Self-adjoint Operators and Cones

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Abstract

Suppose that K is a cone in a real Hilbert space $\mathcal{H}$ with $K^\perp = \{0\}$ and that $A : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator which maps K into itself. If $\| A \|$ is an eigenvalue of A, it is shown that it has an eigenvector in the cone. As a corollary it follows that if $\| A \|^n$ is an eigenvalue of A^n then $\| A \|$ is an eigenvalue of A which has an eigenvector in K. The role of the support-boundary of K in the simplicity of the principal eigenvalue $\| A \|$ is investigated. If H is a separable Hilbert space it is shown that $\| A \| \in \sigma(A)$; i.e. the spectral radius of A lies in the spectrum of A. When A is compact we obtain a very elementary proof of the Krein-Rutman Theorem in the self-adjoint case without assuming that $K^\perp = \{0\}$.

1 Introduction

Suppose that K is a closed convex cone in a real Hilbert space $\mathcal{H}$. For each $x \in \mathcal{H}$, let $\pi(x)$ denote the point of K nearest to x. We make the simple observation that if $A : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator which leaves K invariant and $Ax = \| A \| x$ for $x \in \mathcal{H}$, then $A\pi(x) = \| A \| \pi(x)$ (Lemma 2). If $K^\perp = \{0\}$ and $x \neq 0$, then at least one of $\pi(x)$ and

$\pi(-x)$ must be non-zero. With this in mind, suppose that $-\parallel A\parallel$ is an eigenvalue of A with eigenvector x. Then $\{x,-x\}\cap K=\emptyset$, because K is a cone (see Section 2). Since $A^2x=\parallel A\parallel^2 x=\parallel A^2\parallel x$, our observation now gives that $\pi(x)$ is an eigenvector of A^2. Hence $(\parallel A\parallel I+A)\pi(x)\in K$ is an eigenvector of A corresponding to the eigenvalue $\parallel A\parallel$. Therefore, if $K^\perp=\{0\}$ and $-\parallel A\parallel$ is an eigenvalue of A, then $\parallel A\parallel$ is an eigenvalue of A with an eigenvector in K.

To discuss multiplicity of the eigenvalue $\parallel A\parallel$ of A we use the support-boundary, $\partial_\pi K=\pi(\mathcal{H}\backslash K)$, of K. The Bishop-Phelps Theorem [1] shows that $\partial_\pi K$ is dense in the metric boundary ∂K of K, but for practical purposes it is often much smaller. For example, in $L_2(\Omega)$, when K is the cone $L_2^+(\Omega)$ of non-negative functions, $\partial_\pi K$ consists precisely of functions which are zero on a set of positive measure, while $\partial K=K$. We show, amongst other things, that if $K^\perp=\{0\}$ and $\parallel A\parallel$ is an eigenvalue of A with no eigenvectors in $\partial_\pi K$, then $\parallel A\parallel$ is simple.

To understand further the significance of an invariant cone for the spectrum $\sigma(A)$ of a self-adjoint operator the following functionals on the set of all self-adjoint operators A which leave K invariant ($K^\perp=\{0\}$ is not assumed) are useful:

$$\rho(A)=\sup\{\langle Ax,x\rangle:\parallel x\parallel=1, x\in\mathcal{H}\}$$

$$\chi(A)=\sup\{\langle Aw,w\rangle:\parallel w\parallel=1, w\in K\}.$$

It is well-known that $\rho(A)$ is the supremum of $\sigma(A)$, and clearly $\chi(A)\le\rho(A)$. We show that χ is convex, continuous, and $\chi(A)>0$ if $A\not\equiv 0$ on K. We show that if $\rho(A)$ or $\chi(A)$ is attained, then it is attained at a normalized eigenvector of A which lies in K. In the light of elementary spectral theory, these observations yield, for compact self-adjoint operators leaving a cone invariant, the following version of the Krein-Rutman Theorem [8]. (Note $K^\perp\neq\{0\}$ is allowed.)

Suppose that $K(\neq \{0\}) \subset \mathcal{H}$ is a cone, $A : \mathcal{H} \to \mathcal{H}$ is a compact self-adjoint operator and $A : K \to K$. Then

(i) $\chi(A)$ is a non-negative eigenvalue of A and $\chi(A)$ has an eigenvector in K.

(ii) If μ is another eigenvalue of A with an eigenvector in $H = \overline{\mathrm{span} K}$, then $|\mu| \leq \chi(A)$.

(iii) If $\partial_\pi K$ contains no eigenvector of A with eigenvalue $\chi(A)$ then $\chi(A)$ is a simple eigenvector of A in H. (If $K^\perp = \{0\}$ then $H = \mathcal{H}$ and $\chi(A)$ is simple.)

Finally, we show that if $\mathcal{H}$ is separable then $\| A \|$ lies in the spectrum $\sigma(A)$ of A. In other words the spectral radius of A lies in its spectrum. While it is well-known that when an operator leaves a cone in a Banach space invariant the spectral radius is in the spectrum provided that the cone is reproducing and normal (see Bonsall [2] and Krasnosel'skii *et al* [7]), the general result is false. There is a beautiful example due to Bonsall (Example (iv), [2]) of a reproducing cone with non-empty interior which is mapped into itself by an operator whose spectral radius is not in its spectrum. This example can easily be adapted to yield, in a separable Hilbert space, a reproducing cone which is mapped to itself by a non-self-adjoint operator, the spectral radius of which is not in its spectrum. In our theorem on self-adjoint operators the condition that $K^\perp = \{0\}$ is weaker than reproducing (which says that $K - K = \mathcal{H}$) and is unavoidable since the result is clearly false if K lies in a proper closed subspace of $\mathcal{H}$. No other hypotheses on K are adopted.

Note that the converse is false. For example, if -1 is an eigenvalue and $+1$ is not an eigenvalue of a self-adjoint operator A with $\| A \| = 1$, then the results discussed earlier mean that there is no cone K with $K^\perp = \{0\}$ which A maps into itself, even though 1 may be in the spectrum of A. Our proof depends on a result about approximation by polynomials with non-negative coefficients and the multiplicative version of the spectral theorem.

In the proofs of these and other related results, no use is made of the notion of a partial ordering induced on $\mathcal{H}$ by K. In this sense the approach is different from ones used earlier ([7] is a very good account of the modern theory). We do not give results which involve special kinds of cones (solid, minihedral, normal etc.) even when they are obviously corollaries of the method. The only purpose is to examine the consequences for a self-adjoint operator of the existence of an invariant cone K with $K^\perp = \{0\}$. To give a complete theory, it is necessary to study invariant wedges and adjoint wedges before specialising to cones.

In applications to integral operators A on $L_2(\Omega)$ with positive symmetric kernels it is significant that such operators have no eigenfunctions in $\partial_\pi K$ when $K = L_2^+(\Omega)$, because no non-negative eigenfunction is zero on a set of positive measure when the kernel of A is positive. An easy consequence is that $-\|A\|$ is not an eigenvalue of such an integral operator and if $\|A\|$ is an eigenvalue then it is simple.

