loaded by equal and opposite surface tractions with components $P_i(t, x_1, x_2)$ on the upper face $(x_3 = +0)$, and $-P_i$ on the lower face $(x_3 = -0)$. Thus, the problem is to solve the equations of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{2.2}$$

subject to "initial conditions" that $u_i(t, \mathbf{x}) \to 0$ as $t \to -\infty$, and boundary conditions

$$\sigma_{i3}(t, x_1, x_2, x_3) \to -P_i(t, x_1, x_2) \quad \text{as} \quad x_3 \to \pm 0; \quad x_1 < Vt. \tag{2.3}$$

Stress σ_{ij} and strain e_{ij} are related by

$$\sigma_{ij} = c_{ijkl} e_{kl}, \tag{2.4}$$

where $c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Strain and displacement components e_{ij} and u_i are related by $e_{ij} = (1/2)(u_{i,j} + u_{j,i})$. The displacement is continuous everywhere except across the surface (2.1) occupied by the crack.

It is known (see, for example, Freund 1990) that the stresses and strains in the vicinity of the edge of the crack are expressible completely in terms of the stress intensity factors K_I, K_{II} and K_{III}; in particular,

$$\begin{aligned} \sigma_{33}(t, x_1, x_2, 0) &\sim K_I(t, x_2)/(\sqrt{2\pi X}), \\ \sigma_{13}(t, x_1, x_2, 0) &\sim K_{II}(t, x_2)/(\sqrt{2\pi X}), \\ \sigma_{23}(t, x_1, x_2, 0) &\sim K_{III}(t, x_2)/(\sqrt{2\pi X}) \\ \text{as} \quad X = x_1 - Vt &\to +0. \end{aligned} \tag{2.5}$$

A major concern, therefore, is to calculate these stress intensity factors.

3. WEIGHT FUNCTIONS

It is convenient to express the problem described above relative to coordinates that move with the crack: (t, X, x_2, x_3), where $X = x_1 - Vt$, so that the crack now occupies the fixed region $X < 0$. When the equations of motion are expressed relative to these coordinates, in terms of displacements they become

$$C_{ijkl} u_{k,lj} + 2\rho V \dot{u}_{i,1} = \rho \ddot{u}_i, \tag{3.1}$$

where now the suffix ,1 represents differentiation with respect to X, a superposed dot represents differentiation with respect to t keeping X fixed, and

$$C_{ijkl} = c_{ijkl} - \rho V^2 \delta_{ik} \delta_{j1} \delta_{l1}. \tag{3.2}$$

In the sequel, equations (3.1) will be Fourier transformed; this motivates discussion of the system

$$(C_{ijkl}\hat{u}_{k,l})_{,j} + (D_{ijk}\hat{u}_k)_{,j} + E_{ik}\hat{u}_k = 0, \tag{3.3}$$

where

$$D_{ijk} = -2\mathrm{i}\rho V\omega\delta_{ik}\delta_{j1} \quad \text{and} \quad E_{ik} = \rho V^2\omega^2\delta_{ik} \tag{3.4}$$

and $\hat{u}_i$ denotes the Fourier transform with respect to time of u_i. The system of equations adjoint to (3.3) is

$$(\hat{u}^*_{i,j}C_{ijkl})_{,l} - \hat{u}^*_{i,j}D_{ijk} + \hat{u}^*_i E_{ik} = 0. \tag{3.5}$$

If S is any surface enclosing a region within which (3.3) and (3.5) are both satisfied, it follows from the divergence theorem that

$$\int_S \left\{\hat{u}^*_i(C_{ijkl}\hat{u}_{k,l} + D_{ijk}\hat{u}_k)n_j - (\hat{u}^*_{i,j}C_{ijkl}\hat{u}_k)n_l\right\} \mathrm{d}S = 0, \tag{3.6}$$

where the outward normal to S has components n_i.

Now apply the result (3.6) to the elastodynamic equations (3.1), Fourier transformed with respect to time, where S encloses a hemispherical domain in $x_3 > 0$, centred just above the origin, whose plane boundary is $x_3 = +0$ and whose radius R will be allowed to tend to infinity. If the fields $\hat{u}_i$ and $\hat{u}^*_i$ decay suitably as $R \to \infty$, equation (3.6) implies that

$$\int_{(x_3=+0)} \{\hat{u}^*_i\hat{\sigma}_{i3} - \hat{\sigma}^*_{k3}\hat{u}_k\} \,\mathrm{d}S = 0, \tag{3.7}$$

where

$$\hat{\sigma}^*_{k3} = \hat{u}^*_{i,j}C_{ijk3} \equiv \hat{u}^*_{i,j}c_{ijk3}, \tag{3.8}$$

in view of the definitions (3.4) of D_{ijk} and (3.2) of C_{ijkl}. Now note that, if $\hat{u}^*_i(\omega, X, x_2, x_3)$ satisfies (3.5), then $\hat{U}_i(\omega, X - X', x_2 - x'_2, x_3) = -\hat{u}^*_i(\omega, X' - X, x'_2 - x_2, -x_3)$ satisfies equations (3.1) (Fourier transformed with respect to time), and

$$\hat{\sigma}^*_{k3} = \hat{\Sigma}_{k3} := \hat{U}_{i,j}c_{ijk3} \tag{3.9}$$

(differentiation being with respect to the unprimed variables). Choosing $\hat{u}^*_i$ in this way, and applying the inverse Fourier transform to replace ω by t, now gives, from (3.7),

$$-\left\{U_i(x_3 = -0) * \sigma_{i3}(x_3 = +0) + \Sigma_{k3}(x_3 = -0) * u_k(x_3 = +0)\right\}(t, X', x'_2) = 0. \tag{3.10}$$

Here, $*$ signifies convolution over t, X and x_2, interpreted in the sense of generalized functions. A similar equation can be developed by integrating over a hemisphere in $x_3 < 0$. For this, the designations $+0$ and -0 are interchanged. If, now, one representation is subtracted from the other, and u_i^* is constructed, like u_i, so that σ_{k3}^* is continuous across $x_3 = 0$, while u_i^* may have a discontinuity when $X < 0$, the corresponding fields U_i and Σ_{k3} provide

$$\{[U_i] * \sigma_{i3} - \Sigma_{k3} * [u_k]\}(t, X', x_2') = 0, \tag{3.11}$$

where [.] denotes the jump across $x_3 = 0$ of the quantity indicated and stresses are evaluated (unambiguously) for $x_3 = 0$.

Equation (3.11) provides the basic identity upon which the remainder of this work is founded. A field with components $U_i(t, X, x_2, x_3)$ will give rise to a *weight function* for the given crack problem if:

- it satisfies the equations of motion (3.1);
- $[U_i] = 0$ when $X < 0$;
- the associated Σ_{k3} is continuous and $\Sigma_{k3} = 0$ when $x_3 = 0$ and $X > 0$;
- U_i is homogeneous of degree $-5/2$, and $[U_i] \to k_i X^{-1/2}\delta(x_2)\delta(t)$ as $X \to +0$, where k_i $(i = 1, 2, 3)$ are constants.

Note that the displacement and traction conditions for U_i are *opposite* to those for u_i; in contrast, the associated vector function with components u_i^* satisfies the *adjoint* equations (3.5), together with traction boundary conditions when $X < 0$, while u_i^* is continuous when $X > 0$. The identity corresponding to (3.11), written in terms of u_i^* instead of U_i, is then not in convolution form. The problem of constructing such weight functions is addressed next.

4. CONSTRUCTION OF WEIGHT FUNCTIONS

The displacement field $U_i(t, \mathbf{x})$ is that of a (dynamic) Somigliana dislocation with displacement jump $[U_i]$ across the plane $x_3 = 0$. The displacement field, and hence also the stresses, can be expressed as a convolution (with respect to t, x_1 and x_2) of a kernel function related to the infinite-body Green's function, with the jump $[U_i]$. Then, it is possible to deduce the relation between the Fourier transforms (with respect to t, x_1 and x_2) of Σ_{k3} on the plane $x_3 = 0$ and $[U_i]$. The calculation is summarized in Appendix A. It turns out that Σ_{33} on the plane depends only on $[U_3]$, while Σ_{13} and

Σ_{23} each depend on both $[U_1]$ and $[U_2]$. The "opening" mode involving $[U_3]$ is termed Mode I, while the other two, Modes II and III, are coupled. For the Mode I case, the result is

$$\tilde{\Sigma}_{33}(\omega', \xi_1, \xi_2) = \frac{\mathrm{i}\mu D(\omega', \xi_1, \xi_2)}{2\omega'^2/b^2 \left(\omega'^2/a^2 - |\xi|^2\right)^{1/2}} [\tilde{U}_3](\omega', \xi_1, \xi_2). \tag{4.1}$$

Here, $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$,

$$D(\omega', \xi_1, \xi_2) = 4|\xi|^2 \left(\omega'^2/a^2 - |\xi|^2\right)^{1/2} \left(\omega'^2/b^2 - |\xi|^2\right)^{1/2} + \left(\omega'^2/b^2 - 2|\xi|^2\right)^2 \tag{4.2}$$

and a, b denote the speeds of dilatational and shear waves:

$$a^2 = (\lambda + 2\mu)/\rho, \quad b^2 = \mu/\rho. \tag{4.3}$$

The equation $D = 0$ is satisfied when $\omega' = \pm c|\xi|$, where c is the speed of Rayleigh waves. There is also a double zero at $\omega' = 0$.

