The Effect of Small Time-Delays on the Stability of Feedback Systems Described by Partial Differential Equations *

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Abstract: It has been observed that for many stable feedback systems, the introduction of arbitrarily small time delays into the loop causes instability. In this paper we present a systematic treatment of this phenomenon for a large class of systems described by partial differential equations. Our approach is based on a combination of input-output methods and modal analysis. We give a number of sufficient conditions for robustness/nonrobustness of closed loop input-output stability with respect to delays. Our framework includes a large class of ill-posed systems, i.e. systems whose open-loop transfer function is unbounded on any right half plane. We then analyze the relationship between the poles of the transfer function and the exponential modes of the underlying boundary value problem to derive internal stability properties from external ones.

Keywords: Small time delays; stability; robustness; partial differential equations; transfer functions.

AMS subject classifications: 93C20, 93C80, 93D09, 93D15, 93D25.

* This work was supported by the British Council/Daad (ARC project 464), by the Human Capital and Mobility programme (Project number CHRX-CT93-0402) and by the National Science Foundation (Grant DMS-9206986).

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1. Introduction

It is well known that for many stable feedback systems the introduction of small delays into the feedback loop can cause instability. While this phenomenon even occurs in finite-dimensional feedback systems, it becomes considerably more complicated in infinite dimensions due to the rich high-frequency behaviour exhibited by many distributed parameter systems. In the last decade various authors have presented specific partial differential equations (PDEs), or classes of related PDEs, which are exponentially stabilized by some feedback, but destabilized when arbitrarily small delays are introduced into the feedback loop, see, for example, Datko [4, 5, 6], Datko *et al.* [7], Desch and Wheeler [8], and Hannsgen *et al.* [11]. More precisely, it was shown that there exists a sequence of delays $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that the feedback system with delay ε_n in the loop has an unstable mode. By "mode" we mean a solution of the form $e^{s_0 t}\phi$, where $\phi \not\equiv 0$ is a function of the space variable. A mode is called unstable if the exponent s_0 satisfies $\mathrm{Re}\, s_0 \geq 0$. We say that a system described by partial differential equations is modally stable if it has no unstable modes. While the above papers are based on PDE and related techniques, the problem of robustness/nonrobustness in the presence of small delays has also been investigated using frequency-domain approaches, see Barman *et al.* [1] and Logemann *et al.* [15]. Closely related to these papers is the work by Georgiou and Smith [10]. However, their concept of w-stability is considerably stronger than robust stability with respect to small delays; it covers a large class of perturbations which represent high-frequncy modelling uncertainties including small time delays.

In the frequency-domain a linear time-invariant system is described by a transfer function matrix $\mathbf{H}$. Applying negative unity feedback in the presence of a delay $\varepsilon \geq 0$ in the loop leads to the closed-loop transfer function $\mathbf{G}_\varepsilon(s) = \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1}$. A transfer function is called spectrally stable if it is holomorphic in the closed right-half plane and it is called L^2-stable if it is holomorphic and bounded in the open right-half plane. If $\mathbf{G}_0$ is spectrally stable or L^2-stable, then either form of stability is called robust with respect to small delays if it is preserved for all sufficiently small delays, i.e. if there exists $\varepsilon^* > 0$ such that $\mathbf{G}_\varepsilon$ is stable for all $\varepsilon \in (0, \varepsilon^*)$. In [15] conditions are given (in terms of the high-frequency behaviour of $\mathbf{H}$) for robustness and nonrobustness of L^2-stability if the open-loop transfer function $\mathbf{H}$ is in the class of regular transfer functions, defined in Section 3. This class contains all well-posed transfer functions which are relevant in the applications to PDEs and functional differential equations, where well-posedness means that $\mathbf{H}$ is bounded on some right-half plane. While the frequency-domain approach in [15] is quite general in the sense that it is not tied to specific classes of PDEs and boundary conditions or specific feedback laws, the results obtained are external in the sense that the conclusions are in terms of $\mathbf{G}_\varepsilon$, and not in terms of the solutions of the PDE.

The purpose of this paper is twofold. The results in [15] apply to regular transfer functions. However, examples in the PDE literature (in particular, see [5] and [8]) seem to indicate that systems with open-loop transfer functions which are ill-posed (that is, not well-posed) are even more likely to be destabilized by delays than well-posed systems. Thus the first goal in this paper is to present destabilization results for systems with ill-posed open-loop transfer functions. We find that if the open-loop transfer function is ill-posed, then $\mathbf{G}_\varepsilon$ is not well-posed, and hence not L^2-stable, for all $\varepsilon > 0$. Moreover, in Theorem 3.7 we identify a class of ill-posed transfer functions for which there exist delays $\varepsilon_n \to 0$ and complex numbers s_n with $\operatorname{Re} s_n \to \infty$ such that $\mathbf{G}_{\varepsilon_n}$ has a pole at s_n.

The second goal of this paper is to identify a large class of multivariable systems described by PDEs for which results about robustness/nonrobustness of spectral stability translate to results about robustness/nonrobustness of modal stability. The class we consider consists of linear PDEs of spatial dimension 1, where on different parts of the space interval different partial differential equations are satisfied. The coefficients may depend on the spatial variable. The boundary conditions are general enough to allow all natural coupling conditions and in-span control, including some dynamic control. It is not assumed that the controlled, observed systems has any useful (A, B, C, D) state-space representation, and in fact many systems in our class do not. Moreover, we do not even assume that the free dynamics are described by a strongly continuous semigroup. In order to relate modal stability of the PDE to the spectral stability of the transfer function, a relationship between the modes of a PDE and the poles of the associated transfer function is needed. A natural relationship, expected for systems which have a state-space representation, is that a pole s_0 of the transfer function leads to a mode with exponent s_0 of the corresponding free dynamics and we find this to be the case for our class of systems. Of course, as in finite dimensions, it is possible that a mode will not appear as a pole of the associated transfer function, since any possible effect of the mode on the output might be anihilated by the observation and control operators. However, the results in Section 2 will be used to show that for any system in our class the exponents of the unstable modes and the unstable poles of the transfer function coincide, provided the closed-loop system (without delay) is modally stable. This result will in turn be applied to prove that for a large class of systems robustness of spectral stability implies robustness of modal stability.

The paper is organized as follows. Section 2 contains preliminaries about factorizations of matrix-valued meromorphic functions, which are needed later in the paper. In Section 3 we prove a number of robustness and destabilization results for ill-posed systems in a frequency-domain setting. Moreover, we give a summary of results from [15]. We emphasize that most of the results in Section 3 cover not only single-delay perturbations but also multi-delay perturbations of the form $\operatorname{diag}(e^{-\varepsilon_1 s}, \ldots, e^{-\varepsilon_m s})$. In Section 4 we discuss robustness/nonrobustness of modal stability for a class of systems described by PDEs with boundary control, and in Section 5 we consider distributed control for this class. Finally, in Section 6 we present three examples illustrating the results in the previous sections.

2. Coprime and bi-coprime factorizations of matrix-valued meromorphic functions

Let $\alpha \in \mathbb{R}$. We will use the notation $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \operatorname{Re} s > \alpha\}$. Furthermore, we define

$$\mathcal{H}_- := \{f : \Omega_f \to \mathbb{C} \,|\, \Omega_f \text{ open}, \Omega_f \supset \mathbb{C}_0^{cl} \text{ and } f \text{ holomorphic}\},$$

$$\mathcal{M}_- := \{f : \Omega_f \to \mathbb{C} \,|\, \Omega_f \text{ open}, \Omega_f \supset \mathbb{C}_0^{cl} \text{ and } f \text{ meromorphic}\}.$$

Let $m, l \in \mathbb{N}$ and let $\mathbf{H} \in \mathcal{M}_-^{m\times l}$. A number $p \in \mathbb{C}_0^{cl}$ is called a *pole* of $\mathbf{H}$, if p is a pole of at least one entry of $\mathbf{H}$. We say that $\mathbf{H} \in \mathcal{M}_-^{m\times m}$ is *invertible* if $\det \mathbf{H}(s) \not\equiv 0$. The inverse $\mathbf{H}^{-1}$ is then in $\mathcal{M}_-^{m\times m}$ and is given by Cramer's rule, i.e. $\mathbf{H}^{-1} = (1/\det \mathbf{H}) \operatorname{adj}(\mathbf{H})$, where $\operatorname{adj}(\mathbf{H})$ denotes the adjugate of $\mathbf{H}$. Obviously, $\mathbf{H}$ is invertible if $\det \mathbf{H}(s_0) \neq 0$ for some $s_0 \in \mathbb{C}_0^{cl}$.

A matrix $\mathbf{H} \in \mathcal{H}_-^{m\times m}$ is called *unimodular* if it is invertible and $\mathbf{H}^{-1}$ belongs to $\mathcal{H}_-^{m\times m}$ as well. Clearly, $\mathbf{H}$ is unimodular if and only if $\det \mathbf{H}(s) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$. Two matrices $\mathbf{N} \in \mathcal{H}_-^{m\times l}$ and $\mathbf{D} \in \mathcal{H}_-^{l\times l}$ are called *right-coprime* if there exist $\mathbf{X} \in \mathcal{H}_-^{l\times m}$ and $\mathbf{Y} \in \mathcal{H}_-^{l\times l}$ such that $\mathbf{XN} + \mathbf{YD} = I$. Similarly, two matrices $\tilde{\mathbf{N}} \in \mathcal{H}_-^{m\times l}$ and $\tilde{\mathbf{D}} \in \mathcal{H}_-^{m\times m}$ are called *left-coprime* if there exist $\tilde{\mathbf{X}} \in \mathcal{H}_-^{l\times m}$ and $\tilde{\mathbf{Y}} \in \mathcal{H}_-^{m\times m}$ such that $\tilde{\mathbf{N}}\tilde{\mathbf{X}} + \tilde{\mathbf{D}}\tilde{\mathbf{Y}} = I$.

The following lemma can be found in Logemann *et al.* [15].

Lemma 2.1 *Suppose* $\mathbf{H} \in \mathcal{M}_-^{m\times l}$. *Then the following statements hold:*

(i) $\mathbf{H}$ *admits a right-coprime factorization* $(\mathbf{N}, \mathbf{D})$ *over* $\mathcal{H}_-$, *i.e. there exist right-coprime matrices* $\mathbf{N} \in \mathcal{H}_-^{m\times l}$ *and* $\mathbf{D} \in \mathcal{H}_-^{l\times l}$ *such that* $\det \mathbf{D}(s) \not\equiv 0$ *and* $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}$. *The matrices* $\mathbf{N}$ *and* $\mathbf{D}$ *are unique up to multiplication from the right by a unimodular factor. A number* $p \in \mathbb{C}_0^{cl}$ *is a pole of* $\mathbf{H}$ *if and only if* $\det \mathbf{D}(p) = 0$.

(ii) $\mathbf{H}$ *admits a left-coprime factorization* $(\tilde{\mathbf{D}}, \tilde{\mathbf{N}})$ *over* $\mathcal{H}_-$, *i.e. there exist left-coprime matrices* $\tilde{\mathbf{D}} \in \mathcal{H}_-^{m\times m}$ *and* $\tilde{\mathbf{N}} \in \mathcal{H}_-^{m\times l}$ *such that* $\det \tilde{\mathbf{D}}(s) \not\equiv 0$ *and* $\mathbf{H} = \tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}$. *The matrices* $\tilde{\mathbf{D}}$ *and* $\tilde{\mathbf{N}}$ *are unique up to multiplication from the left by a unimodular factor. A number* $p \in \mathbb{C}_0^{cl}$ *is a pole of* $\mathbf{H}$ *if and only if* $\det \tilde{\mathbf{D}}(p) = 0$.

(iii) If $(\mathbf{N}, \mathbf{D})$ *is a right-coprime factorization of* $\mathbf{H}$ *over* $\mathcal{H}_-$ *and if* $(\tilde{\mathbf{D}}, \tilde{\mathbf{N}})$ *is a left-coprime factorization of* $\mathbf{H}$ *over* $\mathcal{H}_-$, *then the zeros of* $\det \mathbf{D}$ *and* $\det \tilde{\mathbf{D}}$ *in* $\mathbb{C}_0^{cl}$ *coincide (counting multiplicities).*

For $h \in \mathcal{H}_-$ and $s \in \mathbb{C}_0^{cl}$ we define

$$\operatorname{zero}_s(h) = \begin{cases} 0 & \text{if } \; h(s) \neq 0 \\ \text{multiplicity of } s & \text{if } \; h(s) = 0 \end{cases}.$$

Let $\mathbf{H} \in \mathcal{M}_-^{m\times l}$ and let $(\mathbf{N}, \mathbf{D})$ be a right-coprime factorization of $\mathbf{H}$ over $\mathcal{H}_-$. For $s \in \mathbb{C}_0^{cl}$ we define

$$\mathrm{pole}_s(\mathbf{H}) := \mathrm{zero}_s(\det \mathbf{D})\,.$$

If $p \in \mathbb{C}_0^{cl}$ is a pole of $\mathbf{H}$, then we define its *multiplicity* to be $\mathrm{pole}_p(\mathbf{H})$. Lemma 2.1 shows that we obtain the same concept if in the definition of $\mathrm{pole}_s(\mathbf{H})$ the matrix $\mathbf{D}$ is replaced by the "denominator" $\tilde{\mathbf{D}}$ of a left-coprime factorization of $\mathbf{H}$.

Suppose that $\mathbf{H} \in \mathcal{M}_-^{m\times l}$, $\mathbf{N} \in \mathcal{H}_-^{m\times n}$, $\mathbf{D} \in \mathcal{H}_-^{n\times n}$ and $\tilde{\mathbf{N}} \in \mathcal{H}_-^{n\times l}$, where $n \in \mathbb{N}$. The triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is called *bi-coprime* if $\mathbf{N}$ and $\mathbf{D}$ are right-coprime and if $\tilde{\mathbf{N}}$ and $\mathbf{D}$ are left-coprime. We say that the triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is a *bi-coprime factorization* of $\mathbf{H}$ over $\mathcal{H}_-$ if $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime, $\det \mathbf{D}(s) \not\equiv 0$ and $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}\tilde{\mathbf{N}}$. Let $s_0 \in \mathbb{C}_0^{cl}$. If

$$\mathrm{rank}\left(\mathbf{N}^T(s_0), \mathbf{D}^T(s_0)\right) = \mathrm{rank}\left(\tilde{\mathbf{N}}(s_0), \mathbf{D}(s_0)\right) = n\,, \tag{2.1}$$

then we say that $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ satisfies the *generalized Hautus conditions* in s_0.