In Section 2 we discuss cones, wedges and their adjoints, and give criteria for membership of $\partial_\pi K$. Section 3 is then devoted to the geometric relations between a self-adjoint operator and the cones and wedges which it leaves invariant. Section 4 is concerned with the case of compact operators, and Sections 5 and 6 give the proof that, in a separable Hilbert space, $\|A\| \in \sigma(A)$ when $A : K \to K$, $K^\perp = \{0\}$ and A is self adjoint. The proofs in Sections 5 and 6 can be read independently of the rest of the paper.

The only treatment of self-adjoint operators in Hilbert spaces which leave abstract cones invariant known to us is that of Faris [4], [5]. He deals with a class of narrow cones K, called Hilbert cones, which have the property that $\langle u, v\rangle \geq 0$ for all $u, v \in K$. His general viewpoint and conclusions are somewhat different from ours.

If $\mathcal{H}$ is a complex Hilbert space with inner product $(\ ,\ )$, then $\mathcal{H}$ is a real Hilbert space with a real inner product defined by

$$\langle u, v\rangle = \text{Real } (u, v), \quad u, v \in \mathcal{H}.$$

If A acts in the complex space $\mathcal{H}$, $(\ ,\ )$, then

$$(Au, v) = (u, Av) \text{ for all } u, v \in \mathcal{H}$$

if and only if

$$\langle Au, v\rangle = \langle u, Av\rangle \text{ for all } u, v \in \mathcal{H}.$$

Therefore, with this understanding, we deal with self-adjoint operators on a real Hilbert space throughout. Cones and wedges are all defined using multiplication by real scalars.

Acknowledgement This work was done while visiting Ecole Polytechnique Fédérale, Lausanne. It is a pleasure to thank my host, Professor C.A. Stuart, whose interest did much to stimulate the results presented here.

2 The support-boundary

The purpose of this section is to establish notation and to note some elementary facts about cones and, more generally, wedges.

Throughout $\mathcal{H}$ is a real Hilbert space with inner-product $\langle\ ,\ \rangle$. Following Krasnosel'skii *et al*, [7], [6], $W \subset \mathcal{H}$ is a *wedge* if it is closed, convex, and $\alpha W \subset W$ for all $\alpha \geq 0$ where

$$\alpha W = \{\alpha w : w \in W\}.$$

If $-W$ denotes $(-1)W$, then *a cone* is a wedge which in addition satisfies $-W \cap W = \{0\}$. For each $x \in \mathcal{H}$ let $\pi_W(x)$ be the unique point of the wedge W which is closer to x than any other point of W. It is easy to see that if $x \in \mathcal{H}$ and $w \in W$ then $w = \pi_W(x)$ if, and only if,

$$\langle z - w, x - w\rangle \leq 0 \text{ for all } z \in W. \tag{2.1}$$

Lemma 1 *If W is a wedge, $x \in \mathcal{H}$ and $w \in W$ then $w = \pi_W(x)$ if, and only if,*

$$\sup_{z \in W}\langle z, x - w\rangle = \langle w, x - w\rangle = 0. \tag{2.2}$$

Proof. If $w = \pi_W(x)$ then (2.1) holds. If, in particular, we put $z = 0$ and $z = 2w$ then there follows that $\langle w, x - w\rangle = 0$, and (2.2) is immediate. Conversely, if (2.2) holds then (2.1) follows and so $w = \pi_W(x)$.

□

The following corollary motivates the rest of the paper.

Lemma 2 *Suppose W is a wedge in $\mathcal{H}$ and A is a self-adjoint operator on $\mathcal{H}$ which maps W into itself. Let $Ax = \|A\|x$ and let $w = \pi_W(x)$. Then $Aw = \|A\|w$.*

Proof. With $z = Aw$, (2.2) and the self-adjointness of A gives

$$\begin{aligned} 0 &\geq \langle Aw, x - w\rangle = \langle Aw, x\rangle - \langle Aw, w\rangle \\ &= \langle w, Ax\rangle - \langle Aw, w\rangle = \|A\|\,\|w\|^2 - \langle Aw, w\rangle. \end{aligned}$$

Therefore

$$\|A\|\,\|w\|^2 \leq \langle Aw, w\rangle \leq \|A\|\,\|w\|^2$$

and hence

$$\|(Aw - \|A\|w)\| = \|Aw\|^2 - 2\,\|A\|\,\langle Aw, w\rangle + \|A\|^2\,\|w\|^2 = 0.$$

□

For any wedge W, let

$$W_- = \{y \in \mathcal{H} : \langle z, y\rangle \leq 0 \text{ for all } z \in W\}.$$

It is clear that W_- is also a wedge, and there is a reciprocal relation between W and W_- which can be expressed like this.

Lemma 3 *If W is a wedge then*

$$W = \{z \in \mathcal{H} : \langle z, y \rangle \leq 0 \text{ for all } y \in W_-\}. \tag{2.3}$$

Proof. If $\widehat{W}$ denotes the right-hand side of (2.3), then $W \subset \widehat{W}$ because of the definition of W_-. If $x \in \widehat{W}$, let $\pi_W(x) = w$. By Lemma 1, $x - w \in W_-$ and $\langle w, x - w \rangle = 0$. Therefore

$$\| x - w \|^2 = \langle x, x - w \rangle - \langle w, x - w \rangle \leq 0.$$

Hence $x = w \in W$ which proves that $W = \widehat{W}$.

$\square$

Because of Lemma 3, $W = (W_-)_-$. To express this natural duality between W and W_- in the notation, write W_+ for W (we say that W_+, W_- is an *adjoint-pair* of wedges) and, for convenience, write π_+ and π_- instead of π_{W_+} and π_{W_-}. We see in the next theorem that $\pi_+ + \pi_- = I$ and $\langle \pi_+(x), \pi_-(x) \rangle = 0$ for all $x \in \mathcal{H}$.

The following is a record of some elementary, yet significant, observations.

Theorem 4 *Suppose that W_+, W_- is an adjoint-pair of wedges.*

(a) $-W_- \cap W_- = \{W_+\}^\perp$.

(b) *W_- is a cone if, and only if, $\{W_+\}^\perp = \{0\}$.*

(c) *If $x \in \mathcal{H}$ then $\pi_+(x) + \pi_+(-x) \in -W_- \cap W_+$.*

(d) *If $\{W_+\}^\perp = \{0\}$ and $x \neq 0$, $\pi_+(x)$ and $\pi_+(-x)$ cannot both be zero.*

(e) *If W_+ is not a linear subspace then $-W_- \cap W_+ \neq \{0\}$.*

(f) $\pi_+(x) + \pi_-(x) = x$ *and* $\langle \pi_+(x), \pi_-(x)\rangle = 0$ *for all* $x \in \mathcal{H}$.

The roles of the subscripts + and − can be interchanged in (a)-(f) above.

Proof. (a) If W_+ is given and W_- is defined in terms of it then $z \in -W_- \cap W_-$ means that $\langle z, w\rangle = 0$ for all $w \in W_+$, i.e. $w \in \{W_+\}^\perp$. Conversely, if $w \in \{W_+\}^\perp$ then $\langle z, w\rangle = 0$ for all $z \in W_+$ and hence $z \in -W_- \cap W_-$.

(b) This follows immediately from part (a).

