DYNAMIC WEIGHT FUNCTIONS FOR A MOVING CRACK
II. SHEAR LOADING

A.B. Movchan[1] and J.R. Willis[2]

[1]School of Mathematical Sciences
University of Bath
Bath BA2 7AY, U.K.

[2]University of Cambridge
Department of Applied Mathematics and Theoretical Physics
Silver Street, Cambridge CB3 9EW, U.K.

ABSTRACT

Dynamic weight functions are constructed for general time–dependent shear loading of a plane semi–infinite crack propagating with constant speed in an infinite isotropic elastic body. The use of Fourier transforms reduces the problem to the analysis of a matrix Wiener–Hopf equation. The solution of the Wiener–Hopf problem is presented. An expression is derived for the first–order perturbation to the stress intensity factors induced by a small time–dependent deviation from straightness of the crack front. The asymptotic procedure requires consideration of two terms of the asymptotic expansions of the displacement and stress tensor components in a neighbourhood of the crack front.

1. INTRODUCTION

This is a continuation of a study of dynamic weight functions for three-dimensional loading of a plane, semi-infinite crack, moving with speed V. Part I (Willis and Movchan, 1994) presented the basic framework and developed the solution in detail, for the case of Mode I loading. This required the solution of a scalar equation of Weiner-Hopf type. In contrast, even though the crack edge is straight, three-dimensional shear loading inevitably produces coupling between Modes II and III. The Fourier transform formulation correspondingly generates, for shear loading, a

2×2 matrix Wiener-Hopf problem. This problem has particular structure that permits its solution in closed form; this is obtained.

Next, the Fourier transform of the weight function *matrix* is employed, in a manner similar to that presented in Part I, to obtain a (Fourier transformed) relation between the first-order perturbations to the Mode II and Mode III stress intensity factors, induced by a small deviation from straightness of the crack front. Inversion of the transforms is performed employing a technique, alternative to that of Cagniard (1939), developed by Willis (1973), to obtain the relation in integral form. A by-product of the analysis is a set of explicit relations between second-order terms in the expansions of relative displacement just behind, and traction just ahead of, the crack edge, in the three-dimensional time-dependent case.

It is confirmed in an Appendix that, in the limiting case of statics, the formulae derived reproduce results that are already known (Gao and Rice, 1986).

Since the basic formulation has already been presented in Part I, formulae are usually not repeated: reference to (I, 3.3), for example, means equation (3.3) of Part I. However, just to set the scene, the reader is reminded that the basic problem is to solve the equations of isotropic elastodynamics, for an infinite space which contains a crack occupying the region

$$X := x_1 - Vt < 0, \quad -\infty < x_2 < \infty, \quad x_3 = 0. \tag{1.1}$$

The crack faces are loaded, with tractions prescribed as $\sigma_{i3}(t, X, x_2, 0) = -P_i(t, X, x_2)$ for $X < 0$. Attention is focussed, initially, on the relative displacement of the crack faces, $[u_i](t, X, x_2)$ (for $X < 0$) and the traction components on the plane $x_3 = 0$, $\sigma_{i3}(t, X, x_2, 0)$ (for $X > 0$). In particular, the weight functions generate the associated stress intensity factors. Later, the crack is taken to occupy the region $X < \varepsilon\phi(t, x_2)$, $-\infty < x_2 < \infty$, $x_3 = 0$, and the perturbations to the stress intensity factors are found, to first order in ε.

2. THE MATRIX WIENER–HOPF PROBLEM

Part I (see Willis and Movchan (1994)) introduced the notation $[U_i]$ and Σ_{i3} for the relative displacement and traction components on the plane $x_3 = 0$, associated with a weight function, and derived, in Appendix A, a relation between their Fourier transforms $[\tilde{U}_i]$ and $\tilde{\Sigma}_{i3}$ with respect to t, x_1, and x_2. Mode I uncouples from Modes II and III and was discussed in Part I. For the coupled Modes II and III, the relation takes the form given by (I, A.16), which is equivalent to

$$\begin{pmatrix} \tilde{\Sigma}_{13} \\ \tilde{\Sigma}_{23} \end{pmatrix} (\omega', \xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{-1} \mathbf{R}^t \boldsymbol{\mathcal{D}} \mathbf{R} \begin{pmatrix} [\tilde{U_1}] \\ [\tilde{U_2}] \end{pmatrix} (\omega', \xi_1, \xi_2), \tag{2.1}$$

where

$$\mathbf{R} = \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}, \tag{2.2}$$

and $\boldsymbol{\mathcal{D}}$ denotes the diagonal matrix

$$\boldsymbol{\mathcal{D}} = \mathrm{diag}\left\{ \frac{\mathrm{i}\mu D(\omega', \xi_1, \xi_2)}{2(\omega'/b)^2((\omega'/b)^2 - |\xi|^2)^{1/2}}, \frac{\mathrm{i}\mu((\omega'/b)^2 - |\xi|^2)^{1/2}}{2} \right\}. \tag{2.3}$$

Here, notation is as defined in Part I: in particular, $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$ and D is given by (I, 4.2).

The matrix Wiener–Hopf problem is obtained, as described in Part I, by introducing transforms $\overline{\Sigma}_{i3}$ and $[\overline{U}_i]$, which are the Fourier transforms of Σ_{i3} and $[U_i]$ with respect to variables t, $X = x_1 - Vt$ and x_2. It is easily verified that, for a function $f(t, x_1, x_2)$,

$$\overline{f}(\omega, \xi_1, \xi_2) = \tilde{f}(\omega - V\xi_1, \xi_1, \xi_2). \tag{2.4}$$

Thus, (2.1) relates $\overline{\Sigma}_{i3}$ and $[\overline{U}_i]$, $(i = 1, 2)$, if $\omega' = \omega - V\xi_1$. Furthermore, since $[U_i] = 0$ when $X < 0$ and $\Sigma_{i3} = 0$ when $X > 0$, it follows that $\overline{\Sigma}_{i3}$ is an analytic function of ξ_1 (for each fixed ω with $Im(\omega) > 0$ and each real ξ_2) in a half–plane $Im(\xi_1) < k_+$, while $[\overline{U}_i]$ is analytic in a half–plane $Im(\xi_1) > k_-$, for some k_-, k_+ satisfying $-k_- < 0 < k_+$.

In preparation for the solution of the Wiener–Hopf equation, factorize $\boldsymbol{\mathcal{D}}(\omega - V\xi_1, \xi_1, \xi_2)$ as follows:

$$\boldsymbol{\mathcal{D}}(\omega - V\xi_1, \xi_1, \xi_2) = \boldsymbol{\mathcal{D}}_+ \boldsymbol{\mathcal{D}}_-, \tag{2.5}$$

where $\boldsymbol{\mathcal{D}}_\pm$ are the diagonal matrices

$$\begin{aligned} \boldsymbol{\mathcal{D}}_+ &= \mathrm{diag}\left\{ \frac{[\xi_1 + \omega V/(c\gamma)^2 + \mathrm{i}q_c]T_+}{[\xi_1 + \omega V/(b\beta)^2 + \mathrm{i}q_b]^{1/2}}, [\xi_1 + \omega V/(b\beta)^2 + \mathrm{i}q_b]^{1/2} \right\}, \\ \boldsymbol{\mathcal{D}}_- &= -\frac{\mu}{2}\mathrm{diag}\left\{ \frac{(\xi_1 + \omega V/(c\gamma)^2 - \mathrm{i}q_c)b^2 R(V) T_-}{V^2\beta[\xi_1 + \omega V/(b\beta)^2 - \mathrm{i}q_b]^{1/2}}, \beta[\xi_1 + \omega V/(b\beta)^2 - \mathrm{i}q_b]^{1/2} \right\}. \end{aligned} \tag{2.6}$$

Notation is as defined in Part I; $\boldsymbol{\mathcal{D}}_+$ is analytic in $Im(\xi_1) > k_-$ and $\boldsymbol{\mathcal{D}}_-$ is analytic in $Im(\xi_1) < k_+$.