The reason for employing the variable ω' for the reciprocal time is that, next, a relation will be deduced from (4.1) between transforms related to the variables t, X and x_2. It is readily deduced that, if

$$\overline{f}(\omega, \xi_1, \xi_2) = \int \mathrm{d}t \int \mathrm{d}X \int \mathrm{d}x_2 f(t, X + Vt, x_2) \exp\{\mathrm{i}[\omega t + \xi_1 X + \xi_2 x_2]\}, \tag{4.4}$$

then

$$\overline{f}(\omega, \xi_1, \xi_2) = \tilde{f}(\omega - V\xi_1, \xi_1, \xi_2), \tag{4.5}$$

where $\tilde{f}$ denotes the Fourier transform relative to t, x_1 and x_2. Thus, the relation (4.1) yields immediately a relation between $\overline{\Sigma}_{33}$ and $[\overline{U}_3]$ by setting $\omega' = \omega - V\xi_1$.

It is noted now that, since $[U_3] = 0$ when $X < 0$, its transform $[\overline{U}_3]$ is an analytic function of ξ_1 in an upper half-plane, $Im(\xi_1) > k_-$, say, where k_- is real and negative (its magnitude depending on ω and ξ_2). Similarly, since $\Sigma_{33} = 0$ when $X > 0$, its transform $\overline{\Sigma}_{33}$ is an analytic function of ξ_1 in a lower half-plane, $Im(\xi_1) < k_+$, say. It will emerge that $k_+ > k_-$, so that the two half-planes overlap. Within the strip where overlap occurs, the relation (4.1) holds. A Wiener-Hopf problem is thus defined. In preparation for its solution, note that, as $\xi_1 \to \infty$,

$$D(\omega - V\xi_1, \xi_1, \xi_2) \sim -\xi_1^4 R(V), \tag{4.6}$$

where

$$R(V) = 4\alpha\beta - (1 + \beta^2)^2, \tag{4.7}$$

with

$$\alpha = (1 - V^2/a^2)^{1/2}, \quad \beta = (1 - V^2/b^2)^{1/2} \tag{4.8}$$

(so that $R(c) = 0$). This motivates writing

$$D(\omega - V\xi_1, \xi_1, \xi_2) = \left((\omega - V\xi_1)^2/c^2 - |\xi|^2\right)(\xi_1 - \omega/V)^2 R(V) T(\omega, \xi_1, \xi_2)/\gamma^2, \tag{4.9}$$

where

$$\gamma = (1 - V^2/c^2)^{1/2}. \tag{4.10}$$

The function $T(\omega, \xi_1, \xi_2)$ tends to 1 as $\xi_1 \to \infty$ and is analytic and non-zero in a strip containing the real axis in the complex ξ_1-plane, for any fixed ω (with positive imaginary part, for causality) and real ξ_2. It follows that

$$T(\omega, \xi_1, \xi_2) = T_+(\omega, \xi_1, \xi_2) T_-(\omega, \xi_1, \xi_2), \tag{4.11}$$

where

$$\begin{aligned} T_+(\omega, \xi_1, \xi_2) &= \exp\left\{\frac{-1}{2\pi \mathrm{i}} \int_{C_-} \frac{\ln T(\omega, \xi_1', \xi_2)}{\xi_1' - \xi_1} \mathrm{d}\xi_1'\right\}, \\ T_-(\omega, \xi_1, \xi_2) &= \exp\left\{\frac{-1}{2\pi \mathrm{i}} \int_{C+} \frac{\ln T(\omega, \xi_1', \xi_2)}{\xi_1' - \xi_1} \mathrm{d}\xi_1'\right\}. \end{aligned} \tag{4.12}$$

The function $\ln T(\omega, \xi_1', \xi_2)$ is analytic in the complex ξ_1'-plane, with branch cuts B_- joining branch points ξ_a^- and ξ_b^- in the half-plane $Im(\xi_1') < k_-$, and B_+ joining branch points ξ_a^+ and ξ_b^+ in the half-plane $Im(\xi_1') > k_+$. The contours C_- and C_+ enclose these branch cuts and the integrals are taken in the usual (anticlockwise) sense. Further detail, including explicit formulae for $\xi_{a,b}^{\pm}$, is provided in Section 9.

Other terms appearing in (4.1), with $\omega' = \omega - V\xi_1$, can be factorized at sight into "+" and "−" functions. For instance,

$$\left(\omega'^2/a^2 - |\xi|^2\right)^{1/2} = \mathrm{i}\alpha\left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} - \mathrm{i}q_a\right)^{1/2}\left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a\right)^{1/2}, \tag{4.13}$$

where

$$\mathrm{i}q_a = \alpha^{-1}\left(\frac{\omega^2}{a^2 - V^2} - \xi_2^2\right)^{1/2}. \tag{4.14}$$

Equation (4.1) can now be written

$$\frac{\left(\xi_1 + \frac{\omega V}{\gamma^2 c^2} + \mathrm{i}q_c\right) T_+[\overline{U}_3]}{\left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a\right)^{1/2}} = \frac{-2\alpha\,(V^2/b^2)\left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} - \mathrm{i}q_a\right)^{1/2} \overline{\Sigma}_{33}}{\mu R(V)\left(\xi_1 + \frac{\omega V}{\gamma^2 c^2} - \mathrm{i}q_c\right) T_-}, \tag{4.15}$$

the terms q_b, q_c being defined similarly to q_a, with β^{-1}, γ^{-1} respectively replacing α^{-1}. The left side of (4.15) is analytic in a half-plane $Im(\xi_1) > k_-$ (and so is a "+" function), while the right side is analytic in a lower half-plane $Im(\xi_1) < k_+$ (and so is a "$-$" function). Together, therefore, they define an entire function of ξ_1 which, furthermore, is required to be a homogeneous function of degree 0 in the transform variables, consistent with U_i being homogeneous of degree $-5/2$ and, correspondingly, $[\overline{U}_i]$ being homogeneous of degree $-1/2$. It suffices to take the function to be a constant, C. Then, (4.15) gives

$$\begin{aligned} [\overline{U}_3] &= \frac{C\left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a\right)^{1/2}}{\left(\xi_1 + \frac{\omega V}{\gamma^2 c^2} + \mathrm{i}q_c\right) T_+}, \\ \overline{\Sigma}_{33} &= \frac{-\mu R(V) C\left(\xi_1 + \frac{\omega V}{\gamma^2 c^2} - \mathrm{i}q_c\right) T_-}{2\alpha\left(V^2/b^2\right)\left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} - \mathrm{i}q_a\right)^{1/2}}. \end{aligned} \tag{4.16}$$

Champion (1988) constructed a Green's function for normal loading of the faces of an extending crack and deduced from this an expression for the stress intensity factor. His result is closely related to the first equation (4.16). His application, however, was to time–independent loading; here, the development follows an entirely different path.

5. TRANSFORM INVERSIONS

The solution (4.16) is useful directly in the transform domain. It is, nevertheless, of interest to see the corresponding solution in terms of t, X and x_2. The homogeneity of the transforms permits inversion by the technique of Cagniard (1939) or some variant: here, the method introduced by Willis (1973) is employed. First, it may be noted that the function

$$\overline{\Phi}_+ = C\left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a\right)^{-1/2} \tag{5.1}$$

is the transform of

$$\begin{aligned} \Phi_+(t, X, x_2) &= (2\pi)^{-3} C \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}\xi_2 \int_{-\infty}^{\infty} \mathrm{d}\xi_1 \times \\ &\times \left(\xi_1 + \frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a\right)^{-1/2} \exp[-\mathrm{i}(\omega t + \xi_1 X + \xi_2 x_2)]. \end{aligned} \tag{5.2}$$

The integral with respect to ξ_1 can be evaluated by deforming the contour: it gives zero when $X < 0$ whereas, when $X > 0$, it is transformed to an integral around the branch cut and then evaluated by elementary means. The result is

$$\begin{aligned}\Phi_+(t,X,x_2) &= 2(2\pi)^{-3}C\left(\frac{\pi\mathrm{i}}{X}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}\xi_2\int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}}\mathrm{d}\omega\,\times\\ &\quad\times\exp\left\{-\mathrm{i}\left[(\omega(t-VX/\alpha^2a^2)+\xi_2x_2\right]-q_aX\right\}H(X),\end{aligned} \tag{5.3}$$

where $H(\cdot)$ denotes Heaviside's unit step function. The integrals remaining in (5.3) can be evaluated by Cagniard's method, or by the alternative proposed by Willis (1973). In the latter method, the variable ω is replaced by $\Omega = |\xi_2|\omega$ and then the resulting integral with respect to ξ_2 is evaluated. Finally, the integral with respect to Ω follows from a direct application of Cauchy's theorem. The calculation is summarized in Appendix B. The result is

$$\begin{aligned}&\Phi_+(t,X,x_2) = -\mathrm{i}\pi^{-2}C\left(\frac{\pi\mathrm{i}}{X}\right)^{1/2}\left(\frac{X/\alpha}{X^2/\alpha^2+x_2^2}\right)H(X)\,\times\\ &\times\frac{\partial}{\partial t}\left[\left(t-\frac{VX}{\alpha^2a^2}\right)Re\left\{\left[\left(t-\frac{VX}{\alpha^2a^2}\right)^2-\frac{X^2/\alpha^2+x_2^2}{\alpha^2a^2}\right]^{-1/2}\right\}\right].\end{aligned} \tag{5.4}$$