(c) Now for $x \in \mathcal{H}$ and $z \in W_+$ it follows by Lemma 1 that

$$\langle z, x - \pi_+(x)\rangle \leq 0 \text{ and } \langle z, -x - \pi_+(-x)\rangle \leq 0.$$

Adding these gives, for all $z \in W_+$,

$$\langle z, -\pi_+(x) - \pi_+(-x)\rangle \leq 0.$$

Hence $(\pi_+(x) + \pi_+(-x)) \in -W_-$. But $\pi_+(x) + \pi_+(-x) \in W_+$ because W_+ is a wedge. This proves (c).

(d) Suppose that $\pi_+(x) = \pi_+(-x) = 0$ and $\{W_+\}^\perp = \{0\}$. Then, by Lemma 1,

$$\langle z, x\rangle = \langle z, -x\rangle = 0 \text{ for all } z \in W_+.$$

Hence $x \in \{W_+\}^\perp = \{0\}$. Therefore $\pi_+(x)$ or $\pi_+(-x)$ is non-zero if $x \neq \{0\}$.

(e) Suppose W_+ is not a linear subspace. Then there exists $w \in W_+$ such that $-w \notin W_+$. By part (c), $w + \pi_+(-w) = \pi_+(w) + \pi_+(-w) \in -W_- \cap W_+$, and $w + \pi_+(-w) \neq 0$ because $-w \notin W_+$. This proves (e).

(f) Let $x \in \mathcal{H}$. Then, by Lemma 1,

$$\langle z, x - \pi_+(x)\rangle \leq 0 \text{ for all } z \in W_+,$$

whence $x - \pi_+(x) \in W_-$, by Lemma 3. Also by Lemma 1,

$$\langle x - \pi_+(x), x - (x - \pi_+(x))\rangle = \langle \pi_+(x), x - \pi_+(x)\rangle = 0$$

and for all $z \in W_-$,

$$\langle z, x - (x - \pi_+(x))\rangle = \langle z, \pi_+(x)\rangle \leq 0.$$

Therefore $x - \pi_+(x) = \pi_-(x)$ by Lemma 1. This proves (f).

It is clear from the duality between W_+ and W_- that the subscripts $+$ and $-$ can be interchanged in the Theorem.

□

If $w \in W_+$, $x \in \mathcal{H}\backslash W_+$ and $w = \pi_+(x)$ then w is called a *support-point* of W_+. The reason is that, by Lemma 1, $\{x - w\}^\perp + w$ is then a supporting hyperplane for W_+ which intersects it at every point of the ray $\{\alpha w : \alpha \geq 0\}$. We denote the set of all support-points of W_+ by $\partial_\pi W_+$ and refer to it as the *support-boundary* of W_+. The usual metric-space boundary of W_+ is denoted by ∂W_+. In important examples ∂W_+ may coincide with the set W_+, while $\partial_\pi W_+$ is much smaller. However, it is known from a theorem of Bishop and Phelps [1] that $\partial_\pi W_+$ is dense in ∂W_+.

An immediate corollary of the next theorem is that $\partial_\pi W_+ = W_+$ if $\{W_+\}^\perp \neq \{0\}$.

Theorem 5 *Suppose W_+, W_- is an adjoint-pair of wedges and $w \in W_+$. Then*

$$w \in \partial_\pi W_+ \textit{ if, and only if, } \{w\}^\perp \cap W_- \neq \{0\}.$$

Proof. Suppose $w \in \partial_\pi W_+$, $x \in \mathcal{H}\backslash W_+$ and $w = \pi_+(x)$. Then $0 \neq (x - w) \in W_- \cap \{w\}^\perp$, by Lemma 1.

Conversely, suppose $0 \neq z \in \{w\}^\perp \cap W_-$ and $w \in W_+$. Let $x = z + w$. If $x \in W_+$ then

$$\| z \|^2 = \langle w + z, z\rangle = \langle x, z\rangle \leq 0$$

since $z \in W_- \cap \{w\}^\perp$. But $z \neq 0$, so $x \notin W_+$. Also for all $y \in W_+$,

$$\langle y, x - w\rangle = \langle y, z\rangle \leq 0$$

since $z \in W_-$ and

$$\langle w, x - w \rangle = \langle w, z \rangle = 0$$

since $z \in \{w\}^\perp$. Hence, by Lemma 1, $w = \pi_+(x)$, $x \notin W_+$ and so $w \in \partial_\pi W_+$.

$\square$

The role of $+$ and $-$ in the preceding discussion can be interchanged since W_+, W_- is an adjoint-pair of wedges. We will say that an adjoint-pair of wedges if *non-trivial* if $\{W_+, W_-\} \neq \{\{0\}, \mathcal{H}\}$. When $W^\perp = \{0\}$ the metric boundary ∂W and the support-boundary do not coincide in significant examples such as the following.

Example. $\mathcal{H} = L_2(\Omega, \mathcal{M}, m)$ for any measure space $(\Omega, \mathcal{M}, m)$ with $m(\Omega) < \infty$ and let K_+ denote the cone of all non-negative, real-valued functions in $\mathcal{H}$. Then K_+ is a cone. Also $K_- = -K_+$ in this case. Therefore, from Theorem 5, it follows that $\partial_\pi K_+$ consists precisely of those functions w for which there exists a non-negative, non-zero L^2-function v orthogonal to w. If w is positive almost everywhere on Ω, this is impossible.

In this example, therefore, the support-boundary of K_+ consists precisely of those functions w which are zero on a set of positive measure in Ω.

When Ω is an open set in $\mathbb{R}^n$ with Lebesgue measure, the metric-boundary ∂K_+ of K_+ is K_+ itself and when $L_2(\Omega, \mathcal{M}, m)$ is a finite dimensional space (when Ω is finite and m is counting measure) $\partial K_+ = \partial_\pi K_+$.

Observe that the general result that $\pi_+ + \pi_- = I$ is the abstract version of the usual decomposition of an L_2-function into its positive and negative parts:

$$u = u_+ - u_- \text{ were } u_+(x) = \max\{u(x), 0\}, \quad x \in \Omega.$$

To see this note that for any $v \geq 0$ with $v \in L_2(\Omega, \mathcal{M}, m)$ with u_+ defined above

$$\langle v, u - u_+ \rangle = -\langle v, u_- \rangle \leq 0$$

and

$$\langle u_+, u - u_+ \rangle = \langle u_+, u_- \rangle = 0.$$

Lemma 1 now confirms this observation.

3 Self-adjoint operators

An operator A is *self-adjoint* if, and only if,

$$\langle Ax, y \rangle = \langle x, Ay \rangle \text{ for all } x, y \in \mathcal{H}.$$

We assume throughout that

$$W_+,\ W_- \textit{ is a non-trivial adjoint-pair of wedges.}$$

Let $\mathcal{A}$ denote the set of all self-adjoint linear operators A on $\mathcal{H}$ with the property that

$$A : W_+ \to W_+.$$

(It is easy to see from Lemma 3 that $A : W_+ \to W_+$ if, and only if, $A : W_- \to W_-$ since A is self-adjoint.) Let $\mathcal{A}_0^{\pm}$ denote the set of elements of $\mathcal{A}$ which are identically zero on $W_{\pm}$. For $A \in \mathcal{A}$ define

$$\rho(A) = \sup\{\langle Ax, x \rangle : \| x \| = 1,\ x \in \mathcal{H}\}$$

$$\chi_+(A) = \sup\{\langle Ay, y \rangle : \| y \| = 1, y \in W_+\}$$

$$\chi_-(A) = \sup\{\langle Az, z \rangle : \| z \| = 1,\ z \in W_-\}.$$

Recall that every self-adjoint operator on a Hilbert space is bounded (this follows from the Closed Graph Theorem) and

$$\| A \| = \sup_{\| x \| = 1} |\langle Ax, x \rangle|.$$

In particular, if $B = \| A \| I + A$, then B is self-adjoint, $\langle Bx, x\rangle \geq 0$ for all x and

$$\| B \| = \sup_{\|x\|=1} \langle Bx, x\rangle.$$

Lemma 6 **(a)** *Suppose* $\{x_n\} \subset \mathcal{H}, \| x_n \| = 1$ *and* $\langle Ax_n, x_n\rangle \to \rho(A)$. *Then*

$$\| Ax_n - \rho(A)x_n \| \to 0 \textit{ as } n \to \infty.$$

(b) *If there exists* $x \in \mathcal{H}$ *with* $\| x \| = 1$ *and* $\langle Ax, x\rangle = \rho(A)$ *then* $Ax = \rho(A)x$.