Thus, in the strip $k_- < Im(\xi_1) < k_+$,

$$\overline{\boldsymbol{\Sigma}}_- = (\xi_1^2 + \xi_2^2)^{-1}\mathbf{R}^t\boldsymbol{\mathcal{D}}_+\boldsymbol{\mathcal{D}}_-\mathbf{R}\overline{\mathbf{U}}_+, \tag{2.7}$$

where $\overline{\boldsymbol{\Sigma}}_- := (\overline{\Sigma}_{13},\ \overline{\Sigma}_{23})^t$ and $\overline{\mathbf{U}}_+ := ([\overline{U}_1], [\overline{U}_2])^t$.

The objective is to deduce from (2.7) the vector functions $\overline{\boldsymbol{\Sigma}}_-$, analytic in $Im(\xi_1) < k_+$, and $\overline{\mathbf{U}}_+$, analytic in $Im(\xi_1) > k_-$, such that $\overline{\mathbf{U}}_+$ is a homogeneous vector function of degree $-1/2$ and, correspondingly, $\overline{\boldsymbol{\Sigma}}_-$ is homogeneous of degree $1/2$ in (ω, ξ_1, ξ_2).

3. SOLUTION OF THE WIENER–HOPF EQUATION

It is noted that, as $\xi_2 \to 0$ (or, equivalently, as $\xi_1/|\xi_2| \to \infty$),

$$\mathbf{R} \sim \xi_1\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix, and the Wiener–Hopf equation (2.7) uncouples, asymptotically, to give

$$\overline{\boldsymbol{\Sigma}}_- \sim \boldsymbol{\mathcal{D}}_+\boldsymbol{\mathcal{D}}_-\overline{\mathbf{U}}_+, \tag{3.1}$$

for which a candidate solution with the desired homogeneity is

$$\overline{\boldsymbol{\Sigma}}_- \sim \boldsymbol{\mathcal{D}}_-\mathbf{C}, \quad \overline{\mathbf{U}}_+ \sim (\boldsymbol{\mathcal{D}})_+^{-1}\mathbf{C}, \tag{3.2}$$

for some vector vector function $\mathbf{C}$ of ω and ξ_2, which is homogeneous of degree zero. This provides at least partial motivation for seeking a solution of (2.7) in the form

$$\begin{aligned}\overline{\boldsymbol{\Sigma}}_- &= \boldsymbol{\mathcal{D}}_-\mathbf{C}^{(1)} + \mathbf{R}^t\boldsymbol{\sigma}_-^{(1)},\\ \overline{\mathbf{U}}_+ &= (\boldsymbol{\mathcal{D}})_+^{-1}\mathbf{C}^{(2)} + \mathbf{R}^t\mathbf{u}_+^{(1)},\end{aligned} \tag{3.3}$$

where $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ are vector functions of ω and ξ_2, homogeneous of degree zero.

Substituting (3.3) into (2.7) and multiplying by $\mathbf{R}$ from the left give

$$\mathbf{R}\boldsymbol{\mathcal{D}}_-\mathbf{C}^{(1)} + (\xi_1^2+\xi_2^2)\boldsymbol{\sigma}_-^{(1)} = \boldsymbol{\mathcal{D}}_+\boldsymbol{\mathcal{D}}_-\mathbf{R}(\boldsymbol{\mathcal{D}}_+)^{-1}\mathbf{C}^{(2)} + (\xi_1^2+\xi_2^2)\boldsymbol{\mathcal{D}}_+\boldsymbol{\mathcal{D}}_-\mathbf{u}_+^{(1)}, \tag{3.4}$$

or, upon rearrangement,

$$(\xi_1^2+\xi_2^2)^{-1}\left\{\xi_1\left(\mathbf{C}^{(1)} - \mathbf{C}^{(2)}\right) + \xi_2\boldsymbol{\mathcal{F}}(\omega,\xi_1,\xi_2)\right\} + (\boldsymbol{\mathcal{D}}_+)^{-1}\boldsymbol{\sigma}_-^{(1)} = \boldsymbol{\mathcal{D}}_+\mathbf{u}_+^{(1)}, \tag{3.5}$$

where

$$\mathcal{F}(\omega,\xi_1,\xi_2) = (\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-\mathbf{C}^{(1)} - \boldsymbol{\mathcal{D}}_+\mathbf{E}(\boldsymbol{\mathcal{D}}_+)^{-1}\mathbf{C}^{(2)}, \tag{3.6}$$

with

$$\mathbf{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.7}$$

To construct a general solution, two cases are considered: (i) $\mathbf{C}^{(2)} = \mathbf{0},\ \mathbf{C}^{(1)} \neq \mathbf{0}$ and (ii) $\mathbf{C}^{(1)} = \mathbf{0},\ \mathbf{C}^{(2)} \neq \mathbf{0}$.

First, suppose that $\mathbf{C}^{(2)} = \mathbf{0}$. Equation (3.5) takes the form

$$(\xi_1^2+\xi_2^2)^{-1}\Big[\xi_1\mathbf{I} + \xi_2(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-\Big]\mathbf{C}^{(1)} + (\boldsymbol{\mathcal{D}}_-)^{-1}\boldsymbol{\sigma}_-^{(1)} = \boldsymbol{\mathcal{D}}_+\mathbf{u}_+^{(1)}. \tag{3.8}$$

The following additive split is obvious:

$$\begin{aligned}&\Bigg\{\frac{1}{2}\Bigg[\frac{1}{\xi_1+\mathrm{i}|\xi_2|} + \frac{1}{\xi_1-\mathrm{i}|\xi_2|}\Bigg] + \frac{\mathrm{sign}(\xi_2)}{2\mathrm{i}}\frac{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\mathbf{D}_-}{\xi_1-\mathrm{i}|\xi_2|}\\ &-\frac{\mathrm{sign}(\xi_2)}{2\mathrm{i}}\Bigg[\frac{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_- - \{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-\}(\omega,-\mathrm{i}|\xi_2|,\xi_2)}{\xi_1+\mathrm{i}|\xi_2|}\Bigg]\\ &-\frac{\mathrm{sign}(\xi_2)}{2\mathrm{i}}\frac{\{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-\}(\omega,-\mathrm{i}|\xi_2|,\xi_2)}{\xi_1+\mathrm{i}|\xi_2|}\Bigg\}\mathbf{C}^{(1)} = \boldsymbol{\mathcal{D}}_+\mathbf{u}_+^{(1)} - (\boldsymbol{\mathcal{D}}_-)^{-1}\boldsymbol{\sigma}_-^{(1)}.\end{aligned} \tag{3.9}$$

The usual Wiener–Hopf argument then gives

$$\mathbf{u}_+^{(1)} = \frac{1}{2(\xi_1+\mathrm{i}|\xi_2|)}(\boldsymbol{\mathcal{D}}_+)^{-1}\Big\{\mathbf{I} + \mathrm{i}\ \mathrm{sign}(\xi_2)\{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-\}(\omega,-\mathrm{i}|\xi_2|,\xi_2)\Big\}\mathbf{C}^{(1)}, \tag{3.10}$$

$$\begin{aligned}\boldsymbol{\sigma}_-^{(1)} = -\frac{1}{2}\boldsymbol{\mathcal{D}}_-\Bigg\{&\frac{1}{\xi_1-\mathrm{i}|\xi_2|} - \frac{\mathrm{i}\ \mathrm{sign}(\xi_2)(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-}{\xi_1-\mathrm{i}|\xi_2|}\\ &+\mathrm{i}\ \mathrm{sign}(\xi_2)\frac{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_- - \{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-\}(\omega,-\mathrm{i}|\xi_2|,\xi_2)}{\xi_1+\mathrm{i}|\xi_2|}\Bigg\}\mathbf{C}^{(1)},\end{aligned} \tag{3.11}$$

and, consequently,

$$\overline{\mathbf{U}}_+ = \frac{\mathbf{R}^t(\boldsymbol{\mathcal{D}}_+)^{-1}}{2(\xi_1+\mathrm{i}|\xi_2|)}\Big\{\mathbf{I} + \mathrm{i}\ \mathrm{sign}(\xi_2)\{(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_-\}(\omega,-\mathrm{i}|\xi_2|,\xi_2)\Big\}\mathbf{C}^{(1)}, \tag{3.12}$$

$$\overline{\boldsymbol{\Sigma}}_{-} = \frac{1}{2(\xi_1 + \mathrm{i}|\xi_2|)}\mathbf{R}^t\boldsymbol{\mathcal{D}}_{-}\left\{\mathbf{I} + \mathrm{i}\ \mathrm{sign}(\xi_2)\{(\boldsymbol{\mathcal{D}}_{-})^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_{-}\}(\omega, -\mathrm{i}|\xi_2|, \xi_2)\right\}\mathbf{C}^{(1)}. \quad (3.13)$$