Inversion of the transform for $[U_3]$ is less straightforward but it can be represented, employing the convolution theorem, in the form

$$[U_3] = \Phi_+ * W_+, \tag{5.5}$$

where $W_+(t,X,x_2)$ has transform

$$\overline{W}_+(\omega,\xi_1,\xi_2) = \frac{(\xi_1+\omega V/\alpha^2a^2+\mathrm{i}q_a)}{(\xi_1+\omega V/\gamma^2c^2+\mathrm{i}q_c)T_+}. \tag{5.6}$$

The inversion of this transform (also summarized in Appendix B) gives

$$\begin{aligned}W_+(t,X,x_2) &= -(2\pi)^{-2}\frac{\partial^2}{\partial t^2}\int_{|\eta|=1}\frac{\mathrm{d}s}{(\eta\cdot x/t-0\mathrm{i})^2}\,\times\\ &\quad\times\frac{\left[\eta_1-\frac{\eta\cdot x}{t}\frac{V}{\alpha^2a^2}+\mathrm{i}q_a\right]}{\left[\eta_1+\left(\frac{-\eta\cdot x}{t}+0\mathrm{i}\right)\frac{V}{\gamma^2c^2}+\mathrm{i}q_c\right]T_+(-\eta\cdot x/t+0\mathrm{i},\eta_1,\eta_2)}.\end{aligned} \tag{5.7}$$

Here, $\eta\cdot x = \eta_1X+\eta_2x_2$.

No attempt at further simplification of this expression is made, because it is likely to remain complicated and, furthermore, it is not needed in the sequel. One useful result is, however, noted. First, reverting to the expression (5.4) for Φ_+, as $X \to +0$,

$$\frac{X/\alpha}{X^2/\alpha^2 + x_2^2} \sim \pi\delta(x_2) \tag{5.8}$$

and

$$\left(t - \frac{VX}{\alpha^2 a^2}\right) Re \left\{ \left[\left(t - \frac{VX}{\alpha^2 a^2}\right)^2 - \frac{X^2}{\alpha^4 a^2} \right]^{-1/2} \right\} \sim H(t). \tag{5.9}$$

It follows that, as $X \to 0$,

$$\Phi_+(t, X, x_2) \sim -\mathrm{i}C \left(\frac{\mathrm{i}}{\pi X}\right)^{1/2} H(X)\delta(t)\delta(x_2), \tag{5.10}$$

which is the desired singular behaviour for a weight function. The result (5.10) can also be obtained directly from the asymptotic behaviour of its transform, as $\xi_1 \to \infty$. It follows from (5.1) that, as $\xi_1 \to \infty$,

$$\overline{\Phi}_+ \sim C(\xi_1 + 0\mathrm{i})^{-1/2}, \tag{5.11}$$

and this yields (5.10) immediately, upon inversion. The point of this observation is that, even though the weight function $[U_3]$ is hard to find explicitly, it follows directly from (4.16) that its transform has the *same* asymptotic form as $\overline{\Phi}_+$ as $\xi_1 \to \infty$, and hence that $[U_3]$ has the asymptotic form shown in (5.11) as $X \to 0$. The asymptotic form of Σ_{33} can be derived similarly. This is taken up later.

6. STRESS INTENSITY FACTOR

The result (3.11) is now applied, when the field u_i solves the problem described in Section 2 and $[U_3]$ is the field just derived. For this, the components $[U_1]$, $[U_2]$, Σ_{13} and Σ_{23} are zero. If the convolution is evaluated when $X' > 0$ it follows, since Σ_{33} is a "$-$" function, as is $[u_3]$, that $\Sigma_{33} * [u_3] = 0$ and so (3.11) reduces to

$$\{[U_3] * \sigma_{33}\}(t, X', x_2') = 0. \tag{6.1}$$

But when $X < 0$, $\sigma_{33} = -P_3$ and, when $X \to +0$, $\sigma_{33} \sim K_I/\sqrt{2\pi X}$. Therefore, from (6.1), evaluated as $X' \to +0$,

$$K_I(t, x_2') = \{[U_3] * P_3\}(t, 0, x_2') =: F(t, x_2'), \text{ say}, \tag{6.2}$$

choosing the constant C so that $-\mathrm{i}C(\mathrm{i}/2)^{1/2}=1$. The stress intensity factor is thus expressed as an integral of the applied loads.

In preparation for reasoning to follow, it is helpful to derive the result (6.2) by an alternative route, which allows X' to be positive or negative. In this case, it is necessary also to allow for the other convolution in (3.11), which was zero when the restriction $X'>0$ was made. It is convenient to express σ_{33} in the form

$$\sigma_{33}(t,X,x_2,0)=-P_3(t,X,x_2)H(-X)+\sigma_{33}^{(+)}(t,X,x_2), \tag{6.3}$$

so that $\sigma_{33}^{(+)}=0$ when $X<0$. Its asymptotic form has already been given; including also further terms for future reference,

$$\sigma_{33}^{(+)}\sim\left\{\frac{K_I(t,x_2)}{\sqrt{2\pi X}}-P_3(t,0,x_2)+AX^{1/2}\right\}H(X). \tag{6.4}$$

Also, from Freund (1990),

$$[u_3](t,X,x_2)\sim\left\{\frac{2\alpha(V^2/b^2)}{\mu R(V)}\left(\frac{2}{\pi}\right)^{1/2}K_I(t,x_2)\sqrt{(-X)}-B(-X)^{3/2}\right\}H(-X). \tag{6.5}$$

Convolutions of $[U_3]$ with $\sigma_{33}^{(+)}$, and Σ_{33} with $[u_3]$, are required, when X' is small. Since $[U_3]$ and $\sigma_{33}^{(+)}$ are both "+" functions, their convolution can be evaluated, as above, from the small-X asymptotic forms of each. Also, the small-X' asymptotic form of the convolution can be deduced from the large-ξ_1 asymptotic form of its transform which is, in turn, the product of the corresponding asymptotic forms of the transforms of $[U_3]$ and $\sigma_{33}^{(+)}$. Thus,

$$\overline{[U_3]*\sigma_{33}^{(+)}}\sim\left\{C(\xi_1+0\mathrm{i})^{-1/2}\right\}\left\{(\mathrm{i}/2)^{1/2}\overline{K}_I(\xi_1+0\mathrm{i})^{-1/2}\right\}\equiv\frac{C(\mathrm{i}/2)^{1/2}\overline{K}_I}{(\xi_1+0\mathrm{i})}. \tag{6.6}$$

It may be noted that the inverse transform contains the factor $H(X')$. Notation is slightly distorted here, because K_I depends only on t and x_2 and its transform (still indicated with an overline) is with respect to these two variables only. The convolution of Σ_{33} with $[u_3]$ can be worked out similarly:

$$[\overline{u}_3]\sim\frac{-2\alpha(V^2/b^2)}{\mu R(V)}\left(\frac{\mathrm{i}}{2}\right)^{1/2}\frac{\overline{K}_I}{(\xi_1-0\mathrm{i})^{3/2}} \tag{6.7}$$

and, from (4.16),

$$\overline{\Sigma}_{33} \sim \frac{-\mu R(V) C(\xi_1 - 0\mathrm{i})^{1/2}}{2\alpha(V^2/b^2)}. \tag{6.8}$$

Hence,

$$\overline{\Sigma_{33} * [u_3]} \sim \frac{C(\mathrm{i}/2)^{1/2}\overline{K}_I}{(\xi - 0\mathrm{i})}, \tag{6.9}$$

whose inverse transform contains the factor $H(-X')$. The identity (3.11) thus yields

$$C(\mathrm{i}/2)^{1/2}\overline{K}_I \left\{ \frac{1}{\xi_1 + 0\mathrm{i}} - \frac{1}{\xi_1 - 0\mathrm{i}} \right\} = \mathrm{i}\overline{F} \left\{ \frac{1}{\xi_1 + 0\mathrm{i}} - \frac{1}{\xi_1 - 0\mathrm{i}} \right\}, \tag{6.10}$$

which is exactly consistent with (6.2). The derivation of (6.10) has been laboured because, superficially, it was derived from asymptotic forms when ξ_1 is large, in which case the bracketed term is *zero*. The complete argument underlying the formal calculation shows, however, that the terms involving $(\xi_1 + 0\mathrm{i})^{-1}$, $(\xi_1 - 0\mathrm{i})^{-1}$ do *not* cancel, since they relate, respectively, to $X' > 0$ and $X' < 0$. The study of large-argument asymptotic forms of transforms provides an efficient machine for evaluating convolutions which satisfy present conditions and will be exploited in sections to follow.