(c) *If there exists* $w \in W_+$ *with* $\langle Aw, w\rangle = \chi_+(A)$, $\| w \| = 1$, *then* $Aw = \chi_+(A)w$.

(d) *If there exists* $w \in W_-$ *with* $\langle Aw, w\rangle = \chi_-(A)$, $\| w \| = 1$, *then* $Aw = \chi_-(A)w$.

Proof. (a) In the above notation,

$$\lim_{n\to\infty} \langle Bx_n, x_n\rangle = \| A \| + \rho(A) = \sup_{\|x\|=1} \langle Bx, x\rangle = \| B \| .$$

Hence

$$\begin{aligned} \| Ax_n - \rho(A)x_n \|^2 &= \| Bx_n - (\| A \| + \rho(A))x_n \|^2 \\ &= \| Bx_n - \| B \| x_n \|^2 = \| Bx_n \|^2 - 2 \| B \| \langle Bx_n, x_n\rangle + \| B \|^2 \| x_n \|^2 \\ &\leq \| B \|^2 - 2 \| B \| \langle Bx_n, x_n\rangle + \| B \|^2 \to 0 \text{ as } n \to \infty. \end{aligned}$$

(b) This is immediate from (a) with $x_n = x$ for all n.

(c), (d) It will suffice to prove (c). The proof of (d) is identical. If $\langle Aw, w\rangle = \chi_+(A)$ and $\| w \| = 1$, then for all $k \in W_+$ and $t > 0$

$$\frac{1}{t}\left\{\frac{\langle A(w+tk), w+tk\rangle}{\| w+tk \|^2} - \frac{\langle Aw, w\rangle}{\| w \|^2}\right\} \leq 0.$$

In the limit as $t \to 0$ this gives

$$\langle Aw, k\rangle \leq \langle Aw, w\rangle\langle k, w\rangle$$

for all $k \in W_+$, since $\| w \| = 1$. In particular, when $k = Aw$ this gives

$$\| w \|^2 \| Aw \|^2 = \| Aw \|^2 \leq \langle Aw, w \rangle^2 \leq \| Aw \|^2 \| w \|^2 .$$

Hence

$$\| Aw - \langle Aw, w \rangle w \|^2 = \| Aw \|^2 - \langle Aw, w \rangle^2 = 0.$$

This proves the result.

□

Theorem 7 **(a)** *ρ, χ_+ and χ_- are continuous and convex on $\mathcal{A}$.*

(b) *$\rho(A) = \max\{\chi_+(A), \chi_-(A)\}$ for all $A \in \mathcal{A}$.*

(c) *If W_+ is a cone, $\chi_+(A) > 0$ for $A \in \mathcal{A} \backslash \mathcal{A}_0^+$.*

(d) *If W_- is a cone, $\chi_-(A) > 0$ for $A \in \mathcal{A} \backslash \mathcal{A}_0^-$.*

(e) *If $\{W_+\}^\perp = \{0\}$ and $\rho(A) = \langle Ax, x \rangle$ for some $x \in \mathcal{H}$ with $\| x \| = 1$, then $\rho(A)$ is an eigenvalue of A with an eigenvector in W_+. If $\{+x, -x\} \cap W_+ = \emptyset$ then the eigenvalue $\rho(A)$ of A has an eigenvector in $\partial_\pi W_+$.*

(f) *If W_+ is a cone and $\rho(A) = \langle Ax, x \rangle$ for some $x \in \mathcal{H}$ with $\| x \| = 1$ then the eigenvalue $\rho(A)$ of A has an eigenvector in W_-. If $\{+x, -x\} \cap W_- = \emptyset$ then the eigenvalue $\rho(A)$ of A has an eigenvector in $\partial_\pi W_-$.*

(g) *Suppose W_+ is a cone, $\{W_+\}^\perp = \{0\}$ and $\rho(A)$ is an eigenvalue of A which has no eigenvector in $\partial_\pi W_+$. Then $\rho(A)$ is a simple eigenvalue of A with a unique, normalised eigenvector in $W_+ \cap -W_-$.*

Proof. (a) The continuity and convexity of ρ, χ_+ and χ_- follow directly from the definitions.

(b) It is clear that $\rho(A) \geq \max\{\chi_+(A), \chi_-(A)\}$. Let $\{x_n\} \subset \mathcal{H}, \| x_n \| = 1$ be such that

$$\langle Ax_n, x_n \rangle \to \rho(A) \text{ as } n \to \infty.$$

Since, by Theorem 4(f), $1 = \| \pi_+(x_n) + \pi_-(x_n) \|^2 = \| \pi_+(x_n) \|^2 + \| \pi_-(x_n) \|^2$, there is a sequence $\{n_k\}$ of natural numbers for which either $\| \pi_+(x_{n_k}) \| \geq 1/\sqrt{2}$ for all $k \in \mathbb{N}$ or $\| \pi_-(x_{n_k}) \| \geq 1/\sqrt{2}$ for all $k \in \mathbb{N}$. Let $x_{n_k} = y_k$ and suppose $\| \pi(y_k) \| \geq 1/\sqrt{2}$ for all k, where π denotes π_+ or π_-, as appropriate. Let $W = \pi(\mathcal{H})$. Then

$$\langle Ay_k, y_k \rangle \to \rho(A), \ \| y_k \| = 1, \ \| \pi(y_k) \| \geq 1/\sqrt{2} \text{ for all } k \in \mathbb{N}.$$

By Lemma 6, $\| Ay_k - \rho(A) y_k \| \to 0$. By Lemma 1

$$\frac{1}{2} \leq \| \pi(y_k) \|^2 = \langle y_k, \pi(y_k) \rangle$$

and

$$\langle w, y_k - \pi(y_k) \rangle \leq 0 \text{ for all } w \in W.$$

If $w = A\pi(y_k)$ then $w \in W$, since $A : W \to W$, and there results that

$$\begin{aligned} \langle A\pi(y_k), \pi(y_k) \rangle &\geq \langle A\pi(y_k), y_k \rangle = \langle \pi(y_k), Ay_k \rangle \\ &= \langle \pi(y_k), \rho(A) y_k \rangle + \langle \pi(y_k), Ay_k - \rho(A) y_k \rangle \\ &\geq \rho(A) \| \pi(y_k) \|^2 - \epsilon_k \| \pi(y_k) \| \end{aligned}$$

where

$$\epsilon_k = \| \rho(A) y_k - Ay_k \| \to 0 \text{ as } k \to \infty.$$

Since $\| \pi(y_k) \| \geq 1/\sqrt{2}$ and $\pi(y_k) \in W$ it follows that

$$\sup\{\langle Ay, y \rangle : \| y \| = 1, \ y \in W\} \geq \rho(A).$$

This proves (b).