It can be checked by direct calculation that the expression (3.13) does represent a "$-$" function: it does not have a singularity at $\xi_1 = -\mathrm{i}|\xi_2|$, on account of the particular form of $\mathbf{R}$. It should be mentioned that the rank of the matrix

$$\mathfrak{D}^{(1)} := \mathbf{I} + \mathrm{i}\ \mathrm{sign}(\xi_2)\{(\boldsymbol{\mathcal{D}}_{-})^{-1}\mathbf{E}\boldsymbol{\mathcal{D}}_{-}\}(\omega, -\mathrm{i}|\xi_2|, \xi_2) \quad (3.14)$$

is the same as that of $\mathbf{I} + \mathrm{i}\ \mathrm{sign}(\xi_2)\mathbf{E}$ and so is equal to one. Thus, this solution contains, in effect, only one free parameter.

For the case when $\mathbf{C}^{(1)} = \mathbf{0}$, equation (3.5) can be rewritten as follows

$$(\xi_1^2 + \xi_2^2)^{-1}\left[\xi_1\mathbf{I} + \xi_2\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\boldsymbol{\mathcal{D}}_{+})^{-1}\right]\mathbf{C}^{(2)} = (\boldsymbol{\mathcal{D}}_{-})^{-1}\boldsymbol{\sigma}_{-}^{(1)} - \boldsymbol{\mathcal{D}}_{+}\mathbf{u}_{+}^{(1)}, \quad (3.15)$$

which is equivalent to

$$\begin{aligned}&\left\{\frac{1}{2}\left[\frac{1}{\xi_1 + \mathrm{i}|\xi_2|} + \frac{1}{\xi_1 - \mathrm{i}|\xi_2|}\right] - \frac{\mathrm{sign}(\xi_2)}{2\mathrm{i}}\frac{\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\mathbf{D}_{+})^{-1}}{\xi_1 + \mathrm{i}|\xi_2|}\right.\\ &+\frac{\mathrm{sign}(\xi_2)}{2\mathrm{i}}\left[\frac{\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\boldsymbol{\mathcal{D}}_{+})^{-1} - \{\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\boldsymbol{\mathcal{D}}_{+})^{-1}\}(\omega, \mathrm{i}|\xi_2|, \xi_2)}{\xi_1 - \mathrm{i}|\xi_2|}\right]\\ &\left.+\frac{\mathrm{sign}(\xi_2)}{2\mathrm{i}}\frac{\{\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\boldsymbol{\mathcal{D}}_{+})^{-1}\}(\omega, -\mathrm{i}|\xi_2|, \xi_2)}{\xi_1 - \mathrm{i}|\xi_2|}\right\}\mathbf{C}^{(2)} = (\boldsymbol{\mathcal{D}}_{-})^{-1}\boldsymbol{\sigma}_{-}^{(1)} - \boldsymbol{\mathcal{D}}_{+}\mathbf{u}_{+}^{(1)}. \quad (3.16)\end{aligned}$$

Similarly to (3.12), (3.13) it follows in this case that

$$\overline{\mathbf{U}}_{+} = \frac{\mathbf{R}^t(\boldsymbol{\mathcal{D}}_{+})^{-1}}{2(\xi_1 - \mathrm{i}|\xi_2|)}\left\{\mathbf{I} - \mathrm{i}\ \mathrm{sign}(\xi_2)\{\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\boldsymbol{\mathcal{D}}_{+})^{-1}\}(\omega, \mathrm{i}|\xi_2|, \xi_2)\right\}\mathbf{C}^{(2)}, \quad (3.17)$$

and

$$\overline{\boldsymbol{\Sigma}}_{-} = \frac{1}{2(\xi_1 - \mathrm{i}\ |\xi_2|)}\mathbf{R}^t\boldsymbol{\mathcal{D}}_{-}\left\{\mathbf{I} - \mathrm{i}\ \mathrm{sign}(\xi_2)\left(\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\boldsymbol{\mathcal{D}}_{+})^{-1}\right)(\omega, \mathrm{i}|\xi_2|, \xi_2)\right\}\mathbf{C}^{(2)}. \quad (3.18)$$

The particular form of $\mathbf{R}$ ensures that $\mathbf{U}_{+}$ has no singularity at $\xi_1 = \mathrm{i}|\xi_2|$. As in (3.14), the rank of the matrix

$$\mathfrak{D}^{(2)} := \mathbf{I} - \mathrm{i}\ \mathrm{sign}(\xi_2)\left(\boldsymbol{\mathcal{D}}_{+}\mathbf{E}(\boldsymbol{\mathcal{D}}_{+})^{-1}\right)(\omega, \mathrm{i}|\xi_2|, \xi_2) \quad (3.19)$$

is equal to one.

The solution of the Wiener–Hopf equation (3.5) is now

$$\overline{\mathbf{U}}_+ = \frac{1}{2}\mathbf{R}^t(\boldsymbol{\mathcal{D}}_+)^{-1}\left[\frac{1}{\xi_1 + \mathrm{i}|\xi_2|}\mathfrak{D}^{(1)}\mathbf{C}^{(1)} + \frac{1}{\xi_1 - \mathrm{i}|\xi_2|}\mathfrak{D}^{(2)}\mathbf{C}^{(2)}\right], \tag{3.20}$$

$$\overline{\boldsymbol{\Sigma}}_- = \frac{1}{2}\mathbf{R}^t\boldsymbol{\mathcal{D}}_-\left[\frac{1}{\xi_1 + \mathrm{i}|\xi_2|}\mathfrak{D}^{(1)}\mathbf{C}^{(1)} + \frac{1}{\xi_1 - \mathrm{i}|\xi_2|}\mathfrak{D}^{(2)}\mathbf{C}^{(2)}\right]. \tag{3.21}$$

Although $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ are arbitrary vectors (independent of ξ_1), the fact that $\mathfrak{D}^{(1)}$ and $\mathfrak{D}^{(2)}$ have rank one implies that this solution is expressible in terms of two arbitrary scalar parameters. Since $\mathfrak{D}^{(1)}$ can be written in the form

$$\mathfrak{D}^{(1)} = (\boldsymbol{\mathcal{D}}_-)^{-1}\Big[\mathbf{I} + \mathrm{i}\ \mathrm{sign}(\xi_2)\mathbf{E}\Big]\boldsymbol{\mathcal{D}}_-,$$

its eigenvalues are readily evaluated as

$$\lambda_1 = 0, \quad \lambda_2 = 2,$$

and the corresponding eigenvectors are $(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{X}^{(1)}$, $(\boldsymbol{\mathcal{D}}_-)^{-1}\mathbf{X}^{(2)}$, where

$$\mathbf{X}^{(1)} = \begin{pmatrix} 1 \\ \mathrm{i}\ \mathrm{sign}(\xi_2) \end{pmatrix}, \ \mathbf{X}^{(2)} = \begin{pmatrix} \mathrm{i}\ \mathrm{sign}(\xi_2) \\ 1 \end{pmatrix}.$$