7. FURTHER DIRECT IMPLICATIONS

It is evident from the construction in Section 4 that the basic identity (3.11), when Fourier transformed, yields *precisely* the Wiener-Hopf equation satisfied by $[\overline{u}_i]$ and $\overline{\sigma}_{i3}^{(+)}$. Explicitly, appending appropriate superscripts $(+)$ and $(-)$ throughout for emphasis,

$$[\overline{U}_i]^{(+)}\overline{\sigma}_{i3}^{(+)} - \overline{\Sigma}_{k3}^{(-)}[\overline{u}_k]^{(-)} = [\overline{U}_i]^{(+)}\overline{P}_i^{(-)}. \tag{7.1}$$

Implications of (3.11) will be explored, when $X' \sim 0$; as already noted, this corresponds to study of (7.1) when $\xi_1 \to \infty$. The structure is best displayed by introducing abbreviated notation. First write, in correspondence with (6.8),

$$\overline{\Sigma}^{(-)} \sim -C\xi_1^{1/2}[\mathcal{A}(V)]^{-1} \sim -[\overline{U}]^{(+)}[\mathcal{A}(V)]^{-1}\xi_1. \tag{7.2}$$

This notation allows simultaneous treatment of the Mode–I problem and the model problem of Rice *et al.* (1994), with appropriate specialization of $\mathcal{A}(V)$.

It follows, from inspection of (4.1), that

$$\overline{\Sigma}^{(-)} \sim -[\overline{U}]^{(+)}[\mathcal{A}(V - \omega/\xi_1)]^{-1}\xi_1, \tag{7.3}$$

correct to the next order, $O(\xi_1^{-1/2})$. Thus, if

$$[\overline{U}]^{(+)} \sim \frac{C}{(\xi_1 + 0\mathrm{i})^{1/2}} \left\{ 1 + \frac{\mathrm{i}\overline{Q}}{(\xi_1 + 0\mathrm{i})} \right\} \tag{7.4}$$

(the term 0i being added for future convenience), then

$$\overline{\Sigma}^{(-)} \sim -C(\xi_1 - 0\mathrm{i})^{1/2} \left\{ 1 + \frac{1}{\xi_1 - 0\mathrm{i}} \left(\mathrm{i}\overline{Q} + \omega[\mathcal{A}(V)]^{-1}\mathcal{A}'(V) \right) \right\} [\mathcal{A}(V)]^{-1}. \tag{7.5}$$

For the Mode I problem discussed in Sections 4, 5 and 6,

$$\mathrm{i}\overline{Q} = \frac{1}{2}\left(\frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a \right) - \left(\frac{\omega V}{\gamma^2 c^2} + \mathrm{i}q_c \right) - \frac{1}{2\pi\mathrm{i}} \int_{C_-} \ln T(\omega, \xi_1', \xi_2) \mathrm{d}\xi_1' \tag{7.6}$$

and

$$\mathcal{A}(V) = \frac{2\alpha(V)(V^2/b^2)}{\mu R(V)}. \tag{7.7}$$

The relation (7.5) can also be derived by direct expansion of $\overline{\Sigma}_{33}$, as given in (4.16), and then exploiting Cauchy's theorem to replace the integral over C_+ by one over C_-, plus other terms.

Now to exploit relation (7.1), the asymptotic expansion of $[U]^{(+)} * P^{(-)}$ is needed, for small X'. This has the form

$$[U]^{(+)} * P^{(-)} \sim F(t, x_2) - 2(2/\pi)^{1/2} L(t, x_2)(X')^{1/2} H(X') + (\pi/2)^{1/2} M(t, x_2) X', \tag{7.8}$$

where $L(t, x_2) = P(t, 0, x_2)$. The first term, involving F, has been introduced already. That involving L follows from considering the singular term (5.10), which $[U]^{(+)}$ contains, and that involving M is simply the "next term"; in fact,

$$M(t, x_2) = \{[U]^{(+)} * P'^{(-)}\}(t, 0, x_2), \tag{7.9}$$

where $P'^{(-)}(t, X, x_2) = (\partial P/\partial X) H(-X)$. The transform of (7.8) is now

$$\begin{aligned} \mathrm{i}\Bigg\{ &\overline{F} \left[\frac{1}{\xi_1 + 0\mathrm{i}} - \frac{1}{\xi_1 - 0\mathrm{i}} \right] - \frac{(2\mathrm{i})^{1/2} \overline{L}}{(\xi_1 + 0\mathrm{i})^{3/2}} \\ &+ \mathrm{i}(\pi/2)^{1/2} \overline{M} \left[\frac{1}{(\xi_1 + 0\mathrm{i})^2} - \frac{1}{(\xi_1 - 0\mathrm{i})^2} \right] \Bigg\}. \end{aligned} \tag{7.10}$$

Now suppose that, for small X,

$$\sigma^{(+)} \sim \left[K/(2\pi X)^{1/2} - P^* + AX^{1/2}\right] H(X) \tag{7.11}$$

and

$$[u]^{(-)} \sim \left[\Lambda K(-X)^{1/2} - B(-X)^{3/2}\right] H(-X). \tag{7.12}$$

Then,

$$\overline{\sigma}^{(+)} \sim \left(\frac{\mathrm{i}}{2}\right)^{1/2} \frac{\overline{K}}{(\xi_1 + 0\mathrm{i})^{1/2}} - \frac{\mathrm{i}\overline{P^*}}{\xi_1 + 0\mathrm{i}} - \frac{\pi^{1/2}\overline{A}}{2\mathrm{i}^{1/2}(\xi_1 + 0\mathrm{i})^{3/2}} \tag{7.13}$$

and

$$[\overline{u}]^{(-)} \sim \frac{-(\mathrm{i}\pi)^{1/2}\Lambda\overline{K}}{2(\xi_1 - 0\mathrm{i})^{3/2}} - \left(\frac{3\mathrm{i}}{4}\right) \frac{(\mathrm{i}\pi)^{1/2}\overline{B}}{(\xi_1 - 0\mathrm{i})^{5/2}}. \tag{7.14}$$

Fourier transforming the small-X' expansion of (7.1) therefore yields, to order (ξ_1^{-2}),

$$\begin{aligned}
&[\overline{U}]^{(+)}\overline{\sigma}^{(+)} - \overline{\Sigma}^{(-)}[\overline{u}]^{(-)} \sim \\
&C\left(\frac{\mathrm{i}}{2}\right)^{1/2}\left\{\left[\frac{\overline{K}}{\xi_1 + 0\mathrm{i}} - \frac{(2\mathrm{i})^{1/2}\overline{P^*}}{(\xi_1 + 0\mathrm{i})^{3/2}} + \mathrm{i}\frac{\overline{Q}\,\overline{K} + (\pi/2)^{1/2}\overline{A}}{(\xi_1 + 0\mathrm{i})^2}\right]\right. \\
&\left.-\left[\left(\frac{\pi}{2}\right)^{1/2}\frac{\mathcal{A}^{-1}\Lambda\overline{K}}{\xi_1 - 0\mathrm{i}} + \frac{(3\mathrm{i}/4)(2\pi)^{1/2}\mathcal{A}^{-1}\overline{B} + (\pi/2)^{1/2}(\mathrm{i}\overline{Q} + \omega\mathcal{A}^{-1}\mathcal{A}')\mathcal{A}^{-1}\Lambda\overline{K}}{(\xi_1 - 0\mathrm{i})^2}\right]\right\} \\
&= \text{r.h.s.},
\end{aligned} \tag{7.15}$$

where "r.h.s." is given by (7.10).

The terms containing $\xi_1 + 0\mathrm{i}$ contribute when $X' > 0$ while those containing $\xi_1 - 0\mathrm{i}$ contribute when $X' < 0$. Thus, they are independent. Equating coefficients gives

$$\begin{gathered}
\overline{K} = \overline{F},\ \ \overline{P^*} = \overline{L},\ \ \overline{A} = \overline{M} - (2/\pi)^{1/2}\overline{Q}\,\overline{K}, \\
\Lambda = (2/\pi)^{1/2}\mathcal{A},\ \ \overline{B} = (2/3)\mathcal{A}(\overline{A} + (2/\pi)^{1/2}\mathrm{i}\omega\mathcal{A}^{-1}\mathcal{A}'\overline{K}).
\end{gathered} \tag{7.16}$$

The stress intensity factor K is given by F, as it should be, and $P^*(t, x_2) = P(t, 0, x_2)$, as expected. In the case that K is constant, the terms A and B are independent of K because, in physical space, the implied convolutions become ordinary integrals which are, in fact, the Fourier transforms of the kernel functions, evaluated at $\omega = 0$, $\xi_2 = 0$. Since these Fourier transforms are homogeneous functions of degree 1, the value of each is zero at the origin. The deduced value of Λ also confirms the asymptotic form (6.5).

8. APPLICATION TO A WAVY CRACK

The identity (3.11) will now be applied to the problem of relating a small deviation from straightness of the crack to the associated perturbation of the stress intensity factor. The problem is as specified in Section 2, except that the crack now occupies the region

$$x_1 - Vt < \varepsilon\phi(t, x_2), \quad -\infty < x_2 < \infty, \quad x_3 = 0. \tag{8.1}$$

The function $\phi(t, x_2)$ is assumed to be bounded and a solution will be sought, within the framework of first-order perturbation theory, for small ε. The displacement and stress fields associated with the perturbed crack are represented as $u_i + \Delta u_i$, $\sigma_{ij} + \Delta\sigma_{ij}$ respectively, where u_i and σ_{ij} are the fields already discussed, which give the solution when $\varepsilon = 0$. The identity (3.11) applies exactly, both to the original and to the perturbed fields and hence, by subtraction,

$$[U]^{(+)} * \Delta\sigma - \Sigma^{(-)} * [\Delta u] = 0 \tag{8.2}$$

or, in terms of transforms,

$$[\overline{U}]^{(+)}\overline{\Delta\sigma} - \overline{\Sigma}^{(-)}[\overline{\Delta u}] = 0, \tag{8.3}$$

for all values of ω, ξ_1, ξ_2 and ε. The symbolic notation introduced in the preceding section is maintained here. Superscripts $+$ or $-$ are not assigned to the perturbations because their supports are not restricted to $X \geq 0$ or $X \leq 0$.

