Multiplicity of homoclinic orbits in fourth-order conservative systems

B. Buffoni *

School of Mathematics, University of Bath, Bath BA27AY, U.K.

Abstract

We discuss the advantages of using variational methods to study the existence of homoclinic orbits to a saddle-focus equilibrium in a four-dimensional autonomous Hamiltonian system.

1 Introduction

In this paper we present multiplicity results for homoclinic orbits of some classes of autonomous Hamiltonian systems. Let us recall that Devaney [13] (see also [22]) proved that an autonomous Hamiltonian system in dimension 4, with a homoclinic orbit to a saddle-focus equilibrium (i.e. the linearized system around the equilibrium has the eigenvalues $\pm\lambda \pm i\omega, \lambda, \omega > 0$), is chaotic (and has infinitely many other homoclinics), provided the homoclinic orbit is the transverse intersection of the stable and unstable manifolds. Multiplicity results in the real-saddle case (all eigenvalues are real) are much more subtle [22, 1], and there are counterexamples (see below).

When trying to apply Devaney's result to precise examples, one is faced to the difficulty of proving transversality. In Section 2, we present a class of "saddle-focus" autonomous Hamiltonian systems, where infinitely many distinct homoclinic solutions have been found by shooting arguments, without checking transversality. In this paper, "distinct" means different up to

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time-translations. Moreover we refer throughout to non-zero solutions. These systems arise in the theory of water-waves. For more details and proofs of the results stated in Section 2, we refer to [4].

In Section 3, we present a variational approach to the multiplicity question. This approach applies to the water-wave problem studied in Section 2, and improves the results already obtained by shooting. Moreover, the variational approach is more general than the shooting method, and we give some applications to other classes of equations. In Section 3 as well as in Section 2, no transversality assumption is made to obtain multibump solutions. In fact, the variational method allows transversality to be replaced by a much weaker assumption: the stable and unstable manifolds do not coincide. This assumption is clearly much easier to check. Here the stable and unstable manifolds are said to coincide if (i) they are equal as sets of points and (ii) the time for a homoclinic orbit to go from the local unstable manifold to the local stable manifold is uniformly bounded.

2 The shooting argument

We summarize some results concerning the system

$$Tq''(t) + \nabla V(q(t)) = 0,\ \ q \in C^2(\mathbb{R}, \mathbb{R}^2), \tag{1}$$

where T is a symmetric invertible real matrix which is *indefinite*, and $V \in C^2(\mathbb{R}^2, \mathbb{R})$ satisfies $V(0) = 0$ and $\nabla V(0) = 0$. We are concerned with homoclinic solutions, i.e. solutions $q \neq 0$ that are doubly asymptotic to 0. System (1) may be transformed into a Hamiltonian system with Hamiltonian

$$H(q, q') = \frac{1}{2}\langle Tq', q'\rangle + V(q).$$

The value of H is 0 along any homoclinic solution. A particular but important case appears in water-wave theory (see [8, 16] and the references therein) and in the buckling of a strut on a nonlinear foundation [14]. It is given by the equation

$$u^{iv} + Pu'' + u - u^2 = 0,\ \ u \in C^4(\mathbb{R}), \tag{2}$$

where $P \in \mathbb{R}$ is a fixed parameter, and is equivalent to system (1) with $T = \begin{pmatrix} P & 1 \\ 1 & 0 \end{pmatrix}$ and $V(q_1, q_2) = \frac{1}{2}q_1^2 - \frac{1}{3}q_1^3 - \frac{1}{2}q_2^2$, where $q_1 = u$, $q_2 = u''$. Variants of equation (2) are also used to model solitary waves on suspension bridge [19] and solitons in optical fibers [18].

System (1) has a remarkable monotonicity property [21, 15]. Any solution q of (1) with $H = 0$ is monotone as long as $V(q(t)) \neq 0$, in the sense that $q'(t)$

is constrained to lie in one of the four cones of $\mathcal{C} = \{v \in \mathbb{R}^2 : \langle Tv, v\rangle \neq 0\}$. This is the basic ingredient in [21, 15] to get the existence of a symmetric monotonous homoclinic solution, under some geometrical hypotheses on $V_0 = \{q : V(q) = 0\}$ which imply in particular that 0 is a real-saddle. Equation (2) belongs to this setting exactly when $P \leq -2$ and, additionally to the existence result, uniqueness has been proved in this special case [2, 6].

