A global condition for quasi-random behavior in a class of conservative systems

B. Buffoni*   E. Séré†

Abstract.

As shown by Devaney, an autonomous Hamiltonian system in dimension 4, with an orbit homoclinic to a saddle-focus equilibrium, admits a chaotic behavior as soon as the homoclinic orbit is the transverse intersection of the stable and unstable manifolds. In this paper we deal with two classes of "saddle-focus" systems: Lagrangian systems defined on a 2-manifold in the presence of a gyroscopic force, and fourth order systems arising in water-wave theory. We first establish, by a standard variational method, the existence of a homoclinic orbit. Then, under a weak "non-degeneracy" condition, we show that it gives rise to an infinite family of multibump homoclinic solutions and that the dynamics is chaotic. Our condition is much easier to check than transversality. For example, it is automatically satisfied for gyroscopic systems on a 2-torus, for topological reasons.

AMS classification: 34C37, 58E05, 58F05.

1 Introduction

Let us consider the following system

$$\begin{cases} x''(t) + 2\alpha y'(t) - x(t) + F_1'(x(t), y(t)) = 0 \\ y''(t) - 2\alpha x'(t) - y(t) + F_2'(x(t), y(t)) = 0 \end{cases} \tag{1}$$

*School of Mathematics, University of Bath, Bath BA27AY, U.K. Supported by a EPSRC research grant (No 85757).

†Courant Institute of Mathematical Sciences, 251 Mercer Street, New York NY 10012, USA. Partially supported by NSF grant DMS-9114456.

where x, y are real valued, $F \in C^5(\mathbb{R}^2, \mathbb{R})$ and $\alpha \in (0,1)$ is a parameter. Writing $z = (x, y)$, we assume F to satisfy

(H1) F is not identically zero,

(H2) $\lim_{|z| \to 0} F(z)/|z|^2 = 0$,

(H3) there exists $p > 2$ such that $(F'(z), z) \geq pF(z) \geq 0$ for all z in $\mathbb{R}^2$,

(H4) if $F(z) > 0$, then $(F''(z)z, z) > (F'(z), z)$.

These conditions are satisfied, for instance, by $F(z) = |z|^4$.

System (1) admits the invariant

$$H(x, y, x', y') = \frac{1}{2}(x'^2 + y'^2) - \frac{1}{2}(x^2 + y^2) + F(x, y).$$

Identifying $(x, y) \in \mathbb{R}^2$ with $z \in \mathbf{C}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with i, the linearization of (1) at the origin is simply $z'' - 2\alpha i z' - z = 0$, whose characteristic equation has $\pm\sqrt{1 - \alpha^2} \pm i\alpha$ as eigenvalues, i.e. 0 is a saddle-focus equilibrium. The fact that the general solution of the linearization oscillates around 0 when tending to 0 will be crucial. Hence the case $\alpha = 0$ is excluded from our discussion; but see the paper [3], where some multiplicity results are discussed for conservative systems near completely integrable ones. The implications of 0 to be a saddle-focus are presented in Appendix A.

The homoclinic solutions of (1) are exactly the critical points of the following functional defined on the Sobolev space $H^1(\mathbb{R}, \mathbb{R}^2)$

$$J(z) = \frac{1}{2}\int_{\mathbb{R}}(|z'|^2 + |z|^2)dt + \alpha\int_{\mathbb{R}}(z, iz')dt - \int_{\mathbb{R}} F(z)dt,$$

where the parentheses $(.,.)$ denote the scalar product in $\mathbb{R}^2$. In the second section, we shall see that J has a mountain-pass structure and we will find a critical point $\hat{z}$ of mountain-pass type. Then we will do a partial Morse reduction using the "broken geodesics" technique (see [13], and [6] for the application to homoclinic problems). The reduced functional $J \circ E_1$ admits an isolated critical point of mountain-pass type, at which the degree of $(J \circ E_1)'$ is -1, *provided $\hat{z}$ is an isolated critical point of J up to time-translations.*

In the third section, we adapt the technique of Bessi [6] to glue many copies of $\hat{z}$ and obtain our first result

Theorem 1 *Under conditions (H1) to (H4), system (1) admits infinitely many homoclinic solutions geometrically distinct. One, denoted by $\hat{z}$, is given by the mountain-pass lemma. If it is isolated in $H^1(\mathbb{R},\mathbb{R}^2)$ up to time-translations (non-degeneracy condition), the others are multibump solutions, i.e. for any $\delta > 0$ small, we can find $\gamma > 0$ such that, for any integers $k \geq 2$ and $n_1, ..., n_{k-1} \geq 1$, there exists a homoclinic solution in the ball of radius δ for the L^∞-norm, centered at*

$$\sum_{i=1}^{k} \hat{z}(. - s_1 - \ldots - s_{i-1}) \, .$$

Here, s_i is some real number such that $\mid s_i - n_i \alpha^{-1} 2\pi - \gamma \mid < \delta$, for $i = 1, \ldots, k-1$.

The γ provided by the proof tends to $+\infty$ when $\delta \to 0$. The proof of Theorem 1 is based on the study of the interaction between bumps. Such an approach is inspired by the theory of critical points at infinity (see [4]). It is also related to the stability problem for solitons in optical fibers (see [11]).

Let us recall that Devaney [20] (see also [34]) proved that an autonomous Hamiltonian system in dimension 4, with a homoclinic orbit to a saddle-focus equilibrium (i.e. the linearized system around the equilibrium has the eigenvalues $\pm\lambda \pm i\omega, \lambda, \omega > 0$), is chaotic provided the homoclinic orbit is the transverse intersection of the stable and unstable manifolds. As a consequence, the system admits infinitely many other homoclinic orbits [5]. The saddle-saddle case (all eigenvalues are real) is much more subtle (see [34]).

For various examples of "saddle-focus" autonomous Hamiltonian systems, multibump solutions have been found by shooting arguments, without checking transversality. This is the case for a class of water-wave problems which have been studied in [8] (see also [12]). We also refer to [30] for a discussion of the three-body problem.

The non-degeneracy condition of Theorem 1 is weaker than transversality. In Section 4, we discuss this condition when F is real analytic. We show that it is satisfied if the global stable and unstable manifolds do not coincide. This is the "global condition" announced in the title. As an application, we treat the case of the fourth order equation already discussed in [8]. This problem is not of the form (1), but it admits a variational formulation which is very similar. A mountain-pass point can be found, and a reduction can be performed, exactly as for (1). We do not give the details and just state a result which improves those of [8].

This paper ends with another application, to the case where the configuration space M is a compact analytic Riemannian 2-manifold. Let us consider the equation

$$- D_t q'(t) + B(q)q' = \text{ grad } V(q), \tag{2}$$

where D_t is the covariant derivative, $V : M \to \mathbb{R}$ is a real analytic potential, grad $V \in T_q M$ is the gradient of V defined with the scalar product at q, and $B(q) \in L(T_q M, T_q M)$ is real analytic and such that $(u, v) \to \beta(u, v) := (B(q)u, v)$ is an exact two-form. We suppose (2) to admit a saddle-focus equilibrium $q_0 \in M$ with $V(q_0) = 0$. Let α be a 1-form such that $d\alpha = \beta$ and $\alpha(q_0) = 0$. The homoclinic solutions of (2) are the stationary points of the functional $J(q) = \int_{\mathbb{R}} L(q(t), q'(t))dt$, where

$$L(q, q') = (1/2)(q', q') + \alpha(q)q' - V(q),$$

defined on an appropriate functional space.

Theorem 2 *If*

1. *q_0 is a saddle-focus equilibrium,*

2. *there exists $\epsilon > 0$ such that $L(q, u) \geq \epsilon(|u|^2 +$ dist $(q, q_0)^2)$ for all $q \in M$ and $u \in T_q M$, dist $(.,.)$ being the geodesic distance,*

3. *$\pi_1(M)$ is not generated as a group by only one element,*

then (2) admits infinitely many homoclinic solutions to q_0 geometrically distinct and its topological entropy is positive.

For a definition of the topological entropy, see Appendix B, where its relation with multibump solutions is also given.

Theorem 2 is to be compared with the results in [25] connecting nonintegrability of natural dynamical systems and the topology of the configuration space. An example of an exact two-form β is given by the Coriolis force on a rotating surface.