Similarly, $\mathfrak{D}^{(2)}$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$ and corresponding eigenvectors $\boldsymbol{\mathcal{D}}_+\mathbf{X}^{(2)}$, $\boldsymbol{\mathcal{D}}_+\mathbf{X}^{(1)}$. Therefore, in the solution given by (3.20), (3.21) there is no loss in choosing

$$\mathbf{C}^{(1)} = \frac{c_1}{2}(\boldsymbol{\mathcal{D}}_-)^{-1}(\omega, -\mathrm{i}|\xi_2|, \xi_2)\mathbf{X}^{(2)}, \ \mathbf{C}^{(2)} = \frac{c_2}{2}\boldsymbol{\mathcal{D}}_+(\omega, \mathrm{i}|\xi_2|, \xi_2)\mathbf{X}^{(1)}, \tag{3.22}$$

where c_1, c_2 are arbitrary scalar functions of ω, ξ_2, homogeneous of degree zero. Then, (3.20) takes the form

$$\overline{\mathbf{U}}_+ = \mathbf{R}^t(\boldsymbol{\mathcal{D}}_+)^{-1}\begin{pmatrix} \mathrm{i}\dfrac{\mathrm{sign}(\xi_2)}{(\xi_1 + \mathrm{i}|\xi_2|)a_-} & \dfrac{a_+}{\xi_1 - \mathrm{i}|\xi_2|} \\ \dfrac{1}{(\xi_1 + \mathrm{i}|\xi_2|)b_-} & \mathrm{i}\dfrac{\mathrm{sign}(\xi_2)b_+}{\xi_1 - \mathrm{i}|\xi_2|} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{3.23}$$

where $a_\pm$, $b_\pm$ are defined so that

$$\mathrm{diag}\{a_-,\ b_-\} := \boldsymbol{\mathcal{D}}_-(\omega, -\mathrm{i}|\xi_2|, \xi_2), \ \ \mathrm{diag}\{a_+,\ b_+\} := \boldsymbol{\mathcal{D}}_+(\omega, \mathrm{i}|\xi_2|, \xi_2). \tag{3.24}$$

It is convenient to introduce, starting from (3.23), a "fundamental matrix" solution, with prescribed behaviour as $\xi_1 \to \infty$. This is done by expressing $(c_1,\ c_2)^t$ in the form

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{a_- b_-}{a_+ a_- + b_+ b_-} \begin{pmatrix} \mathrm{i}\ \mathrm{sign}(\xi_2) b_+ & -a_+ \\ -1/b_- & \mathrm{i}\ \mathrm{sign}(\xi_2)/a_- \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \tag{3.25}$$

for some constant l_1, l_2.

Then

$$\overline{\mathbf{U}}_+ = \frac{1}{a_+ a_- + b_+ b_-} \mathbf{R}^t (\boldsymbol{\mathcal{D}}_+)^{-1} \begin{pmatrix} \dfrac{b_+ b_-}{\xi_1 + \mathrm{i}|\xi_2|} + \dfrac{a_+ a_-}{\xi_1 - \mathrm{i}|\xi_2|} & \dfrac{2a_+ b_- \xi_2}{\xi_1^2 + \xi_2^2} \\ -\dfrac{2a_- b_+ \xi_2}{\xi_1^2 + \xi_2^2} & \dfrac{a_+ a_-}{\xi_1 + \mathrm{i}|\xi_2|} + \dfrac{b_+ b_-}{\xi_1 - \mathrm{i}|\xi_2|} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \tag{3.26}$$

and, as $\xi_1 \to \infty$,

$$\overline{\mathbf{U}}_+ \sim (\xi_1 + 0\mathrm{i})^{-1/2} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}. \tag{3.27}$$

The expression for $\overline{\boldsymbol{\Sigma}}_-$ is similar: it is necessary only to replace $(\boldsymbol{\mathcal{D}}_+)^{-1}$ by $\boldsymbol{\mathcal{D}}_-$ on the right side of (3.26). The matrix multiplying $(l_1,\ l_2)^t$ in (3.26) is the fundamental matrix that was sought on account of its simple form as $\xi_1 \to \infty$.

This section is concluded with the observation that, in correspondence with (3.27), the asymptotic form of $\mathbf{U}_+$ as $X \to 0$ is

$$\mathbf{U}_+(t, X, x_2) \sim -\mathrm{i} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \left(\frac{\mathrm{i}}{\pi X} \right)^{1/2} H(X) \delta(t) \delta(x_2). \tag{3.28}$$

4. STRESS–INTENSITY FACTORS

It was shown in Part I that

$$\Big\{ [U_i] * \sigma_{i3} - \Sigma_{k3} * [u_k] \Big\} (t, X', x_2') = 0, \tag{4.1}$$

where σ_{i3}, Σ_{k3} denote stress components for $\mathbf{u}$ and $\mathbf{U}$ respectively, evaluated at $x_3 = 0$; as before, $[.]$ denotes the jump across $x_3 = 0$ of the quantity indicated, and $*$ signifies convolution with respect to t, X and x_2. Implications of the identity

(4.1) are now explored in the case that $([U_1],\ [U_2])^t = \mathbf{U}_+,\ (\Sigma_{13},\ \Sigma_{23})^t = \mathbf{\Sigma}_-$ and $[U_3] = \Sigma_{33} = 0$. The convolution (4.1) will first be evaluated when $X' > 0$. Since $[u_k]$ and Σ_{k3} are $-$ functions, their convolution has the value zero and (4.1) reduces to

$$\left\{\mathbf{U}_+^t * \boldsymbol{\sigma}\right\}(t, X', x_2') = 0, \tag{4.2}$$

where $\boldsymbol{\sigma}$ represents $(\sigma_{13},\ \sigma_{23})^t$, evaluated when $x_3 = 0$. Now,

$$\sigma_{i3}(t, X, x_2', 0) = -P_i(t, X, x_2)H(-X) + \sigma_{i3}^{(+)}(t, X, x_2), \tag{4.3}$$

where $P_i(t, X, x_2)$ $(i = 1, 2)$ are prescribed. Furthermore, as $X \to 0$,

$$\begin{pmatrix} \sigma_{13}^{(+)} \\ \sigma_{23}^{(+)} \end{pmatrix} (t, X, x_2) \sim (2\pi X)^{-1/2} \begin{pmatrix} K_{II} \\ K_{III} \end{pmatrix} H(X). \tag{4.4}$$

The convolution in (4.2) is now readily evaluated, in the limit as $X' \to +0$, employing the asymptotic forms (3.28) and (4.4):

$$\mathrm{i}\left(\frac{\mathrm{i}}{2}\right)^{1/2} (l_1,\ l_2) \begin{pmatrix} K_{II}(t, x_2) \\ K_{III}(t, x_2) \end{pmatrix} + \{\mathbf{U}_+^t * \mathbf{P}^{(-)}\}(t, 0, x_2) = 0, \tag{4.5}$$

where $\mathbf{P}^{(-)} = (P_1, P_2)^t H(-X)$.