Remark 2.2 It is not difficult to show that $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ satisfies the generalized Hautus conditions in $s_0 \in \mathbb{C}_0^{cl}$ if and only if there exist an open neighbourhood U of s_0 and matrices $\mathbf{X}$, $\mathbf{Y}$, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ whose entries are holomorphic on U and such that the Bezout identities $\mathbf{X}(s)\mathbf{N}(s) + \mathbf{Y}(s)\mathbf{D}(s) = I$ and $\tilde{\mathbf{N}}(s)\tilde{\mathbf{X}}(s) + \mathbf{D}(s)\tilde{\mathbf{Y}}(s) = I$ hold for all $s \in U$. In fact, while it is trivial that the Hautus conditions are necessary for the solvability of the Bezout equations, sufficiency follows from the fact that the elementary divisor theorem holds for holomorphic matrices, i.e. any holomorphic matrix is equivalent to its Smith form (see Narasimhan [16], p. 139). Moreover, the same argument applies globally, and thus $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime if and only if it satisfies the generalized Hautus conditions in s for all $s \in \mathbb{C}_0^{cl}$. $\diamond$

Proposition 2.3 *Let* $\mathbf{N} \in \mathcal{H}_-^{m\times n}$, $\mathbf{D} \in \mathcal{H}_-^{n\times n}$ *and* $\tilde{\mathbf{N}} \in \mathcal{H}_-^{n\times l}$, *suppose that* $\det \mathbf{D}(s) \not\equiv 0$ *and set* $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}\tilde{\mathbf{N}}$. *If (2.1) holds for some* $s_0 \in \mathbb{C}_0^{cl}$, *then*

$$\mathrm{pole}_{s_0}(\mathbf{H}) > 0 \quad \Longleftrightarrow \quad \mathrm{zero}_{s_0}(\det \mathbf{D}) > 0\ . \tag{2.2}$$

In particular, if the triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *is bi-coprime, then (2.2) holds for all* $s_0 \in \mathbb{C}_0^{cl}$.

The above proposition follows from a considerably stronger result by Pandolfi [18] which says that for any $s_0 \in \mathbb{C}_0^{cl}$

$$\mathrm{pole}_{s_0}(\mathbf{H}) = \mathrm{zero}_{s_0}(\det \mathbf{D}) \quad \Longleftrightarrow \quad (2.1)\ \text{holds}\ .$$

For sake of completeness we give a simple direct proof of Proposition 2.3 which does not rely on Pandolfi's result.

Proof of Proposition 2.3: Clearly, if $\mathrm{pole}_{s_0}(\mathbf{H}) > 0$, then $\mathrm{zero}_{s_0}(\det \mathbf{D}) > 0$. Conversely assume that $\mathrm{zero}_{s_0}(\det \mathbf{D}) > 0$. By Remark 2.2 there exist an open neighbourhood U of s_0 and holomorphic matrices $\mathbf{X}$, $\mathbf{Y}$, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ on U such that

$$\mathbf{X}(s)\mathbf{N}(s) + \mathbf{Y}(s)\mathbf{D}(s) = I \qquad \text{and} \qquad \tilde{\mathbf{N}}(s)\tilde{\mathbf{X}}(s) + \mathbf{D}(s)\tilde{\mathbf{Y}}(s) = I \qquad \forall\, s \in U\,. \tag{2.3}$$

By the first equation, $\mathbf{X}\mathbf{N}\mathbf{D}^{-1} + \mathbf{Y} = \mathbf{D}^{-1}$, and it follows that

$$\mathrm{pole}_{s_0}(\mathbf{N}\mathbf{D}^{-1}) > 0\,. \tag{2.4}$$

Multiplying the second equation in (2.3) with $\mathbf{N}\mathbf{D}^{-1}$ from the left we obtain $\mathbf{H}\tilde{\mathbf{X}} + \mathbf{N}\tilde{\mathbf{Y}} = \mathbf{N}\mathbf{D}^{-1}$. Combining this with (2.4) shows that $\mathrm{pole}_{s_0}(\mathbf{H}) > 0$. $\square$

Finally, we prove a simple lemma which will be needed in Section 4.

Lemma 2.4 *Let* $\mathbf{N} \in \mathcal{H}_-^{m\times n}$, $\mathbf{D} \in \mathcal{H}_-^{n\times n}$, $\tilde{\mathbf{N}} \in \mathcal{H}_-^{n\times l}$ *and* $\mathbf{F} \in \mathcal{H}_-^{l\times m}$, *suppose that* $\det \mathbf{D}(s) \not\equiv 0$ *and set* $\mathbf{H} = \mathbf{N}\mathbf{D}^{-1}\tilde{\mathbf{N}}$. *Then the following statements are true.*

(i) The matrix $I + \mathbf{F}\mathbf{H}$ *is invertible if and only if* $\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N}$ *is invertible. If this is the case, then*

$$\mathbf{H}(I + \mathbf{F}\mathbf{H})^{-1} = \mathbf{N}(\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N})^{-1}\tilde{\mathbf{N}}\,. \tag{2.5}$$

(ii) Let $s_0 \in \mathbb{C}_0^{cl}$. *The triple* $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *satisfies the generalized Hautus conditions in* s_0 *if and only if* $(\mathbf{N}, \mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N}, \tilde{\mathbf{N}})$ *does.*

Statement (ii) implies that if $\det(\mathbf{D}(s_0) + \tilde{\mathbf{N}}(s_0)\mathbf{F}(s_0)\mathbf{N}(s_0)) \neq 0$, *then* $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *satisfies the generalized Hautus conditions in* s_0. *In particular, if* $\det(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{F}(s)\mathbf{N}(s)) \neq 0$ *for all* $s \in \mathbb{C}_0^{cl}$, *then* $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *is bi-coprime.*

Proof: (i) If $I + \mathbf{F}\mathbf{H}$ is invertible, then it is not difficult to show that $\mathbf{D}^{-1}(I - \tilde{\mathbf{N}}(I + \mathbf{F}\mathbf{H})^{-1}\mathbf{F}\mathbf{N}\mathbf{D}^{-1})$ is the inverse of $\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N}$. Conversely, if $\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N}$ is invertible then $I - \mathbf{F}\mathbf{N}(\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N})^{-1}\tilde{\mathbf{N}}$ is the inverse of $I + \mathbf{F}\mathbf{H}$. The proof of formula (2.5) is straightforward and is left to the reader.

(ii) Let $x \in \mathbb{C}^n$. The claim follows from the following two equivalences

$$\begin{aligned}
\begin{pmatrix}\mathbf{N}(s_0)\\ \mathbf{D}(s_0)\end{pmatrix} x = 0 \quad &\Longleftrightarrow \quad \begin{pmatrix}\mathbf{N}(s_0)\\ \mathbf{D}(s_0) + \tilde{\mathbf{N}}(s_0)\mathbf{F}(s_0)\mathbf{N}(s_0)\end{pmatrix} x = 0\,,\\
x^T\left(\tilde{\mathbf{N}}(s_0), \mathbf{D}(s_0)\right) = 0 \quad &\Longleftrightarrow \quad x^T\left(\tilde{\mathbf{N}}(s_0), \mathbf{D}(s_0) + \tilde{\mathbf{N}}(s_0)\mathbf{F}(s_0)\mathbf{N}(s_0)\right) = 0\,.
\end{aligned}$$

$\square$

The above lemma is purely algebraic. It has, however, a control interpretation: If $\mathbf{H}$ is the transfer function of a linear system and if $\mathbf{F}$ is the transfer function of a feedback law, then (2.5) is the corresponding closed-loop transfer function. Moreover, in many cases the stability of the feedback system is determined by the zeros of $\det(\mathbf{D} + \tilde{\mathbf{N}}\mathbf{F}\mathbf{N})$. In Sections 4 and 5 we combine Proposition 2.3 and Lemma 2.4 to relate external and modal stability.

3. Robustness and nonrobustness with respect to small delays in the frequency-domain

Let $\alpha \in \mathbb{R}$. The field of all meromorphic functions on $\mathbb{C}_\alpha$ is denoted by $\mathcal{M}_\alpha$, while $\mathcal{H}_\alpha$ denotes the algebra of all holomorphic functions on $\mathbb{C}_\alpha$. H^∞_α denotes the algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$. We write H^∞ for H^∞_0. If $f \in \mathcal{M}_\alpha$ and $g \in \mathcal{M}_\beta$, where $\alpha < \beta$, and if $f(s) = g(s)$ for all $s \in \mathbb{C}_\beta$, then we shall identify f and g. Consequently, we have

$$\mathcal{M}_\alpha \subset \mathcal{M}_\beta\,, \quad \mathcal{H}_\alpha \subset \mathcal{H}_\beta\,, \quad H^\infty_\alpha \subset H^\infty_\beta \qquad \text{if } \ \alpha < \beta\ .$$

Let $\Omega \subset \mathbb{C}$. A function $\mathbf{H} : \Omega \to \mathbb{C}^{m\times m}$ is called a ($\mathbb{C}^{m\times m}$-valued) *transfer function* † if there exists $\alpha \in \mathbb{R}$ such that $\mathbb{C}_\alpha \subset \Omega$ and $\mathbf{H}|_{\mathbb{C}_\alpha} \in \mathcal{M}^{m\times m}_\alpha$.

Let $\mathbf{H}$ be a transfer function and for $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in [0,\infty)^m$ set

$$\mathbf{E}_{\vec{\varepsilon}}(s) := \operatorname{diag}_{1\le j\le m}(e^{-\varepsilon_j s})\,.$$

Consider the feedback system shown in Figure 3.1, where u is the input function, y is the output function and the block with transfer function $\mathbf{E}_{\vec{\varepsilon}}(s)$ represents a delay by $\varepsilon_j \ge 0$ in the the j-th feedback loop, $j = 1, \ldots, m$.

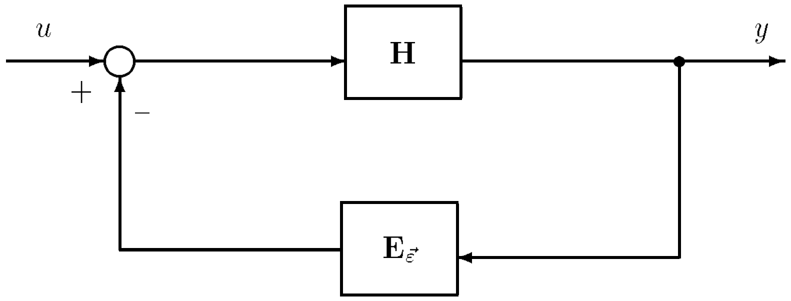

Figure 3.1: Feedback system with delay

If $\det(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)) \not\equiv 0$, then the function $\mathbf{G}_{\vec{\varepsilon}}$ defined by

$$\mathbf{G}_{\vec{\varepsilon}}(s) := \mathbf{H}(s)(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s))^{-1} \tag{3.1}$$

is a transfer function, the so-called closed-loop transfer function of the feedback system shown in Figure 3.1.

† We will assume that all transfer functions are square. If we want to apply feedback and the plant $\mathbf{H}$ is not square, we use a compensator $\mathbf{K}$ such that $\mathbf{HK}$ is square, and absorb $\mathbf{K}$ into $\mathbf{H}$.

Definition 3.1 $\mathbf{G}_{\vec{\varepsilon}}$ is called *L^2-stable* if $\mathbf{G}_{\vec{\varepsilon}} \in (H^\infty)^{m\times m}$. $\mathbf{G}_{\vec{\varepsilon}}$ is called *spectrally stable* if $\mathbf{G}_{\vec{\varepsilon}} \in \mathcal{H}_-^{m\times m}$. $\diamond$

It is well known that L^2-stability of $\mathbf{G}_{\vec{\varepsilon}}$ is equivalent to the existence of a constant $K > 0$ such that

$$\|y\|_{L^2(0,\infty)} \leq K\|u\|_{L^2(0,\infty)} \qquad \forall\, u \in L^2(0,\infty;\mathbb{C}^m)$$

when u and y are related as in Figure 3.1. Moreover, notice that $\mathbf{G}_{\vec{\varepsilon}}$ is spectrally stable if and only if $\mathbf{G}_{\vec{\varepsilon}} \in \mathcal{M}_-^{m\times m}$ and $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in $\mathbb{C}_0^{cl}$.

It is easy to see that if $\mathbf{G}_{\vec{\varepsilon}_0}$ is L^2-stable for some $\vec{\varepsilon}_0 \in [0,\infty)^m$, then $\mathbf{H} \in \mathcal{M}_0^{m\times m}$. Similarly, if there exists $\vec{\varepsilon}_0 \in [0,\infty)^m$ such that $\mathbf{G}_{\vec{\varepsilon}_0}$ is spectrally stable, then $\mathbf{H} \in \mathcal{M}_-^{m\times m}$.

Definition 3.2 Let $\mathbf{H}$ be a transfer function. $\mathbf{H}$ is called *well-posed* if $\mathbf{H} \in (H_\alpha^\infty)^{m\times m}$ for some $\alpha \in \mathbb{R}$. Moreover, $\mathbf{H}$ is called *regular* if it is well-posed and if the limit $\lim_{\xi\to+\infty} \mathbf{H}(\xi) = D$ exists (where $\xi \in \mathbb{R}$). The matrix D is called the *feedthrough matrix* of $\mathbf{H}$. If $\mathbf{H}$ is not well-posed, we say that it is *ill-posed.* $\diamond$

Well-posed transfer functions correspond to shift-invariant operators on $L^2_{loc}(0,\infty;\mathbb{C}^m)$ with finite growth bound, see Weiss [22], section 2. In particular, the transfer function of any abstract linear system (in the sense of Weiss [21]) is well-posed, as follows from [22], proposition 4.1. Conversely, for any well-posed transfer function $\mathbf{H}$ we can find an abstract linear system whose transfer function is $\mathbf{H}$, as follows from results in Salamon [20].

Practically all well-posed transfer functions of interest are regular. (In fact, it is a non-trivial exercise in complex analysis to construct an example of a well-posed and non-regular transfer function.) If the transfer function of an abstract linear system is regular (such systems are called regular), then the system has a simple and convenient state-space representation, analogous to a finite-dimensional state space system, see [22], section 2.

Nonrobustness results

For a transfer function $\mathbf{H}$ let $\mathfrak{P}_\mathbf{H}$ denote the set of its poles. We define

$$\gamma(\mathbf{H}) := \limsup_{|s|\to\infty,\, s\in\mathbb{C}_0\setminus\mathfrak{P}_\mathbf{H}} r(\mathbf{H}(s))\,, \tag{3.2}$$

where $r(\mathbf{H}(s))$ denotes the spectral radius of $\mathbf{H}(s)$. As usual, the spectrum of $\mathbf{H}(s)$ is denoted by $\sigma(\mathbf{H}(s))$.

The following destabilization result for regular transfer functions was proved by Logemann *et al.* [15]. We write $\mathbf{G}_\varepsilon$ for $\mathbf{G}_{\vec{\varepsilon}}$ if the components of ε satisfy $\varepsilon_i = \varepsilon \geq 0$ for all $i = 1,\ldots,m$.

Theorem 3.3 *Let* $\mathbf{H}$ *be a transfer function and suppose that* $\mathbf{H}$ *is regular with feedthrough matrix* D*. Then the following statements hold true:*

(i) If $\mathbf{G}_0$ *is* L^2*-stable, and if* $\gamma(\mathbf{H}) > 1$*, then there exists sequences* (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad p_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} |\mathrm{Im}\, p_n| = \infty\,,$$

and such that for any $n \in \mathbb{N}$*,* p_n *is a pole of* $\mathbf{G}_{\varepsilon_n}$*.*

(ii) If $r(D) > 1$*, then there exists sequences* (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad p_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} \mathrm{Re}\, p_n = \infty\,, \quad \lim_{n\to\infty} |\mathrm{Im}\, p_n| = \infty\,,$$

and such that for any $n \in \mathbb{N}$*,* p_n *is a pole of* $\mathbf{G}_{\varepsilon_n}$*.*

There are many partial differential equation models of physical and technical systems which have transfer function which are not well-posed. In the following we will show that, if $\mathbf{H}$ is not well-posed, then the robustness properties of $\mathbf{G}_0$ with respect to small delays are extremely poor (as has been observed by Desch and Wheeler [8] for a class of systems governed by skew-adjoint generators), in fact they are worse than in the regular case. First we show that if $\mathbf{H}$ is not well-posed, then $\mathbf{G}_{\vec{\varepsilon}}$ is not well-posed (and hence in particular is not L^2-stable) for any $\vec{\varepsilon} \in (0,\infty)^m$.