In the last few years, many results have been obtained on homoclinic solutions in Hamiltonian systems, by variational methods. The first existence results, in the case of Lagrangian systems, are due to Bolotin (see the survey [26]). For more general Hamiltonian systems, see the later works [15, 24, 33, 21, 14]. In the case of time-periodic Hamiltonians, multiplicity results are also known. In [15], two solutions are found, and in [31], a new variational method is introduced, to prove the existence of infinitely many solutions. This method has been later improved and generalized to elliptic PDEs, in several works (see e.g. [17, 18, 27, 32, 6, 1]). The results obtained in these papers look like Theorem 1, but the translations s_k are integer multiples of the period, and $\hat{z}$ is assumed isolated up to *integer* translations in time. Note that this condition, although weaker than the classical transversality, is not easier to check on practical examples. Note also that this theory gives trivial results in the autonomous case, because the primary homoclinic solution is not isolated (it belongs to a continuous family of solutions, namely the one given by time-translations).

For autonomous systems, there are already some variational results on multiplicity (see [3]) but the solutions obtained are not multibump. For fourth order systems of saddle-focus type, arising in water-wave theory, we already mentioned that multibump solutions have been obtained without checking transversality, by a shooting argument (see [8]). This fact has been the starting point of our work. The idea was to adapt the variational construction of multibump solutions, to the situation of fourth order autonomous systems. The advantage of variational methods is that they apply to much more general situations.

Before beginning the proof of Theorem 1, let us precise that the assumptions (H1-4) could be considerably weakened. We have made rather restrictive assumptions, because we wanted to have a mountain-pass argument that can also be viewed as a minimization under constraint. Such a fact is not essential, but it simplifies the proof, thanks to the use of degree theory (see [6]). It would also be possible to relax the analyticity assumption in Theorem 2. These technicalities, as well as other applications of the method, will be discussed elsewhere.

Notations. For an interval $I \subset \mathbb{R}$, $H^1(I,\mathbb{R}^N)$ denotes the set of absolutely continuous curves $\gamma : I \to \mathbb{R}^N$ such that

$$\|\gamma\|^2_{H^1(I,\mathbb{R}^N)} := \int_I \left(|\gamma(t)|^2 + |\gamma'(t)|^2\right) dt < \infty.$$

The duality between a Banach space and its dual is denoted by $\langle .,. \rangle$. Finally we use $|.|$ and $(.,.)$ for the norm and scalar product in $\mathbb{R}^N$ and on a Riemannian manifold.

2 The primary orbit

We set $Cr = \{z \in H^1(\mathbb{R},\mathbb{R}^2) : J'(z) = 0,\ z \neq 0\}$. It is easy to check that $c := \inf\{J(z) : z \in Cr\} > 0$. One of the difficulties arising when dealing with homoclinic solutions is a lack of compactness. Fortunately the following concentration-compactness result is available [15, 28]. We recall that $\{u_n\} \subset H^1(\mathbb{R},\mathbb{R}^2)$ is a Palais-Smale sequence if $J'(u_n) \to 0$ and $J(u_n) \to a$ for some $a \in \mathbb{R}$.

Lemma 3 *Let $\{u_n\}$ be a Palais-Smale sequence with $J(u_n) \to a > 0$. Then there exist $m > 0$, a subsequence $\{u_{n_p}\}$ and $u^1, \ldots, u^m$ in Cr, not necessarily distinct, such that*

$$\lim_{p\to\infty} \|u_{n_p} - \sum_{i=1}^{m} u^i(. - k_p^i)\|_{H^1(\mathbb{R},\mathbb{R}^2)} = 0$$

for some $k_p^i \in \mathbb{R}$ satisfying $\lim_{p\to\infty} |k_p^i - k_p^j| = \infty$ if $i \neq j$. Moreover $a = \sum_{i=1}^m J(u^i)$.

Corollary 4 *Let $\{u_n\}$ be a Palais-Smale sequence with $J(u_n) \to a \in (0, 2c)$. Then there exist a subsequence $\{u_{n_p}\}$ and $\hat{u}$ in Cr such that $J(\hat{u}) = a$ and*

$$\lim_{p\to\infty} \|u_{n_p}(. + k_p) - \hat{u}\|_{H^1(\mathbb{R},\mathbb{R}^2)} = 0$$

for some $\{k_p\} \subset \mathbb{R}$.

Lemma 5 *There exists $\hat{z} \in Cr$ such that $J(\hat{z}) = c$.*

PROOF. First we show that J has a mountain-pass structure, namely that $0 \in H^1(\mathbb{R},\mathbb{R}^2)$ is a strict local minimum of J, and that $J(z_0) < 0$ for some $z_0 \in H^1(\mathbb{R},\mathbb{R}^2)$. Clearly

$$\left|\int_{\mathbb{R}}(z, iz')dt\right| \leq \|z\|_{L^2}\|z'\|_{L^2} \leq \frac{1}{2}\|z\|_{L^2}^2 + \frac{1}{2}\|z'\|_{L^2}^2$$

and we deduce thanks to $\alpha \in (0,1)$ and (H2) that 0 is a strict local minimum of J. Now we choose $z(t)$ such that $\int_{\mathbb{R}} F(z)dt > 0$ (it is possible by (H1)). Condition (H3) implies that $s^{-p}\int_{\mathbb{R}} F(sz)dt$ is increasing in $s > 0$ and therefore $\int_{\mathbb{R}} F(sz)dt$ increases more than the quadratic part of J. Hence we can set $z_0 = sz$ for some $s > 0$ large enough.

Consider the number $a = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s))$, where

$$\Gamma = \{h \in C([0,1], H^1(\mathbb{R},\mathbb{R}^2)) : h(0) = 0, (J \circ h)(1) < 0\}.$$

As 0 is a strict local minimum of J, we get $a > 0$. The mountain-pass lemma as stated in [7] gives a Palais-Smale sequence $\{u_n\}$ with $J(u_n) \to a$. By Lemma 3, $Cr \neq \emptyset$ and $a \geq c$.

Let $z \neq 0$ satisfies $\langle J'(z), z\rangle = 0$. By (H4),

$$\langle J''(z)z, z\rangle = \langle J'(z), z\rangle - \int_{\mathbb{R}}\{F''(z)(z,z) - F'(z)z\}dt < 0 \tag{3}$$

from which we deduce that $J(z)$ is the (unique) strict maximum of $J(sz)$, $s > 0$. As $J(sz) < 0$ for s large, we deduce that $J(z) \geq a$. As z may be arbitrarily chosen in Cr, we get $a = c$. ■

Applying (3) to $\hat{z}$, we conclude that the index of J at $\hat{z}$ is at least 1. Moreover $\hat{z}$ is of mountain-pass type as defined in [23], i.e. the set $\{z \in H^1(\mathbb{R},\mathbb{R}^2) : J(z) < J(\hat{z})\}$ is disconnected and non-empty in all small neighbourhoods of $\hat{z}$. This results from the fact that $\hat{z}$ minimizes J on the one-codimensional C^1 manifold $\{z \in H^1(\mathbb{R},\mathbb{R}^2) : \langle J'(z), z\rangle = 0\}$.

For $v \in \mathbb{R}^2$ small enough, $\{(z, z') : z = v\}$ is transverse to W^s_{loc} and W^u_{loc} (the stable and unstable manifolds: see Appendix A). Hence there exist solutions $\rho_s(.,v)$ and $\rho_u(.,v)$ of (1) such that $\rho_{s/u}(0,v) = v$ and $(v, \partial_1\rho_{s/u}(0,v)) \in W^{s/u}_{loc}$. In addition they are C^1 with respect to their variables, as well as $\partial_1\rho_s$ and $\partial_1\rho_u$.