This result motivates defining a weight function *matrix*, still called $\mathbf{U}_+$, so that it has the Fourier transform

$$\overline{\mathbf{U}}_+ = \frac{(2\mathrm{i})^{1/2}}{a_+a_- + b_+b_-}\mathbf{R}^t(\boldsymbol{\mathcal{D}}_+)^{-1} \begin{pmatrix} \dfrac{b_+b_-}{\xi_1 + \mathrm{i}|\xi_2|} + \dfrac{a_+a_-}{\xi_1 - \mathrm{i}|\xi_2|} & \dfrac{2a_+b_-\xi_2}{\xi_1^2 + \xi_2^2} \\ -\dfrac{2a_-b_+\xi_2}{\xi_1^2 + \xi_2^2} & \dfrac{a_+a_-}{\xi_1 + \mathrm{i}|\xi_2|} + \dfrac{b_+b_-}{\xi_1 - \mathrm{i}|\xi_2|} \end{pmatrix}. \tag{4.6}$$

This is nothing but the fundamental matrix introduced in the preceding section, multiplied by $(2\mathrm{i})^{1/2}$. The corresponding matrix $\mathbf{\Sigma}_-$ has Fourier transform

$$\overline{\mathbf{\Sigma}}_- = \frac{(2\mathrm{i})^{1/2}}{a_+a_- + b_+b_-}\mathbf{R}^t\boldsymbol{\mathcal{D}}_- \begin{pmatrix} \dfrac{b_+b_-}{\xi_1 + \mathrm{i}|\xi_2|} + \dfrac{a_+a_-}{\xi_1 - \mathrm{i}|\xi_2|} & \dfrac{2a_+b_-\xi_2}{\xi_1^2 + \xi_2^2} \\ -\dfrac{2a_-b_+\xi_2}{\xi_1^2 + \xi_2^2} & \dfrac{a_+a_-}{\xi_1 + \mathrm{i}|\xi_2|} + \dfrac{b_+b_-}{\xi_1 - \mathrm{i}|\xi_2|} \end{pmatrix}. \tag{4.7}$$

The result (4.5) then gives

$$\begin{pmatrix} K_{II}(t,x_2) \\ K_{III}(t,x_2) \end{pmatrix} = \left\{ \mathbf{U}_+^t * \mathbf{P}^{(-)} \right\}(t,0,x_2). \tag{4.8}$$

It is also possible to develop the relation (4.8) by evaluating the convolutions $\mathbf{U}_+^t * \boldsymbol{\sigma}^{(+)}$ and $\mathbf{\Sigma}_-^t * [\mathbf{u}]$, as $X \to \pm 0$, from the corresponding asymptotic forms of the transforms as $\xi_1 \to \infty$, as was done in Part I for the case of Mode-I loading. This will emerge from the more general reasoning presented next.

5. ASYMPTOTIC EXPANSIONS TO HIGHER ORDER

As noted in Part I, the identity (4.1) induces a corresponding identity for its Fourier transform,

$$[\overline{U}_i]\overline{\sigma}_{i3} - \overline{\Sigma}_{k3}[\overline{u}_k] = 0, \tag{5.1}$$

or

$$\overline{\mathbf{U}}_+^t \overline{\boldsymbol{\sigma}}^{(+)} - \overline{\mathbf{\Sigma}}_-^t [\overline{\mathbf{u}}] = -\overline{\mathbf{U}_+^t * \mathbf{P}^{(-)}}. \tag{5.2}$$

The identity (5.2) will now be expanded as $\xi_1 \to \infty$. It is noted first that, as $X \to 0$,

$$\boldsymbol{\sigma}^{(+)} \sim \left\{ \mathbf{K}(2\pi X)^{-1/2} - \mathbf{P}^* + \mathbf{A}X^{1/2} \right\} H(X), \tag{5.3}$$

and

$$[\mathbf{u}] \sim \left\{ \mathbf{\Lambda}(V)\mathbf{K}(-2X/\pi)^{1/2} - \mathbf{B}(-X)^{3/2} \right\} H(-X), \tag{5.4}$$

for some vectors $\mathbf{P}^*$, $\mathbf{A}$ and $\mathbf{B}$, and some matrix $\mathbf{\Lambda}(V)$. Here, $\mathbf{K}$ denotes $(K_{II},\ K_{III})^t$. Correspondingly, as $\xi_1 \to \infty$,

$$\overline{\sigma}^{(+)} \sim \left(\frac{\mathrm{i}}{2}\right)^{1/2} \frac{\overline{\mathbf{K}}}{(\xi_1 + 0\mathrm{i})^{1/2}} - \frac{\mathrm{i}\overline{\mathbf{P}}^*}{\xi_1 + 0\mathrm{i}} - \frac{\pi^{1/2}\overline{\mathbf{A}}}{2\mathrm{i}^{1/2}(\xi_1 + 0\mathrm{i})^{3/2}}, \tag{5.5}$$

and

$$[\overline{\mathbf{u}}] \sim -\frac{(\mathrm{i}\pi)^{1/2}\mathbf{\Lambda}\overline{\mathbf{K}}}{2(\xi_1 - 0\mathrm{i})^{3/2}} - \left(\frac{3\mathrm{i}}{4}\right) \frac{(\mathrm{i}\pi)^{1/2}\overline{\mathbf{B}}}{(\xi_1 - 0\mathrm{i})^{5/2}}. \tag{5.6}$$

Next,

$$\mathbf{U}_+^t * \mathbf{P}^{(-)} \sim \mathbf{F}(t,x_2) - 2(2/\pi)^{1/2}\mathbf{L}(t,x_2)X^{1/2}H(X) + (\pi/2)^{1/2}\mathbf{M}(t,x_2)X, \tag{5.7}$$

where

$$\mathbf{F}(t,x_2) = (\mathbf{U}_+^t * \mathbf{P}^{(-)})(t,0,x_2), \tag{5.8}$$
$$\mathbf{L}(t,x_2) = \mathbf{P}(t,0,x_2), \tag{5.9}$$

and

$$\mathbf{M}(t,x_2) = (\mathbf{U}_+^t * \mathbf{P}'^{(-)})(t,0,x_2), \tag{5.10}$$

with $\mathbf{P}'^{(-)} = (\partial \mathbf{P}/\partial X)H(-X)$. Fourier transforming (5.7) gives, as $\xi_1 \to \infty$,

$$\overline{\mathbf{U}_+^t * \mathbf{P}^{(-)}} \sim \mathrm{i}\Big\{ \overline{\mathbf{F}}\Big[\frac{1}{\xi_1 + 0\mathrm{i}} - \frac{1}{\xi_1 - 0\mathrm{i}}\Big] - \frac{(2\mathrm{i})^{1/2}\overline{\mathbf{L}}}{(\xi_1+0\mathrm{i})^{3/2}} + \mathrm{i}(\pi/2)^{1/2}\overline{\mathbf{M}}\Big[\frac{1}{(\xi_1+0\mathrm{i})^2} - \frac{1}{(\xi_1-0\mathrm{i})^2}\Big]\Big\}. \tag{5.11}$$

All of these results are direct counterparts of corresponding results in Part I for the case of Mode I loading.

Now corresponding expansions of $\overline{\mathbf{U}}_+$ and $\overline{\mathbf{\Sigma}}_-$ are required. First, for the diagonal matrices $\mathcal{D}_\pm$, as $\xi_1 \to \infty$,

$$\mathcal{D}_- \sim -\frac{\mu}{2}(\xi_1 - 0\mathrm{i})^{1/2}\Big\{\mathfrak{A}_-^{(1)} + \frac{1}{\xi_1 - 0\mathrm{i}}\mathfrak{A}_-^{(2)}\Big\}, \tag{5.12}$$
$$\mathcal{D}_+^{-1} \sim (\xi_1 + 0\mathrm{i})^{-1/2}\Big\{\mathbf{I} + \frac{1}{\xi_1 + 0\mathrm{i}}\mathfrak{B}_+^{(2)}\Big\}, \tag{5.13}$$

where

$$\mathfrak{A}_-^{(1)} = \beta\ \mathrm{diag}\{\mathcal{B}, 1\}, \tag{5.14}$$
$$\mathfrak{A}_-^{(2)} = \beta\ \mathrm{diag}\Big\{\mathcal{B}\Big(\frac{\omega V}{(c\gamma)^2} - \mathrm{i}q_c - \frac{1}{2}(\frac{\omega V}{(b\beta)^2} - \mathrm{i}q_b) + t_-\Big),\ \frac{1}{2}(\frac{\omega V}{(b\beta)^2} - \mathrm{i}q_b)\Big\}, \tag{5.15}$$
$$\mathfrak{B}_+^{(2)} = -\mathrm{diag}\Big\{\frac{\omega V}{(c\gamma)^2} + \mathrm{i}q_c - \frac{1}{2}(\frac{\omega V}{(b\beta)^2} + \mathrm{i}q_b) + t_+,\ \frac{1}{2}(\frac{\omega V}{(b\beta)^2} + \mathrm{i}q_b)\Big\}, \tag{5.16}$$

with

$$\mathcal{B} = b^2 R(V)/(\beta V)^2, \tag{5.17}$$

and

$$t_\pm = -\frac{1}{2\pi\mathrm{i}}\int_{C_\mp} \ln T(\omega, \xi_1', \xi_2)\, d\xi_1'. \tag{5.18}$$

Notation not explicitly defined is as in Part I.