For $P \in]-2, 2[$, 0 is a saddle-focus, and the situation becomes more intricate. By a continuation method, Amick and Toland [2] followed a branch of homoclinic solutions from $P = -2$ to $P = \epsilon$ for some small $\epsilon > 0$. In [11, 6] it is shown that for $P = -2$ the homoclinic orbit is the transverse intersection of the stable and unstable manifolds. Therefore, by Devaney theorem, the dynamics of (2) is chaotic for $P \in]-2, -2+\delta[$, $\delta > 0$ small. Moreover, in this range of values of P, equation (2) has infinitely many distinct homoclinic solutions. When P tends to 2, it is numerically observed that they disappear in pairs (coalescence). Finally, still in [11, 10], the homoclinic solutions are described in the framework of system (1) and the way they rotate around the equilibrium is explained in terms of V_0 and $\mathcal{C}$.

By a shooting argument, paper [10] directly deals with system (1) in the saddle-focus case and gives geometrical conditions on V_0 to ensure the existence of a homoclinic solution. This gives another proof of the existence of a homoclinic solution of (2) when $P \in]-2, 0]$. The shooting argument neither requires nor implies the transversality of the stable and unstable manifolds. In [4] an infinite family of distinct homoclinic solutions is constructed, still by shooting. As a consequence, equation (2) has infinitely many distinct homoclinics for all $P \in]-2, 0]$.

Finally, by a normal form approach, it is known that at least two branches of homoclinic solutions bifurcate from $P = 2$ towards the region $P < 2$ (see [17]). Let us summarize the results about equation (2) in the form of a theorem:

Theorem 1 *The equilibrium 0 is a real-saddle for $P \leq -2$ and a saddle-focus for $P \in]-2, 2[$. For some small $\epsilon > 0$ and all $P \in]-\infty, \epsilon[$, equation (2) admits at least one homoclinic solution. If $P \in]-\infty, -2]$, this solution is unique (up to time-translations), is even, and is the transverse intersection of the stable and unstable manifolds. For some small $\delta > 0$, Devaney theorem holds for $P \in]-2, -2+\delta[$ and hence (2) is chaotic. For all $P \in]-2, 0]$, equation (2) has an infinite family $\{u_n\}_{n\geq 2}$ of homoclinic solutions, u_n being a solution with n bumps without any oscilltions between two consecutive bumps. There are two branches of homoclinic solutions bifurcating from $P = 2$ towards the region $P < 2$.*

The family $\{u_n\}$ is not sufficiently rich to deduce that equation (2) is chaotic for $P \in]-2, 0]$. In the next section, we will apply a variational method to complete the discussion. We will obtain

Theorem 2 *For all $P \in]-2, 2[$, equation (2) admits a homoclinic solution. The dynamics of (2) is chaotic as soon as the stable and unstable manifolds do not coincide. Hence equation (2) has infinitely many distinct homoclinic solutions for all $P \in]-2, 2[$ and its dynamics is chaotic for all $P \in]-2, 0]$.*

We see that equation (2) is a paradigm for fourth-order Hamiltonian systems with a saddle-focus. We refer to [15, 7] for existence results of periodic solutions of system (1).

3 The variational approach

Let us now consider the following system

$$\begin{cases} x''(t) + 2\alpha y'(t) - x(t) + F_1'(x(t), y(t)) = 0 \\ y''(t) - 2\alpha x'(t) - y(t) + F_2'(x(t), y(t)) = 0 \end{cases} \tag{3}$$

where x, y are real valued, $F \in C^5(\mathbb{R}^2, \mathbb{R})$, $\nabla F = (F_1', F_2')$, and $\alpha \in (0, 1)$ is a parameter. Writing $z = (x, y)$, we assume F to satisfy

(H1) F is not identically zero,

(H2) $\lim_{|z|\to 0} F(z)/|z|^2 = 0$,

(H3) there exists $p > 2$ such that $(F'(z), z) \geq pF(z) \geq 0$ for all z in $\mathbb{R}^2$,

(H4) if $F(z) > 0$, then $(F''(z)z, z) > (F'(z), z)$.

These conditions are satisfied, for instance, by $F(z) = |z|^4$.

Identifying $(x, y) \in \mathbb{R}^2$ with $z \in \mathbf{C}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with i, the linearization of (3) at the origin is simply $z'' - 2\alpha i z' - z = 0$, whose characteristic equation has $\pm\sqrt{1-\alpha^2} \pm i\alpha$ as eigenvalues, i.e. 0 is a saddle-focus equilibrium.

The homoclinic solutions of (3) are exactly the critical points of the following functional defined on the Sobolev space $H^1(\mathbb{R}, \mathbb{R}^2)$

$$J(z) = \frac{1}{2}\int_{\mathbb{R}} (|z'|^2 + |z|^2)dt + \alpha \int_{\mathbb{R}} (z, iz')dt - \int_{\mathbb{R}} F(z)dt.$$

J has a mountain-pass structure (see [9]) and one can find a critical point $\hat{z}$ of mountain-pass type. The parentheses $(.,.)$ denote the scalar product in $\mathbb{R}^2$.