Now we perform a reduction. After a time-translation if necessary, we may suppose that $\hat{z}(0) \neq 0$ and $\hat{z}'(0) \not\parallel \hat{z}(0)$ (here we already use the oscillating character of the general solution of $z'' - 2\alpha i z' - z = 0$). We choose $T > 0$ such that $(\hat{z}(-T), \hat{z}'(-T)) \in W^u_{loc}$ and $(\hat{z}(T), \hat{z}'(T)) \in W^s_{loc}$. We define X as the space of $z \in H^1([-T,T],\mathbb{R}^2)$ such that $z(0) \parallel \hat{z}(0)$. We extend elements of X, which are defined only on $[-T,T]$, to all $\mathbb{R}$ as follows. For $z \in X$ near $\hat{z}$, we set

$$(E_1 z)(t) = \begin{cases} \rho_u(t+T, z(-T)) & \text{if } t \leq -T \\ z(t) & \text{if } -T \leq t \leq T \\ \rho_s(t-T, z(T)) & \text{if } t \geq T \end{cases}$$

We find

$$\begin{aligned} \langle (J \circ E_1)'(z), u \rangle &= \langle J'(E_1 z), E_1'(z) u \rangle \\ &= \int_{-T}^{T} \{(z', u') + (z + 2\alpha i z' - F'(z), u)\} dt \\ &+ (v_\infty^{arr}(z), u(-T)) - (v_\infty^{dep}(z), u(T)) \\ &= \langle J'(E_1 z), E_1 u \rangle \end{aligned}$$

where we have set

$$v_\infty^{arr}(z) = \partial_1 \rho_u(0, z(-T)) \text{ and } v_\infty^{dep}(z) = \partial_1 \rho_s(0, z(T)).$$

Hence if $z \in X$ is a critical point of $J \circ E_1$, then z is solution of (1) on $]-T, 0[$ and $]0, T[$, $(z'(t), \hat{z}(0))$ is continuous at $t = 0$, $z'(T) = v_\infty^{dep}(z)$ and $z'(-T) = v_\infty^{arr}(z)$. Therefore $(E_1 z)'$ is continuous at $-T$ and T, and $E_1 z$ is a solution of (1) on $\mathbb{R}^*$ with $((E_1 z)'(t), \hat{z}(0))$ continuous at $t = 0$. Note that $H(E_1 z, (E_1 z)')$ is identically zero and hence $|(E_1 z)'(t)|$ is continuous at $t = 0$. Supposing $\|z - \hat{z}\|_{H^1([-T,T])} \leq \mu$ with μ sufficiently small, we deduce that $E_1 z$ is a solution of (1).

In the remaining of this section and in the next one we will suppose $\hat{z}$ to be an isolated solution in $H^1(\mathbb{R},\mathbb{R}^2)$ of (1) up to time-translations. It immediately follows that $\hat{z}|_{[-T,T]}$ is an isolated critical point of $J \circ E_1$. Moreover $\hat{z}|_{[-T,T]}$ minimizes $J \circ E_1$ on $\{z : \langle J'(E_1 z), E_1 z \rangle = 0\}$, which is equal to $\{z : \langle (J \circ E_1)'(z), z \rangle = 0\}$, and for $z = \hat{z}|_{[-T,T]}$

$$\langle (J \circ E_1)''(z) z, z \rangle = \int_{-T}^{T} \{(z', z') + (z + 2\alpha i z', z) - (F''(z) z, z)\} dt$$

$$
\begin{aligned}
&+ \quad (\partial_{12}\rho_u(0, z(-T))z(-T), z(-T)) - (\partial_{12}\rho_s(0, z(T))z(T), z(T)) \\
&\to \quad \int_{\mathbb{R}} \{(\hat{z}', \hat{z}') + (\hat{z} + 2\alpha i \hat{z}', \hat{z}) - (F''(\hat{z})\hat{z}, \hat{z})\} dt \\
&= \quad \langle J''(\hat{z})\hat{z}, \hat{z}\rangle < 0
\end{aligned}
$$

as $T \to \infty$. Choosing T large enough, the index of $J \circ E_1$ at $\hat{z}|_{[-T,T]}$ is at least 1 and, as above, $\hat{z}|_{[-T,T]}$ is a critical point of $J \circ E_1$ of mountain-pass type. If we endow X with the scalar product

$$
\langle u, v\rangle = \int_{-T}^{T} \{(u', v') + \alpha(u, iv') + \alpha(v, iu') + (u, v)\} dt
$$

and identify X' with X, then $(J \circ E_1)'$ has the form identity+compact. Therefore $\deg((J \circ E_1)', \hat{z}|_{[-T,T]}, 0) = -1$ [23].

3 Multibump solutions

We need the following result:

Lemma 6 *There exist $\delta_0, \nu > 0$ such that if $0 < \delta < \delta_0$ and $|\hat{z}(-T)|, |\hat{z}(T)| < \delta$, then for all $v, w \in \mathbb{R}^2$ and $s > 0$ satisfying $|v - \hat{z}(T)|, |w - \hat{z}(-T)|, 1/s \leq \epsilon < \delta$ for some ϵ dependent only on δ, $\hat{z}(-T)$ and $\hat{z}(T)$, we can find a solution $\sigma(., v, w, s)$ of (1) with*

1. *$\sigma(0, v, w, s) = v$ and $\sigma(s, v, w, s) = w$,*

2. *$|\sigma(t, v, w, s)|, |\partial_1\sigma(t, v, w, s)| \leq O(\delta)$ for $0 \leq t \leq s$,*

3.
$$
\partial_1\sigma(0, v, w, s) - \partial_1\rho_s(0, v) \to 0
$$
and
$$
\partial_1\sigma(s, v, w, s) - \partial_1\rho_u(0, w) \to 0
$$
uniformly in v and w as $s \to \infty$,

4. *there exists $\tilde{\gamma} \in [0, 2\pi\alpha^{-1})$ dependent only on $\hat{z}(-T)$ and $\hat{z}(T)$ such that the sequence $s_n := n\alpha^{-1}2\pi + \tilde{\gamma}$ satisfies*

$$
H(\sigma(., v, w, s_n - \nu\delta), \partial_1\sigma(., v, w, s_n - \nu\delta)) > 0,
$$

$$H(\sigma(.,v,w,s_n+\nu\delta),\partial_1\sigma(.,v,w,s_n+\nu\delta))<0$$

for $s_n \geq 1/\epsilon$,

5. σ, $\partial_1\sigma$ *are* C^1 *with respect to all their variables.*

PROOF. For v and w small, the sets $M_1 = \{(z,z') : z = v\} \subset \mathbb{R}^4$ and $M_2 = \{(z,z') : z = w\} \subset \mathbb{R}^4$ are transverse to the local stable and unstable manifolds respectively. It is sufficient to apply the results of Appendix A, with $\lambda = \sqrt{1-\alpha^2}$ and $\omega = \alpha$. Lemma 12, with $\kappa = (v,w)$, gives the existence of σ, and the fourth assertion is a consequence of Lemma 11. ∎

Take $T > 0$ such that $|\hat{z}(-T)|, |\hat{z}(T)| < \delta$ with $\delta < \delta_0$. For $\mu \leq \epsilon$, we set $U_\mu = \{z \in X : \|z-\hat{z}\|_{H^1([-T,T])} \leq \mu\}$. Given $k \geq 2$, $z_1,\dots,z_k \in U_\epsilon$ and $t_1,\dots,t_{k-1} \geq 1/\epsilon$, we define $z = E_k(z_1,\dots,z_k,t_1,\dots,t_{k-1}) \in H^1(\mathbb{R},\mathbb{R}^2)$ by

1. $z(t) = \rho_u(t, z_1(-T))$ for $t \leq 0$,
2. $z(2(j-1)T + \sum_{l=1}^{j-1} t_l + t) = z_j(t-T)$ for $t \in [0,2T]$ and $j = 1,\dots,k$,
3. $z(2jT + \sum_{l=1}^{j-1} t_l + t) = \sigma(t, z_j(T), z_{j+1}(-T), t_j)$ for $t \in [0,t_j]$ and $j = 1,\dots,k-1$,
4. $z(2kT + \sum_{l=1}^{k-1} t_l + t) = \rho_s(t, z_k(T))$ for $t \geq 0$.

It is a function with k bumps, the time-interval between the jth and $(j+1)$th bumps being of length t_j. The computation of

$$G_k := (J\circ E_k)' \in C(U_\mu^k \times]1/\epsilon,+\infty[^{k-1},\ (X')^k \times \mathbb{R}^{k-1})$$

gives, for $\mu < \epsilon$,

$$\begin{aligned}
&\langle G_k(z_1,\dots,z_k,t_1,\dots,t_{k-1}),(u_1,\dots,u_k,\tau_1,\dots,\tau_{k-1})\rangle\\
= \ & \sum_{j=1}^{k}\langle (J\circ E_1)'(z_j),u_j\rangle + \sum_{j=1}^{k-1}\Big(v_\infty^{dep}(z_j) - v^{dep}(z_j,z_{j+1},t_j),u_j(T)\Big)\\
&-\sum_{j=2}^{k}(v_\infty^{arr}(z_j) - v^{arr}(z_{j-1},z_j,t_{j-1}),u_j(-T))\\
&-\sum_{j=1}^{k-1}\tau_j H\left(\sigma(.,z_j(T),z_{j+1}(-T),t_j),\ \partial_1\sigma(.,z_j(T),z_{j+1}(-T),t_j)\right)
\end{aligned}$$

for all $u_1, \dots, u_k \in X$ and $\tau_1, \dots, \tau_{k-1} \in \mathbb{R}$. For z_1, z_2 in U_ϵ and $s \geq 1/\epsilon$, we have set

$$v^{dep}(z_1, z_2, s) = \partial_1 \sigma(0, z_1(T), z_2(-T), s)$$

and

$$v^{arr}(z_1, z_2, s) = \partial_1 \sigma(s, z_1(T), z_2(-T), s).$$

For μ small (independently of k), $(z_1, \dots, z_k, t_1, \dots, t_{k-1})$ is a solution of $G_k = 0$ exactly when $z = E(z_1, \dots, z_k, t_1, \dots, t_{k-1})$ is a critical point of J. Here again the invariance of H is used to prove the continuity of z'.