(c), (d) It will suffice to prove (c), since (d) then follows by duality. Suppose that $\chi_+(A) \leq 0$. Then

$$\langle Aw, w\rangle \leq 0 \text{ for all } w \in W_+.$$

Let $z \in W_+$ and let $k = z + \pi_+(-z)$. By Theorem 4(c), $k \in -W_-$ and hence

$$\langle k, w\rangle \geq 0 \text{ for all } w \in W_+.$$

Also $k \in W_+$. So with $w = Ak$, there results that

$$\langle Ak, k\rangle \geq 0.$$

Since $k \in W_+$,

$$0 = \langle Ak, k\rangle \geq \langle Aw, w\rangle \text{ for } w \in W_+.$$

Therefore for any $t > 0$ and $w \in W_+$,

$$\frac{1}{t}\{\langle A(k+tw), k+tw\rangle - \langle Ak, k\rangle\} \leq 0$$

which, in the limit as $t \to 0$, gives

$$\langle Ak, w\rangle \leq 0 \text{ for all } w \in W_+.$$

But $Ak \in W_+$ and hence $Ak = 0$. Therefore, for any $z \in W_+$, $Ak = 0$ where $k = z + \pi_+(-z)$, and hence

$$Az = -A\pi_+(-z) \in -W_+.$$

Since W_+ is a cone, this implies that $Az = 0$ for all $z \in W_+$.

(e) Suppose $\rho(A) = \langle Ax, x\rangle$ with $\| x \| = 1$. Then, by Lemma 6, $Ax = \rho(A)x$. Now one of $\pi_+(x)$ and $\pi_+(-x)$ (at least) is non-zero, by Theorem 4(d). Without loss of generality, suppose

$$Ax = \rho(A)x, \ \| x \| = 1, \ w = \pi_+(x) \neq 0.$$

Then, by Lemma 1,

$$\| w \|^2 = \langle x, w \rangle \text{ and } \langle z, x - w \rangle \leq 0 \text{ for all } z \in W_+.$$

Hence, if $z = Aw$, it follows that

$$\begin{aligned} \langle Aw, w \rangle &\geq \langle Aw, x \rangle = \langle w, Ax \rangle \\ &= \rho(A) \langle w, x \rangle = \rho(A) \| w \|^2 . \end{aligned}$$

Hence, by Lemma 6, $Aw = \rho(A)w$ and $w \in W_+$. If $\{x, -x\} \cap W_+ = \emptyset$ then $w = \pi_+(x) \in \partial_\pi W_+$.

(f) Since W_+ is a cone, W_- is a wedge with $\{W_-\}^\perp = \{0\}$. Therefore part (e) yields the required result.

(g) Since W_+ is a cone, $\{W_-\}^\perp = \{0\}$. Hence, by parts (e) and (f) the eigenvalue $\rho(A)$ has eigenvectors in W_+ and in W_-. Now to show that $\rho(A)$ is a simple eigenvalue.

First note that since A is self-adjoint it follows easily that

$$\ker(\rho(A)I - A)^n = \ker(\rho(A)I - A) \text{ for all } n \in \mathbb{N}.$$

Now suppose that $\ker(\rho(A)I - A)$ has dimension greater than 1. By part (e), $\ker(\rho(A)I - A)\backslash\{0\}$ intersects both $-W_+$ and W_+. Since $-W_+ \cap W_+ = \{0\}$ and $\ker(\rho(A)I - A)\backslash\{0\}$ is connected, there exists $e \in \ker(\rho(A)I - A)\backslash(-W_+ \cup W_+)$. Hence neither e nor $-e$ is in W_+ and, in particular, $e \neq 0$. Since $\rho(A)e = Ae$ we infer from the last sentence of part (e) that $\rho(A)$ has an eigenvector in $\partial_\pi W_+$. This contradiction proves that $\rho(A)$ is a simple eigenvalue.

This completes the proof.

$\square$

Remarks

1. Suppose that the Hilbert space H is decomposed, using the spectral theorem, into $H_1 \oplus H_2$ where H_1 and H_2 are invariant subspaces for A on which A is positive definite and negative definite, respectively. When W_+ is a cone, the result of (b), applied to A restricted to H_2, shows that $A|_{H_2 \cap W_+} = 0$. (If A is injective then $H_2 \cap W_+ = \{0\}$.)

2. In (c), the assumption that W_+ is a cone is necessary since if the wedge W_+ is a proper subspace and $A = -I$, the result is clearly false.

3. In (e), if W_+ is a cone with $\{W_+\}^\perp = \{0\}$, then the results of (e) and (f) both hold. If W_+ is a wedge and not a cone then $\rho(A)$ can have an eigenvector in W_+, but perhaps not one in W_-. An example where this happens is $\mathcal{H} = \mathbb{R}^2, W_+ = \text{span}\{(1,1)\}, W_- = \text{span}\{(-1,1)\}$ and $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

4. If W_+ has additional structure relative to the operator A, then part (b) can be sharpened. For example, suppose the following.

$$If\ \{w_n\} \subset W_+,\ \| w_n \| = 1\ and\ \langle Aw_n, w_n\rangle \to \rho(A),\ then\ \ \| w_n + \pi_+(-w_n) \| \not\to 0.$$

Then the result of (b) can be strengthened to say $\rho(A) = \chi_-(A)$. To see why this is so, suppose that in the proof of part (b), the map π is π_+ (if it is π_- then $\chi_-(A) = \rho(A)$ is already proved.) Then, by part (b), there exists $\{w_n\} \subset W_+, \| w_n \| = 1$ with

$$\langle Aw_n, w_n\rangle \to \rho(A) \text{ as } n \to \infty.$$

Let $v_n = -w_n$. Then

$$\langle Av_n, v_n\rangle \to \rho(A),\ \| v_n \| = 1.$$

Now, by Theorem 4(f),

$$\pi_-(v_n) = v_n - \pi_+(v_n) = -(w_n + \pi_+(-w_n)) \not\to 0$$

as $n \to \infty$. A repeat of the argument for part (b), starting with $\{v_n\}$ instead of $\{x_n\}$ and now taking $\pi = \pi_-$ gives the required result.

The following is an immediate corollary.

Theorem 8 *Suppose that W_+ is a cone with $\{W_+\}^\perp = 0$ and that $-\|A\|$ is an eigenvalue of A with eigenvector $x, \|x\| = 1$.*

(a) $\{+x, -x\} \cap (W_+ \cup W_-) = \emptyset$.

(b) *$\| A \|^2$ is an eigenvalue of A^2 with an eigenvector in $\partial_\pi W_+$ and an eigenvector in $\partial_\pi W_-$.*

(c) *$\|A\|$ is an eigenvalue of A with an eigenvector in W_+ and an eigenvector in W_-.*

Proof. Since both W_+ and W_- are cones by Theorem 4(b) no eigenvector with a negative eigenvalue is in W_+ or W_-. This proves (a).