It follows now, from (3.26) and (5.13), that

$$\overline{\mathbf{U}}_+ \sim (2\mathrm{i})^{1/2}(\xi_1 + 0\mathrm{i})^{-1/2}\left\{\mathbf{I} + \frac{\mathrm{i}}{\xi_1 + 0\mathrm{i}}\overline{\mathbf{Q}}\right\}, \tag{5.19}$$

where

$$\mathrm{i}\overline{\mathbf{Q}} = \mathfrak{B}^{(2)} + \mathrm{i}|\xi_2|\mathbf{q} + \xi_2(2\mathbf{f} - \mathbf{E}), \tag{5.20}$$

with

$$\mathbf{q} = \frac{a_+a_- - b_+b_-}{a_+a_- + b_+b_-}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.21}$$

and

$$\mathbf{f} = (a_+a_- + b_+b_-)^{-1}\begin{pmatrix} 0 & a_+b_- \\ -a_-b_+ & 0 \end{pmatrix}. \tag{5.22}$$

Inspection of the Wiener–Hopf equation (2.1) reveals that, correct to order $|\xi_1|^{-1/2}$,

$$\overline{\mathbf{\Sigma}}_- \sim -\xi_1\left(\mathbf{I} - \frac{\xi_2}{\xi_1}\mathbf{E}\right)\left[\boldsymbol{\mathcal{A}}(V - \frac{\omega}{\xi_1})\right]^{-1}\left(\mathbf{I} + \frac{\xi_2}{\xi_1}\mathbf{E}\right)\overline{\mathbf{U}}_+, \tag{5.23}$$

where

$$\boldsymbol{\mathcal{A}}(V) = \mathrm{diag}\left\{\frac{2\beta(V)(V/b)^2}{\mu R(V)}, \frac{2}{\mu\beta(V)}\right\}. \tag{5.24}$$

Thus, as $\xi_1 \to \infty$,

$$\begin{aligned}\overline{\mathbf{\Sigma}}_- \sim {} & -(2\mathrm{i})^{1/2}(\xi_1 - 0\mathrm{i})^{1/2}[\boldsymbol{\mathcal{A}}(V)]^{-1}\left\{\mathbf{I} + \frac{1}{\xi_1 - \mathrm{i}0}\left(\omega\boldsymbol{\mathcal{A}}'(V)[\boldsymbol{\mathcal{A}}(V)]^{-1}\right.\right.\\ & \left.\left. + \xi_2(\mathbf{E} - \boldsymbol{\mathcal{A}}(V)\mathbf{E}[\boldsymbol{\mathcal{A}}(V)]^{-1}) + \mathrm{i}\overline{\mathbf{Q}}\right)\right\}.\end{aligned} \tag{5.25}$$

These expressions suffice to expand the identity (5.2) to three terms as $\xi_1 \to \infty$:

$$\frac{\mathrm{i}\overline{\mathbf{K}}}{\xi_1 + 0\mathrm{i}} - \frac{\mathrm{i}(2\mathrm{i})^{1/2}\mathbf{P}^*}{(\xi_1 + 0\mathrm{i})^{3/2}} - \frac{(\pi/2)^{1/2}\overline{\mathbf{A}} + \overline{\mathbf{Q}}^t\overline{\mathbf{K}}}{(\xi_1 + 0\mathrm{i})^2} - \frac{\mathrm{i}(\pi/2)^{1/2}[\boldsymbol{\mathcal{A}}(V)]^{-1}\mathbf{\Lambda}\overline{\mathbf{K}}}{\xi_1 - 0\mathrm{i}}$$

$$-\frac{\mathrm{i}(\pi/2)^{1/2}\left(\omega[\boldsymbol{\mathcal{A}}(V)]^{-1}\boldsymbol{\mathcal{A}}'(V) - \xi_2(\mathbf{E} - [\boldsymbol{\mathcal{A}}(V)]^{-1}\mathbf{E}\boldsymbol{\mathcal{A}}(V)) + \mathrm{i}\overline{\mathbf{Q}}^t\right)[\boldsymbol{\mathcal{A}}(V)]^{-1}\mathbf{\Lambda}\overline{\mathbf{K}}}{(\xi_1 - 0\mathrm{i})^2}$$

$$+\frac{(3/2)(\pi/2)^{1/2}[\boldsymbol{\mathcal{A}}(V)]^{-1}\overline{\mathbf{B}}}{(\xi_1-0\mathrm{i})^2}=\mathrm{i}\overline{\mathbf{F}}\left[\frac{1}{\xi_1+0\mathrm{i}}-\frac{1}{\xi_1-0\mathrm{i}}\right]-\frac{\mathrm{i}(2\mathrm{i})^{1/2}\overline{\mathbf{L}}}{(\xi_1+0\mathrm{i})^{3/2}}$$
$$-(\pi/2)^{1/2}\overline{\mathbf{M}}\left[\frac{1}{(\xi_1+0\mathrm{i})^2}-\frac{1}{(\xi_1-0\mathrm{i})^2}\right]. \tag{5.26}$$

It follows, therefore, that

$$\overline{\mathbf{K}}=\overline{\mathbf{F}},\ \boldsymbol{\Lambda}=(\tfrac{2}{\pi})^{1/2}\boldsymbol{\mathcal{A}}(V),$$
$$\overline{\mathbf{A}}=\overline{\mathbf{M}}-(\tfrac{2}{\pi})^{1/2}\overline{\mathbf{Q}}^t\overline{\mathbf{K}},$$
$$\overline{\mathbf{B}}=\tfrac{2}{3}\boldsymbol{\mathcal{A}}(V)\Bigg\{\overline{\mathbf{A}}+\mathrm{i}(\tfrac{2}{\pi})^{1/2}\Bigg[\omega[\boldsymbol{\mathcal{A}}(V)]^{-1}\boldsymbol{\mathcal{A}}'(V)$$
$$-\xi_2\Big(\mathbf{E}-\boldsymbol{\mathcal{A}}(V)\mathbf{E}[\boldsymbol{\mathcal{A}}(V)]^{-1}\Big)^t\Bigg]\overline{\mathbf{K}}\Bigg\}. \tag{5.27}$$

The first of relations (5.27) confirms the result (4.8). The second is quoted, for example, by Freund (1990). The expressions for $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ were not known previously to the authors.

6. PERTURBATION OF THE CRACK FRONT

As in the Mode I analysis of Part I, the identity (5.1) can also be applied when the stress and displacement fields solve the problem of a crack occupying the region

$$X=x_1-Vt<\varepsilon\phi(t,x_2),\quad -\infty<x_2<\infty,\quad x_3=0, \tag{6.1}$$

where ε is a small positive constant and the function ϕ is smooth and bounded. The displacement and stress fields for this problem are written as $\mathbf{u}+\Delta\mathbf{u}$, $\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}$, where $\mathbf{u}$, $\boldsymbol{\sigma}$ solve the problem for the unperturbed crack ($\varepsilon=0$). Since (5.1) is satisfied for both the perturbed and unperturbed fields, it follows that

$$\overline{\mathbf{U}}_+^t\overline{\Delta\boldsymbol{\sigma}}-\overline{\boldsymbol{\Sigma}}_-^t[\overline{\Delta\mathbf{u}}]=0. \tag{6.2}$$

This identity applies for all ξ_1 and all ε; however, information on the first–order perturbation of the stress–intensity factors will be sought by considering the lowest–order non–trivial asymptotic expansion of (6.2) as $\xi_1\to\infty$ and $\varepsilon\to 0$.