Proposition 3.4 *Suppose that* $\mathbf{H}$ *is a transfer function. If* $\mathbf{H}$ *is well-posed, then* $\mathbf{G}_{\vec{\varepsilon}}$ *is well-posed for all* $\vec{\varepsilon} \in (0,\infty)^m$*. Conversely, if* $\mathbf{G}_{\vec{\varepsilon}_0}$ *is well-posed for some* $\vec{\varepsilon}_0 \in (0,\infty)^m$*, then* $\mathbf{H}$ *is well-posed.*

Proof: If $\mathbf{H}$ is well-posed, then it is clear that for any $\vec{\varepsilon} \in (0,\infty)^m$ the transfer function $\mathbf{G}_{\vec{\varepsilon}}$ is also well-posed, since $\|\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)\| < 1$ in $\mathbb{C}_\alpha$ for large enough α. Conversely, assume that $\mathbf{G}_{\vec{\varepsilon}_0}$ is well-posed for some $\vec{\varepsilon}_0 \in (0,\infty)^m$. Seeking a contradiction, let us assume that $\mathbf{H}$ is not well-posed. Then there exist $s_n \in \mathbb{C}$, $s_n \notin \mathfrak{P}_{\mathbf{H}}$, with $\mathrm{Re}\, s_n \to \infty$ and such that

$$\lim_{n\to\infty} \|\mathbf{H}(s_n)\| = \infty\,.$$

By assumption $\mathbf{G}_{\vec{\varepsilon}_0}$ is holomorphic in some right-half plane $\mathbb{C}_\alpha$. Without loss of generality we may assume that $s_n \in \mathbb{C}_\alpha$ for all $n \in \mathbb{N}$. Consequently, $-1 \notin \sigma(\mathbf{E}_{\vec{\varepsilon}_0}(s_n)\mathbf{H}(s_n))$ for all $n \in \mathbb{N}$.

Case 1: $\det(\mathbf{H}(s)) \not\equiv 0$.
In this case we may assume without loss of generality that $\det(\mathbf{H}(s_n)) \neq 0$ for all $n \in \mathbb{N}$. Therefore

$$\|\mathbf{G}_{\vec{\varepsilon}_0}(s_n)\| = \|\mathbf{H}(s_n)(I + \mathbf{E}_{\vec{\varepsilon}_0}(s_n)\mathbf{H}(s_n))^{-1}\| = \|(\mathbf{H}^{-1}(s_n) + \mathbf{E}_{\vec{\varepsilon}_0}(s_n))^{-1}\|\,,$$

and hence, denoting the smallest singular value by σ_{min}, we obtain

$$\begin{aligned}
\|\mathbf{G}_{\vec{\varepsilon}_0}(s_n)\| &= \frac{1}{\sigma_{min}(\mathbf{H}^{-1}(s_n)+\mathbf{E}_{\vec{\varepsilon}_0}(s_n))} \\
&\geq \frac{1}{|\sigma_{min}(\mathbf{H}^{-1}(s_n)+\mathbf{E}_{\vec{\varepsilon}_0}(s_n)) - \sigma_{min}(\mathbf{H}^{-1}(s_n))| + \sigma_{min}(\mathbf{H}^{-1}(s_n))} \\
&\geq \frac{1}{\|\mathbf{E}_{\vec{\varepsilon}_0}(s_n)\| + \sigma_{min}(\mathbf{H}^{-1}(s_n))} \\
&= \frac{1}{\|\mathbf{E}_{\vec{\varepsilon}_0}(s_n)\| + 1/\|\mathbf{H}(s_n)\|}, \qquad (3.3)
\end{aligned}$$

where, in the last inequality, we have used a well-known estimate for singular values, see e.g. Horn and Johnson [12], p. 419. From (3.3) it follows that $\|\mathbf{G}_{\vec{\varepsilon}_0}(s_n)\| \to \infty$ as $n \to \infty$, contradicting the well-posedness of $\mathbf{G}_{\vec{\varepsilon}_0}$

CASE 2: $\det(\mathbf{H}(s)) \equiv 0$.
Clearly, for any $n \in \mathbb{N}$, there exist matrices $H_n^j \in \mathbb{C}^{m\times m}$ such that $\lim_{j\to\infty} \|\mathbf{H}(s_n) - H_n^j\| = 0$, $\det(H_n^j) \neq 0$ and $-1 \notin \sigma(\mathbf{E}_{\vec{\varepsilon}_0}(s_n)H_n^j)$ for all $j, n \in \mathbb{N}$. As in CASE 1 we then can show that

$$\|H_n^j(I + \mathbf{E}_{\vec{\varepsilon}_0}(s_n)H_n^j)^{-1}\| \geq \frac{1}{\|\mathbf{E}_{\vec{\varepsilon}_0}(s_n)\| + 1/\|H_n^j\|}$$

for all $j, n \in \mathbb{N}$. Now, letting j tend to ∞, we see that (3.3) remains true in the present case for all $n \in \mathbb{N}$, and hence we obtain a contradiction as in CASE 1. □

The following corollary is a trivial consequence of Proposition 3.4.

Corollary 3.5 *Let* $\mathbf{H}$ *be a transfer function. If* $\mathbf{H}$ *is not well-posed, then* $\mathbf{G}_{\vec{\varepsilon}} \notin (H^\infty)^{m\times m}$ *for all* $\vec{\varepsilon} \in (0,\infty)^m$.

The corollary says in particular that if $\mathbf{H}$ is not well-posed and if $\mathbf{G}_0$ is L^2-stable, then any positive time-delay will destroy the L^2-stability of the undelayed closed-loop system. Note that Corollary 3.5 does not ensure the existence of poles of $\mathbf{G}_{\vec{\varepsilon}}$ in the open-right half plane. The following simple example shows that there exist transfer functions which are not well-posed and which have the property that the delayed closed-loop transfer function is spectrally stable for all sufficiently small delays.

Example 3.6 Consider the transfer function $\mathbf{H}(s) = 2e^s$ which clearly is not well-posed. The delayed closed-loop transfer function

$$\mathbf{G}_\varepsilon(s) = \frac{\mathbf{H}(s)}{1 + e^{-\varepsilon s}\mathbf{H}(s)} = \frac{2e^s}{1 + 2e^{(1-\varepsilon)s}}$$

is in $\mathcal{H}_-$ for any $\varepsilon \in (0,1)$. ◇

The next theorem shows that for a large class of ill-posed transfer functions arbitarily small delays lead to closed-loop poles with arbitrarily large real parts. In order to state the theorem we introduce some more notation. For $\delta \in (0, \pi]$ define the open sector $\mathcal{S}(\delta)$ by

$$\mathcal{S}(\delta) := \{\lambda e^{\imath\psi} \mid \lambda \in (0, \infty),\ \psi \in (-\delta, \delta)\},$$

and for $\psi_0 \in [-\pi, \pi)$ set

$$\mathcal{S}(\psi_0, \delta) := e^{\imath\psi_0}\mathcal{S}(\delta).$$

Theorem 3.7 *Let* $\mathbf{H}$ *be a transfer function and asssume the following:*

(1) *There exist* $\theta \in (0, \pi/2)$ *and* $\alpha > 0$ *such that* $\mathbf{H}(s)$ *is holomorphic in* $\mathcal{S}(\theta) \cap \mathbb{C}_\alpha$.

(2) *There exist numbers* $\rho > \alpha$, $\nu > 0$, $\psi_0 \in [-\pi, \pi)$, $\delta \in (0, \theta)$ *and* $\eta \in (0, \pi/2)$ *such that*

$$\lim_{|s|\to\infty,\, s\in\mathcal{S}(\delta)} r(\mathbf{H}(s)) = \infty, \tag{3.4}$$

$$r(\mathbf{H}(s)) \le |s|^\nu \qquad \forall\, s \in \mathcal{S}(\delta) \cap \mathbb{C}_\rho, \tag{3.5}$$

$$\sigma(\mathbf{H}(s)) \subset \mathbb{C} \setminus \mathcal{S}(\psi_0, \eta) \qquad \forall\, s \in \mathcal{S}(\delta) \cap \mathbb{C}_\rho. \tag{3.6}$$

Then there exist sequences (ε_n) *and* (p_n) *with*

$$\varepsilon_n > 0, \quad \varepsilon_n \to 0, \quad p_n \in \mathbb{C}_0, \quad \operatorname{Im} p_n \to \infty, \quad \operatorname{Re} p_n \to \infty$$

and such that for any $n \in \mathbb{N}$, p_n *is a pole of* $\mathbf{G}_{\varepsilon_n}$.

Condition (3.4) guarantees that $\mathbf{H}$ is not well-posed. If $\mathbf{G}_0$ is L^2-stable, then it follows from lemma 6.3 in [15] that

$$\lim_{|s|\to\infty,\, s\in\mathcal{S}(\delta)} \|\mathbf{H}(s)\| = \infty \qquad \Longrightarrow \qquad \lim_{|s|\to\infty,\, s\in\mathcal{S}(\delta)} r(\mathbf{H}(s)) = \infty\,.$$

Therefore Theorem 3.7 remains true if $r(\mathbf{H}(s))$ is replaced by $\|\mathbf{H}(s)\|$, provided that $\mathbf{G}_0$ is L^2-stable. Condition (3.6) says that the spectrum of $\mathbf{H}(s)$ does not spiral around the origin as s moves in a sector of sufficiently small angle.

Proof of Theorem 3.7: Let $\mathbf{H}(s)$ be of size $m \times m$ and let $s \in \mathcal{S}(\theta) \cap \mathbb{C}_\alpha$. Moreover, let ℓ be the largest non-negative integer such that $\det(\lambda I - \mathbf{H}(s))$ can be written in the form

$$\det(\lambda I - \mathbf{H}(s)) = \lambda^\ell \left(\lambda^{m-\ell} + h_{m-\ell-1}(s)\lambda^{m-\ell-1} + \ldots + h_0(s)\right),$$

where the h_i are holomorphic functions defined on $\mathcal{S}(\theta) \cap \mathbb{C}_\alpha$. By (3.4), $\ell \le m-1$, and by the maximality of ℓ we have that $h_0(s) \not\equiv 0$. Clearly, $\ell > 0$ if and only if $\det \mathbf{H}(s) \equiv 0$. For $s \in \mathcal{S}(\theta) \cap \mathbb{C}_\alpha$ define

$$\mathbf{h}_s(\lambda) := \lambda^{m-\ell} + h_{m-\ell-1}(s)\lambda^{m-\ell-1} + \ldots + h_0(s), \qquad \zeta(\mathbf{h}_s) := \{\lambda \in \mathbb{C} \mid \mathbf{h}_s(\lambda) = 0\}.$$

Obviously, for given $s_0 \in \mathcal{S}(\theta) \cap \mathbb{C}_\alpha$ we have

$$0 \in \zeta(\mathbf{h}_{s_0}) \quad \Longleftrightarrow \quad h_0(s_0) = 0\,. \tag{3.7}$$

Moreover

$$\zeta(\mathbf{h}_s) \subset \sigma(\mathbf{H}(s)) \qquad \forall\, s \in \mathcal{S}(\theta) \cap \mathbb{C}_\alpha\,. \tag{3.8}$$

Set $\Psi_0 := \{\lambda e^{\imath\psi_0} \mid \lambda \geq 0\}$ and let $\arg : \mathbb{C} \setminus \Psi_0 \to (\psi_0, \psi_0 + 2\pi)$ be a continuous branch of the argument on $\mathbb{C} \setminus \Psi_0$. Then, by (3.6), $\arg \lambda$ is well defined for any $\lambda \in \sigma(\mathbf{H}(s)) \setminus \{0\}$, where $s \in \mathcal{S}(\delta) \cap \mathbb{C}_\rho$. For all such s we define

$$A_s := \{\arg \lambda \mid \lambda \in \sigma(\mathbf{H}(s)) \setminus \{0\}\}\,.$$

In the following let $\beta \in (0, \tan\delta)$. By (3.4) there exists $R \geq \rho$ such that $A_s \neq \emptyset$ for all $s \in \mathcal{S}(\delta) \cap \mathbb{C}_R$, and

$$\log r(\mathbf{H}(s)) - \frac{\sup A_s + 3\pi}{\beta} > 0 \qquad \forall\, s \in \mathcal{S}(\delta) \cap \mathbb{C}_R\,. \tag{3.9}$$

For $a, b \in \mathbb{C}$, let $[a, b]$ denote the closed segment in the complex plane with endpoints a and b. We define $\mu := \inf_{s \in \mathcal{S}(\delta) \cap \mathbb{C}_R} (\inf A_s + 3\pi) > 0$ and

$$\alpha_n := \frac{\mu}{2\nu \log(\sqrt{1+\beta^2}\, x_n)}\,, \tag{3.10}$$

where (x_n) is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = \infty$. We choose the x_n such that $x_n > \max(\alpha, R)$ and $\alpha_n < \beta$ for all $n \in \mathbb{N}$ and

$$h_0(s) \neq 0 \qquad \forall\, s \in [x_n, x_n + \imath\beta x_n]\,, \quad \forall\, n \in \mathbb{N}\,. \tag{3.11}$$

Note that by the choice of β and x_n

$$x_n + \imath\alpha_n x_n \in [x_n, x_n + \imath\beta x_n] \subset \mathcal{S}(\delta) \cap \mathbb{C}_R \qquad \forall\, n \in \mathbb{N}\,,$$

Using (3.5) and (3.10) we obtain that for all $s \in [x_n, x_n + \imath\beta x_n]$

$$\begin{aligned} \log r(\mathbf{H}(s)) - \frac{\inf A_s + 3\pi}{\alpha_n} &\leq \log |s|^\nu - \frac{\mu}{\alpha_n} \\ &\leq \nu \log(\sqrt{1+\beta^2}\, x_n) - 2\nu \log(\sqrt{1+\beta^2}\, x_n) \\ &< 0\,. \end{aligned} \tag{3.12}$$

Setting $z_n' := (1 + \imath\alpha_n) x_n$ and $z_n'' := (1 + \imath\beta) x_n$ we have that

$$[z_n', z_n''] \subset [x_n, x_n + \imath\beta x_n] \subset \mathcal{S}(\delta) \cap \mathbb{C}_R \qquad \forall\, n \in \mathbb{N}\,,$$

and moreover
$$\frac{\mathrm{Re}\ z_n'}{\mathrm{Im}\ z_n'} = \frac{1}{\alpha_n}, \qquad \frac{\mathrm{Re}\ z_n''}{\mathrm{Im}\ z_n''} = \frac{1}{\beta}.$$
Hence it follows from (3.12) and (3.9) that

$$\log r(\mathbf{H}(z_n')) - \frac{(\inf A_{z_n'} + 3\pi)\mathrm{Re}\ z_n'}{\mathrm{Im}\ z_n'} < 0 \qquad \forall n \in \mathbb{N}, \tag{3.13}$$
$$\log r(\mathbf{H}(z_n'')) - \frac{(\sup A_{z_n''} + 3\pi)\mathrm{Re}\ z_n''}{\mathrm{Im}\ z_n''} > 0 \qquad \forall n \in \mathbb{N}. \tag{3.14}$$

An application of proposition 5.2 in [15] shows that there exists a holomorphic function λ_n defined on open set $U_n \supset [z_n', z_n'']$ such that $|\lambda_n(z_n'')| = r(\mathbf{H}(z_n''))$ and $\lambda_n(s) \in \zeta(\mathbf{h}_s)$ for all $s \in U_n$.† A combination of (3.7) and (3.11) shows that $\lambda_n(s) \neq 0$ for all $s \in U_n$, provided we choose U_n small enough. It then follows from (3.13) and (3.14) that

$$\log|\lambda_n(z_n')| - \frac{(\arg\lambda(z_n') + 3\pi)\mathrm{Re}\ z_n'}{\mathrm{Im}\ z_n'} < 0 \qquad \forall n \in \mathbb{N}, \tag{3.15}$$
$$\log|\lambda_n(z_n'')| - \frac{(\arg\lambda(z_n'') + 3\pi)\mathrm{Re}\ z_n''}{\mathrm{Im}\ z_n''} > 0 \qquad \forall n \in \mathbb{N}. \tag{3.16}$$

We show that there exist sequences (ε_n) in $(0,\infty)$ and (p_n) in $\mathbb{C}_0$ with $\varepsilon_n \downarrow 0$ and $p_n \in [z_n', z_n'']$ and such that

$$\log\lambda_n(p_n) - \varepsilon_n p_n = -3\imath\pi \qquad \forall n \in \mathbb{N}, \tag{3.17}$$

where we choose the branch of the logarithm to be $\log z = \log|z| + \imath\arg z$, with $z \in \mathbb{C}\setminus\Psi_0$ and $\log|z| \in \mathbb{R}$. Since, by (3.8), $\lambda_n(s) \in \sigma(\mathbf{H}(s))$ for all $s \in U_n$, it is easy to see that (3.17) is a sufficient condition for $\mathbf{G}_{\varepsilon_n}(s) = \mathbf{H}(s)(I + e^{-\varepsilon_n s}\mathbf{H}(s))^{-1}$ to have a pole at p_n.