We can now construct a homoclinic solution of (1) with $k \geq 2$ bumps. Taking μ small enough, $\hat{z}|_{[-T,T]}$ is the unique critical point of $J \circ E_1$ in U_μ and $\inf \|(J \circ E_1)'(z)\| > 0$ on the boundary of this set. Moreover taking the t_j sufficiently large, we may suppose the terms involving $v^{dep}_\infty - v^{dep}$ and $v^{arr}_\infty - v^{arr}$ to be as small as necessary. Hence setting

$$V_{k-1,\vec{n},\delta} = \bigotimes_{j=1}^{k-1} [s_{n_j} - \nu\delta, s_{n_j} + \nu\delta],$$

where $\vec{n}$ denotes the vector $(n_1, \dots, n_{k-1})$, there exists $N > 0$ (independent of k) such that for $n_j > N$, $j = 1, \dots, k-1$, the application G_k is homotopic in the sense of degree theory to $\tilde{G}_k$ given by

$$\begin{aligned} &\langle \tilde{G}_k(z_1, \dots, z_k, t_1, \dots, t_{k-1}), (u_1, \dots, u_k, \tau_1, \dots, \tau_{k-1}) \rangle \\ = \ &\sum_{j=1}^{k} \langle (J \circ E_1)'(z_j), u_j \rangle - \sum_{j=1}^{k-1} \tau_j H\left(\sigma(., \hat{z}(T), \hat{z}(-T), t_j),\ \partial_1 \sigma(., \hat{z}(T), \hat{z}(-T), t_j)\right). \end{aligned}$$

Therefore

$$\deg(G_k, U_\mu^k \times V_{k-1,\vec{n},\delta}, 0) = \deg(\tilde{G}_k, U_\mu^k \times V_{k-1,\vec{n},\delta}, 0) = (-1)^k$$

and hence G_k takes the value 0 at some $(z_1, \dots, z_k, t_1, \dots, t_{k-1}) \in U_\mu^k \times V_{k-1,\vec{n},\delta}$.

We obtain in this way a countable family of homoclinic solutions parametrized by k and $\vec{n}$. This proves Theorem 1.

4 The real analytic case

Our aim is to prove the following

Theorem 7 *If F is real analytic, then either $\hat{z}$ is an isolated homoclinic solution in $H^1(\mathbb{R},\mathbb{R}^2)$ up to time-translations or the global stable and unstable manifolds of 0 coincide, the time for a homoclinic orbit to go from the local unstable manifold to the local stable manifold being uniformly bounded.*

Proof. We can of course regard (1) as a four-dimensional Hamiltonian system of the form (6) in Appendix A with $U = \mathbb{R}^4$ by setting $z = (x_1, x_2)$ and $z' = (x_3, x_4)$. After an appropriate local analytical change of coordinates (c.f. Appendix A), the equations are given in some ball $B(0, \delta)$ by

$$x' = \begin{pmatrix} -\lambda & -\omega & 0 & 0 \\ \omega & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -\omega \\ 0 & 0 & \omega & \lambda \end{pmatrix} x + f(x), \tag{4}$$

where f is real analytic, $f(0) = 0$, $f'(0) = 0$, $f_u(x_s, 0) \equiv 0$, and $f_s(0, x_u) \equiv 0$. We use the notation $x = (x_s, x_u)$ with $x_s = (x_1, x_2)$ and $x_u = (x_3, x_4)$. The local unstable manifold is simply given by $x_s = 0$ and, writing $x_3 = r_u \cos\theta_u$ and $x_4 = r_u \sin\theta_u$, the dynamics on W^u_{loc} is described by $r'_u/r_u = \lambda + O(\delta)$ and $\theta'_u = \omega + O(\delta)$ (see Appendix A). For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we denote by z_θ the solution of (1) corresponding to $\xi(t, (0, 0, (1/2)\delta\cos\theta, (1/2)\delta\sin\theta))$, $\xi(t, x)$ being the flow of (4). Reciprocally, for any non-trivial homoclinic solution $\tilde{z}$ of (1), we denote the corresponding θ by $\Theta(\tilde{z})$, i.e. $z_{\Theta(\tilde{z})} = \tilde{z}$.

Let $\mathcal{C} \subset \mathbb{R}/2\pi\mathbb{Z}$ be the image by Θ of the set of homoclinic solutions which are critical points of J for the critical value c, but which are not isolated in $H^1(\mathbb{R},\mathbb{R}^2)$, up to time-translations, from all the other homoclinic solutions. We will show that $\mathcal{C}$ is open and closed. Hence either $\mathcal{C} = \mathbb{R}/2\pi\mathbb{Z}$ or $\mathcal{C} = \emptyset$. In the first case the stable and unstable manifolds coincide and the time for a homoclinic orbit to go from W^u_{loc} to W^s_{loc} is uniformly bounded (by the concentration-compactness lemma). In the second case $\hat{z}$ is an isolated critical point of J in $H^1(\mathbb{R},\mathbb{R}^2)$ up to time-translations.

First we prove that $\mathcal{C}$ is open, using the analyticity. Suppose that $\theta \in \mathcal{C}$, i.e. $J'(z_\theta) = 0$, $J(z_\theta) = c$ and there exists $\{\theta_n\} \subset \mathbb{R}/2\pi\mathbb{Z} - \{\theta\}$ such that $J'(z_{\theta_n}) = 0$ and $\lim_{n\to\infty} \|z_{\theta_n} - z_\theta\|_{H^1} = 0$. As z_{θ_n} and z_θ are homoclinic solutions, we have $\lim_{n\to\infty}\|z_{\theta_n}-z_\theta\|_{H^2}=0$ and $\lim_{n\to\infty}\|z_{\theta_n}-z_\theta\|_{W^{1,\infty}}=0$. Hence there exist $T, n_0>0$ large such that $(z_\theta(t), z'_\theta(t))$ and $(z_{\theta_n}(t), z'_{\theta_n}(t))$ remain in a small neighbourhood of $0\in\mathbb{R}^4$ for $t\geq T$ and $n>n_0$. By the stable manifold theorem, $(z_\theta(T), z'_\theta(T))$ and $(z_{\theta_n}(T), z'_{\theta_n}(T))$ are in W^s_{loc}, that is $(z_\theta(0), z'_\theta(0))$ and $(z_{\theta_n}(0), z'_{\theta_n}(0))$ are in the embedded analytical manifold $W^s_T=\xi(-T, W^s_{loc})$ for $n>n_0$. We deduce that

$$W^s_T\cap W^u_{loc}\supset\{\xi(t,(0,0,(1/2)\delta\cos\theta_n,(1/2)\delta\sin\theta_n)) : t\geq 0 \text{ small}, n>n_0\}.$$

W^s_T and W^u_{loc} being real analytic two-dimensional manifolds, $W^s_T\cap W^u_{loc}$ is also a two-dimensional manifold. Hence $\mathcal{C}$ contains a neighbourhood of θ.

To prove that $\mathcal{C}$ is closed, we rely on the concentration-compactness lemma. Given $\{\theta_n\}\subset\mathcal{C}$ with $\theta_n\to\theta$ in $\mathbb{R}/2\pi\mathbb{Z}$, we must show that $\theta\in\mathcal{C}$. The sequence $\{z_{\theta_n}\}$ satisfies $J'(z_{\theta_n})=0$ and $J(z_{\theta_n})=c$. Hence

$$\|z_{\theta_n}-z_{\hat\theta}(.-t_n)\|_{H^1}\to 0$$

for some $\hat\theta\in\mathbb{R}/2\pi\mathbb{Z}$ and some sequence $\{t_n\}\subset\mathbb{R}$, where $z_{\hat\theta}$ is a homoclinic solution corresponding to the critical value c (see Corollary 4). As $\{z_{\theta_n}\}$ is convergent in $H^1(\mathbb{R}_-,\mathbb{R}^2)$, we necessarily have $t_n\to 0$ and $\hat\theta=\theta$. Thus $\theta\in\mathcal{C}$. ∎

In the real analytic case, we thus have a simple way to check the non-degeneracy condition. As an example, consider the following fourth order equation

$$u^{iv}+Pu''+u-u^2=0,\ u\in C^4(\mathbb{R},\mathbb{R}),\tag{5}$$

where $P\in(-2,2)$ is a parameter. In [9] it is shown that (5) is the Euler-Lagrange equation of a functional defined on $H^2(\mathbb{R},\mathbb{R})$ that has a nice mountain-pass structure. Hence there exists a non-trivial homoclinic solution to 0. Moreover 0 is a saddle-focus of (5) seen as a Hamiltonian system. Applying our method, we can deduce the existence of infinitely many homoclinic solutions. In [8], a sequence of solutions $\{u_n\}_{n\geq 2}$ is constructed by a shooting argument for $P\in(-2,0]$, u_n being a homoclinic solution with n bumps without any oscillations between two consecutive bumps. Moreover the time for u_n to go from W^s_{loc} to W^u_{loc} increases as $O(n)$. Hence we can construct a whole family of multibump homoclinic solutions for $P\in(-2,0]$ which shows that the dynamic is chaotic. The relevance of equation (5) to the theory of water waves is explained in [10]. A similar equation is also used to model solitary waves on suspension bridges (see [29]).