Now

$$\{\sigma + \Delta\sigma\} \sim \left\{ \frac{K + \Delta K}{\sqrt{2\pi(X - \varepsilon\phi)}} + (A + \Delta A)(X - \varepsilon\phi)^{1/2} \right\} H(X - \varepsilon\phi) - (P^* + \Delta P^*), \tag{8.4}$$

when X is small. It is remarked that $P^* + \Delta P^* = P$ when $X < \varepsilon\phi$. The corresponding perturbed relative displacement has the asymptotic form

$$[u + \Delta u] \sim \left\{ (\Lambda + \Delta\Lambda)(K + \Delta K)(\varepsilon\phi - X)^{1/2} - (B + \Delta B)(\varepsilon\phi - X)^{3/2} \right\} H(\varepsilon\phi - X). \tag{8.5}$$

Since (8.2) is an identity, the strategy now is simply to construct its formal expansion, in ε and $1/\xi_1$, to lowest non-trivial order. First, from (8.4) and (7.11) (to which must be added $P^{(-)}$ to produce σ),

$$\overline{\Delta\sigma} \sim \left(\frac{\mathrm{i}}{2}\right)^{1/2} \frac{\overline{\Delta K} + \mathrm{i}\varepsilon\xi_1\overline{\phi K}}{(\xi_1 + 0\mathrm{i})^{1/2}} - \frac{\mathrm{i}\overline{\Delta P^*}}{\xi_1 + 0\mathrm{i}} - \frac{\pi^{1/2}(\overline{\Delta A} + \mathrm{i}\varepsilon\xi_1\overline{\phi A})}{2\mathrm{i}^{1/2}(\xi_1 + 0\mathrm{i})^{3/2}}. \tag{8.6}$$

Next, from (8.5) and (7.12),

$$[\overline{\Delta u}] \sim \frac{-(\mathrm{i}\pi)^{1/2}(\Lambda\overline{\Delta K} + \overline{\Delta\Lambda K} + \mathrm{i}\varepsilon\xi_1\Lambda\overline{\phi K})}{2(\xi_1 - 0\mathrm{i})^{3/2}} - \left(\frac{3\mathrm{i}}{4}\right)\frac{(\mathrm{i}\pi)^{1/2}(\overline{\Delta B} + \mathrm{i}\varepsilon\xi_1\overline{\phi B})}{(\xi_1 - 0\mathrm{i})^{5/2}}. \qquad (8.7)$$

From (7.4) and (8.6),

$$[\overline{U}]^{(+)}\overline{\Delta\sigma} \sim C\left(\frac{\mathrm{i}}{2}\right)^{1/2}\left\{\mathrm{i}\varepsilon\overline{\phi K} + \frac{1}{\xi_1 + 0\mathrm{i}}\left[\overline{\Delta K} - \varepsilon\overline{Q}\;\overline{\phi K} - \left(\frac{\pi}{2}\right)^{1/2}\varepsilon\overline{\phi A}\right]\right\} \qquad (8.8)$$

and from (7.5) and (8.7),

$$\begin{aligned}\overline{\Sigma}^{(-)}[\overline{\Delta u}] \quad \sim \quad & \frac{C(\mathrm{i}\pi)^{1/2}}{2}\Big\{\mathrm{i}\varepsilon\mathcal{A}^{-1}\Lambda\overline{\phi K} \\ & + \frac{1}{\xi_1 - 0\mathrm{i}}\Big[\mathcal{A}^{-1}(\Lambda\overline{\Delta K} + \overline{\Delta\Lambda K}) - (\overline{Q} - \mathrm{i}\omega\mathcal{A}^{-1}\mathcal{A}')\mathcal{A}^{-1}\varepsilon\Lambda\overline{\phi K} \\ & - \frac{3}{2}\varepsilon\mathcal{A}^{-1}\overline{\phi B}\Big]\Big\}. \end{aligned} \qquad (8.9)$$

The identity (8.3) requires that the expressions (8.8) and (8.9) should be equal. The terms that are independent of ξ_1 are the same, since $\Lambda = (2/\pi)^{1/2}\mathcal{A}$, from (7.16). There remains the requirement that the coefficients of $1/(\xi_1 \pm 0\mathrm{i})$ should separately be zero. Considering the '+' term is equivalent to evaluating the convolution (8.2) for small positive X', expanding the result to first order in ε with X' fixed, and then considering the limit as $X' \to 0$. There is a corresponding interpretation for the '−' term. These are *mathematical* operations, which lead to an answer because (8.2) is an *identity*: the result does not depend upon any physical interpretation of these operations. Thus, considering $1/(\xi_1 + 0\mathrm{i})$, (8.8) yields

$$\overline{\Delta K} = \varepsilon\left[\overline{Q}\;\overline{\phi K} + \left(\frac{\pi}{2}\right)^{1/2}\overline{\phi A}\right]. \qquad (8.10)$$

The coefficient of $1/(\xi_1 - 0\mathrm{i})$ in (8.9) in fact is *exactly the same* as that of $1/(\xi_1 + 0\mathrm{i})$ in (8.8), in view of the relations (7.16). Thus, the result (8.10) follows unambiguously, whether the basic convolution (8.2) is considered when X' is positive or negative.

Taking the value for A from (7.16), the result (8.10) gives

$$\Delta K = \varepsilon\left[Q * (\phi K) - \phi(Q * K) + \left(\frac{\pi}{2}\right)^{1/2}\phi M\right]. \qquad (8.11)$$

The function $Q(t,x_2)$ is easily deduced from its transform $\overline{Q}(\omega,\xi_2)$ (which is a homogeneous function of degree 1), by Cagniard's method or by the equivalent method proposed by Willis (1973):

$$Q(t,x_2) = -\frac{1}{2\pi}\frac{\partial^2}{\partial t^2}\left\{\left(\frac{\overline{Q}(-x_2+0\mathrm{i},t)}{(-x_2+0\mathrm{i})^2} + \frac{\overline{Q}(x_2+0\mathrm{i},-t)}{(x_2+0\mathrm{i})^2}\right)H(t)\right\}. \tag{8.12}$$

9. SOME EXPLICIT FORMULAE

Consider first the problem for the scalar wave equation considered by Rice *et al.* (1994). The analogue of the displacement is the field u, which satisfies a scalar wave equation. Calling the wave speed b and the associated modulus μ, the boundary condition corresponding to (2.3) is

$$\sigma = \mu u_{,3} \to -P \quad \text{as} \quad x_3 \to \pm 0; \quad x_1 < Vt. \tag{9.1}$$

It is easily shown that

$$[\overline{U}] = C\left(\xi_1 + \frac{\omega V}{\beta^2 b^2} + \mathrm{i}q_b\right)^{1/2}, \tag{9.2}$$

which is $\overline{\Phi}_+$, with the replacement $a \to b$. Also,

$$\mathcal{A}(V) = \beta^{-1}. \tag{9.3}$$

It follows from (9.2) that

$$\mathrm{i}\overline{Q} = \frac{\omega V}{\beta^2 b^2} + \mathrm{i}q_b. \tag{9.4}$$

Therefore, from (8.12),

$$Q(t,x_2) = Q_b(t,x_2) := \frac{V}{\beta^2 b^2}\delta'(t)\delta(x_2)$$
$$-\frac{1}{2\pi\beta}\frac{\partial^2}{\partial t^2}\left\{\left[\frac{1}{(x_2-0\mathrm{i})^2} + \frac{1}{(x_2+0\mathrm{i})^2}\right]\left(t^2 - x_2^2/\beta^2 b^2\right)^{1/2} H(t - |x_2|/\beta^2 b^2)\right\} \tag{9.5}$$

or, equivalently,

$$Q(t,x_2) = Q_b(t,x_2) = \frac{V}{\beta^2 b^2}\delta'(t)\delta(x_2) - \frac{1}{\pi\beta}\frac{\partial}{\partial t}\left\{\frac{tH(t-|x_2|/\beta^2 b^2)}{x_2^2(t^2 - x_2^2/\beta^2 b^2)^{1/2}}\right\}, \tag{9.6}$$

the generalized function $1/x_2^2$ having the meaning shown in (9.5) as a Hadamard finite part.

It is easily verified that the relation (8.11), with (9.5), is consistent with the formula of Rice *et al.*, in the case of constant K and $M = 0$ that they considered. It should, perhaps, be remarked that, even for loading that follows the crack, so that K and M are constant, the case $M = 0$ is rather special. The significance in practice of M would depend upon the particular problem under discussion.