For each $n \in \mathbb{N}$ and each $s \in [z_n', z_n'']$ the ray

$$R_n(s) := \{\log\lambda_n(s) - es \,|\, e \in [0,\infty)\}$$

intersects the horizontal line

$$L := \{s \in \mathbb{C} \,|\, \mathrm{Im}\ s = -3\pi\}.$$

Indeed, for $e = 0$ the corresponding point of $R_n(s)$ is above L, while for large $e > 0$ the corresponding point is below L.

† This is only true if $U_n \cap \mathfrak{C}_{\mathbf{H}} = \emptyset$, where $\mathfrak{C}_{\mathbf{H}}$ denotes the set of critical points of $\mathbf{H}$ as defined in [15]. However, since $\mathfrak{C}_{\mathbf{H}}$ is a discrete set, we can choose the x_n such that $[x_n, x_n + \imath\beta x_n] \cap \mathfrak{C}_{\mathbf{H}} = \emptyset$. Consequently, $U_n \cap \mathfrak{C}_{\mathbf{H}} = \emptyset$ for sufficiently small sets U_n.

Thus, for each $n \in \mathbb{N}$, we can define real-valued functions $w_n(s)$ and $e_n(s) > 0$ for $s \in [z'_n, z''_n]$ such that

$$\log \lambda_n(s) - e_n(s)s = w_n(s) - 3\imath\pi \,. \tag{3.18}$$

Taking real and imaginary parts in (3.18) it follows that for all $n \in \mathbb{N}$ and for all $s \in [z'_n, z''_n]$

$$e_n(s) = \frac{\arg \lambda_n(s) + 3\pi}{\operatorname{Im}\, s}\,, \tag{3.19}$$

$$w_n(s) = \log|\lambda_n(s)| - \frac{(\arg \lambda_n(s) + 3\pi)\operatorname{Re}\, s}{\operatorname{Im}\, s}\,. \tag{3.20}$$

Using (3.15) and (3.16) we see that for all $n \in \mathbb{N}$

$$w_n(z'_n) < 0\,, \quad w_n(z''_n) > 0\,.$$

Hence, since w_n is continuous on $[z'_n, z''_n]$, there exists $p_n \in [z'_n, z''_n]$ such that $w_n(p_n) = 0$ for all $n \in \mathbb{N}$. Setting $\varepsilon_n := e_n(p_n) > 0$ it follows from (3.18) that p_n and ε_n satisfy (3.17). Finally, Re $p_n = x_n \to \infty$, and moreover

$$\operatorname{Im}\, p_n \geq \alpha_n x_n = \frac{\mu x_n}{2\nu \log(\sqrt{1+\beta^2}\, x_n)} \to \infty\,. \tag{3.21}$$

Combining (3.19) and (3.21) shows that $\varepsilon_n \to 0$. □

The next result gives a sufficient condition for assumption (2) in Theorem 3.7 to be satisfied.

Proposition 3.8 *Let* $\mathbf{H}$ *be a transfer function and suppose that assumption (1) in Theorem 3.7 is satisfied. Moreover, assume that there exist a number* $\alpha \in \mathbb{R}$ *and a transfer function* $\tilde{\mathbf{H}}$ *such that*

$$\mathbf{H}(s) = (a + bs)^\nu \tilde{\mathbf{H}}(s) \qquad \forall\, s \in \mathbb{C}_\alpha\,,$$

where $a, b \in \mathbb{C}$, $b \neq 0$ *and* $\nu > 0$. *If there exists* $\delta > 0$ *such that*

$$\lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} \tilde{\mathbf{H}}(s) = D\,, \quad D \in \mathbb{C}^{m\times m}\,,$$

with $\det D \neq 0$, *then* $\mathbf{H}(s)$ *satisfies assumption (2) in Theorem 3.7.*

The proof of Proposition 3.8 is straightforward and is left to the reader.

Finally, we give a simple instability result which applies to regular, well-posed as well as to ill-posed transfer functions.

Proposition 3.9 *Let $\mathbf{H}$ be a transfer function and assume that $\mathbf{H} \in \mathcal{M}_{-}^{m\times m}$. If*

$$\liminf_{\omega\to\infty,\, \imath\omega\notin\mathfrak{P}_{\mathbf{H}}} \sigma_{max}(\mathbf{H}(\imath\omega)) < 1 \quad \textit{and} \quad \limsup_{\omega\to\infty,\, \imath\omega\notin\mathfrak{P}_{\mathbf{H}}} \sigma_{min}(\mathbf{H}(\imath\omega)) > 1\,, \tag{3.22}$$

where σ_{max} and σ_{min} denote the maximal † *and the minimal singular value, respectively, then there exist sequences (ε_n) and (ω_n) in $(0,\infty)$ with*

$$\lim_{n\to\infty} \varepsilon_n = 0\,, \quad \lim_{n\to\infty} \omega_n = \infty\,,$$

and such that for any $n \in \mathbb{N}$, $\imath\omega_n$ is a pole of $\mathbf{G}_{\varepsilon_n}$.

We obtain a similar result if in (3.22) ∞ is replaced by $-\infty$. For the proof of Proposition 3.9 we need the following lemma.

Lemma 3.10 *Suppose that $\mathbf{H}$ is a transfer function and suppose there exists $\varepsilon \geq 0$ such that $\mathbf{G}_\varepsilon \in \mathcal{M}_{-}^{m\times m}$. If $s_0 \in \mathbb{C}_0^{cl}$ is a pole of $\mathbf{H}$, but not a pole of $\mathbf{G}_\varepsilon$, then $\lim_{s\to s_0} r(\mathbf{H}(s)) = \infty$.*

The above lemma is a slight generalization of lemma 6.3 in [15] whose proof is extended easily to the present situation.

Proof of Proposition 3.9: First of all notice that

$$\sigma_{min}(\mathbf{H}(s)) \leq r(\mathbf{H}(s)) \leq \sigma_{max}(\mathbf{H}(s))\,, \qquad \forall\, s \in \mathbb{C}_0^{cl}\,,\ s \notin \mathfrak{P}_{\mathbf{H}}\,. \tag{3.23}$$

By assumption there exist sequences (ω_n') and (ω_n'') in $(0,\infty)$ with

$$\omega_n' < \omega_n''\,, \quad \lim_{n\to\infty} \omega_n' = \lim_{n\to\infty} \omega_n'' = \infty\,,$$

and such that

$$\sigma_{max}(\mathbf{H}(\imath\omega_n')) < 1\,, \quad \sigma_{min}(\mathbf{H}(\imath\omega_n'')) > 1\,. \tag{3.24}$$

There are two cases: either the transfer function $\mathbf{H}$ is continuous on $[\imath\omega_n', \imath\omega_n'']$ or it is not. In the latter case the segment $[\imath\omega_n', \imath\omega_n'']$ contains poles of $\mathbf{H}$.

Case 1: If $\mathbf{H}(s)$ is continuous on $[\imath\omega_n', \imath\omega_n'']$, then so is $r(\mathbf{H}(s))$. Therefore, it follows from (3.23) and (3.24) that there exists $\omega_n \in (\omega_n', \omega_n'')$ such that $r(\mathbf{H}(\imath\omega_n)) = 1$. This shows that there exist numbers $\psi_n \in [0, 2\pi)$ such that $e^{\imath\psi_n} \in \sigma(\mathbf{H}(\imath\omega_n))$. Defining $\varepsilon_n := (\psi_n + \pi)/\omega_n$, we obtain

$$-1 = e^{\imath\psi_n} e^{-\imath\varepsilon_n\omega_n} \in \sigma(e^{-\imath\varepsilon_n\omega_n}\mathbf{H}(\imath\omega_n))\,,$$

† Recall that the maximal singular value of a matrix is equal to the operator norm induced by the Euclidean norm.

which in turn implies that $\mathbf{G}_{\varepsilon_n}$ has a pole at $\imath\omega_n$. By construction, $\lim_{n\to\infty}\omega_n = \infty$, and therefore $\lim_{n\to\infty}\varepsilon_n = 0$.

CASE 2: If $\mathbf{H}$ is not continuous on $[\imath\omega_n', \imath\omega_n'']$, then there exists $\tilde{\omega}_n \in (\omega_n', \omega_n'')$ such that $\imath\tilde{\omega}_n$ is a pole of $\mathbf{H}$. Moreover, we may choose $\tilde{\omega}_n$ such that $\mathbf{H}$ is continuous on $[\imath\omega_n', \imath\tilde{\omega}_n)$. If $\lim_{\omega\to\tilde{\omega}_n} r(\mathbf{H}(\imath\omega)) = \infty$, then, by (3.23) and (3.24), there exists $\omega_n \in (\omega_n', \tilde{\omega}_n)$ such that $r(\mathbf{H}(\imath\omega_n)) = 1$ and we can argue as in CASE 1. If $\lim_{\omega\to\tilde{\omega}_n} r(\mathbf{H}(\imath\omega)) \neq \infty$ or if this limit does not exist, then an application of Lemma 3.10 shows that $\imath\tilde{\omega}_n$ is a pole of $\mathbf{G}_\varepsilon$ for any $\varepsilon \geq 0$. □

Robustness results

The following result complements Theorem 3.3.

Theorem 3.11 *Let $\mathbf{H}$ be a transfer function and suppose that $\mathbf{G}_0$ is L^2-stable (spectrally stable). If $\gamma(\mathbf{H}) < 1$, then there exists $\varepsilon^* > 0$ such that $\mathbf{G}_\varepsilon$ is L^2-stable (spectrally stable) for all $\varepsilon \in (0, \varepsilon^*)$.*

The part of the theorem which relates to L^2-stability is proved in [15], theorem 6.1. Robustness of spectral stability can be shown in a similar way and is therefore left to the reader.

Theorem 3.11 deals with perturbations of the form $e^{-\varepsilon s}$. In a multivariable setting it is natural to consider the more general class of *multi-delay perturbations* of the form $\mathbf{E}_{\vec{\varepsilon}}$. There are simple examples which show that Theorem 3.11 does not remain true for multi-delay perturbations, cf. [15], example 6.4. It is not difficult to prove that if in Theorem 3.11 $\gamma(\mathbf{H})$ is replaced by the number

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0\setminus\mathfrak{P}_{\mathbf{H}}} \|\mathbf{H}(s)\|\,,$$

then we obtain a sufficient condition for robustness in the presence of multi-delay perturbations. However, it can be shown that this condition can be very conservative (which is not surprising since we are dealing with a highly structured class of perturbations). A better result can be derived by using structured singular values (see e.g. Packard and Doyle [17]). To this end set

$$\mathbf{\Delta} := \{\mathrm{diag}_{1\leq j\leq m}(s_j) \,|\, s_j \in \mathbb{C}\} \subset \mathbb{C}^{m\times m}\,.$$

The structured singular value $\mu_{\mathbf{\Delta}}(M)$ of $M \in \mathbb{C}^{m\times m}$ with respect to $\mathbf{\Delta}$ is defined by

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\{\|\Delta\| \,|\, \Delta \in \mathbf{\Delta}\,,\ \det(I - M\Delta) = 0\}}\,,$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$. For any $M \in \mathbb{C}^{m\times m}$ we have that $r(M) \leq \mu_{\mathbf{\Delta}}(M) \leq \|M\|$, see [17].

Theorem 3.12 *Let* $\mathbf{H}$ *be a transfer function and suppose that* $\mathbf{G}_0$ *is spectrally stable (L^2-stable). If*

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0\setminus\mathfrak{P}_{\mathbf{H}}} \mu_{\mathbf{\Delta}}(\mathbf{H}(s)) < 1\,, \tag{3.25}$$

then there exists $\varepsilon^* > 0$ *such that the transfer function* $\mathbf{G}_{\vec{\varepsilon}}$ *is spectrally-stable (L^2-stable) for all* $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in [0,\infty)^m$ *satisfying* $\|\vec{\varepsilon}\| < \varepsilon^*$.

Proof: In the following sections we will not use the L^2-stability part of the theorem. Therefore we prove only the part which relates to spectral stability. So let us assume that (3.25) holds and that $\mathbf{G}_0$ is spectrally stable, i.e. $\mathbf{G}_0 \in \mathcal{H}_-^{m\times m}$. As already mentioned, the latter implies that $\mathbf{H} \in \mathcal{M}_-^{m\times m}$. We proceed in three steps.

STEP 1: By (3.25) there exist numbers $R > 0$ and $\rho \in (0,1)$ such that

$$r(\mathbf{H}(s)) \le \mu_{\mathbf{\Delta}}(\mathbf{H}(s)) \le \rho \qquad \forall\, s \in E_R \setminus \mathfrak{P}_{\mathbf{H}}\,, \tag{3.26}$$

where $E_R := \{s \in \mathbb{C}_0 \mid |s| > R\}$. Since $\mathbf{G}_0$ is spectrally stable, it follows from Lemma 3.10 that $\mathbf{H}$ has no poles in E_R^{cl}.

STEP 2: By STEP 1, $E_R^{cl} \cap \mathfrak{P}_{\mathbf{H}} = \emptyset$, and hence we obtain from (3.26) using the continuity of $\mu_{\mathbf{\Delta}}(\cdot)$ (cf. Packard and Doyle [17])

$$\mu_{\mathbf{\Delta}}(\mathbf{H}(s)) \le \rho < 1 \qquad \forall\, s \in E_R^{cl}\,. \tag{3.27}$$

Clearly, $\|\mathbf{E}_{\vec{\varepsilon}}(s)\| \le 1$ for all $\vec{\varepsilon} \in [0,\infty)^m$ and all $s \in E_R^{cl}$, and hence

$$\det(I + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)) \neq 0 \qquad \forall\, \vec{\varepsilon} \in [0,\infty)^m\,,\ \forall\, s \in E_R^{cl}\,.$$

Combining this with the result of STEP 1, we see that for any $\vec{\varepsilon} \in [0,\infty)^m$, $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in E_R^{cl}.