If $-\|A\|$ is an eigenvalue of A, then $\|A\|^2$ is an eigenvalue of A^2. Hence $\rho(A^2) = \|A\|^2$. By Theorem 7(e),(f), A^2 has an eigenvector, $w_\pm$ say, in $\partial_\pi W_\pm \backslash \{0\}$, and

$$(\|A\| I - A)(\|A\| I + A)w_\pm = 0.$$

Since $(\|A\| I + A)w \in W_\pm \backslash \{0\}$ (because $W_\pm$ is a cone) it follows that $\|A\|$ is an eigenvalue of A with an eigenvector in $W_\pm$.

□

Remark. The argument for part (b) shows that if $\|A\|^n$ is an eigenvalue of A^n, then $\|A\|$ is an eigenvalue of A with eigenvector in the cone.

Theorem 9 *If W_+ is a wedge and λ, μ is a pair of distinct eigenvalues of A with eigenvectors $w \in W_+$ and $k \in W_-$, respectively, then*

$$\langle w, k\rangle = 0, \;\; w \in \partial_\pi W_+ \textit{ and } k \in \partial_\pi W_-.$$

Proof. Because A is self-adjoint, the orthogonality of eigenvectors for distinct eigenvalues is well-known. But it is then immediate that

$$0 \neq k \in \{w\}^\perp \cap W_- \text{ and } 0 \neq w \in \{k\}^\perp \cap W_+.$$

Hence, by Theorem 5, $w \in \partial_\pi W_+$ and $k \in \partial_\pi W_-$.

$\square$

4 Compact operators

This section is concerned with hypotheses under which the existence of eigenvectors in W_+ can be established. If $A \in \mathcal{A}$, let

$$J_A(w) = \langle Aw, w \rangle, \; w \in W_+.$$

Then J_A is *weakly upper semi-continuous on* W_+ if, and only if, when $\{w_n\} \subset W_+$ and $w_n \rightharpoonup w$ in $\mathcal{H}$ then

$$J_A(w) \geq \limsup J_A(w_n).$$

Note that if $A : W_+ \to W_+$ is compact then J_A is weakly upper semi-continous.

Theorem 10 *Suppose that W_+ is a wedge and J_A is weakly upper semi-continuous on W_+. Then there exists $w \in W_+$ such that*

$$Aw = \chi_+(A)w, \quad \| w \| = 1.$$

Proof. By Theorem 7, we need only consider the case $\chi_+(A) > 0$. Let $\{w_n\} \subset W_+$, $\| w_n \| = 1$ be such that $\langle Aw_n, w_n \rangle \to \chi_+(A)$. Then $\{w_n\}$ has a weakly convergent subsequence $\{w_{n_k}\}$. Since W_+ is closed and convex, it is weakly closed and so there exists $w \in W_+$ with

$$\| w_{n_k} \| = 1, \quad w_{n_k} \rightharpoonup w, \quad J_A(w_{n_k}) \to \chi_+(A).$$

By hypothesis,

$$J_A(w) \geq \chi_+(A) > 0,$$

and $\| w \| \leq 1$ since the norm in $\mathcal{H}$ is weakly lower semi-continuous. If $\| w \| < 1$, then $J_A(w/ \| w \|) > \chi_+(A)$, which is a contradiction. Hence $\| w \| = 1$ and $J_A(w) = \chi_+(A)$. The result now follows from Theorem 6(c).

$\square$

In the next theorem we suppose that

$$W_+ \text{ is a cone in } \mathcal{H}, \quad A \in \mathcal{A},$$

$$A : H \to H \text{ is compact where } H = \overline{\text{span} W_+}.$$

Note, in particular, that no other assumptions are made about W_+. (It could be very thin, for example a ray $\{\alpha e : \alpha \geq 0\}$ where $Ae = \mu e$ for some $\mu > 0$. Of course the cones one has in mind for applications are not determined by the spectral structure of A; rather they may help to determine it.) In the context of parts (b) and (c) of the next theorem it is worth recalling the general result in Theorems 8 and 9. Let $A|_H$ denote the restriction of A to H.

Theorem 11 **(a)** *$\chi_+(A) = \chi_-(A) = \|A|_H\|$ is a non-negative eigenvalue of A with an eigenvector in W_+ and an eigenvector in W_-.*

(b) *If $\{W_+\}^\perp = 0$, then $\| A \| = \rho(A) = \chi_+(A) = \chi_-(A)$ is an eigenvalue of A. If $\| A \|$ has no eigenvector in $\partial_\pi(W_+)$ then $\rho(A)$ is a simple eigenvalue of A with a unique, normalised eigenvector in $W_+ \cap -W_-$.*

(c) *If $\{W_+\}^\perp = \{0\}$ and there is no eigenvector of A^2 in $\partial_\pi W_+$ corresponding to the eigenvalue $\| A \|^2$ then $- \| A \|$ is not an eigenvalue of A in $\mathcal{H}$.*

Proof. (a) Since $A : H \to H$ is compact and self-adjoint either $-\|A|_H\|$ or $\|A|_H\|$ is an eigenvalue of $A|_H$. Now the set orthogonal to W_+ in H is zero, by definition of H. Hence, if $-\|A|_H\|$ is an eigenvalue then so is $\|A|_H\|$, and there is a corresponding eigenvector in W_+, by Theorem 8(c). If $\| A|_H\|$ is an eigenvalue, then it has an eigenvector in W_+, and an eigenvector in W_-, by Theorem 7(e),(f). (This is because in H the wedge adjoint to W_+ is a cone, since the set orthogonal to W_+ in H is $\{0\}$. Also the adjoint wedge in H is $\{z \in H : \langle z, w\rangle \leq 0 \text{ for all } w \in W_+\} \subset W_-$.)

Since $\| A|_H\|$ is an eigenvalue with an eigenvector in W_+ and an eigenvector in W_-, it follows that $\chi_+(A) = \chi_-(A) = \| A|_H \|$.

(b) Since $\{W_+\}^\perp = \{0\}$, in part (a) H and $\mathcal{H}$ coincide. Hence $\chi_+(A) = \chi_-(A) = \rho(A) = \| A \|$ is an eigenvalue of A with an eigenvector in W_+. The simplicity result then follows from Theorem 7(g).

(c) Suppose $-\| A \|$ is an eigenvalue of A with eigenvector x. Then by Theorem 8(b) there is an eigenvector of the eigenvalue $\| A \|^2$ of A^2 in $\partial_\pi W_+$, which contradicts the hypotheses.

This completes the proof.

□

Remark. In part (a) of the preceding theorem it is not sufficient to assume that A is compact on W_+. For example, if $\mathcal{H} = \ell_2$, let $\mathbf{x} = \{x_i\}_{i\in\mathbb{N}}$ and

$$W_+ = \{\mathbf{x} = 0 \leq x_k \leq x_1/k\}.$$

Then bounded subsets of W_+ are compact so that $I : W_+ \to W_+$ is compact. But $H = \operatorname{span} W_+ = \ell_2$ and I is not compact on H.

5 Muiltiplication Operators on Cones

Here we prove that $\| A \| \in \sigma(A)$ when K, in a separable Hilbert space, is a cone which A maps to itself and $K^{\perp} = \{0\}$. The proof depends on the spectral theorem in the following multiplicative form.