5 On compact surfaces

We consider equation (2) with the same hypotheses as in Theorem 2. For a while, we may work under the assumption of C^∞ regularity. The analyticity will be used only when discussing the non-degeneracy of the primary orbit.

By the Nash embedding theorem, M can be embedded in $\mathbb{R}^N$ for large N, and the Riemannian structure on M is induced from the standard Euclidian metric in $\mathbb{R}^N$. For simplicity, we suppose q_0 to correspond to $0 \in \mathbb{R}^N$ by this embedding. The Sobolev space $\Omega := H^1(\mathbb{R}, M) \subset H^1(\mathbb{R}, \mathbb{R}^N)$ is a Hilbert manifold and its tangent space at $q \in \Omega$ is given by

$$T_q\Omega = \{v \in H^1(\mathbb{R}, \mathbb{R}^N) : v(t) \in T_{q(t)}M \text{ for all } t \in \mathbb{R}\}.$$

On $T_q\Omega$, we have the norm $\|v\|^2_{T_q\Omega} = \int_{\mathbb{R}}(|D_t v|^2 + |v|^2)dt$. It is easily seen that J is smooth in Ω and that $J(q) \geq \tilde{\epsilon}\|q\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}$ for some $\tilde{\epsilon} > 0$ and all $q \in \Omega$. Note that $J'(q) \in (T_q\Omega)'$ for $q \in \Omega$. We denote by Cr the set of non-trivial homoclinic solutions and we have

$$\inf\{\|q\|_{H^1(\mathbb{R},\mathbb{R}^N)} : q \in Cr\} > 0\ .$$

We now state the concentration-compactness lemma in this setting, following [14, 22], and refer to these articles for further details. The main new difficulty is to define the "sum" of two functions in Ω. Let $\phi : U \subset \mathbb{R}^2 \to M \cap B(0, \epsilon_0)$ be a local map with $\phi(0) = 0$ and let $f : \mathbb{R} \to [0, 1]$ be a smooth function such that $f(]-\infty, 0]) = 1$ and $f([1, +\infty[) = 0$. Given $q \in \Omega$, there is $R(q)$ such that $|q| < \epsilon_0$ for any $t \in \mathbb{R} \setminus [-R(q), R(q)]$. So, for any $R \geq R(q)$, we can define $q_R(t) = q(t)$ for $|t| \leq R$, and $q_R(t) = \phi(f(|t| - R)\phi^{-1}(q(t)))$ for $|t| \geq R$. Now, given m homoclinic solutions $q^1, \ldots, q^m$ and m integers $p^1, \ldots, p^m$ such that

$$\forall i : p^{i+1} - p^i \geq 2\max(R(q^i), R(q^{i+1})) + 2,$$

we define the formal sum of bumps $\bar{p} \star \bar{q}$ as the function satisfying

$$\bar{p} \star \bar{q} = q^i_{R_i}(t - p^i) \text{ for } t \in \left[\frac{p^{i-1} + p^i}{2}, \frac{p^i + p^{i+1}}{2}\right]$$

with $p_0 = -\infty$, $p_{m+1} = +\infty$, and

$$R_i = \min\left(\frac{p^i - p^{i-1}}{2}, \frac{p^{i+1} - p^i}{2}\right) - 1.$$

Lemma 8 *Let $\{q_n\}$ be a sequence so that $J(q_n) \to a > 0$ and $J'(q_n) \to 0$. Then there exist an integer m, a m-uple $\bar{q} = (q^1, \ldots, q^m) \in (Cr)^m$ and a sequence $\bar{p}_n = (p_n^1, \ldots, p_n^m) \in \mathbb{R}^m$ satisfying*

$$\forall i : p_n^{i+1} - p_n^i \to +\infty$$

as $n \to \infty$, and such that, for some subsequence $\{q_{\phi(n)}\}$ of $\{q_n\}$,

$$\|q_{\phi(n)} - \bar{p}_n \star \bar{q}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \to 0.$$

Moreover $a = \sum_{i=1}^m J(q^i)$.

Using the hypothesis that $\pi_1(M)$ is not generated by only one of its elements, we obtain the well known

Corollary 9 *There exist at least two non-trivial homoclinic solutions $\hat{q}$ and $\tilde{q}$ homotopically distinct. They minimize J on their respective classes of homotopy.*

Proof. Define

$$\hat{c} = \inf\{J(q) : q \in \Omega \text{ is not homotopically trivial}\} > 0$$

and take a minimizing sequence $\{q_n\}$. By the above lemma, as $n \to \infty$, $q_{\phi(n)}$ will be near $\bar{p_n} \star \bar{q}$, $\bar{q} = (q^1, \ldots, q^m)$. Hence at least one of these q^i is not homotopically trivial and we take it as $\hat{q}$. We find $\tilde{q}$ by a similar argument, considering

$$\tilde{c} = \inf\{J(q) : q \in \Omega \text{ and } [q] \neq [\hat{q}]^k \text{ for all } k \in \mathbb{Z}\} > 0,$$

where $[q]$ is the homotopy class of $q \in \Omega$. ■

The reduction of the problem around $\hat{q}$ is similar to what has been done for equation (1). For convenience, we perform in this case a complete Morse reduction to a finite dimensional space. We suppose $\hat{q}$ to be isolated in $H^1(\mathbb{R},\mathbb{R}^N)$ up to time-translations. Let $L \in U$ be a stright line containing $0 \in \mathbb{R}^2$ in its interior. We may suppose that $\hat{q}(0) \in \phi(L)$ with transverse intersection. We choose $t_{-k} < \ldots < t_0 = 0 < \ldots < t_k$ such that $\hat{q}(t_{-k}) \in W^u_{loc}$, $\hat{q}(t_k) \in W^s_{loc}$, there are neighbourhoods $N_i \subset M$ of $\hat{q}(t_i)$ such that for all $x \in N_{i-1}$, $y \in N_i$, there exists a unique solution $S_i(., x, y)$ of (2) near $\hat{q}|_{[t_{i-1},t_i]}$

such that $S_i(t_{i-1},x,y)=x$ and $S_i(t_i,x,y)=y$; moreover S_i and $\partial_1 S_i$ are C^1 with respect to their variables ($i=-k+1,\ldots,k$). For $x_i\in N_i$, $-k\le i\le k$, with $x_0\in\phi(L)$, we set

$$E_1(x_{-k},\ldots,x_k)(t)=\begin{cases}\rho_u(t-t_{-k},x_0), & t\le t_{-k}\\ S_i(t,x_{i-1},x_i), & t_{i-1}\le t\le t_i,\ -k+1\le i\le k\\ \rho_s(t-t_k,x_k), & t\ge t_k\end{cases},$$

where ρ_u and ρ_s are defined in a similar way as in Section 2. If the N_i are chosen small enough, then $(x_{-k},\ldots,x_k)$ is a critical point of $J\circ E_1$ exactly when $E_1(x_{-k},\ldots,x_k)$ is a critical point of J. Moreover it is clear that $J\circ E_1$ is minimum at $\hat{x}_i=\hat{q}(t_i)$ and that [2]

$$\deg((J\circ E_1)',(\hat{x}_{-k},\ldots,\hat{x}_k),0)=1.$$

The gluing procedure to find multibump homoclinics is now the same as the one presented in Section 3. Under the analyticity assumption, we obtain as in Section 4 the following alternative. Either $\hat{q}$ is isolated in $H^1(\mathbb{R},\mathbb{R}^N)$ up to time-translations or the stable and unstable manifolds of q_0 coincide, the time needed for a homoclinic solution to go from W^u_{loc} to W^s_{loc} being uniformly bounded. In the second case all non-trivial homoclinics belong to the same homotopy class. As Corollary 9 gives us two non-trivial homoclinic solutions homotopically distinct, we deduce that $\hat{q}$ is isolated up to time-translations.