Recently, the variational construction of multibump solutions has been investigated in several works, for Hamiltonian systems as well as for elliptic PDEs (see [20] and the references therein). In all these works, the equation contains a non-autonomous term. For autonomous systems, it is more difficult to find multibump solutions, because the functional presents a stronger lack of compactness. In [9], a Morse reduction in the spirit of [12, 3] is used to overcome this lack of compactness. The first result of [9] is the following:

Theorem 3 *Under conditions (H1) to (H4), system (3) admits infinitely many geometrically distinct homoclinic solutions. One, denoted by $\hat{z}$, is given by the mountain-pass lemma. If it is isolated in $H^1(\mathbb{R},\mathbb{R}^2)$ up to time-translations (non-degeneracy condition), the others are multibump solutions, i.e. for any $\delta > 0$ small, we can find $\gamma > 0$ such that, for any integers $k \geq 2$ and $n_1, ..., n_{k-1} \geq 1$, there exists a homoclinic solution in the ball of radius δ for the L^∞-norm, centered at*

$$\sum_{i=1}^{k} \hat{z}(. - s_1 - \ldots - s_{i-1}) .$$

Here, s_i is some real number such that $\mid s_i - n_i\alpha^{-1}2\pi - \gamma \mid < \delta$, for $i = 1, \ldots, k-1$.

The γ provided by the proof tends to $+\infty$ when $\delta \to 0$.

In the real analytic case, we have a simple way to check the non-degeneracy condition.

Theorem 4 *If F is real analytic, then either $\hat{z}$ is an isolated homoclinic solution in $H^1(\mathbb{R},\mathbb{R}^2)$ up to time-translations or the global stable and unstable manifolds of 0 coincide, the time for a homoclinic orbit to go from the local unstable manifold to the local stable manifold being uniformly bounded.*

As an example, consider the fourth order equation (2) of Section 2 with $P \in]-2, 2[$. In [5] it is shown that (2) is the Euler-Lagrange equation of a functional defined on $H^2(\mathbb{R},\mathbb{R})$ that has a nice mountain-pass structure. Hence there exists a non-trivial homoclinic solution to 0. Moreover 0 is a saddle-focus of (2) seen as a Hamiltonian system. As already mentioned, in [4], a sequence of solutions $\{u_n\}_{n\geq 2}$ is constructed by a shooting argument for $P \in (-2, 0]$, u_n being a homoclinic solution with n bumps without any oscillations between two consecutive bumps. Moreover the time for u_n to go from W^s_{loc} to W^u_{loc} increases as $O(n)$. Hence the stable and unstable manifold do not coincide. Since we are in an analytic case, an analogue of Theorem 4 gives the existence of a whole family of multibump homoclinic solutions for $P \in (-2, 0]$, which shows that the dynamic is chaotic. Note that the family

$\{u_n\}$ is too small to give, for example, any information about the topological entropy of the system. Even if the stable and unstable manifolds coincide, the analogue of Theorem 4 ensures the existence of infinitely many distinct homoclinic solutions for all $P \in]-2,2[$.

Let us end with another application, to the case where the configuration space M is a compact analytic Riemannian 2-manifold. Let us consider the equation

$$-D_t q'(t) + B(q)q' = \text{ grad } V(q), \tag{4}$$

where D_t is the covariant derivative, $V : M \to \mathbb{R}$ is a real analytic potential, grad $V \in T_qM$ is the gradient of V defined with the scalar product at q, and $B(q) \in L(T_qM, T_qM)$ is real analytic and such that $(u,v) \to \beta(u,v) := (B(q)u, v)$ is an exact two-form. An example of such β is given by the Coriolis force on a rotating surface. We suppose (4) to admit a saddle-focus equilibrium $q_0 \in M$ with $V(q_0) = 0$.

Theorem 5 *If*

1. q_0 *is a saddle-focus equilibrium,*
2. *there exists* $\epsilon > 0$ *such that* $L(q,u) \geq \epsilon(|u|^2 + \text{ dist } (q,q_0)^2)$ *for all* $q \in M$ *and* $u \in T_qM$, *dist* $(.,.)$ *being the geodesic distance,*
3. $\pi_1(M)$ *is not generated as a group by only one element,*

then (4) admits infinitely many homoclinic solutions to q_0 *geometrically distinct and its topological entropy is positive.*

For a definition of the topological entropy, see e.g. [9], Appendix B.

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