Now consider the case of Mode I loading. It is convenient to express $\overline{Q}$, as given by (7.6), in the form

$$\mathrm{i}\overline{Q} = \frac{1}{2}\left(\frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a\right) - \left(\frac{\omega V}{\gamma^2 c^2} + \mathrm{i}q_c\right) + \mathrm{i}|\xi_2|\mathcal{T}(\omega/|\xi_2|), \tag{9.7}$$

where

$$\mathcal{T}(\omega/|\xi_2|) = \frac{1}{2\pi|\xi_2|}\int_{C_-} \ln T(\omega, \xi_1, \xi_2)\mathrm{d}\xi_1. \tag{9.8}$$

The function $\ln T$ depends on ξ_2 only through its magnitude. Considered as a function of the complex variable ξ_1, it is analytic in the complex plane with cuts

$$B_- = \{\xi_a^- + s(\xi_b^- - \xi_a^-);\ 0 \le s \le 1\} \text{ and } B_+ = \{\xi_a^+ + s(\xi_b^+ - \xi_a^+);\ 0 \le s \le 1\}, \tag{9.9}$$

where

$$\begin{aligned}
\xi_a^- &= -\frac{\omega V}{\alpha^2 a^2} - \mathrm{i}q_a, \qquad & \xi_b^- &= -\frac{\omega V}{\beta^2 b^2} - \mathrm{i}q_b,\\
\xi_a^+ &= -\frac{\omega V}{\alpha^2 a^2} + \mathrm{i}q_a, \qquad & \xi_b^+ &= -\frac{\omega V}{\beta^2 b^2} + \mathrm{i}q_b.
\end{aligned} \tag{9.10}$$

Treating the contour integral more explicitly,

$$\mathcal{T}(\omega/|\xi_2|) = \frac{(\xi_b^- - \xi_a^-)}{2\pi|\xi_2|}\int_0^1 \ln\left(\frac{Y+Z}{Y-Z}\right)\mathrm{d}s, \tag{9.11}$$

where

$$\begin{aligned}
Y &= [\xi_1^2 + \xi_2^2 + \beta^2(\xi_1 - \xi_b^-)(\xi_1 - \xi_b^+)]^2,\\
Z &= 4(\xi_1^2 + \xi_2^2)\alpha\beta[(\xi_1 - \xi_a^-)(\xi_1 - \xi_b^-)(\xi_1 - \xi_a^+)(\xi_1 - \xi_b^+)]^{1/2},\\
\xi_1 &= \xi_a^- + (s - 0\mathrm{i})(\xi_b^- - \xi_a^-).
\end{aligned} \tag{9.12}$$

The formal inversion of $\overline{Q}$ is given by (8.12):

$$Q(t,x_2) = \tfrac{1}{2}Q_a(t,x_2) - Q_c(t,x_2) - \frac{1}{2\pi}\frac{\partial^2}{\partial t^2}\left\{\left(\frac{\mathcal{T}(-x_2/t+0\mathrm{i})}{(-x_2+0\mathrm{i})^2} + \frac{\mathcal{T}(x_2/t+0\mathrm{i})}{(x_2+0\mathrm{i})^2}\right) tH(t)\right\}, \tag{9.13}$$

where Q_a and Q_c are defined like Q_b but with the replacements $b \to a$ or $b \to c$. It can be shown that the two bracketed terms are complex conjugates of one another and, furthermore, that $\mathcal{T}$ is imaginary when $t < |x_2|/\alpha a$. With the notation $\mathcal{T}(-x_2/t+0\mathrm{i}) = \mathcal{T}_r(x_2/t) + \mathrm{i}\mathcal{T}_i(x_2/t)$, (9.13) can be written

$$Q(t,x_2) = \tfrac{1}{2}Q_a(t,x_2) - Q_c(t,x_2) - \frac{\partial^2}{\partial t^2}\left\{\frac{\mathcal{T}_r(x_2/t)tH(t-|x_2|/\alpha a)}{\pi x_2^2}\right\} - \mathcal{T}_i'(0)\delta'(t)\delta(x_2). \tag{9.14}$$

Figure 1 shows a two-dimensional plot of $\mathcal{T}_r(x_2/t)H(t-|x_2|/\alpha a)$ against x_2/at and V/a, for the case $a^2 = 3b^2$. Figure 2 shows three cross-sections at fixed values of V/a, with x_2/at varying. It is assumed that $V < c$, the speed of Rayleigh waves ($c = 0.53851a = 0.93273b$ in the case considered), and values $V/c = 0.1,\ 0.5$ and 0.9 were chosen.

10. THE INVERSE RELATION

For the scalar wave equation, in the case of constant K and zero M, Rice *et al.* also addressed the problem of identifying the crack perturbation ϕ that would generate a prescribed ΔK. The same problem is now discussed, in the case of Mode I loading. Consideration of mixed-mode loading is deferred to Part II. The term $Q * K$ vanishes and so $\overline{\Delta K}$ and $\overline{\phi}$ satisfy

$$\varepsilon\overline{\phi} = \overline{\Delta K}/K\overline{Q}. \tag{10.1}$$

The transform inverse to $1/\overline{Q}$ is easily found by the method of Willis (1973):

$$Q^{-1}(t,x_2) = \frac{1}{2\pi}\left\{\frac{1}{\overline{Q}(-x_2+0\mathrm{i},t)} + \frac{1}{\overline{Q}(x_2+0\mathrm{i},-t)}\right\}. \tag{10.2}$$

For the case of the scalar wave equation considered by Rice *et al.* (1994), equation (10.2) yields the simple result

$$Q^{-1}(t,x_2) = \frac{\beta(t^2 - x_2^2/\beta^2 b^2)^{1/2}H(t-|x_2|/\beta b)}{\pi(t^2 - x_2^2/b^2)}. \tag{10.3}$$

For the Mode I problem, (10.2) gives

$$Q^{-1}(t,x_2) \;=\; \frac{1}{\pi}Re\left\{\left[\frac{\mathrm{i}x_2V}{2\alpha^2a^2} - \frac{\mathrm{i}}{2\alpha}\left(\frac{(-x_2+0\mathrm{i})^2}{\alpha^2a^2} - t^2\right)^{1/2} - \frac{\mathrm{i}x_2V}{\gamma^2c^2}\right.\right.$$

$$+\frac{\mathrm{i}}{\gamma}\left(\frac{(-x_2+0\mathrm{i})^2}{\gamma^2c^2}-t^2\right)^{1/2}+t[\mathcal{T}_r(x_2/t)+\mathrm{i}\mathcal{T}_i(x_2/t)]\Bigg]^{-1}\Bigg\}. \tag{10.4}$$

Admission of a non-zero M would significantly complicate the result, because the transform would no longer be a homogeneous function. This will be addressed elsewhere.

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REFERENCES

Bueckner, H.F., 1970, A Novel Principle for the Computation of Stress Intensity Factors, *Zeit. Angew. Math. Mech.* **50**, 529–546.

Burridge, R., 1976, An Influence function for the Intensity factor in Tensile Fracture, *Int. J. Eng. Sci.* **14**, 725–734.

Cagniard, L., 1939, *Reflexion et Refraction des Ondes Seismiques Progressives*, Gauthier–Villars, Paris.

Champion, C., 1988, The Stress Intensity Factor History for an Advancing Crack under Three–Dimensional Loading, *Int. J. Solids Structures* **24**, 285–300.

Freund, L.B., 1990, *Dynamic Fracture Mechanics*, Cambridge Univ. Press.

Kassir, M.K. and Sih, G.C., 1973, Application of Papkovich–Neuber Potentials to a Crack Problem, *Int. J. Solids Struct.* **9**, 643–654.

Rice, J.R., 1972, Some Remarks on Elastic Crack Tip Stress Fields, *Int. J. Solids Struct.* **8**, 751–758.

Rice, J.R., Ben-Zion, Y. and Kim, K., 1994, Three–Dimensional Perturbation Solution for a Dynamic Planar Crack Moving Unsteadily in a Model Elastic Solid, *J. Mech. Phys. Solids* **42**, 813–843.

Willis, J.R., 1973, Self–Similar Problems in Elastodynamics, *Phil. Trans. R. Soc. Lond.* **A274**, 435–491.

Appendix A

This Appendix develops the relation between the Fourier transforms, taken with respect to t, x_1 and x_2, for the components Σ_{3k} on the plane $x_3 = 0$ and components $[U_i]$ of the displacement jump.

First, consider the equations of motion and take the Fourier transform with respect to time:

$$c_{ijkl}\hat{U}_{k,lj} + \rho\omega^2\hat{U}_i = 0. \tag{A.1}$$

Similarly, for components of Green's tensor,

$$c_{ijkl}\hat{\mathcal{G}}_{kp,lj}(\mathbf{x},\mathbf{x}') + \delta_{ip}\delta(\mathbf{x}-\mathbf{x}') + \rho\omega^2\hat{\mathcal{G}}_{ip}(\mathbf{x},\mathbf{x}') = 0. \tag{A.2}$$

Relations (A.1), (A.2) yield

$$\hat{\mathcal{G}}_{ip}(\mathbf{x},\mathbf{x}')c_{ijkl}\hat{U}_{k,lj} - \hat{U}_i c_{ijkl}\hat{\mathcal{G}}_{kp,lj} - \hat{U}_p\delta(\mathbf{x}-\mathbf{x}') = 0,$$

and therefore, for any region Ω, where equations (A.1), (A.2) are satisfied,

$$\int_\Omega \left\{\hat{\mathcal{G}}_{ip}(\mathbf{x},\mathbf{x}')c_{ijkl}\hat{U}_{k,lj}(\mathbf{x}) - \hat{U}_i(\mathbf{x})c_{ijkl}\hat{\mathcal{G}}_{kp,lj}(\mathbf{x},\mathbf{x}')\right\}d\mathbf{x} = \hat{U}_p(\mathbf{x}').$$

Integrating by parts one can derive

$$\int_{\partial\Omega} \left\{\hat{\mathcal{G}}_{ip}(\mathbf{x},\mathbf{x}')c_{ijkl}\hat{U}_{k,l}(\mathbf{x}) - \hat{U}_i(\mathbf{x})c_{ijkl}\hat{\mathcal{G}}_{kp,l}(\mathbf{x},\mathbf{x}')\right\}n_j(\mathbf{x})d\mathbf{x} = \hat{U}_p(\mathbf{x}'),$$

where n_j, $j = 1,2,3$, are components of the unit outward normal to $\partial\Omega$.