As in Part I, since $\xi_1 \to \infty$ corresponds to $X \to 0$, it suffices to note that, as $X \to 0$ and $\varepsilon \to 0$,

$$\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma} \sim \left\{\frac{\mathbf{K} + \Delta\mathbf{K}}{[2\pi(X - \varepsilon\phi)]^{1/2}} + (X - \varepsilon\phi)^{1/2}(\mathbf{A} + \Delta\mathbf{A})\right\} H(X - \varepsilon\phi)$$

$$- (\mathbf{P}^* + \Delta\mathbf{P}^*) + \varepsilon\phi'\mathbf{E}\left\{\frac{\mathbf{K}}{(2\pi X)^{1/2}} - \mathbf{P}^* + X^{1/2}\mathbf{A}\right\} H(X), \tag{6.3}$$

and

$$[\mathbf{u} + \Delta\mathbf{u}] \sim \left\{(\varepsilon\phi - X)^{1/2}(\boldsymbol{\Lambda} + \Delta\boldsymbol{\Lambda})(\mathbf{K} + \Delta\mathbf{K}) - (\varepsilon\phi - X)^{3/2}(\mathbf{B} + \Delta\mathbf{B})\right\} H(\varepsilon\phi - X)$$

$$+\varepsilon\phi'\mathbf{E}\left\{(-X)^{1/2}\boldsymbol{\Lambda}\mathbf{K} - (-X)^{3/2}\mathbf{B}\right\} H(-X). \tag{6.4}$$

In comparison with the Mode I problem the expansions (6.3), (6.4) include additional terms which arise from the local rotation of the crack edge.

Fourier transforming (6.3), (6.4) and performing formal expansions with respect to ε and $1/\xi_1$ gives

$$\overline{\Delta\boldsymbol{\sigma}} \sim \left(\frac{\mathrm{i}}{2}\right)^{1/2} (\xi_1 + 0\mathrm{i})^{-1/2}(\overline{\Delta\mathbf{K}} + \varepsilon\mathbf{E}\overline{\phi'\mathbf{K}} + \mathrm{i}\varepsilon\xi_1\overline{\phi\mathbf{K}}) - \frac{\mathrm{i}}{\xi_1 + 0\mathrm{i}}(\overline{\Delta\mathbf{P}^*} + \varepsilon\mathbf{E}\overline{\phi'\mathbf{P}^*})$$

$$-\frac{\pi^{1/2}}{2\mathrm{i}^{1/2}(\xi_1 + 0\mathrm{i})^{3/2}}(\overline{\Delta\mathbf{A}} + \mathrm{i}\varepsilon\xi_1\overline{\phi\mathbf{A}} + \varepsilon\mathbf{E}\overline{\phi'\mathbf{A}}), \tag{6.5}$$

and

$$[\overline{\Delta\mathbf{u}}] \sim \frac{-(\mathrm{i}\pi)^{1/2}}{2(\xi_1 - 0\mathrm{i})^{3/2}}\left(\boldsymbol{\Lambda}\overline{\Delta\mathbf{K}} + \varepsilon\mathbf{E}\boldsymbol{\Lambda}\overline{\phi'\mathbf{K}} + \overline{(\Delta\boldsymbol{\Lambda})\mathbf{K}} + \mathrm{i}\varepsilon\xi_1\boldsymbol{\Lambda}\overline{\phi\mathbf{K}}\right)$$

$$-\frac{3\mathrm{i}}{4}\frac{(\mathrm{i}\pi)^{1/2}}{(\xi_1 - 0\mathrm{i})^{5/2}}\left(\overline{\Delta\mathbf{B}} + \varepsilon\mathbf{E}\overline{\phi'\mathbf{B}} + \mathrm{i}\varepsilon\xi_1\overline{\phi\mathbf{B}}\right). \tag{6.6}$$

These formulae are vector analogues of (I, 8.6) and (I, 8.7).

The asymptotic relations (5.19) and (6.5) yield

$$\overline{\mathbf{U}}_+^t\overline{\Delta\boldsymbol{\sigma}} \sim -\varepsilon\overline{\phi\mathbf{K}} + \frac{\mathrm{i}}{\xi_1 + 0\mathrm{i}}\left[\overline{\Delta\mathbf{K}} + \varepsilon\mathbf{E}\overline{\phi'\mathbf{K}}\right.$$

$$\left.-\varepsilon\overline{\mathbf{Q}}^t\overline{\phi\mathbf{K}} - \left(\frac{\pi}{2}\right)^{1/2}\varepsilon\overline{\phi\mathbf{A}}\right], \tag{6.7}$$

and (5.25) and (6.4) give

$$\overline{\boldsymbol{\Sigma}}^t_-[\overline{\Delta\mathbf{u}}] \sim -\varepsilon\overline{\phi\mathbf{K}} + \frac{\mathrm{i}}{\xi_1 - 0\mathrm{i}}\Bigg[\overline{\Delta\mathbf{K}} + \varepsilon[\boldsymbol{\mathcal{A}}(V)]^{-1}\mathbf{E}\boldsymbol{\mathcal{A}}(V)\overline{\phi'\mathbf{K}} + (\pi/2)^{1/2}[\boldsymbol{\mathcal{A}}(V)]^{-1}\overline{(\Delta\boldsymbol{\Lambda})\mathbf{K}}$$

$$+\varepsilon\left(\mathrm{i}\omega[\boldsymbol{\mathcal{A}}(V)]^{-1}\boldsymbol{\mathcal{A}}'(V) - \mathrm{i}\xi_2(\mathbf{E} - [\boldsymbol{\mathcal{A}}(V)]^{-1}\mathbf{E}\boldsymbol{\mathcal{A}}(V)) - \overline{\mathbf{Q}}^t\right)\overline{\phi\mathbf{K}}$$

$$-\varepsilon\frac{3}{2}\left(\frac{\pi}{2}\right)^{1/2}[\boldsymbol{\mathcal{A}}(V)]^{-1}\overline{\phi\mathbf{B}}\Bigg]. \tag{6.8}$$

The terms that are independent of ξ_1 in (6.7) and (6.8) are the same; hence, they cancel, upon substitution into (6.2). It also follows, from the identities (5.27), that the coefficient of $(\xi_1 + 0\mathrm{i})^{-1}$ in (6.7) is the same as the coefficient of $(\xi_1 - 0\mathrm{i})^{-1}$ in (6.8). Hence, the identity (6.2) implies to first order in ε, that

$$\overline{\Delta\mathbf{K}} = \varepsilon\left\{\overline{\mathbf{Q}}^t\overline{\phi\mathbf{K}} - \mathbf{E}\overline{\phi'\mathbf{K}} + (\pi/2)^{1/2}\overline{\phi\mathbf{A}}\right\}, \tag{6.9}$$

where $\mathbf{A}$ is given by the third of relations (5.27). Inverting the transforms yields

$$\Delta\mathbf{K} = \varepsilon\left\{\mathbf{Q}^t * (\phi\mathbf{K}) - \mathbf{E}\phi'\mathbf{K} + (\pi/2)^{1/2}\phi\mathbf{M} - \phi(\mathbf{Q}^t * \mathbf{K})\right\}. \tag{6.10}$$

Here, the convolution $*$ is over the variables t, x_2 and, since $\overline{\mathbf{Q}}(\omega, \xi_2)$ is homogeneous of degree one, it follows from (I, 8.12) that

$$\mathbf{Q}(t, x_2) = -\frac{1}{2\pi}\frac{\partial^2}{\partial t^2}\left\{\left[\frac{\overline{\mathbf{Q}}(-x_2 + 0\mathrm{i}, t)}{(-x_2 + 0\mathrm{i})^2} + \frac{\overline{\mathbf{Q}}(x_2 + 0\mathrm{i}, -t)}{(x_2 + 0\mathrm{i})^2}\right]H(t)\right\}. \tag{6.11}$$

The result (6.10) is the mixed Mode II/III analogue of the Mode I formula (I, 8.11).