STEP 3: It remains to show that there exists $\varepsilon^* > 0$ such that for all $\vec{\varepsilon} \in \mathbb{R}_+^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$, $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in the set $D_R := \{s \in \mathbb{C}_0^{cl} \mid |s| \le R\}$. To this end let $(\mathbf{N}, \mathbf{D})$ be a right-coprime factorization of $\mathbf{H}$ over $\mathcal{H}_-$. Then, $\mathbf{G}_{\vec{\varepsilon}} = \mathbf{N}(\mathbf{D} + \mathbf{E}_{\vec{\varepsilon}}\mathbf{N})^{-1}$, and it is easy to show that $(\mathbf{N}, \mathbf{D} + \mathbf{E}_{\vec{\varepsilon}}\mathbf{N})$ is a right-coprime factorization of $\mathbf{G}_{\vec{\varepsilon}}$ over $\mathcal{H}_-$. Since $\mathbf{G}_0 \in \mathcal{H}_-^{m\times m}$, it follows via Lemma 2.1, part (i), that

$$\inf_{s\in D_R} |\det(\mathbf{D}(s) + \mathbf{N}(s))| > 0\,. \tag{3.28}$$

By the compactness of D_R it is clear that $\mathbf{E}_{\vec{\varepsilon}}(s)$ converges uniformly to I on D_R as $\|\vec{\varepsilon}\| \to 0$. Consequently, (3.28) yields that there exists $\varepsilon^* > 0$ such that for all $\vec{\varepsilon} \in \mathbb{R}_+^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$

$$\inf_{s\in D_R} |\det(\mathbf{D}(s) + \mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s))| > 0\,,$$

which in turn implies that for all $\vec{\varepsilon}$ as above $\mathbf{G}_{\vec{\varepsilon}}$ has no poles in D_R. $\square$

It seems to be a difficult open problem whether the condition

$$\limsup_{|s|\to\infty,\, s\in\mathbb{C}_0\setminus\mathfrak{P}_{\mathbf{H}}} \mu_{\mathbf{\Delta}}(\mathbf{H}(s)) > 1\,,$$

implies a lack of robustness with respect to small multi-delay perturbations.

4. Robustness and nonrobustness results for PDE's with boundary control

In the following we introduce a fairly general class of distributed parameter systems, linear in time and of spatial dimension 1, where on different parts of the space interval different PDEs are satisfied. We suppose that the space variable x belongs to some closed interval $[a,b]$. Without loss of generality we assume that $[a,b]=[0,1]$. Let $\lambda\in\mathbb{N}$, and $\{x_l\}_{l=1}^{\lambda}\subset(0,1)$, where $x_1<x_2<\ldots<x_\lambda$. These numbers determine a decomposition of $(0,1)$ into $\lambda+1$ open intervals $\{I_k\}_{k=0}^{\lambda}$. Let $\iota\in\mathbb{N}$. For $j=0,\ldots,\iota$ and $k=0,\ldots,\lambda$, let p_j^k be polynomials and a_j^k be continuous functions on I_k^{cl}. The class of PDEs we consider is of the form

$$\sum_{j=0}^{\iota} a_j^k(x)p_j^k(\mathfrak{D})\frac{\partial^j w}{\partial t^j}(x,t)=0\,,\qquad x\in I_k,\quad t>0\,, \tag{4.1}$$

where $\mathfrak{D}$ denotes differentiation with respect to x. Set $n_k:=\max_{0\le j\le\iota}\deg p_j^k$ and let $p_k(x,s)$ be the coefficient of $\mathfrak{D}^{n_k}$ in the expression

$$\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(\mathfrak{D})\,,$$

where s is a complex variable. We introduce the following assumption

(**A 1**) There exists an open set $\Omega\supset\mathbb{C}_0^{cl}$ such that for any $k=0,\ldots,\lambda$

$$p_k(x,s)\neq 0\qquad\forall\,x\in[0,1]\,,\ \forall\,s\in\Omega\,. \qquad\diamond$$

This condition guarantees that when the Laplace transform of the PDE in I_k are taken, the resulting ordinary differential equation is not degenerate.

To define boundary operators for the PDE (4.1), we note that, by (**A 1**), the PDE has spatial order n_k in I_k, so we need

$$n:=\sum_{k=0}^{\lambda} n_k$$

boundary conditions. While the boundary for $\bigcup_{k=0}^{\lambda} I_k$ is $\{x_l\}_{l=1}^{\lambda}\cup\{0,1\}$, for the purpose of defining boundary conditions each x_j should be represented by x_j^+ and x_j^-. This allows coupling conditions (for example $\mathfrak{D}w(0.5^-,t)=\mathfrak{D}w(0.5^+,t)$) and in-span control (see for instance Chen *et al.* [2], Liu *et al.* [14] and Rebarber [19]). Therefore, we consider the boundary set to be $\{x_l^-\}_{l=1}^{\lambda}\cup\{x_l^+\}_{l=1}^{\lambda}\cup\{0,1\}$, which we rename as $\{z_l\}_{l=1}^{\mu}$, where $\mu=2(\lambda+1)$. For any piecewise continuous function $f:[0,1]\to\mathbb{C}$ and for $l=1,\ldots,\lambda$ we define

$$f(x_l^-):=\lim_{x\nearrow x_l} f(x)\,,\quad f(x_l^+):=\lim_{x\searrow x_l} f(x)\,,$$

so that $f(z_l)$ is a well-defined complex number for all $l = 1, \ldots, \mu$.

Let $q^i_{l,j}$ be polynomials for $i = 1, \ldots, n$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$. We define boundary operators B_i on solutions $w(x,t)$ of (4.1) by

$$(B_i w)(t) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} q^i_{l,j}(\mathcal{D}) \frac{\partial^j}{\partial t^j} w(z_l, t) \,. \tag{4.2}$$

We need to impose bounds on the order of the spatial derivatives in (4.2). In particular, we do not wish to take spatial derivatives at the boundary of I_k which are of order larger than $n_k - 1$. To this end it is useful to introduce the function $\kappa : \{1, \ldots, \mu\} \to \{0, \ldots, \lambda\}$ given by

$$\kappa(l) := \begin{cases} 0 & \text{if } z_l = 0 \\ \lambda & \text{if } z_l = 1 \\ l_0 & \text{if } z_l = x^+_{l_0} \\ l_0 - 1 & \text{if } z_l = x^-_{l_0} \,. \end{cases} \tag{4.3}$$

We assume that for any $i = 1, \ldots, n$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$

$$\deg q^i_{l,j} \le n_{\kappa(l)} - 1 \,. \tag{4.4}$$

The boundary operators (4.2) are sufficiently general to allow higher order differential equations at the boundary, as in hybrid systems, see Littman and Markus [13].

Let m be a positive integer with $m \le n$. We consider the following boundary conditions for the PDE (4.1)

$$(B_i w)(t) = u_i(t) \,, \qquad i = 1, \ldots, m \,, \tag{4.5a}$$

$$(B_i w)(t) = 0 \,, \qquad i = m+1, \ldots, n \,, \tag{4.5b}$$

where $u(t) := [u_1(t), \ldots, u_m(t)]^T \in \mathbb{C}^m$ is the control input. We need to impose a suitable independence condition on the boundary operators. This will be done later in the section, when its relevance can be made clear.

The observation operators for the system will be more general than the boundary operators B_i, since we will allow distributed observation as well. Distributed control is more complicated in these examples than boundary control, and will be discussed in the next section. For $i = 1, \ldots, m$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$, let $r^i_{l,j}$ be polynomials, let $f^i_j \in L^2(0,1;\mathbb{C})$, and define the observation operators by

$$(C_i w)(t) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} r^i_{l,j}(\mathcal{D}) \frac{\partial^j}{\partial t^j} w(z_l, t) + \sum_{j=0}^{\iota} \int_0^1 f^i_j(x) \frac{\partial^j}{\partial t^j} w(x,t) \, dx \,. \tag{4.6}$$

We assume that for any $i = 1, \ldots, m$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$

$$\deg r^i_{l,j} \le n_{\kappa(l)} - 1 \,. \tag{4.7}$$

The observation for the system given by the PDEs (4.1) and the boundary conditions (4.5) is $y(t) = [y_1(t), \ldots, y_m(t)]^T \in \mathbb{C}^m$, where

$$y_i(t) = (C_i w)(t)\,, \quad i = 1, \ldots, m\,. \tag{4.8}$$

We refer to the observed boundary control system given by (4.1), (4.5), and (4.8) as the *open-loop system*. In the following it will be denoted by (**OBC**).

Application of output feedback of the form $u_i(t) = v_i(t) - y_i(t - \varepsilon_i)$, $i = 1, \ldots, m$, leads to

$$(B_i w)(t) + (C_i w)(t - \varepsilon_i) = v_i(t)\,, \quad i = 1, \ldots, m\,, \tag{4.9}$$

where the $\varepsilon_i \geq 0$ are time-delays and $v(t) = [v_1(t), \ldots v_m(t)]^T$ denotes the input of the feedback system. We set $\vec{\varepsilon} := (\varepsilon_1, \ldots \varepsilon_m)$ and refer to system given by (4.1), (4.5b), (4.9), and (4.8) as the *closed-loop system with delay* $\vec{\varepsilon}$. In the following it will be denoted by ($\mathbf{CBC}_{\vec{\varepsilon}}$). If $\vec{\varepsilon} = 0$, then we will call ($\mathbf{CBC}_0$) the *undelayed closed-loop system*. If $u(t) \equiv 0$ (resp. $v(t) \equiv 0$), then we refer to (**OBC**) (resp. ($\mathbf{CBC}_{\vec{\varepsilon}}$)) as the *uncontrolled* open-loop system (resp. *uncontrolled* closed-loop system with delay $\vec{\varepsilon}$).

Existence of exponential solutions

Let $s \in \mathbb{C}$. We will be looking for exponential solutions of the form

$$w(x,t) = e^{st}\phi(x)\,, \quad \text{where } \phi \in L^2(0,1;\mathbb{C}),\ \phi \not\equiv 0\,, \tag{4.10}$$

of the uncontrolled open and closed-loop systems. A solution of the form (4.10) will be called a *mode* of (**OBC**) (resp. ($\mathbf{CBC}_{\vec{\varepsilon}}$)). The complex number s is called the *exponent* of the mode. We say that a mode is *stable* if Re $s < 0$, otherwise we say it is *unstable*.

In the following we first derive a necessary and sufficient condition for the existence of modes for (**OBC**). To this end suppose that (4.10) is a solution of the uncontrolled system (**OBC**). Since ϕ will in general depend on s, we write $\phi(x) = \phi(x,s)$. Now $w(x,t)$ solves the underlying PDE (4.1), and hence it follows that $\phi(x,s)$ satisfies

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(\mathcal{D})\right) \phi(x,s) = 0\,, \quad x \in I_k\,. \tag{4.11}$$

Using assumption (**A 1**), we see that for every $s \in \Omega \supset \mathbb{C}_0^{cl}$ and every $k = 0, \ldots, \lambda$, (4.11) is a ordinary differential equation of order n_k on I_k^{cl}. Let $\{e_j^k\}_{j=1}^{n_k}$ be a basis of $\mathbb{C}^{n_k}$ and let $\{\phi_j^k(\cdot,s)\}_{j=1}^{n_k}$ be solutions of (4.11) on I_k^{cl} satisfying

$$\left(\phi_j^k(x_k,s), \mathcal{D}\phi_j^k(x_k,s), \ldots, \mathcal{D}^{n_k-1}\phi_j^k(x_k,s)\right) = e_j^k\,, \qquad j = 1, \ldots, , n_k;\ s \in \Omega\,,$$

where $x_0 := 0$. Clearly, for any $k = 0, \ldots, \lambda$ and any $s \in \Omega$, the functions $\phi_j^k(\cdot, s)$ are linear independent, and hence span the solution space of (4.11) on I_k^{cl}. In particular, for every $s \in \Omega$, $\phi_j^k(\cdot, s) \in L^2(I_k^{cl}, \mathbb{C})$, and every solution of (4.11) in I_k can be written in the form

$$\sum_{j=1}^{n_k} A_j^k(s)\phi_j^k(x,s)\,, \qquad x \in I_k\,,$$

for some coefficients $A_j^k(s)$. For $j = 1, \ldots, n$ and $l = 1, \ldots, \lambda$, let us define

$$\Phi_j^k(x,s) := \begin{cases} \phi_j^k(x,s) & \text{for } x \in I_k^{cl} \\ 0 & \text{for } x \in [0,1] \setminus I_k^{cl}\ . \end{cases}$$

It will be convenient to rename these n vectors as $\{\Phi_p(x,s)\}_{p=1}^n$. It is clear that any solution of (4.11) can be written as

$$\phi(x,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s)\,, \quad x \in \bigcup_{k=0}^{\lambda} I_k\,, \tag{4.12a}$$

$$\phi(z_l,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(z_l,s)\,, \quad l = 1, \ldots \mu\,, \tag{4.12b}$$

for some coefficients $A_p(s)$.

Lemma 4.1 *The following statements hold true:*

(i) For any $p = 1, \ldots, n$ and any $l = 1, \ldots, \mu$, the function

$$\Omega \to \mathbb{C}, \quad s \mapsto \mathcal{D}^i\Phi_p(z_l,s)$$

is holomorphic, provided that $i \leq n_{\kappa(l)} - 1$, where κ is given by (4.3).

(ii) For any $p = 1, \ldots, n$ and any function $f \in L^2(0,1;\mathbb{C})$, the function

$$\Omega \to \mathbb{C}, \quad s \mapsto \int_0^1 f(x)\Phi_p(x,s)\,dx$$

is holomorphic.

Proof: There exists $k \in \{0, \ldots, \lambda\}$ and $p_0 \in \{1, \ldots, n_k\}$ such that $\Phi_p = \Phi_{p_0}^k$. It is convenient to define $x_0^+ := 0$ and $x_{\lambda+1}^- := 1$.

In order to prove statement (i), assume first that $z_l \in \{x_k^+, x_{k+1}^-\} = \partial I_k$.Then $\kappa(l) = k$, and therefore $i \leq n_k - 1$ by hypothesis. Consequently, the claim follows from a well-known result on parameter-dependent ordinary differential equations - see Coddington and Levinson [3], chapter 1, theorem 8.4. If $z_l \notin \{x_k^+, x_{k+1}^-\}$, then there exists $x^* \in \{x_1, \ldots, x_\lambda\} \cup \{0,1\}$ such that for all $i \in \mathbb{N}$ and all $s \in \Omega$

$$\mathcal{D}^i \Phi_p(z_l, s) = \mathcal{D}^i \Phi_{p_0}^k(z_l, s) = \lim_{x \to x^*, x \notin I_k^{cl}} \mathcal{D}^i \Phi_{p_0}^k(x, s) = 0\,,$$

showing that the claim is true in this case also.