6 Appendix A: dynamics near the equilibrium

We deal with Hamiltonian systems that have the following form near an equilibrium, say 0:

$$Jx'=\nabla H(x),\ x\in U\subset\mathbb{R}^4, \tag{6}$$

where $J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, U is an open set, $0\in U$ and $H\in C^5(U,\mathbb{R})$. We suppose that $H(0)=0$, $\nabla H(0)=0$ and the eigenvalues of $J^{-1}H''(0)$ are $\pm\lambda\pm i\omega$, $\lambda,\omega>0$. We denote the flow corresponding to (6) by $\xi(t,x)$:

$$\begin{cases}J\partial_t\xi(t,x)=\nabla H(\xi(t,x)),\\ \xi(0,x)=x.\end{cases}$$

For each $x \in U$, $\xi(t,x)$ is defined on a maximal open interval $D(x)$ containing $t=0$. The global stable and unstable manifolds are defined by

$$W^s_{glob} = \{x \in U : \mathbb{R}_+ \subset D(x) \text{ and } \lim_{t\to+\infty} \xi(t,x) = 0\}$$

and

$$W^u_{glob} = \{x \in U : \mathbb{R}_- \subset D(x) \text{ and } \lim_{t\to-\infty} \xi(t,x) = 0\}.$$

The linear stable (resp. unstable) space, denoted by L^s (resp. L^u) is the global stable (resp. unstable) manifold corresponding to the linearization of (6): $Jx' = H''(0)x$.

Theorem 10 *There exists η_s such that if $\mathbb{R}_+ \subset D(x)$ and $\limsup_{t\to+\infty} |\xi(t,x)| < \eta_s$, then $x \in W^s_{glob}$. Moreover*

$$W^s_{loc} = \{x \in W^s_{glob} : \ |\xi(x,t)| < \eta_s \textit{ for all } t \geq 0\}$$

is an embedded two-dimensional manifold of class C^4 tangent to L^s at 0. It is called the local stable manifold. If H is real analytic, so is W^s_{loc}.

Proof. This version of the "stable manifold theorem" may be found in [16]. ■

Note that $W^s_T := \{x \in U : \xi(T,x) \in W^s_{loc}\}$, $T > 0$, is also an embedded two-dimensional manifold of class C^4 and $W^s_{glob} = \cup_{T>0} W^s_T$. The same results hold for the unstable manifold:

$$W^u_{loc} = \{x \in U : \mathbb{R}_- \subset D(x) \text{ and } |\xi(x,t)| < \eta_u \text{ for all } t \leq 0\}$$

for some $\eta_u > 0$, $W^u_T := \{x \in U : \xi(-T,x) \in W^u_{loc}\}$, $T > 0$, are embedded two-dimensional manifolds of class C^4 and $W^u_{glob} = \cup_{T>0} W^u_T$.

Doing a linear transformation, we may suppose that $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$,

$$J^{-1}H''(0) = \begin{pmatrix} -\lambda & -\omega & 0 & 0 \\ \omega & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -\omega \\ 0 & 0 & \omega & \lambda \end{pmatrix}$$

and

$$H(x) = \lambda(x_1x_3 + x_2x_4) + \omega(x_2x_3 - x_1x_4) + O(|x|^3). \tag{7}$$

Clearly $L^s = \{x \in \mathbb{R}^4 : x_3 = x_4 = 0\}$ and $L^u = \{x \in \mathbb{R}^4 : x_1 = x_2 = 0\}$. We adopt the following notations. For $x = (x_1, \ldots, x_4) \in \mathbb{R}^4$, $x_s = (x_1, x_2)$, $x_u = (x_3, x_4)$, $B_2(\delta) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < \delta\}$ and $B_4(\delta) = B_2(\delta) \times B_2(\delta)$.

Near 0 we can describe W^s_{loc} and W^u_{loc} as graphs of functions

$$W^s_{loc} = \{x \text{ small } : x_u = h_u(x_s)\},\ W^u_{loc} = \{x \text{ small } : x_s = h_s(x_u)\},$$

with $h_{u/s}(0) = 0$ and $h'_{u/s}(0) = 0$. This allows one to perform a further change of coordinates:

$$(\hat{x}_s, \hat{x}_u) = (x_s - h_s(x_u), x_u - h_u(x_s)).$$

We now have $W^{s/u}_{loc} = \{\hat{x} \text{ small } : \hat{x}_{u/s} = 0\}$ and the linearization of this transformation at 0 is the identity. Forgetting the hats, we obtain in a neighbourhood of the origin

$$x' = \begin{pmatrix} -\lambda & -\omega & 0 & 0 \\ \omega & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -\omega \\ 0 & 0 & \omega & \lambda \end{pmatrix} x + f(x), \tag{8}$$

where f is C^3, $f(0) = 0$, $f'(0) = 0$, $f_u(x_s, 0) \equiv 0$, and $f_s(0, x_u) \equiv 0$. The Hamiltonian is C^4 and still of the form (7). In the polar coordinates

$$\left\{ \begin{array}{l} x_1 = r_s \cos\theta_s \\ x_2 = r_s \sin\theta_s \end{array} \right. ,\ \left\{ \begin{array}{l} x_3 = r_u \cos\theta_u \\ x_4 = r_u \sin\theta_u \end{array} \right. ,\ r_s, r_u \geq 0,$$

the Hamiltonian becomes $H = \sqrt{\lambda^2 + \omega^2} r_s r_u \cos(\theta_s - \theta_u - \beta) + O(r_s^3 + r_u^3)$, where $\beta \in]0, \pi/2[$ is defined by $\lambda = \sqrt{\lambda^2 + \omega^2} \cos\beta$ and $\omega = \sqrt{\lambda^2 + \omega^2} \sin\beta$.

Lemma 11 *For $\delta > 0$ small and $T > 0$, if $x \in C^1([0, T], B_4(\delta))$ is a solution of (8) with $r_s(0) r_u(T) > 0$ and $r_s(0)/r_u(T) + r_u(T)/r_s(0) \leq (1/2)\exp(\lambda T)$, then its Hamiltonian is estimated by*

$$H = \sqrt{\lambda^2 + \omega^2}\, \tilde{r}^2 \cos(\theta_s(0) - \theta_u(T) - \beta + T\omega + O(\delta)) + O(\tilde{r}^3),$$

$$\tilde{r} = r_s(0)^{1/2} r_u(T)^{1/2} (1 + O(\delta)) \exp \frac{-\lambda T}{2}.$$

PROOF. As $f_s(x) = r_s O(r_s + r_u)$ and $f_u(x) = r_u O(r_s + r_u)$, we obtain

$$\begin{cases} r_s' \cos\theta_s - r_s \sin\theta_s \theta_s' = x_1' = -\lambda r_s \cos\theta_s - \omega r_s \sin\theta_s + r_s O(r_s + r_u) \\ r_s' \sin\theta_s + r_s \cos\theta_s \theta_s' = x_2' = -\lambda r_s \sin\theta_s + \omega r_s \cos\theta_s + r_s O(r_s + r_u) \end{cases}$$

and

$$r_s'/r_s = -\lambda + O(r_s + r_u),\ \theta_s' = \omega + O(r_s + r_u). \tag{9}$$

Similarly

$$r_u'/r_u = \lambda + O(r_s + r_u),\ \theta_u' = \omega + O(r_s + r_u). \tag{10}$$

Hence $r_s(t) = r_s(0)\exp\{(-\lambda + O(\delta))t\}$ and $r_u(t) = r_u(T)\exp\{(-\lambda + O(\delta))(T - t)\}$, $0 \le t \le T$. As

$$\int_0^T r_s(t)dt = O(\delta) \text{ and } \int_0^T r_u(t)dt = O(\delta)$$

for small δ, we get from (9) and (10)

$$\begin{cases} r_s(t) = (1 + O(\delta)) r_s(0) \exp(-\lambda t), \\ r_u(t) = (1 + O(\delta)) r_u(T) \exp(-\lambda(T - t)), \end{cases} \tag{11}$$

$$\theta_s(t) = \theta_s(0) + \omega t + O(\delta),\ \theta_u(t) = \theta_u(T) + \omega t - \omega T + O(\delta)$$

for $0 \le t \le T$. Choose $\tilde{t}$ such that $r_s(\tilde{t}) = r_u(\tilde{t}) := \tilde{r}$. We find

$$r_s(0)/r_u(T) = (1 + O(\delta)) \exp(2\lambda\tilde{t} - \lambda T),$$

$$\tilde{t} = \frac{T}{2} + \frac{1}{2\lambda} \ln\{(1 + O(\delta)) r_s(0)/r_u(T)\} \in]0, T[,$$

$$\tilde{r} = r_s(0)^{1/2} r_u(T)^{1/2} (1 + O(\delta)) \exp\frac{-\lambda T}{2}.$$

We conclude by evaluating H at $x(\tilde{t})$. ∎

The next theorem, which is related with the λ-lemma, says that given a manifold transverse to W^s_{loc} and another one transverse to W^u_{loc}, there are solutions starting from the first manifold and reaching the second one. The functions g_s and g_u are parametrizations of the transverse manifolds and κ is an additional parameter.