In particular for an elastic space with a crack S the region Ω can be chosen to be $\{\mathbf{x} : \|\mathbf{x}\| < R\} \setminus \overline{S}$ for $R \gg 1$. Assuming that components $\hat{U}_k$ decay at infinity, we obtain, as $R \to \infty$,

$$\hat{U}_p(\mathbf{x}') = -\int_S [\hat{U}_i](\mathbf{x})c_{i3kl}\hat{\mathcal{G}}_{kp,l}(\mathbf{x},\mathbf{x}')ds_{\mathbf{x}}. \tag{A.3}$$

For the case of an elastic space

$$\mathcal{G}_{ij}(\mathbf{x},\mathbf{x}') := G_{ij}(\mathbf{x}-\mathbf{x}') = G_{ij}(\mathbf{x}'-\mathbf{x}).$$

Components of the traction vector on the crack surface can be represented by

$$\hat{T}_r(\mathbf{x}') = \int_S c_{r3pq}[\hat{U}_i](\mathbf{x}) c_{i3kl} \hat{G}_{kp,lq}(\mathbf{x}' - \mathbf{x}) ds,$$

and the Fourier transform with respect to space variables yields

$$\overline{T}_r = -c_{r3pq} c_{i3kl} \xi_l \xi_q \overline{G}_{kp} [\overline{U}_i]. \tag{A.4}$$

This can be written in the matrix form

$$\overline{\mathbf{T}} = -\mathcal{C}\overline{\mathbf{G}}\mathcal{C}^t[\overline{\mathbf{U}}], \tag{A.5}$$

where

$$\mathcal{C} = \begin{pmatrix} \mu\xi_3 & 0 & \mu\xi_1 \\ 0 & \mu\xi_3 & \mu\xi_2 \\ \lambda\xi_1 & \lambda\xi_2 & (\lambda + 2\mu)\xi_3 \end{pmatrix},$$

and

$$\overline{\mathbf{G}} = \frac{1}{\mu\|\xi\|^2 - \rho\omega^2}\mathbf{I} - \frac{\lambda + \mu}{(\mu\|\xi\|^2 - \rho\omega^2)((\lambda + 2\mu)\|\xi\|^2 - \rho\omega^2)} \xi \otimes \xi.$$

Here λ and μ are the Lamé constants and $\|\xi\|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$.

The next step is to consider the inverse Fourier transform of components of the symmetric matrix in the right–hand side of (A.5) with respect to ξ_3 and to take the limit as $x_3 \to +0$. The columns of the matrix $\mathcal{C}\overline{\mathbf{G}}\mathcal{C}^t$ admit the representation

$$\left(\mathcal{C}\overline{\mathbf{G}}\mathcal{C}^t\right)^{(1)} = \frac{1}{\mu\|\xi\|^2 - \rho\omega^2} \begin{pmatrix} \mu^2(\xi_3^2 + \xi_1^2) \\ \mu^2\xi_1\xi_2 \\ 2\mu(\lambda + \mu)\xi_1\xi_3 \end{pmatrix}$$

$$- \frac{\lambda + \mu}{(\mu\|\xi\|^2 - \rho\omega^2)((\lambda + 2\mu)\|\xi\|^2 - \rho\omega^2)} \begin{pmatrix} 4\mu^2\xi_3^2\xi_1^2 \\ 4\mu^2\xi_3^2\xi_1\xi_2 \\ 2\mu\xi_3\xi_1(\lambda(\xi_1^2 + \xi_2^2) + (\lambda + 2\mu)\xi_3^2) \end{pmatrix}, \tag{A.6}$$

$$\left(\mathcal{C}\overline{\mathbf{G}}\mathcal{C}^t\right)^{(2)} = \frac{1}{\mu\|\xi\|^2 - \rho\omega^2} \begin{pmatrix} \mu^2\xi_1\xi_2 \\ \mu^2(\xi_3^2 + \xi_2^2) \\ 2\mu(\lambda + \mu)\xi_2\xi_3 \end{pmatrix}$$

$$- \frac{\lambda + \mu}{(\mu\|\xi\|^2 - \rho\omega^2)((\lambda + 2\mu)\|\xi\|^2 - \rho\omega^2)} \begin{pmatrix} 4\mu^2\xi_3^2\xi_1\xi_2 \\ 4\mu^2\xi_2^2\xi_3^2 \\ 2\mu\xi_2\xi_3(\lambda(\xi_1^2 + \xi_2^2) + (\lambda + 2\mu)\xi_3^2) \end{pmatrix}, \tag{A.7}$$

$$
\left(\mathcal{C}\overline{\mathbf{G}}\mathcal{C}^t\right)^{(3)} = \frac{1}{\mu\|\xi\|^2-\rho\omega^2}\begin{pmatrix} 2\mu(\lambda+\mu)\xi_1\xi_3 \\ 2\mu(\lambda+\mu)\xi_2\xi_3 \\ \lambda^2(\xi_1^2+\xi_2^2)+(\lambda+2\mu)^2\xi_3^2 \end{pmatrix}
$$

$$
-\frac{\lambda+\mu}{(\mu\|\xi\|^2-\rho\omega^2)((\lambda+2\mu)\|\xi\|^2-\rho\omega^2)}\begin{pmatrix} 2\mu\xi_3\xi_1(\lambda(\xi_1^2+\xi_2^2)+(\lambda+2\mu)\xi_3^2) \\ 2\mu\xi_2\xi_3(\lambda(\xi_1^2+\xi_2^2)+(\lambda+2\mu)\xi_3^2) \\ (\lambda(\xi_1^2+\xi_2^2)+(\lambda+2\mu)\xi_3^2)^2 \end{pmatrix}. \quad (A.8)
$$

Consider the third column (A.8) and rewrite its first component in the form

$$
\frac{2\mu(\lambda+\mu)\xi_1\xi_3}{(\mu\|\xi\|^2-\rho\omega^2)((\lambda+2\mu)\|\xi\|^2-\rho\omega^2)}\left\{2\mu(\xi_1^2+\xi_2^2)-\rho\omega^2\right\}. \quad (A.9)
$$

Clearly, the quantity (A.9) decays at infinity like $O(|\xi_3|^{-3})$. Apply the inverse Fourier transform with respect to ξ_3. For positive values of x_3 the contour of integration can be closed in the lower half–plane; due to the fast decay of (A.9) the integral along the big semi–circle vanishes as the radius tends to infinity. The contour encloses the two simple poles in the lower half–plane

$$
\xi_3^{(b)} = -\mathrm{i}(\xi_1^2+\xi_2^2-\omega^2/b^2)^{1/2},
$$

and

$$
\xi_3^{(a)} = -\mathrm{i}(\xi_1^2+\xi_2^2-\omega^2/a^2)^{1/2}.
$$

Using Cauchy's theorem one can obtain that, as $x_3 \to +0$, the inverse Fourier transform for the first element of the column (A.8) is

$$
-\mathrm{i}\frac{2(\lambda+\mu)}{\lambda+2\mu}\xi_1\left\{2\mu(\xi_1^2+\xi_2^2)-\rho\omega^2\right\}\left\{\frac{\xi_3^{(a)}}{2\xi_3^{(a)}((\xi_3^{(a)})^2-(\xi_3^{(b)})^2)}\right.
$$

$$
\left.+\frac{\xi_3^{(b)}}{2\xi_3^{(b)}((\xi_3^{(b)})^2-(\xi_3^{(a)})^2)}\right\} = 0.
$$

Similarly, the inverse Fourier transform of the second element of the column (A.8) has zero limit as $x_3 \to +0$, and, therefore, the Mode–I problem is uncoupled with respect to the shear Modes II and III.

Now consider the (33) element of the matrix in the right–hand side of (A.5); it equals, asymptotically,

$$
\lambda+2\mu+O(|\xi_3|^{-2}) \ \text{ as } \ \xi_3\to\infty. \quad (A.10)
$$

Inversion of the first term produces $\delta(x_3)$ multiplied by a scalar factor. The variable x_3 is assumed to be positive, and, therefore, the first term in (A.10) does not contribute to the final answer. The remaining part of (A.10) has order $|\xi_3|^{-2}$ and one can close the contour of integration in the lower half plane; the corresponding integral along the large semi–circle vanishes as its radius tends to infinity. Thus, the inverse Fourier transform of the (33) element with respect to ξ_3 has the following limit as $x_3 \to +0$

$$\frac{\mathrm{i}\mu}{2(\omega^2/b^2)(\omega^2/a^2 - \xi_1^2 - \xi_2^2)^{1/2}} D(\omega, \xi_1, \xi_2), \tag{A.11}$$

where

$$D(\omega, \xi_1, \xi_2) = 4(\xi_1^2 + \xi_2^2)(\omega^2/a^2 - \xi_1^2 - \xi_2^2)^{1/2}(\omega^2/b^2 - \xi_1^2 - \xi_2^2)^{1/2} \\ + (\omega^2/b^2 - 2(\xi_1^2 + \xi_2^2)^2)^2.$$

Therefore, for the Fourier transforms (with respect to x_1, x_2, t) of Σ_{33} and of the displacement jump $[U_3]$ across the plane $x_3 = 0$ we have

$$\tilde{\Sigma}_{33} = \frac{\mathrm{i}\mu}{2(\omega^2/b^2)(\omega^2/a^2 - \xi_1^2 - \xi_2^2)^{1/2}} D(\omega, \xi_1, \xi_2)[\tilde{U}_3], \tag{A.12}$$

which is the same as (4.1), except that $\omega \to \omega'$.