To prove statement (ii), recall from [3], chapter 1, theorem 8.4, that $\Phi_{p_0}^k$ is continuous in (x,s) for $x \in I_k^{cl}$ and $s \in \Omega$ and holomorphic in s for each fixed $x \in I_k^{cl}$. The result then follows from a standard argument using theorems of Morera and Fubini. □

Making use of (4.12b), the boundary conditions $(B_i w)(t) = 0$, $i = 1, \ldots, n$, of the uncontrolled open-loop system can be expressed as follows

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i \Phi_p)(s) = 0\,, \qquad i = 1, \ldots, n\,, \tag{4.13}$$

where

$$(\hat{B}_i \Phi_p)(s) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} s^j q_{l,j}^i(\mathcal{D}) \Phi_p(z_l, s)\,. \tag{4.14}$$

Let $\mathbf{D}(s)$ be the $n \times n$ matrix

$$\mathbf{D}(s) := \Big((\hat{B}_i \Phi_p)(s) \Big)\,, \quad i, p = 1, \ldots, n\,. \tag{4.15}$$

Notice that, by (4.4) and Lemma 4.1, $\mathbf{D}(s)$ is holomorphic on Ω. Setting $A(s) := (A_1(s), \ldots, A_n(s))^T$, (4.13) can be written as $\mathbf{D}(s)A(s) = 0$, and we see that if the system (**OBC**) has a mode with exponent $s_0 \in \Omega$, then

$$\det \mathbf{D}(s_0) = 0\,. \tag{4.16}$$

It is easy to show that the above argument can be reversed, i.e. (4.16) guarantees that (**OBC**) has a mode with exponent s_0.

Summarizing our discussion we obtain the following result.

Proposition 4.2 *Suppose that* (**A 1**) *is satisfied and let $s_0 \in \Omega$. Then* (**OBC**) *has a mode with exponent s_0 if and only if (4.16) is satisfied.*

In order to prove a similar result for the closed-loop system ($\mathbf{CBC}_{\vec{\varepsilon}}$), suppose that (4.10) is a solution of the uncontrolled system ($\mathbf{CBC}_{\vec{\varepsilon}}$). Again we will indicate the dependence of ϕ on s explicitly by writing $\phi(x) = \phi(x,s)$. Since the underlying PDE (see (4.1)) is the same as for the open-loop system (**OBC**), it follows that $\phi(x,s)$ is of the form (4.12).

Consequently, the boundary conditions (4.5b) and the closed-loop boundary conditions (4.9) with $v_i \equiv 0$ lead to

$$\sum_{p=1}^{n} A_p(s)\left((\hat{B}_i\Phi_p)(s) + e^{-\varepsilon_i s}(\hat{C}_i\Phi_p)(s)\right) = 0\,, \qquad i = 1,\ldots,m\,, \tag{4.17}$$

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) = 0\,, \qquad i = m+1,\ldots,n\,, \tag{4.18}$$

where

$$(\hat{C}_i\Phi_p)(s) := \sum_{l=1}^{\mu}\sum_{j=0}^{\iota} s^j r^i_{l,j}(\mathfrak{D})\Phi_p(z_l,s) + \sum_{j=0}^{\iota} s^j \int_0^1 f^i_j(x)\Phi_p(x,s)\,dx\,. \tag{4.19}$$

Let $\mathbf{N}(s)$ be the $m \times n$ matrix

$$\mathbf{N}(s) := \left((\hat{C}_i\Phi_p)(s)\right)\,, \quad i = 1,\ldots,m\,;\; p = 1,\ldots,n\,, \tag{4.20}$$

and set

$$\tilde{\mathbf{N}}(s) :\equiv \begin{pmatrix} I_{m\times m} \\ 0_{(n-m)\times m} \end{pmatrix}\,. \tag{4.21}$$

It follows from (4.7) and Lemma 4.1 that $\mathbf{N}(s)$ is holomorphic on Ω. Equations (4.17) and (4.18) can be written in the more compact form

$$\left(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s)\right) A(s) = 0\,, \tag{4.22}$$

where $\mathbf{E}_{\vec{\varepsilon}}(s) = \mathrm{diag}_{1\le i\le m}(e^{-\varepsilon_i s})$. Setting

$$\mathbf{D}_{\vec{\varepsilon}}(s) := \mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s)\,, \tag{4.23}$$

we obtain the following closed-loop counterpart of Proposition 4.2.

Proposition 4.3 *Suppose that* (**A 1**) *is satisfied and let* $s_0 \in \Omega$. *Then* ($\mathbf{CBC}_{\vec{\varepsilon}}$) *has a mode with exponent* s_0 *if and only if* $\det \mathbf{D}_{\vec{\varepsilon}}(s_0) = 0$.

While by construction $\mathbf{D}$ and $\mathbf{D}_{\vec{\varepsilon}}$ depend on the choice of the basis for the solution space of (4.11) on I_k^{cl}, it is a routine exercise to show that the zeros of $\det \mathbf{D}$ and $\det \mathbf{D}_{\vec{\varepsilon}}$ do not. Since the basis $\{\phi_j^k(\cdot,s)\}_{j=1}^{n_k}$ is not always easy to compute, we can use a different, more convenient choice of basis in order to compute the zeros of $\det \mathbf{D}$ and $\det \mathbf{D}_{\vec{\varepsilon}}$.

Transfer functions

We shall need the following assumption

(**A 2**) $$\det \mathbf{D}(s) \not\equiv 0\,. \qquad \diamond$$

For a given $s_0 \in \Omega$, let $\mathcal{E}_{s_0}$ denote the vector space of all functions $\psi : \left(\bigcup_{k=0}^{\lambda} I_k\right) \times [0,\infty) \to \mathbb{C}$ of the form $\psi(x,t) = e^{s_0 t}\tilde{\psi}(x)$, where $\tilde{\psi} : \bigcup_{k=0}^{\lambda} I_k \to \mathbb{C}$ solves the ordinary differential equation (4.11) for $s = s_0$. It is easy to see that assumption (**A 2**) is equivalent to

(**A 2′**) There exists $s_0 \in \Omega$ such that the restricted boundary operators $B_i|_{\mathcal{E}_{s_0}}$ are linearly independent, i.e. if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are such that $\sum_{i=1}^{n} \alpha_i B_i \psi = 0$ for all $\psi \in \mathcal{E}_{s_0}$, then $\alpha_i = 0$ for all $i = 1, \ldots, n$. $\diamond$

Remark 4.4 We illustrate (**A 2**) and (**A 2′**) in a simple situation. Suppose that there are no in-span points (i.e. $I_0 = [0,1]$) and that $n = n_0 = 2$. Let the boundary operators B_1 and B_2 be given by $(B_1 w)(t) = w(0,t)$ and $(B_2 w)(t) = w_t(a,t)$, where $a \in \{0,1\}$. Then it is easy to show that (**A 2′**), or equivalently (**A 2**), is satisfied if and only if $a \neq 0$. Notice however, that B_1 and B_2 are linearly independent on the solution space of (4.1) even if $a = 0$. $\diamond$

If (**A 2**) is satisfied, then we can define

$$\mathbf{H}(s) := \mathbf{N}(s)\mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s)\,, \tag{4.24}$$

where $\mathbf{D}$, $\mathbf{N}$ and $\tilde{\mathbf{N}}$ are given by (4.15), (4.20), and (4.21), respectively. While $\mathbf{N}$ and $\mathbf{D}$ depend on the choice of the basis for the solution space of (4.11) on I_k^{cl}, it is a routine exercise to prove that the product $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ does not (the corresponding transformation matrices cancel). Consequently, $\mathbf{H}$ is independent of the choice of $\{\phi_j^k\}_{j=1}^{n_k}$, $k = 0, \ldots, \lambda$. Since $\mathbf{D}$, $\mathbf{N}$ and $\tilde{\mathbf{N}}$ have all their entries in $\mathcal{H}_-$, it follows from (**A 2**) that $\mathbf{H} \in \mathcal{M}_-^{m\times m}$. In particular, $\mathbf{H}$ is a transfer function in the sense of Section 3. We say that $\mathbf{H}$ is the transfer function of (**OBC**).

In order to show that $\mathbf{H}$ admits the usual dynamical interpretation, let the Laplace transform be denoted by the superscript " ˆ " and let $u(\cdot)$ be a Laplace transformable input function. Moreover, let $w(x,t;u)$ denote the solution of (**OBC**) with initial conditions given by

$$\frac{\partial^j}{\partial t^j} w(\cdot,0;u) = 0\,, \qquad j = 0, \ldots, \iota - 1\,.$$

Then $\hat{w}(x,s;u)$ satisfies the ordinary differential equation (4.11), and consequently, the function $\hat{w}(x,s;u)$ is of the form (4.12). Hence, the boundary conditions (4.5) then imply

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) = \hat{u}_i(s)\,, \qquad i = 1, \ldots, m\,,$$

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) = 0\,, \qquad i = m+1, \ldots, n\,.$$

This is equivalent to

$$\mathbf{D}(s)A(s) = \tilde{\mathbf{N}}(s)\hat{u}(s)\,,$$

so, by assumption (**A 2**)

$$A(s) = \mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s)\hat{u}(s) \tag{4.25}$$

for all $s \in \Omega$ such that $\det \mathbf{D}(s) \neq 0$. Since $\hat{y}_i(s) = \sum_{p=1}^{n} A_p(s)(\hat{C}_i\Phi_p)(s)$, we can write $\hat{y}(s) = \mathbf{N}(s)A(s)$. Therefore, by (4.25), we see that

$$\hat{y}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s)\hat{u}(s) = \mathbf{H}(s)\hat{u}(s)\,.$$

As in Section 3 we define

$$\mathbf{G}_{\vec{\varepsilon}} = \mathbf{H}(I + \mathbf{E}_{\vec{\varepsilon}}\mathbf{H})^{-1}\,. \tag{4.26}$$

Using Lemma 2.4, part (i), $\mathbf{G}_{\vec{\varepsilon}}$ can be written in the form

$$\mathbf{G}_{\vec{\varepsilon}}(s) = \mathbf{N}(s)(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{N}(s))^{-1}\tilde{\mathbf{N}}(s) = \mathbf{N}(s)\mathbf{D}_{\vec{\varepsilon}}^{-1}(s)\tilde{\mathbf{N}}(s)\,. \tag{4.27}$$

Of course, (4.27) only makes sense if $\det \mathbf{D}_{\vec{\varepsilon}}(s) \not\equiv 0$, or equivalently, if $\det(I+\mathbf{E}_{\vec{\varepsilon}}(s)\mathbf{H}(s)) \not\equiv 0$, see Lemma 2.4, part (i). We call $\vec{\varepsilon} \in [0,\infty)^m$ an *admissible* delay for (**OBC**) if $\det \mathbf{D}_{\vec{\varepsilon}}(s) \not\equiv 0$. If $\vec{\varepsilon}$ is an admissible delay for (**OBC**), then $\mathbf{G}_{\vec{\varepsilon}} \in \mathcal{M}_{-}^{m\times m}$, and we say that $\mathbf{G}_{\vec{\varepsilon}}$ is the transfer function of ($\mathbf{CBC}_{\vec{\varepsilon}}$). Going through the above steps with $\mathbf{D}$ replaced by $\mathbf{D}_{\vec{\varepsilon}}$, we see that the response $y(\cdot)$ of ($\mathbf{CBC}_{\vec{\varepsilon}}$) to the input function $v(\cdot)$ under zero initial conditions is given by $\hat{y}(s) = \mathbf{G}_{\vec{\varepsilon}}(s)\hat{v}(s)$.

Modal stability and small delays

We say that ($\mathbf{CBC}_{\vec{\varepsilon}}$) is *modally stable* if ($\mathbf{CBC}_{\vec{\varepsilon}}$) has no unstable modes. This is the kind of internal stability which is considered in the literature on robust stabilization of PDEs, see [4]–[8] and [11]. The following corollary shows that spectral stability combined with bi-coprimeness is equivalent to modal stability.

Corollary 4.5 *Suppose that* (**A 1**) *and* (**A 2**) *are satisfied and let* $\vec{\varepsilon} \in [0,\infty)^m$. *Then the following statements are equivalent*

(i) ($\mathbf{CBC}_{\vec{\varepsilon}}$) *is modally stable.*

(ii) $\vec{\varepsilon}$ *is an admissible delay for* (**OBC**), $\mathbf{G}_{\vec{\varepsilon}}$ *given by (4.26) is spectrally stable and the triple* $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ *is bi-coprime.*

The proof of Corollary 4.5 follows immediately from Proposition 2.3, Lemma 2.4, Proposition 4.3 and equation (4.27). In the following, we write ($\mathbf{CBC}_{\varepsilon}$) for ($\mathbf{CBC}_{\vec{\varepsilon}}$) and $\mathbf{D}_{\varepsilon}$ for $\mathbf{D}_{\vec{\varepsilon}}$ if the components ε_i of $\vec{\varepsilon}$ satisfy $\varepsilon_i = \varepsilon$ for all $i = 1, \ldots, m$.

It follows from (4.27) and the analyticity of $\mathbf{N}$, $\tilde{\mathbf{N}}$ and $\mathbf{D}_{\vec{\varepsilon}}$ that, if $\mathbf{G}_{\vec{\varepsilon}}(s)$ has a pole in $\mathbb{C}_0^{cl}$, then the closed-loop system $(\mathbf{CBC}_{\vec{\varepsilon}})$ has an unstable mode. Therefore, by an application of Theorem 3.3, Theorem 3.7 and Proposition 3.9, we immediately obtain conditions which guarantee the existence of unstable closed-loop modes. We give a precise formulation of these conditions in the following three corollaries.

Corollary 4.6 *Assume that* $(\mathbf{A\,1})$ *and* $(\mathbf{A\,2})$ *hold and suppose that* $\mathbf{H}$ *given by (4.24) satisfies one of the following two conditions:*

(i) $\mathbf{H}$ *is regular and the feedthrough matrix* D *of* $\mathbf{H}$ *satisfies* $r(D) > 1$.

(ii) $\mathbf{H}$ *is not well-posed and satisfies the conditions (1) and (2) in Theorem 3.7.*

Then there exist sequences (ε_n) *and* (s_n) *with*

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad s_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} |\operatorname{Im} s_n| = \infty\,, \quad \lim_{n\to\infty} \operatorname{Re} s_n = \infty \tag{4.28}$$

and such that for any $n \in \mathbb{N}$, $\det(\mathbf{D}_{\varepsilon_n}(s_n)) = 0$, *i.e. for any* $n \in \mathbb{N}$ *the closed-loop system* $(\mathbf{CBC}_{\varepsilon_n})$ *has a mode with exponent* s_n.

Corollary 4.7 *Assume that* $(\mathbf{A\,1})$ *and* $(\mathbf{A\,2})$ *hold and suppose that* $\mathbf{H}$ *given by (4.24) is regular. If* $\mathbf{G}_0$ *given by (4.26) is in* $(H^\infty)^{m\times m}$ *and if* $\gamma(\mathbf{H}) > 1$ *(where* $\gamma(\mathbf{H})$ *is given by (3.2)), then there exist sequences* (ε_n) *and* (s_n) *with*

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad s_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} |\operatorname{Im} s_n| = \infty\,,$$

and such that for any $n \in \mathbb{N}$, $\det(\mathbf{D}_{\varepsilon_n}(s_n)) = 0$, *i.e. for any* $n \in \mathbb{N}$ *the closed-loop system* $(\mathbf{CBC}_{\varepsilon_n})$ *has a mode with exponent* s_n.

Corollary 4.8 *Assume that* $(\mathbf{A\,1})$ *and* $(\mathbf{A\,2})$ *hold and suppose that* $\mathbf{H}$ *given by (4.24) satisfies (3.22). Then there exist sequences* (ε_n) *and* (ω_n) *in* $(0,\infty)$ *with*

$$\lim_{n\to\infty} \varepsilon_n = 0\,, \quad \lim_{n\to\infty} \omega_n = \infty\,,$$

and such that for any $n \in \mathbb{N}$, $\det(\mathbf{D}_{\varepsilon_n}(\imath\omega_n)) = 0$, *i.e. for any* $n \in \mathbb{N}$ *the closed-loop system* $(\mathbf{CBC}_{\varepsilon_n})$ *has a mode with exponent* $\imath\omega_n$.