The Appendix considers the static limit of the solution. This is obtained by setting $V = 0$ and allowing ω to tend to zero. This static limit was addressed directly by Gao and Rice (1986); the agreement that is found provides a partial check on the present analysis.

7. CONCLUDING REMARKS

Parts I and II together provide a complete weight function matrix for any loading of the semi–infinite crack in uniform motion. The matrix partitions into a scalar component corresponding to Mode I, and a 2×2 matrix relating to the coupled Modes II and III. Fourier transforms were employed to derive the weight function matrix. Treatment of the Mode I component required the solution of a scalar Wiener–Hopf problem whose solution is routine; the coupled Modes II and III generated a 2×2 matrix Wiener–Hopf problem for which a closed–form solution was derived. Use of a reciprocal theorem permitted the deduction of the perturbation to the stress intensity factors induced by a small time– and space–dependent deviation from straightness of the crack front. The perturbation is expressed by (I, 8.11) for Mode I, and by (6.10) of the present paper for Modes II and III. Implications of these formulae will be followed up elsewhere.

REFERENCES

Cagniard, L., 1939, *Reflexion et Refraction des Ondes Seismiques Progressives*, Gauthier–Villars, Paris.

Freund, L.B., 1990, *Dynamic Fracture Mechanics*, Cambridge Univ. Press.

Gao, H. and Rice, J.R., 1986, Shear Stress Intensity Factors For a Planar Crack With Slightly Curved Front, *ASME Journal of Applied Mechanics*, **53**, 774–778.

Uflyand, Y.S., 1965, Survey of Articles on the Application of Integral Transforms in the Theory of Elasticity, North Carolina State University, Department of Applied Mathematics, File No. PSR-24/6, Raleigh, N.C.

Willis, J.R., 1973, Self–Similar Problems in Elastodynamics, *Phil. Trans. R. Soc. Lond.* **A274**, 435–491.

Willis, J.R. and Movchan, A.B., 1994, Dynamic Weight Functions For a Moving Crack. I. Mode I Loading, *J. Mech. Phys. Solids*.

APPENDIX

The static limit can be obtained from the full dynamic problem by setting $V = 0$ and allowing ω to tend to zero. The Wiener–Hopf relation (2.1) then reduces to

$$\overline{\boldsymbol{\Sigma}}_{-} = -\frac{\mu}{2(\xi_1^2+\xi_2^2)^{1/2}}\mathbf{R}^t\boldsymbol{\Delta}\mathbf{R}\overline{\mathbf{U}}_{+}, \tag{A.1}$$

where the diagonal matrix $\boldsymbol{\Delta}$ is defined as

$$\boldsymbol{\Delta} = \mathrm{diag}\{(1-\nu)^{-1}, 1\}, \tag{A.2}$$

with ν denoting Poisson's ratio. Comparison with the full dynamic equation (2.7) shows that

$$\boldsymbol{\mathcal{D}} = -\frac{\mu}{2}\boldsymbol{\Delta}(\xi_1^2+\xi_2^2)^{1/2},$$

and, from (2.6),

$$\begin{aligned}\boldsymbol{\mathcal{D}}_{+} &\to (\xi_1 + \mathrm{i}|\xi_2|)^{1/2}\mathbf{I},\\ \boldsymbol{\mathcal{D}}_{-} &\to -\frac{\mu}{2}(\xi_1 - \mathrm{i}|\xi_2|)^{1/2}\boldsymbol{\Delta}.\end{aligned} \tag{A.3}$$

The static limit of the weight function (4.6) is easily calculated, upon noting that

$$\begin{aligned}a_{+} &\to (2\mathrm{i}|\xi_2|)^{1/2}, \quad b_{+} \to (2\mathrm{i}|\xi_2|)^{1/2},\\ a_{-} &\to -\mathrm{i}\frac{(2\mathrm{i}|\xi_2|)^{1/2}}{2(1-\nu)}, \; b_{-} \to -\mathrm{i}\frac{\mu}{2}(2\mathrm{i}|\xi_2|)^{1/2}.\end{aligned} \tag{A.4}$$

Substituting these values into (4.6) gives

$$\overline{\mathbf{U}}_{+} \to \frac{2(2\mathrm{i})^{1/2}}{2-\nu}(\xi_1+\mathrm{i}|\xi_2|)^{-1/2}\begin{pmatrix} 2 - \dfrac{\nu\xi_1}{\xi_1+\mathrm{i}|\xi_2|} & -\dfrac{\nu\xi_2}{\xi_1+\mathrm{i}|\xi_2|} \\ -\dfrac{\nu\xi_2}{\xi_1+\mathrm{i}|\xi_2|} & 2(1-\nu)+\dfrac{\nu\xi_1}{\xi_1+\mathrm{i}|\xi_2|}\end{pmatrix}. \tag{A.5}$$

Inversion of the transforms then yields

$$\mathbf{U}_{+} = \frac{\sqrt{2x_1}H(x_1)}{\pi^{3/2}(x_1^2+x_2^2)}\begin{pmatrix} 1 + \dfrac{2\nu}{2-\nu}\dfrac{1-(x_2/x_1)^2}{1+(x_2/x_1)^2} & \dfrac{4\nu}{2-\nu}\dfrac{x_2/x_1}{1+(x_2/x_1)^2} \\ \dfrac{4\nu}{2-\nu}\dfrac{x_2/x_1}{1+(x_2/x_1)^2} & 1 - \dfrac{2\nu}{2-\nu}\dfrac{1-(x_2/x_1)^2}{1+(x_2/x_1)^2}\end{pmatrix}. \tag{A.6}$$

Allowing for the replacement of x_1 by $-x_1$ because of the present use of notation that permits the use of convolutions, this agrees with the weight function given by equation (5) of Gao and Rice (1986), and attributed by them originally to Uflyand (1965).

Similarly, in the relation (6.9) between $\Delta\mathbf{K}$ and ϕ, the matrix $\overline{\mathbf{Q}}$ reduces to

$$\overline{\mathbf{Q}} = -\frac{|\xi_2|}{2(2-\nu)}\begin{pmatrix} 2-3\nu & 0 \\ 0 & 2+\nu \end{pmatrix} + \frac{\mathrm{i}\xi_2\nu}{2-\nu}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.7}$$

Hence, inverting the transform with respect to ξ_2,

$$\mathbf{Q} = \frac{1}{2\pi(2-\nu)}\begin{pmatrix} 2-3\nu & 0 \\ 0 & 2+\nu \end{pmatrix}\frac{1}{x_2^2} + \frac{\nu}{2-\nu}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\delta'(x_2), \tag{A.8}$$

the generalized function $1/x_2^2$ being interpreted so as to generate Hadamard finite part integrals.

Substitution of (A.8) into (6.10) then gives

$$\begin{aligned}\Delta\mathbf{K}(x_2) = \varepsilon\Bigg\{ & \frac{1}{2\pi(2-\nu)}\begin{pmatrix} 2-3\nu & 0 \\ 0 & 2+\nu \end{pmatrix}\int_{-\infty}^{\infty}\frac{\mathbf{K}(x_2')[\phi(x_2')-\phi(x_2)]}{(x_2-x_2')^2}dx_2' \\ & +\frac{2}{2-\nu}\begin{pmatrix} 0 & -1 \\ 1-\nu & 0 \end{pmatrix}\phi'(x_2)\mathbf{K}(x_2) + \left(\frac{\pi}{2}\right)^{1/2}\phi(x_2)\mathbf{M}(x_2)\Bigg\},\end{aligned} \tag{A.9}$$

This agrees precisely with equation (15) of Gao and Rice (1986), except that the term involving $\mathbf{M}$ was absent from their work, through their choice of local coordinates.