To obtain the relation between the Fourier transforms of the shear stress components and the jump of the longitudinal displacement across the plane $x_3 = 0$, one has to consider the first 2×2 block of the matrix in the right–hand side of (A.5).

Consider the the first component of the column (A.7) (the (12) element of the matrix). As $\xi_3 \to \infty$, it can be specified as

$$-\frac{(3\lambda + 2\mu)\mu}{(\lambda + 2\mu)} \frac{1}{\xi_3^2} + O(|\xi_3|^{-3}).$$

Application of Cauchy's theorem yields, as before, the representation of the inverse Fourier transform with respect to ξ_3 in the limit $x_3 \to +0$

$$\frac{\mathrm{i}\mu\xi_1\xi_2}{2} \Bigg\{ (\frac{\omega^2}{b^2} - \xi_1^2 - \xi_2^2)^{-1/2} \\ - \frac{4b^2}{\omega^2} \Bigg((\frac{\omega^2}{b^2} - \xi_1^2 - \xi_2^2)^{1/2} - (\frac{\omega^2}{a^2} - \xi_1^2 - \xi_2^2)^{1/2} \Bigg) \Bigg\}. \tag{A.13}$$

The diagonal elements (11) and (22) have the asymptotic form

$$\mu + O(|\xi_3|^{-2}), \quad \text{as } \xi_3 \to \infty.$$

Following the same procedure that was used for the calculation of the inverse Fourier transform of the third diagonal element (33), one can obtain that in the limit $x_3 \to +0$ the inverse Fourier transforms of the elements (11) and (22) of the matrix with columns (A.6)–(A.8) are

$$\frac{\mathrm{i}\mu}{2}\Bigg\{(\frac{\omega^2}{b^2} - \xi_2^2)(\frac{\omega^2}{b^2} - \xi_1^2 - \xi_2^2)^{1/2}$$

$$- \frac{4\xi_1^2 b^2}{\omega^2}\left((\frac{\omega^2}{b^2} - \xi_1^2 - \xi_2^2)^{1/2} - (\frac{\omega^2}{a^2} - \xi_1^2 - \xi_2^2)^{1/2}\right)\Bigg\}, \tag{A.14}$$

and

$$\frac{\mathrm{i}\mu}{2}\Bigg\{(\frac{\omega^2}{b^2} - \xi_1^2)(\frac{\omega^2}{b^2} - \xi_1^2 - \xi_2^2)^{1/2}$$

$$- \frac{4\xi_2^2 b^2}{\omega^2}\left((\frac{\omega^2}{b^2} - \xi_1^2 - \xi_2^2)^{1/2} - (\frac{\omega^2}{a^2} - \xi_1^2 - \xi_2^2)^{1/2}\right)\Bigg\}. \tag{A.15}$$

Consequently, using (A.13)–(A.15) for the shear Modes II and III we derive

$$\begin{pmatrix} \tilde{\Sigma}_{13} \\ \tilde{\Sigma}_{23} \end{pmatrix} = \frac{\mathrm{i}\mu}{2(\omega/b)^2((\omega/b)^2 - \xi_1^2 - \xi_2^2)^{1/2}}\mathbf{D}\begin{pmatrix} [\tilde{U}_1] \\ [\tilde{U}_2] \end{pmatrix}, \tag{A.16}$$

where the symmetric matrix $\mathbf{D} = \|D_{ij}\|_{i,j=1}^2$ is given by

$$\mathbf{D} = (\xi_1^2 + \xi_2^2)^{-1}\mathbf{R}^t \operatorname{diag}\left\{D(\omega, \xi_1, \xi_2), \beta\frac{\omega^2}{b^2}\left(\frac{\omega^2}{b^2} - \xi_1^2 - \xi_2^2\right)\right\}\mathbf{R}, \tag{A.17}$$

$$\mathbf{R} = \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}.$$

Thus, equalities (A.12) and (A.16), (A.17) provide the relations between the Fourier transforms of displacements and stresses for all three Modes of the loading.

Appendix B

This Appendix summarises the method that was employed for inverting Fourier transforms of homogeneous functions, following Willis (1973).

Consider first a function $F(t,x)$, whose Fourier transform $\overline{F}(\omega,\xi)$ is a homogeneous function of degree -1. Then

$$F(t,x)=(2\pi)^{-2}\int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}}\mathrm{d}\omega\int_{-\infty}^{\infty}\mathrm{d}\xi\ \overline{F}(\omega,\xi)\exp[-\mathrm{i}(\omega t+\xi x)]. \tag{B.1}$$

Introduce the new variable $\Omega=\omega/|\xi|$. Then,

$$\begin{aligned} F(t,x) \quad &= \quad (2\pi)^{-2}\lim_{\varepsilon\to 0}\int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}}\mathrm{d}\Omega\ \left\{\int_0^{\infty}\ \mathrm{d}\xi\ \overline{F}(\Omega,1)\exp[-\mathrm{i}\xi(\Omega t+x-\mathrm{i}\varepsilon)]\right. \\ &\qquad \left. +\int_{-\infty}^{0}\mathrm{d}\xi\overline{F}(\Omega,-1)\exp[\mathrm{i}\xi(\Omega t-x-\mathrm{i}\varepsilon)]\right\} \end{aligned} \tag{B.2}$$

and so, evaluating the integrals with respect to ξ,

$$\begin{aligned} F(t,x) \quad &= \quad (2\pi\mathrm{i})^{-2}\lim_{\varepsilon\to 0}\int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}}\mathrm{d}\Omega\ \left\{\frac{\overline{F}(\Omega,1)}{\Omega t+x-\mathrm{i}\varepsilon}\right. \\ &\qquad \left. +\frac{\overline{F}(\Omega,-1)}{\Omega t+x-\mathrm{i}\varepsilon}\right\}. \end{aligned} \tag{B.3}$$

The integral with respect to Ω is evaluated by closing the contour in the upper half-plane and employing Cauchy's theorem. It is assumed, for causality, that $\overline{F}(\omega,\xi)$ is analytic for $Im(\omega)>0$. The integral therefore gives zero when $t<0$ while, for $t>0$, the two poles in the upper half-plane give

$$F(t,x)=(2\pi t)^{-1}\left\{\overline{F}(-x/t+0\mathrm{i},1)+\overline{F}(x/t+0\mathrm{i},-1)\right\}. \tag{B.4}$$

This provides, directly, (10.2) and its detailed consequent results (10.3) and (10.4).

For a function $\overline{F}(\omega,\xi)$ that is homogeneous of degree $n-1$,

$$F(t,x)=(2\pi)^{-2}(\mathrm{i})^n\frac{\partial^n}{\partial t^n}\int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}}\mathrm{d}\omega\int_{-\infty}^{\infty}\mathrm{d}\xi\ \frac{\overline{F}(\omega,\xi)}{\omega^n}\exp[-\mathrm{i}(\omega t+\xi x)], \tag{B.5}$$

and this integral can be treated as described above. This provides the results (5.4), (8.12), (9.5) and (9.13).

Finally, for a transform $\overline{F}(\omega, \xi_1, \xi_2)$ which is homogeneous of degree -2,

$$F(t, x_1, x_2) = (2\pi)^{-3} \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\omega \int\int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \overline{F}(\omega, \xi_1, \xi_2) \exp[-\mathrm{i}(\omega t + \xi_1 x_1 + \xi_2 x_2)]. \tag{B.6}$$

This can be treated by employing the new variables $\Omega = \omega/\rho$, $\eta_i = \xi_i/\rho$, where $\rho = (\xi_1^2 + \xi_2^2)^{1/2}$. Then

$$\begin{aligned} F(t, x_1, x_2) &= (2\pi)^{-3} \lim_{\varepsilon \to 0} \int_{|\eta|=1} \mathrm{d}s \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\Omega \; \overline{F}(\Omega, \eta_1, \eta_2) \times \\ & \int_0^\infty \mathrm{d}\rho \; \exp[-\mathrm{i}\rho(\Omega t + \eta_1 x_1 + \eta_2 x_2 - \mathrm{i}\varepsilon)]. \end{aligned} \tag{B.7}$$

The pattern is now similar to that described above. The result (5.7) was obtained in this way.

FIGURE CAPTIONS

1. A two-dimensional plot of $\mathcal{T}_r(x_2/t)H(t-|x_2|/\alpha a)$ against x_2/at and V/a; $(a/b)^2 = 3$.

2. Plots of $\mathcal{T}_r(x_2/t)H(t-|x_2|/\alpha a)$ at fixed values of V/c: (a) solid line, $V/c = 0.1$, (b) dashed line, $V/c = 0.5$, (c) dotted line, $V/c = 0.9$; $(a/b)^2 = 3$.