On the other hand, when there exists $\varepsilon^* > 0$ such that $\mathbf{G}_{\vec{\varepsilon}}$ is spectrally stable for all $\vec{\varepsilon} \in [0,\infty)^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$, we cannot immediately conclude modal stability of $(\mathbf{CBC}_{\vec{\varepsilon}})$ for all such $\vec{\varepsilon}$. However, the following result shows that modal stability can be obtained by using Lemma 2.4 and Corollary 4.5.

Corollary 4.9 *Assume that* $(\mathbf{A\,1})$ *and* $(\mathbf{A\,2})$ *hold and suppose that* $(\mathbf{CBC}_0)$ *is modally stable. If* $\mathbf{H}$ *given by (4.24) satisfies (3.25), then there exists* $\varepsilon^* > 0$ *such that* $(\mathbf{CBC}_{\vec{\varepsilon}})$ *is modally stable for all* $\vec{\varepsilon} \in [0,\infty)^m$ *satisfying* $\|\vec{\varepsilon}\| < \varepsilon^*$.

Proof: Since ($\mathbf{CBC}_0$) is modally stable, it follows from Proposition 4.3 that

$$\det(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{N}(s)) \neq 0 \qquad \forall\, s \in \mathbb{C}_0^{cl}\,, \tag{4.29}$$

and it follows from (4.27) that $\mathbf{G}_0$ is spectrally stable. It follows from (3.25) that $I + \mathbf{E}_{\vec{\varepsilon}}\mathbf{H}$ and hence, by Lemma 2.4, also $\mathbf{D} + \tilde{\mathbf{N}}\mathbf{E}_{\vec{\varepsilon}}\mathbf{N}$ is invertible for all $\vec{\varepsilon} \in [0,\infty)^m$. Thus all $\vec{\varepsilon} \in [0,\infty)^m$ are admissible delays. Moreover, by Theorem 3.12 there exists $\varepsilon^* > 0$ such that $\mathbf{G}_{\vec{\varepsilon}}$ is spectrally stable for all $\vec{\varepsilon} \in [0,\infty)^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$. From Lemma 2.4 and (4.29) we obtain that $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime. An application of Corollary 4.5 shows that ($\mathbf{CBC}_{\vec{\varepsilon}}$) is modally stable for all $\vec{\varepsilon} \in [0,\infty)^m$ with $\|\vec{\varepsilon}\| < \varepsilon^*$ and all $s \in \mathbb{C}_0^{cl}$. □

5. Robustness and nonrobustness results for PDE's with distributed control

In this section we consider partial differential equations which have distributed control instead of boundary control. Unless indicated otherwise, we impose the same assumptions and use the same definitions and notation as in Section 4, except that for purposes of simplifying the exposition we assume that the controlled and observed PDE is a single-input single-output system, i.e. $u(t)$, $y(t) \in \mathbb{C}$. The results in this section have obvious multivariable extensions – the details are left to the reader. For $j = 0, \ldots, \iota$ and $k = 0, \ldots, \lambda$, let $b_j^k \in L^2(0,1;\mathbb{C})$ and consider the controlled PDE

$$\sum_{j=0}^{\iota} a_j^k(x) p_j^k(\mathcal{D}) \frac{\partial^j w}{\partial t^j}(x,t) = \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j u}{\partial t^j}(t)\,, \qquad x \in I_k,\;\; t > 0\,, \tag{5.1}$$

with boundary conditions

$$(B_i w)(t) = 0\,, \qquad i = 1, \ldots, n\,. \tag{5.2}$$

The observation is still of the form (4.8), but since there is just one output, only the observation operator C_1 is needed. Setting $C := C_1$, the observation is given by

$$y(t) = (Cw)(t)\,. \tag{5.3}$$

We refer to the observed distributed control system given by (5.1)–(5.3) as the *open-loop system*. In the following it will be denoted by (**ODC**).

Application of output feedback of the form $u(t) = v(t) - y(t - \varepsilon)$ leads to

$$\sum_{j=0}^{\iota} a_j^k(x) p_j^k(\mathcal{D}) \frac{\partial^j w}{\partial t^j}(x,t) = \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j v}{\partial t^j}(t) - \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j}{\partial t^j}(Cw)(t-\varepsilon)\,, \quad x \in I_k,\; t > 0\,, \tag{5.4}$$

where $\varepsilon \geq 0$ is a time-delay and $v(t)$ denotes the input of the feedback system. We refer to system given by (5.4), (5.2) and (5.3) as the *closed-loop system with delay* ε. In the following it will be denoted by $(\mathbf{CDC}_\varepsilon)$. If $\varepsilon = 0$, then we will call $(\mathbf{CDC}_0)$ the *undelayed closed-loop system.* If $u(t) \equiv 0$ (resp. $v(t) \equiv 0$), then we refer to $(\mathbf{ODC})$ (resp. $(\mathbf{CDC}_\varepsilon)$) as the *uncontrolled* open-loop system (resp. *uncontrolled* closed-loop system with delay ε).

As in Section 4, we first find conditions under which exponential solutions exist, and then find the transfer functions of $(\mathbf{ODC})$ and $(\mathbf{CDC}_\varepsilon)$.

Existence of exponential solutions

We will be looking for exponential solutions of the form (4.10) of the uncontrolled open-loop and closed-loop systems. Such a solution will again be called a mode of $(\mathbf{ODC})$ (resp. $(\mathbf{CDC}_\varepsilon)$). Since the the uncontrolled open-loop system is the same as the uncontrolled open-loop system in Section 4, we know that by Proposition 4.2 the system $(\mathbf{ODC})$ has a mode with exponent $s_0 \in \Omega$ if and only if $\det \mathbf{D}(s_0) = 0$, where $\mathbf{D}$ is defined in (4.15). In order to prove a similar result for the closed-loop system $(\mathbf{CDC}_\varepsilon)$, suppose that

$$w(x,t) = e^{st}\phi(x)\,, \quad \text{where } \phi \in L^2(0,1;\mathbb{C}),\ \phi \not\equiv 0\,,$$

is a solution of the uncontrolled system $(\mathbf{CDC}_\varepsilon)$. Again we will indicate the dependence of ϕ on s explicitly by writing $\phi(x) = \phi(x,s)$. We then obtain from (5.4) with $v(t) \equiv 0$

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(\mathcal{D})\right)\phi(x,s) = -e^{-\varepsilon s} b^k(x,s)(\hat{C}\phi)(s)\,, \qquad x \in I_k\,, \tag{5.5}$$

where

$$b^k(x,s) := \sum_{j=0}^{\iota} b_j^k(x) s^j\,,$$

and $(\hat{C}\phi)(s)$ is defined as in (4.19) with Φ_p replaced by ϕ.

Let η_k be the left endpoint of I_k, and let $\Psi(x,s)$ be the solution of the initial value problem

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(\mathcal{D})\right)\phi(x,s) = b^k(x,s)\,, \qquad x \in I_k\,, \tag{5.6}$$
$$\mathcal{D}^j\phi(\eta_k,s) = 0\,, \qquad j = 0,\ldots,n_k - 1\,.$$

Clearly, the solution of (5.5) can be written in the form

$$\phi(x,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s) + a(s)\Psi(x,s)\,, \tag{5.7}$$

where $a(s), A_p(s) \in \mathbb{C}$ are suitable coefficients. The boundary conditions (5.2) become

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) + a(s)(\hat{B}_i\Psi)(s) = 0\,, \qquad i = 1,\ldots,n\,, \tag{5.8}$$

where $(\hat{B}_i\Psi)(s)$ is defined as in (4.14) with Φ_p replaced by Ψ. Plugging (5.7) into (5.5) and noting that Φ_p satisfies (4.11) and Ψ satisfies (5.6) gives

$$b^k(x,s)\left(\sum_{p=1}^{n} A_p(s)(\hat{C}\Phi_p)(s) + a(s)\left((\hat{C}\Psi)(s) + e^{\varepsilon s}\right)\right) = 0\,, \qquad x \in I_k\,,\ s \in \Omega\,. \tag{5.9}$$

Let

$$\mathbf{c} := \left(\hat{C}\Phi_1,\ldots,\hat{C}\Phi_n\right)^T\,, \qquad \mathbf{b} := -\left(\hat{B}_1\Psi,\ldots,\hat{B}_n\Psi\right)^T\,,$$

and

$$\mathbf{\Delta}_\varepsilon(s) := \begin{pmatrix} \mathbf{D}(s) & -\mathbf{b}(s) \\ \mathbf{c}^T(s) & (\hat{C}\Psi)(s) + e^{\varepsilon s} \end{pmatrix}\,, \tag{5.10}$$

where $\mathbf{D}$ is given by (4.15). We are now in the position to prove the following result.

Proposition 5.1 *Suppose that* $(\mathbf{A\,1})$ *is satisfied and let* $s_0 \in \Omega$. *Then* $(\mathbf{CDC}_\varepsilon)$ *has a mode with exponent* s_0 *if and only if* $\det \mathbf{\Delta}_\varepsilon(s_0) = 0$.

Proof: Suppose that $(\mathbf{CDC}_\varepsilon)$ has a mode with exponent s_0. Then using the above notation, $e^{s_0 t}\phi(x,s_0)$ is a solution of $(\mathbf{CDC}_\varepsilon)$, where $\phi(\cdot,s_0) \not\equiv 0$.

Case 1: Assume that there exists $k_0 \in \{0,\ldots,\lambda\}$ and $\xi_0 \in I_{k_0}$ such that $b^{k_0}(\xi_0,s_0) \neq 0$. Then, by (5.9)

$$\sum_{p=1}^{n} A_p(s_0)(\hat{C}\Phi_p)(s_0) + a(s_0)\left((\hat{C}\Psi)(s_0) + e^{\varepsilon s_0}\right) = 0\,. \tag{5.11}$$

Setting $A(s) := (A_1(s),\ldots,A_n(s))^T$ and $\tilde{A}(s) := (A^T(s), a(s))^T$, then a combination of (5.8) and (5.11) yields that $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$. Since $\phi(\cdot,s_0) \not\equiv 0$, it follows from (5.7) that $\tilde{A}(s_0) \neq 0$ and hence that $\det \mathbf{\Delta}_\varepsilon(s_0) = 0$.

Case 2: Now assume that $b^k(x,s_0) = 0$ for all $x \in I_k$ and for all $k \in \{0,\ldots,\lambda\}$. Then $\Psi(x,s_0) = 0$ for all $x \in [0,1]$ and hence $\mathbf{b}(s_0) = 0$. Moreover, by (5.8)

$$\sum_{p=1}^{n} A_p(s_0)(\hat{B}_i\Phi_p)(s_0) = 0\,, \qquad i = 1,\ldots,n\,. \tag{5.12}$$

Since $\Psi(\cdot,s_0) \equiv 0$ and $\phi(\cdot,s_0) \not\equiv 0$, (5.7) yields that $A(s_0) = (A_1(s_0),\ldots,A_n(s_0))^T \neq 0$. So, combining (5.12) with (4.15), gives $\det \mathbf{D}(s_0) = 0$. Now

$$\mathbf{\Delta}_\varepsilon(s_0) = \begin{pmatrix} \mathbf{D}(s_0) & 0 \\ \mathbf{c}^T(s_0) & e^{\varepsilon s_0} \end{pmatrix}\,,$$

and thus $\det \mathbf{\Delta}_\varepsilon(s_0) = e^{\varepsilon s_0} \det \mathbf{D}(s_0) = 0$.

Conversely, suppose that $\det \mathbf{\Delta}_\varepsilon(s_0) = 0$ for some $s_0 \in \Omega$. Then there exists $\tilde{A}(s_0) = (A^T(s_0), a(s_0))^T \in \mathbb{C}^{n+1}$, $\tilde{A}(s_0) \neq 0$, such that $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$. Writing $A(s_0) = (A_1(s_0), \ldots, A_n(s_0))^T$ we see that (5.9) holds. Moreover, defining $\phi(x, s_0)$ by (5.7), we obtain

$$a(s_0) = -e^{-\varepsilon s_0}(\hat{C}\phi)(s_0)\,. \tag{5.13}$$

Now

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s_0^j\, p_j^k(\mathcal{D})\right) \phi(x, s_0) = a(s_0) b^k(x, s_0)\,, \qquad x \in I_k\,,$$

and so, using (5.13) it follows that $\phi(x, s_0)$ satisfies (5.5) for $s = s_0$. Finally, since $\det \mathbf{\Delta}_\varepsilon(s_0) = 0$, it is clear that (5.8) is satisfied for $s = s_0$. Thus the function $w(x,t) := e^{s_0 t}\phi(x, s_0)$ is a solution of $(\mathbf{CDC}_\varepsilon)$, and hence $(\mathbf{CDC}_\varepsilon)$ has a mode with exponent s_0. $\square$

Transfer functions

Suppose that $(\mathbf{A\,2})$ is satisfied. Then the function

$$\mathbf{H}_d(s) := \mathbf{c}^T(s)\mathbf{D}^{-1}(s)\mathbf{b}(s) + (\hat{C}\Psi)(s) \tag{5.14}$$

is well-defined. While the functions $\mathbf{c}(s)$ and $\mathbf{D}(s)$ depend on the choice of the basis $\{\phi_j^k\}_{j=1}^{n_k}$ of the solution space of (4.11) on I_k^{cl}, it is easy to prove that the product $\mathbf{c}^T(s)\mathbf{D}^{-1}(s)$ does not. If $(\mathbf{A\,1})$ is satisfied, then it can be shown as in Lemma 4.1 that $\hat{C}\Psi \in \mathcal{H}_-$ and that $\mathbf{c}$ and $\mathbf{b}$ have all their entries in $\mathcal{H}_-$. Moreover, we know from Section 4 that $\mathbf{D} \in \mathcal{H}_-^{n\times n}$. It therefore follows that $\mathbf{H}_d \in \mathcal{M}_-^{m\times m}$. In particular, $\mathbf{H}_d$ is a transfer function in the sense of Section 3. We say that $\mathbf{H}_d$ is the transfer function of $(\mathbf{ODC})$. Setting

$$\mathbf{D}_d := \begin{pmatrix} \mathbf{D} & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{pmatrix}, \qquad \tilde{\mathbf{N}}_d := \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix}, \qquad \mathbf{N}_d := (\mathbf{c}^T, \hat{C}\Psi)\,, \tag{5.15}$$

we can write

$$\mathbf{H}_d(s) = \mathbf{N}_d(s)\mathbf{D}_d^{-1}(s)\tilde{\mathbf{N}}_d(s)\,. \tag{5.16}$$

The representation (5.16) enables us to apply the results of Section 2 to $\mathbf{H}_d$.