Lemma 12 *Let g_s and g_u be C^1 from $\bar{B}_2(\epsilon\delta) \times K$ to $B_2(\delta/4)$ with $0 < \epsilon < 1$, δ small and K compact. Then there exists $T_0 > 0$ such that for all $\kappa \in K$ and $T > T_0$, we can find $x(., T, \kappa) \in C^1([0,T], B_4(\delta))$, solution of (8), such that*

$$x(0,T,\kappa) \in \{(x_s, x_u) : x_s = g_s(x_u, \kappa)\},\ x(T,T,\kappa) \in \{(x_s, x_u) : x_u = g_u(x_s, \kappa)\}.$$

Moreover $x(0,T,\kappa)$ and $x(T,T,\kappa)$ are C^1 in κ and T, and

$$|x_u(0,T,\kappa)|, |x_s(T,T,\kappa)| \le \frac{1}{2}\delta \exp(-\lambda T).$$

Proof. For $(y,z) \in \bar{B}_4(\exp(-\lambda T)\delta/2)$, we set

$$F_{T,\kappa}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \xi_u\left(-T, \begin{pmatrix} z \\ g_u(z,\kappa) \end{pmatrix}\right) \\ \xi_s\left(T, \begin{pmatrix} g_s(y,\kappa) \\ y \end{pmatrix}\right) \end{pmatrix},$$

where $\xi(t,x)$ is the flow associated with (8). We shall prove that F is well defined, takes its values in $B_4(\exp(-\lambda T)\delta/2)$ and is a L-contraction for some $L < 1$ independent of $T > T_0$ and $\kappa \in K$ if T_0 is large and δ small enough. Hence $F_{T,\kappa}$ has a fixed point $(y(T,\kappa), z(T,\kappa))$ C^1 with respect to T and κ. The desired solution is then

$$x(t,T,\kappa) = \xi\left(t, \begin{pmatrix} g_s(y(T,\kappa),\kappa) \\ y(T,\kappa) \end{pmatrix}\right).$$

In fact we will prove that $x \to \xi_s(T,x)$ is a contraction for $|x_u| \le \exp(-\lambda T)\delta/2$ and $|x_s| \le \delta/4$, and that $\xi_u(-T,x)$ is a contraction for $|x_s| \le \exp(-\lambda T)\delta/2$ and $|x_u| \le \delta/4$ uniformly in $T > T_0$ if δ is small enough.

Let us suppose $r_s(0) \le \delta/4$ and $r_u(0) \le \exp(-\lambda T)\delta/2$. Arguing as in the derivation of (11), we find for δ small

$$r_s(t) \le (3/2) r_s(0) \exp(-\lambda t),\ r_u(t) \le (3/2) r_u(0) \exp(\lambda t),\ 0 \le t \le T, \quad (12)$$

as long as $\forall \sigma \in [0,t] : x(\sigma) \in B_4(\delta)$. Therefore we see that $x(t) \in B_4(\delta)$ for all $t \in [0,T]$ and (12) holds on this interval. Moreover

$$\left|\xi_s\left(T, \begin{pmatrix} g_s(y,\kappa) \\ y \end{pmatrix}\right)\right| < \exp(-\lambda T)\delta/2$$

for

$$|y| \leq \exp(-\lambda T)\delta/2 < \exp(-\lambda T_0)\delta/2 \leq \epsilon\delta,$$

T_0 large enough. The same holds for the first component of F and so

$$F(\bar{B}_4(\exp(-\lambda T)\delta/2)) \subset B_4(\exp(-\lambda T)\delta/2).$$

Differentiating (8) with respect to x, we find

$$v'(t) = \begin{pmatrix} -\lambda & -\omega & 0 & 0 \\ \omega & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -\omega \\ 0 & 0 & \omega & \lambda \end{pmatrix} v(t) + f'(x(t))v(t).$$

Let $x(t)$, $0 \leq t \leq T$, be a solution of (8) with $|x_u(0)| \leq \exp(-\lambda T)\delta/2$ and $|x_s(0)| \leq \delta/4$. Taking account of (12) and $\partial_s f_s = O(r_u+r_s)$, $\partial_u f_u = O(r_u+r_s)$, $\partial_u f_s = O(r_s)$ and $\partial_s f_u = O(r_u)$, we get

$$v_s' = \left\{ \begin{pmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{pmatrix} + \partial_s f_s(x(t)) \right\} v_s(t) + O(\delta)\exp(-\lambda t) v_u(t),$$

$$v_u' = \left\{ \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} + \partial_u f_u(x(t)) \right\} v_u(t) + O(\delta)\exp(-\lambda T)\exp(\lambda t) v_s(t),$$

$$\int_0^T |\partial_s f_s(x(\sigma))| d\sigma = O(\delta),$$

$$\int_0^T |\partial_u f_u(x(\sigma))| d\sigma = O(\delta).$$

Hence for $0 \leq t \leq T$

$$|v_u(t)| \leq 3\exp(\lambda t) \left\{ |v_u(0)| + O(\delta)\exp(-\lambda T) \int_0^t |v_s(\sigma)| d\sigma \right\}$$

and

$$\begin{aligned} \exp(\lambda t)|v_s(t)| &\leq 3|v_s(0)| + O(\delta)\int_0^t |v_u(\tau)| d\tau \\ &\leq 3|v_s(0)| + O(\delta)|v_u(0)| \int_0^t \exp(\lambda\tau) d\tau \end{aligned}$$

$$
\begin{aligned}
& +O(\delta)\exp(-\lambda T)\int_0^t \exp(\lambda\tau)\int_0^\tau |v_s(\sigma)|d\sigma d\tau \\
\leq\ & 3|v_s(0)| + O(\delta)\exp(\lambda T)|v_u(0)| \\
& +O(\delta)\exp(-\lambda T)\int_0^t |v_s(\sigma)|\frac{\exp(\lambda t)-\exp(\lambda\sigma)}{\lambda}d\sigma \\
\leq\ & 3|v_s(0)| + O(\delta)\exp(\lambda T)|v_u(0)| \\
& +O(\delta)\int_0^t \exp(-\lambda\sigma)\left(\exp(\lambda\sigma)|v_s(\sigma)|\right)d\sigma \\
\leq\ & \{3|v_s(0)| + O(\delta)\exp(\lambda T)|v_u(0)|\}\exp\left\{O(\delta)\int_0^t \exp(-\lambda\sigma)d\sigma\right\}
\end{aligned}
$$

by Gronwall inequality. Finally

$$
|v_s(T)| \leq \frac{1}{10}(|v_s(0)| + |v_u(0)|)(\sup|Dg_u| + \sup|Dg_s| + 1)^{-1}
$$

for T large and δ small. Therefore $x \to \xi_s(T,x)$ is a contraction for $|x_u| \leq \exp(-\lambda T)\delta/2$, $|x_s| \leq \delta/4$, as well as the second component of $F_{T,\kappa}$. The same developments hold for the first component.

7 Appendix B: Remarks

A family of multibump solutions like the ones of Theorems 1 and 2 ensures the positivity of the topological entropy (see [32]). Here we recall the standard definition of the topological entropy, adapting it to our context. We suppose to have expressed equation (1) under the form of system (4) with $U = \mathbb{R}^4$ and we denote by $\xi(t,x)$, $t \in D(x)$, the associated flow. Then we set

$$
h_{top} = \sup_{R,e>0}\left(\limsup_{t\to+\infty}\frac{\log s(t,e,R)}{t}\right),
$$

where

$$
\begin{aligned}
s(t,e,R) &= \max\{\ \mathrm{Card}\ (E)|\ \forall\tau\in[0,t]:\ \xi(t,E)\subset B_4(0,R), \\
& \quad \forall x\neq y\in E, \exists\tau\in[0,t]:\ |\xi(\tau,x)-\xi(\tau,y)|\geq e>0\}.
\end{aligned}
$$

Note that, if $R_1 \leq R_2$, $e_1 \geq e_2$, then $s(t,e_1,R_1) \leq s(t,e_2,R_2)$.