Theorem 12 *(The Spectral Theorem) If A is a bounded, self-adjoint operator on a separable Hilbert space $\mathcal{H}$, then there is a finite measure space (X, M, μ) and a unitary map $U : H \to L_2(X, M, \mu)$ such that*

$$(UAU^{-1}f)(m) = F(m)f(m), \quad f \in L_2(X, M, \mu), \ m \in X$$

for some essentially bounded function F,

$$\inf \sigma(A) = ess\inf_X F, \quad \sup \sigma(A) = ess\sup_X F, \ and \quad \| A \| = ess\sup_X |F| = \| F \|_\infty .$$

(This result is Thm. VII.3, Corollary and the Proposition on page 229, [9])

$\square$

If K is a cone in $\mathcal{H}$ and U is unitary as in the spectral theorem, then $K_2 = U(K)$ is a cone in $L_2(X, M, \mu)$ and the multiplication operator $f \mapsto Ff$ in $L_2(X, M, \mu)$ is a bounded self-adjoint operator which leaves K_2 invariant. Also $K^{\perp} = \{0\}$ if and only if $K_2^{\perp} = \{0\}$ in their respective Hilbert spaces.

Let $F \in L_\infty(X, M, \mu)$ with

$$\alpha = \operatorname{ess}\inf_X F < \beta = \operatorname{ess}\sup_X F,$$

and define a self-adjoint operator B on $L_2(X, M, \mu)$ by multiplication

$$Bf = Ff, \quad f \in L_2(X, M, \mu).$$

Then $\| B \| = \| F \|_\infty = \max\{|\alpha|, |\beta|\}$. Suppose B maps a cone $K_2 \subset L_2(X, M, \mu)$ into itself where $K_2^\perp = \{0\}$. Then $\beta > 0$ by Theorem 7. The aim now is to show that $|\alpha| \leq \beta$ and hence that $\| B \| \in \sigma(B)$. The proof depends on the following observation. Here α and β are defined in terms of F as described above.

Theorem 13 *If $|\alpha| > \beta > 0 > \alpha$ then there exists a continuous function h which is positive in a neighbourhood of α in $[\alpha, \beta]$, and both h and $-h$ are the uniform limits on $[\alpha, \beta]$ of sequences of polynomials with non-negative coefficients.*

□

This result is a special case of the more general one proved in the next section. To see its significance for the spectrum of B, note that if p is any polynomial with non-negative coefficients then

$$p(F)f \in K_2 \text{ for all } f \in K_2,$$

since $B : K_2 \to K_2$. If h is as in the statement of the theorem then $h(F)f$, $-h(F)f \in K_2$ for all $f \in K_2$, since K_2 is closed in $L_2(X, M, \mu)$. Since K_2 is a cone this means that $h(F)f = 0$ for all $f \in K_2$ which implies that $h(F) = 0$, since $K_2^\perp = \{0\}$. Since $h > 0$ in a neighbourhood of $\alpha = \operatorname{ess\,inf}_X F$, this is a contradiction. Hence $|\alpha| \leq \beta$, which proves the required result.

The general construction of a non-zero function which, along with its negative, can be approximated by polynomials with non-negative coefficients on $[\alpha, \beta]$, $\alpha < 0 < \beta < |\alpha|$, has obvious implications for the spectrum of multiplication operators on various other classes of Banach spaces, $L_p(\Omega)$, $C(\Omega)$, etc., and indeed for more general operators in Banach spaces, via Davies's functional calculus [3]. We give it below when β has been normalised to equal 1.

6 Polynomials with non-negative coefficients

Let $a > 1$ be fixed and let $C[-a, 1]$ denote the Banach space of continuous functions f on $[-a, 1]$ with the maximum norm, $\| f \| = \max\{|f(t)| : t \in [-a, 1]\}$. If $d \in [1, a)$ let $C_d[-a, 1]$ denote the Banach subspace of $C[-a, 1]$ consisting of those functions which are zero on $[-d, 1]$.

Let P denote the convex subset of $C[-a, 1]$ consisting of all polynomials p with non-negative coefficients. If $p \in P$ then for $x \in [-a, 1]$,

$$p(x) = \sum_{k=0}^{n} q_k x^k, \quad q_k \geq 0, \ 0 \leq k \leq n.$$

Let $\overline{P}$ denote the closure in $C[-a, 1]$ of P. The existence of the function h in Section 5 is a special case of the following.

Theorem 14 *There exists $d \in [1, a)$ such that*

$$C_d[-a, 1] \subset \overline{P}.$$

The proof is reduced to a simpler question by the following propositions.

Proposition 15 *If there exists $g \in C_d[-a, 1] \cap \overline{P}$ with $g > 0$ on $(-d - \eta, -d)$ for some $\eta \in (0, a - d)$ then $C_d[-a, 1] \subset \overline{P}$.*

Proof. Suppose $f \in C_d[-a, 1] \cap \overline{P}$. Then $f \equiv 0$ on $[-d, 1]$ and there exists a sequence $\{p_n\} \subset P$ with $\| f - p_n \| \to 0$ as $n \to \infty$. We now show that $-f \in \overline{P}$.

Let $\epsilon > 0$ and choose $\alpha > 0$ such that $|e^{\alpha x}| < \epsilon/4 \| f \|$ for all $x \in [-a, -d]$. Now choose m_0 sufficiently large that

$$\left| \sum_{k=0}^{m_0} \frac{\alpha^k}{k!} x^k \right| < \epsilon/2 \| f \| \ \text{ for all } x \in [-a, -d].$$

Then let

$$q(x) = \sum_{k=1}^{m_0} \left(\frac{\alpha^k}{k!} \right) x^k, \; x \in [-a, -d].$$

Then $q \in P$ and since $f \in C_d[-a, 1]$

$$\begin{aligned} \| qf + f \| &= \sup\{|q(t)f(t) + f(t)| : t \in [-a, -d]\} \\ &\leq \epsilon/2. \end{aligned}$$

Hence

$$\begin{aligned} \| qp_n + f \| &\leq \| q(p_n - f) \| + \| qf + f \| \\ &\leq \| q \| \, \| p_n - f \| + \epsilon/2 \\ &< \epsilon \text{ for all } n \text{ sufficiently large .} \end{aligned}$$

Since the product $qp_n \in P$ this shows that $-f \in \overline{P}$. Since $\overline{P} \cap C_d[-a, 1]$ is clearly closed and convex, this shows that it is a closed linear subspace of $C[-a, 1]$ which is non-empty because g is an element of it. Clearly $C_d[-a, 1] \cap \overline{P}$ is closed under multiplication by its elements.

Now $g \in \overline{P} \cap C_d[-a, 1]$, where g is as in the statement of the theorem. For $\tau \in [d/a, 1]$ let

$$g^\tau(x) = g(\tau x), \; x \in [-a, 1].$$

If $p_n \in P$ and $\| p_n - g \| \to 0$ as $n \to \infty$ then clearly for $\tau \in [d/a, 1]$,

$$\begin{aligned} \| p_n^\tau - g^\tau \| &= \sup\{|p_n(\tau x) - g(\tau x)| : x \in [-a, 1]\} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $g^\tau \in \overline{P}$. Also for $x \in [-d, 1]$, $\tau x \in [-d, 1]$ and so $g^\tau(x) = 0$ for $x \in [-d, 1]$. Therefore $g^\tau \in \overline{P} \cap C_d[-a, 1]$ for all $\tau \in [d/a, 1]$.

Now suppose that $x \in [-a, -d)$. If $x \in (-d-\eta, -d)$, where η is given in the statement of the proposition, then $g(x) \neq 0$. If $x \in [-a, -d-\eta]$ let $\tau = -x^{-1}(d+\eta/2)$. Then $\tau \in [d/a, 1]$ and $g^{\tau}(x) = g(-d-\eta/2) \neq 0$.

This shows

$$\{x \in [-a,1] : f(x) = 0 \text{ for all } f \in \overline{P} \cap C_d[-a,1]\} = [-d,1].$$

Similarly $\{(x,y) : f(x) = f(y) \text{ for all } f \in \overline{P} \cap C_d[-a,1]\} = [-d,1] \times [-d,1]$. Hence, by Stone's Approximation Theorem ([10], Theorem 5), $\overline{P} \cap C_d[-a,1]$ generates $C_d[-a,1]$ with scalar multiplication and the ring operations of addition and pointwise multiplication. This proves the proposition.

□

Proposition 16 *There exists a function g as in the statement of the previous Proposition.*

Proof. Let $n \in \mathbb{N}$ be a natural number with $n > 1/2(a-1)$ and define $\phi_n : \mathbb{R} \to \mathbb{R}$ by

$$\phi_n(x) = (a+x)x^{2n}.$$

Note that $\phi_n'(x) = 0$ if, and only if, $x = 0$ or $x = -2na/(1+2n)$. Let

$$c_n = \frac{2na}{1+2n}.$$

Then $-c_n \in (-a, -1)$, because of the previous choice of n. Now

$$\phi_n(1) = (a+1)$$

and

$$\phi_n(-c_n) = \left(\frac{a}{1+2n}\right)\left(\frac{2na}{1+2n}\right)^{2n} \to \infty \text{ as } n \to \infty$$

since $a > 1$. Now choose and fix N, such that

$$N > \frac{1}{2(a-1)} \text{ and } \left(\frac{a}{1+2N}\right)\left(\frac{2Na}{1+2N}\right)^{2N} > 1+a.$$

Let

$$c = c_N \text{ and } \phi(x) = \phi_N(x), \quad x \in \mathbb{R}.$$

Since ϕ has critical points only at $-c$ and 0, $\phi(x) \geq 0$ on $[-a, \infty)$ and since

$$\phi(-c) > \phi(1) > \phi(0) = 0 = \phi(-a),$$

there exists $b > 1$ with the property that

$$\phi(-c) > \phi(x) \geq 0 \text{ for all } x \in [-a, b] \backslash \{-c\}.$$

Now define $\phi_m : [-a, b] \to \mathbb{R}$ as

$$\phi_m(x) = \frac{\phi(x)^m}{\left\{\int_{-a}^{b} \phi(y)^m dy\right\}}.$$

Then $\phi_m(-c) > \phi_m(x) \geq 0$ for all $x \in [-a, b] \backslash \{-c\}$,

$$\int_{-a}^{b} \phi_m(x) dx = 1,$$

and

$$\phi_m(x) = \left\{\frac{\phi(x)}{\left\{\int_{-a}^{b} \phi(y)^m dy\right\}^{1/m}}\right\}^m \to 0$$

as $m \to \infty$, uniformly for $x \in C$, where C is any compact subset of $[-a, b] \backslash \{-c\}$. It follows that

$$\phi_m(-c) \to \infty \text{ as } m \to \infty.$$

Let η be such that $0 < \eta < \min\{a - c, b - 1\}$ and let S be a compact subset of $(-c - \eta, -c) \subset [-a, b]$. Then $x + y \in [-a, b]$ if $x \in S$ and $y \in [0, \eta]$ and, since S is compact,

$$\int_{0}^{\eta} \phi_m(x + y) dy \to 1 \text{ as } m \to \infty,$$

uniformly for $x \in S$. Let $g \in C[-a, 1]$ be a non-negative function with compact support in S. Then

$$q_m(x) = \int_{0}^{\eta} \phi_m(x + y) g(x) dy \to g(x)$$

uniformly for $x \in [-a, 1]$.

Now note that ϕ_m is a polynomial of order $m(2N+1)$ with non-negative coefficients and hence

$$\phi_m(x+y) = \sum_{k=0}^{m(2N+1)} Q_k(y)x^k,$$

where Q_k is a polynomial with non-negative coefficients. Hence a polynomial $p_m \in P$ is defined for $x \in [-a, 1]$ by

$$\begin{aligned} p_m(x) &= \int_0^\eta \phi_m(x+y)g(-c-y)dy \\ &= \sum_{k=0}^{m(2N+1)} \left\{ \int_0^\eta Q_k(y)g(-c-y)dy \right\} x^k \\ &= \sum_{k=0}^{m(2N+1)} p_k x^k, \text{ say }, \end{aligned}$$

where $p_k \geq 0$ for all k because g is non-negative on $[-a, 1]$.

Now, for $x \in [-a, 1]$,

$$p_m(x) - g(x) = p_m(x) - q_m(x) + q_m(x) - g(x),$$

where q_m converges to g uniformly on $[-a, 1]$. Let $\epsilon > 0$ and let δ correspond to ϵ as in the statement of the uniform continuity of g on $[-a, 1]$. Then for any $x \in [-a, 1]$

$$\begin{aligned} |p_m(x) - q_m(x)| &= \left| \int_0^\eta \phi_m(x+y)\{g(-c-y) - g(x)\}dy \right| \\ &\leq \left| \int_{\{y \in (0,\eta): |x+y+c| \geq \delta\}} \phi_m(x+y)\{g(-c-y) - g(x)\}dy \right| + \epsilon \end{aligned}$$

since for $x \in [-a, 1]$,

$$\int_0^\eta |\phi_m(x+y)|dy \leq \int_{-a}^b \phi_m(y)dy = 1.$$

Also

$$\left| \int_{\{y \in (0,\eta): |x+y+c| \geq \delta\}} \phi_m(x+y)\{g(-c-y) - g(x)\}dy \right|$$

$$
\begin{aligned}
&\leq \quad 2 \parallel g \parallel \int_{\{y\in(0,\eta):|x+y+c|\geq\delta\}} \phi_m(x+y)dy \\
&\leq \quad 2 \parallel g \parallel \left\{ \int_{\{z:|z+c|\geq\delta, z\in[-a,b]\}} \phi_m(z)dz \right\} \\
&\to \quad 0 \text{ as } m \to \infty.
\end{aligned}
$$

Therefore $\parallel p_m - g \parallel \to 0$ as $m \to \infty$. Hence $g \in \overline{P}$. To complete the proof, let $[e,d] \subset (-c-\eta, -c)$ and let g be continuous on $[-a,1]$ with support $[e,d]$ and positive on (e,d).

$\square$

Hence there exists a function h which, along with its negative, is the uniform limit on $[-a,1]$ of a sequence of polynomials with non-negative coefficients, and which is non-zero in a neighbourhood of $-a$. This establishes the existence of the function h needed to prove that $\parallel A \parallel \in \sigma(A)$ as discussed in Section 5.

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