Assuming $\hat{z}$ to be isolated up to time-translations, we may formulate Theorem 1 as follows:

Corollary 13 *Set $x = (z, z') \in \mathbb{R}^4$. Then system (6) admits a non-trivial homoclinic solution $\hat{x} \in H^1(\mathbb{R}, \mathbb{R}^4)$ and a family of multibump homoclinic solutions, i.e. for all $\delta > 0$ small, we can find $\gamma > 0$ and a homoclinic solution in the ball in $L^\infty(\mathbb{R}, \mathbb{R}^4)$ of radius δ centered at*

$$\sum_{i=1}^{k} \hat{x}(. - s_1 - \ldots - s_{i-1})$$

where s_i is some real number such that $\mid s_i - n_i\alpha^{-1}2\pi - \gamma \mid < \delta$, for $i = 1, \ldots, k-1$. The integers $k \geq 2$ and $n_i \geq 1$ can be chosen freely.

Choose δ small enough such that any two distinct L^∞-balls given by Corollary 13 are disjoint. Moreover let $R > 0$ satisfy $|\hat{x}(t)| < R - 2\delta$ for all t. Then

$$s(\ (k-1)(2\alpha^{-1}\pi + \gamma + \delta), 2\delta, R) \geq 2^k,$$

where γ is given by the above Corollary and $k \geq 2$ is arbitrary. To see it, just consider the homoclinic solutions given by Corollary 13 and their time-translates. This implies that h_{top} is positive.

Our method also yields multibump periodic solutions for (1). Indeed they correspond to solutions of $G_k = 0$, where now $G_k \in C(U_\mu^k \times V_{k,\vec{n},\delta}, (X')^k \times \mathbb{R}^k)$ is defined by

$$\begin{aligned}
&\langle G_k(z_1, \ldots, z_k, t_1, \ldots, t_k), (u_1, \ldots, u_k, \tau_1, \ldots, \tau_k)\rangle \\
= \ &\sum_{j=1}^{k} \langle (J \circ E_1)'(z_j), u_j\rangle + \sum_{j=1}^{k} \left(v_\infty^{dep}(z_j) - v^{dep}(z_j, z_{j+1}, t_j), u_j(T)\right) \\
&- \sum_{j=1}^{k} (v_\infty^{arr}(z_j) - v^{arr}(z_{j-1}, z_j, t_{j-1}), u_j(-T)) \\
&- \sum_{j=1}^{k} \tau_j H\left(\sigma(., z_j(T), z_{j+1}(-T), t_j),\ \partial_1\sigma(., z_j(T), z_{j+1}(-T), t_j)\right)
\end{aligned}$$

for all $u_1, \ldots, u_k \in X$ and $\tau_1, \ldots, \tau_k \in \mathbb{R}$, and with the convention $z_0 = z_k$, $z_{k+1} = z_1$ and $t_0 = t_k$. See [19] for a similar construction in the time-dependent case.

It is also possible to embed the Bernoulli shift into the dynamics by following the ideas of [32] (but with a slightly more complicated Poincaré section).

∎

Aknowledgements. The first author would like to thank the Swiss National Fund for allowing him a one-year visit to the Scuola Normale Superiore at Pisa. He is also grateful to the SNS for its kind hospitality. He finally thanks the CEREMADE (Paris-Dauphine), where he spent two weeks.

References

[1] S. ALAMA and Y.Y. LI, On multibump bound states for certain semilinear elliptic equations, *J. Diff. Eqns*, **96**, 89-115 (1992)

[2] H. AMANN, A note on the degree theory for gradient mappings, *Proc. AMS*, **84**, 591-595 (1982)

[3] A. AMBROSETTI and V. COTI ZELATI, Multiplicité des orbites homoclines pour des sytèmes conservatifs, *C. R. Acad. Sci. Paris*, **314 II**, 601-604 (1992)

[4] A. BAHRI and J.C. CORON, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, *Comm. Pure Appl. Math.*, **41**, 253-294 (1988)

[5] L.A. BELYAKOV and L.P. SHILNIKOV, Homoclinic curves and complex solitary waves, *Selecta Math. Sov.*, **3**, 219-228 (1990)

[6] U. BESSI, Homoclinic and Period-doubling Bifurcations for Damped Systems. Preprint.

[7] H. BREZIS and L. NIRENBERG, Remarks on finding critical points, *Comm. Pure Appl. Math.*, **44**, 939-963 (1991)

[8] B. BUFFONI, Infinitely many large amplitude homoclinic orbits for a class of autonomous Hamiltonian systems. To appear in *J. Diff. Eqns.*

[9] B. BUFFONI, Periodic and homoclinic orbits for Lorentz-Lagrangian systems via variational methods. To appear in *Nonlinear Analysis, TMA*

[10] B. BUFFONI, M. GROVES and J. TOLAND, A plethora of capillary-gravity waves with near-critical Bond and Froude numbers. Preprint.

[11] A. V. BURYAK and N.N. AKHMEDIEV, Stability criterion for stationary bound states of solitons with radiationless oscillating tails. Preprint (Australian National University Camberra).

[12] A.R. CHAMPNEYS and J.F. TOLAND, Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous Hamiltonian systems, *Nonlinearity*, **6**, 665-722 (1993)

[13] M. CHAPERON, Une idée de type géodésiques brisées pour les systèmes hamiltoniens, *C. R. Acad. Sci. Paris*, **298**, 293-296 (1984)

[14] K. CIELIEBAK and E. SERE, Pseudo-holomorphic curves and multiplicity of homoclinic orbits, to appear in *Duke Math. Journ.*

[15] V. COTI ZELATI, I. EKELAND and E. SERE, A variational approach to homoclinic orbits in Hamiltonian systems, *Math. Ann*, **288**, 133-160 (1990)

[16] E. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations*, Mc Graw-Hill, New-York, 1955.

[17] V. COTI ZELATI and P. H. RABINOWITZ, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, *J. AMS*, **4**, 693-727 (1991)

[18] V. COTI ZELATI and P. H. RABINOWITZ, Homoclinic type solutions for a semilinear elliptic PDE on R^n, *Comm. Pure Appl. Math*, **45**, 1217-1269 (1992)

[19] V. COTI ZELATI and P.H. RABINOWITZ, Multibump periodic solutions of a family of Hamiltonian systems. Preprint.

[20] R. L. DEVANEY, Homoclinic orbits in Hamiltonian systems, *J. Diff. Eq.*, **21**, 431-438 (1976)

[21] P. FELMER, Heteroclinic orbits for spatially periodic Hamiltonian systems, *Annales IHP, anal. non-lin.*, **8 (5)**, 477-497 (1991)

[22] F. GIANNONI and P.H. RABINOWITZ, On the multiplicity of homoclinic orbits on Riemannian manifolds for a class of second order Hamiltonian system, *NoDEA*, **1**, 1-46 (1994)

[23] H. HOFER, A note on the topological degree at a critical point of mountain-pass type, *Proc. of AMS*, **90:2**, 309-315 (1984)

[24] H. HOFER and K. WYSOCKI, First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, *Math. Annalen*, **288**, 483-503 (1990)

[25] V. V. KOZLOV, Integrability and non-integrability in Hamiltonian mechanics, *Russian Math. Surveys*, **38:1**, 1-76 (1983)

[26] V. V. KOZLOV, Calculus of variations in the large and classical mechanics, *Russian Math. Surveys*, **40:2**, 37-71 (1985)

[27] Y. Y. LI, On $-\Delta u = k(x)u^5$ in $\mathbb{R}^3$, *Comm. Pure Appl. Math.*, **46**, 303-340 (1993)

[28] P. L. LIONS, The concentration-compactness principle in the calculus of variations, *Rev. Mat. Iberoam.*, **1**, 145-201 (1994)

[29] P.J. McKENNA and W. WALTER, Travelling waves in a suspension bridge, *SIAM J. Appl. Math.*, **50**, 703-715 (1990)

[30] R. MOECKEL, Chaotic dynamics near triple collisions, *Arch. Rat. Mech. Anal.*, **107**, 37-69 (1989)

[31] E. SÉRÉ, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Ann.*, **209**, 27-42 (1992)

[32] E. SÉRÉ, Looking for the Bernoulli shift, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **10**, 561-590 (1993)

[33] K. TANAKA, Homoclinic orbits in a first order superquadratic Hamiltonian system: convergence of subharmonics. Preprint Nagoya University

[34] S. WIGGINS, *Global Bifurcations and Chaos: Analytical Methods*, Springer, New-York (1988)