In order to show that $\mathbf{H}_d$ admits the usual dynamical interpretation, let $u(\cdot)$ be a sufficiently smooth Laplace transformable input function with zero initial conditions, i.e. $(d^j u/dt^j)(0) = 0$ for all $j = 0, \ldots, \iota - 1$. Denoting, as in Section 4, the corresponding solution of $(\mathbf{ODC})$ with zero initial conditions by $w(x,t;u)$, we see that $\hat{w}(x,s;u)$ satisfies the ordinary differential equation

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(\mathcal{D})\right) \phi(x, s) = b^k(x, s)\hat{u}(s)\,, \qquad x \in I_k\,. \tag{5.17}$$

The general solution of (5.17) is given by

$$\hat{w}(x,s)=\sum_{p=1}^{n}A_p(s)\Phi_p(x,s)+\hat{u}(s)\Psi(x,s) \tag{5.18}$$

The boundary conditions (5.2) then lead to

$$\sum_{p=1}^{n}A_p(s)(\hat{B}_i\Phi_p)(s)+\hat{u}(s)(\hat{B}_i\Psi)(s)=0\,,\qquad i=1,\ldots,n\,.$$

This is equivalent to

$$\mathbf{D}(s)A(s)=\mathbf{b}(s)\hat{u}(s)\,. \tag{5.19}$$

Using (5.18) the observation (5.3) can be expressed as follows

$$\hat{y}(s)=\mathbf{c}^T(s)A(s)+(\hat{C}\Psi)(s)\hat{u}(s)\,.$$

Therefore, using (5.19) and the fact that $\det\mathbf{D}(s)\not\equiv 0$ by (**A 2**), we obtain

$$\hat{y}(s)=\left(\mathbf{c}^T\mathbf{D}^{-1}(s)\mathbf{b}(s)+(\hat{C}\Psi)(s)\right)\hat{u}(s)=\mathbf{H}_d(s)\hat{u}(s)\,.$$

By Lemma 2.4, the transfer function $\mathbf{G}_d^{\varepsilon}(s)$ of $(\mathbf{CDC}_\varepsilon)$ is given by

$$\mathbf{G}_d^{\varepsilon}(s)=\mathbf{H}_d(s)\left(1+e^{-\varepsilon s}\mathbf{H}_d(s)\right)^{-1}=\mathbf{N}_d(s)\left(\mathbf{D}_d(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}_d(s)\mathbf{N}_d(s)\right)^{-1}\tilde{\mathbf{N}}_d(s)\,. \tag{5.20}$$

In contrast to Section 4, it is not true that $\mathbf{\Delta}_\varepsilon(s)=\mathbf{D}_d(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}_d(s)\mathbf{N}_d(s)$. However, we can show the following.

Proposition 5.2 *Suppose that* (**A 1**) *holds. Then the zeros of the functions* $\det\mathbf{\Delta}_\varepsilon(s)$ *and* $\det\left(\mathbf{D}_d(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}_d(s)\mathbf{N}_d(s)\right)$ *in* Ω *coincide (counting multiplicities).*

Proof: Using (5.15) we obtain that

$$\mathbf{D}_d(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}_d(s)\mathbf{N}_d(s)=\begin{pmatrix}\mathbf{D}(s)+e^{-\varepsilon s}\mathbf{b}(s)\mathbf{c}^T(s) & e^{-\varepsilon s}(\hat{C}\Psi)(s)\mathbf{b}(s)\\ e^{-\varepsilon s}\mathbf{c}^T(s) & 1+e^{-\varepsilon s}(\hat{C}\Psi)(s)\end{pmatrix}. \tag{5.21}$$

For any matrices M_1, M_2, M_3, M_4, where M_1 is square and M_4 is square and invertible,

$$\det\begin{pmatrix}M_1 & M_2\\ M_3 & M_4\end{pmatrix}=\det M_4\det(M_1-M_2M_4^{-1}M_3)\,, \tag{5.22}$$

see e.g. Gantmacher [9], p. 46. Setting $h_\varepsilon(s):=1+e^{-\varepsilon s}(\hat{C}\Psi)(s)$ we distinguish between two cases.

Case 1: If $h_\varepsilon(s) \not\equiv 0$, then an application of (5.22) to (5.21) yields, after simplication,

$$\det(\mathbf{D}_d(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}_d(s)\mathbf{N}_d(s)) = \frac{1}{h_\varepsilon^{n-1}(s)} \det\left(h_\varepsilon(s)\mathbf{D}(s) + e^{-\varepsilon s}\mathbf{b}(s)\mathbf{c}^T(s)\right) . \tag{5.23}$$

Combining (5.10) and (5.22) we obtain

$$\begin{aligned}\det \mathbf{\Delta}_\varepsilon(s) &= ((\hat{C}\Psi)(s) + e^{\varepsilon s}) \det\left(\mathbf{D}(s) + \frac{1}{(\hat{C}\Psi)(s) + e^{\varepsilon s}}\mathbf{b}(s)\mathbf{c}^T(s)\right) \\ &= e^{\varepsilon s} h_\varepsilon(s) \det\left(\mathbf{D}(s) + \frac{e^{-\varepsilon s}}{h_\varepsilon(s)}\mathbf{b}(s)\mathbf{c}^T(s)\right) . \end{aligned} \tag{5.24}$$

Comparing (5.23) and (5.24) shows that

$$\det \mathbf{\Delta}_\varepsilon(s) = e^{\varepsilon s} \det\left(\mathbf{D}_d(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}_d(s)\mathbf{N}_d(s)\right) . \tag{5.25}$$

Case 2: If $h_\varepsilon(s) \equiv 0$, then it follows that

$$(\hat{C}\Psi)(s) = -e^{\varepsilon s} . \tag{5.26}$$

Let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty} \varepsilon_n = \varepsilon$ and $\varepsilon_n \neq \varepsilon$ for all $n \in \mathbb{N}$. Clearly, by (5.26), $h_{\varepsilon_n}(s) \not\equiv 0$ for all $n \in \mathbb{N}$, and thus, by Case 1

$$\det \mathbf{\Delta}_{\varepsilon_n}(s) = e^{\varepsilon_n s} \det\left(\mathbf{D}_d(s) + e^{-\varepsilon_n s}\tilde{\mathbf{N}}_d(s)\mathbf{N}_d(s)\right) .$$

Letting $n \to \infty$ we obtain by continuity that (5.25) holds true. □

Using (5.14), (5.20) and combining Propositions 5.1 and 5.2 and Theorems 3.3 and 3.7, we obtain the following result which is the counterpart of Corollary 4.6 for PDEs with distributed control.

Corollary 5.3 *Assume that* (**A 1**) *and* (**A 2**) *hold and suppose that* $\mathbf{H}_d$ *given by (5.14) satisfies one of the following two conditions:*

(i) $\mathbf{H}_d$ *is regular and the feedthrough* D *of* $\mathbf{H}_d$ *satisfies* $|D| > 1$.

(ii) $\mathbf{H}_d$ *is not well-posed and satisfies the conditions (1) and (2) in Theorem 3.7.*

Then there exist sequences (ε_n) *and* (s_n) *with*

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad s_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} |\mathrm{Im}\, s_n| = \infty\,, \quad \lim_{n\to\infty} \mathrm{Re}\, s_n = \infty$$

and such that for any $n \in \mathbb{N}$, $\det \mathbf{\Delta}_{\varepsilon_n}(s_n) = 0$, *i.e. for any* $n \in \mathbb{N}$ *the closed-loop system* $(\mathbf{CDC}_{\varepsilon_n})$ *has a mode with exponent* s_n.

In a similar way counterparts of Corollaries 4.7–4.9 for systems with distributed control can be established. We leave the details to the reader.

6. Examples

In this section we illustrate Theorem 3.7, Proposition 3.9 and the corresponding Corollaries 4.6 (ii) and 4.8 with some simple examples. Theorem 3.3 and Corollaries 4.6 (i) and 4.7 are illustrated by the examples in [15].

Example 6.1 In the first example we consider an equation for two coupled vibrating strings, with two observation and two controls. This example is similar to example 9.3 in [15], except instead of incorporating viscous damping into the model we use Kelvin-Voigt damping, and we draw very different conclusions. We assume that each string satisfies the damped wave equation

$$w_{tt}(x,t) - w_{xx}(x,t) - a w_{xxt}(x,t) = 0\,, \qquad x \in (0,1) \cup (1,2)\,, \quad t > 0\,, \tag{6.1}$$

where $a > 0$. As in [15] we consider the following boundary conditions

$$w(1^-,t) = w(1^+,t)\,, \qquad w(2,t) = 0\,, \tag{6.2}$$

boundary controls

$$w_x(1^-,t) - w_x(1^+,t) = u_1(t)\,, \qquad w_x(0,t) = u_2(t)\,, \tag{6.3}$$

and boundary observations

$$y_1(t) = k_1 w_t(1,t)\,, \qquad y_2(t) = -k_2 w_t(0,t)\,, \tag{6.4}$$

where $k_1, k_2 \geq 0$. Since $a \geq 0$, it is easily checked that assumption (**A 1**) is satisfied.

Let $u(t) := [u_1(t), u_2(t)]^T$ and $y(t) := [y_1(t), y_2(t)]^T$ and set

$$r(s) := \frac{s}{\sqrt{1+as}}\,,$$

where $\sqrt{s}$ denotes the principle branch of the square root. Setting

$$A(s) := \frac{e^{4r(s)} - 1}{2(e^{4r(s)} + 1)}\,, \qquad B(s) := \frac{e^{r(s)} - e^{3r(s)}}{e^{4r(s)} + 1}\,,$$

a routine calculation shows that assumption (**A 2**) is satisfied and that the transfer function for (6.1)–(6.4) is given by

$$\mathbf{H}(s) = \sqrt{1+as}\,\tilde{\mathbf{H}}(s)\,,$$

where

$$\tilde{\mathbf{H}}(s) := \begin{pmatrix} k_1 A(s) & k_1 B(s) \\ k_2 B(s) & 2k_2 A(s) \end{pmatrix}.$$

Note that for any $\delta \in (0, \pi/2)$, $\operatorname{Re} r(s) \to \infty$ as $|s| \to \infty$ in $\mathcal{S}(\delta)$. This implies that

$$\lim_{|s|\to\infty,\, s\in\mathcal{S}(\delta)} A(s) = 1/2\,, \qquad \lim_{|s|\to\infty,\, s\in\mathcal{S}(\delta)} B(s) = 0\,,$$

so

$$\lim_{|s|\to\infty,\, s\in\mathcal{S}(\delta)} \tilde{\mathbf{H}}(s) = \begin{pmatrix} k_1/2 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Consequently, $\mathbf{H}$ satisfies the conditions in Proposition 3.8, and hence conditions (1) and (2) in Theorem 3.9, provided that $k_1 k_2 \neq 0$. Therefore, the conclusions of Corollary 4.6 (ii) hold for this example. In particular, we find that there exist sequences (ε_n) and (s_n) with $\varepsilon_n \downarrow 0$ and $\operatorname{Re} s_n \to \infty$ as $n \to \infty$ and such that the system (6.1)–(6.4) with feedback $u(t) = -y(t - \varepsilon_n)$ has a mode with exponent s_n.

We can contrast this to the situation in [15], where the damping term $aw_{xxt}(x,t)$ is replaced by the viscous damping term $2aw_t(x,t) + a^2 w(x,t)$. In [15], it is found that for every $a > 0$ there are values of k_1 and k_2 such that the input-output system is robust with respect to delays. In the viscous damping case all of the modes oscillate and decay with the same exponential rate, whilst in the Kelvin-Voigt case only finitely many modes oscillate and the rest have exponents which are negative real (pure exponential decay). So, even though, in this sense, Kelvin-Voigt damping is "stronger" than viscous damping, the present example shows that it also causes the system to be ill-posed, destroying robustness of the closed-loop system. $\diamond$

Example 6.2 In this example we consider a one-dimensional beam equation with structural damping (also known as $A^{1/2}$ damping). This type of damping causes the underlying semigroup to be analytic. The partial differential equation for the displacement $w(x,t)$ is

$$w_{tt}(x,t) + 2aw_{xxt}(x,t) + w_{xxxx}(x,t) = 0\,, \qquad x \in (0,1)\,, \quad t > 0\,, \tag{6.5}$$

with the damping parameter $a \in (0,1)$. We consider the following boundary conditions

$$w(0,t) = 0\,, \qquad w(1,t) = 0\,, \qquad w_x(0,t) = 0\,, \tag{6.6}$$

with boundary control

$$w_{xx}(1,t) = u(t) \tag{6.7}$$

and boundary observation

$$y(t) = cw_{xt}(1,t) + dw_t(1,t)\,, \tag{6.8}$$

where we assume that $c, d \in \mathbb{R}$ and $c \neq 0$. Trivially, assumption (**A 1**) is satisfied

If there is no damping, i.e. $a = 0$, it is well-known that this system is ill-posed in the sense that its transfer function is ill-posed. We will find that in spite of a positive damping term $a \in (0,1)$, the system is still ill-posed, and in fact satisfies the conditions in Proposition 3.8. Let $\sqrt{s}$ denote the principle branch of the square root, define $\theta \in (\pi/2, \pi)$ by $e^{\imath\theta} = -a + \imath\sqrt{1-a^2}$, and set $\eta := e^{\imath\theta/2}$. Computing the transfer function $\mathbf{H}$ of (6.5)–(6.8), we find that (**A 2**) holds and we obtain

$$\mathbf{H}(s) = \sqrt{s}\,\tilde{\mathbf{H}}(s)\,,$$

where

$$\begin{aligned}\tilde{\mathbf{H}}(s) = {} & \frac{(c\eta + d/\sqrt{s})e^{\eta\sqrt{s}} + (c\overline{\eta} + d/\sqrt{s})e^{\overline{\eta}\sqrt{s}}g_1(s)}{[\eta^2 e^{\eta\sqrt{s}} + \overline{\eta}^2 e^{\overline{\eta}\sqrt{s}}g_1(s) + \eta^2 e^{-\eta\sqrt{s}}g_2(s) + \overline{\eta}^2 e^{-\overline{\eta}\sqrt{s}}g_3(s)]} \\ & + \frac{(-c\eta + d/\sqrt{s})e^{-\eta\sqrt{s}}g_2(s) + (-c\overline{\eta} + d/\sqrt{s})e^{-\overline{\eta}\sqrt{s}}g_3(s)}{[\eta^2 e^{\eta\sqrt{s}} + \overline{\eta}^2 e^{\overline{\eta}\sqrt{s}}g_1(s) + \eta^2 e^{-\eta\sqrt{s}}g_2(s) + \overline{\eta}^2 e^{-\overline{\eta}\sqrt{s}}g_3(s)]}\,,\end{aligned}$$

and the functions g_1, g_2 and g_3 are given by

$$\begin{aligned} g_1(s) &= \frac{(\eta - \overline{\eta})e^{\eta\sqrt{s}} + (\eta + \overline{\eta})e^{-\eta\sqrt{s}} - 2\eta e^{-\overline{\eta}\sqrt{s}}}{(-\eta + \overline{\eta})e^{\overline{\eta}\sqrt{s}} + (\eta + \overline{\eta})e^{-\overline{\eta}\sqrt{s}} - 2\overline{\eta}e^{-\eta\sqrt{s}}}\,, \\ g_2(s) &= \frac{\eta + \overline{\eta} + 2\overline{\eta}g_1(s)}{\eta - \overline{\eta}}\,, \\ g_3(s) &= \frac{2\eta + (\eta - \overline{\eta})g_1(s)}{\overline{\eta} - \eta}\,. \end{aligned}$$

We first note that $\pi/4 < \theta/2 < \pi/2$, so in particular $\operatorname{Re}\eta > 0$. Since $\operatorname{Re}(\eta - 2\overline{\eta}) < 0$, there exists $\delta > 0$ such that $e^{-\eta\sqrt{s}}$, $e^{-\overline{\eta}\sqrt{s}}$ and $e^{(\eta-2\overline{\eta})\sqrt{s}}$ all go to zero as s goes to infinity in $\mathbb{S}(\delta)$. Setting

$$\varepsilon(s) := g_1(s) + e^{(\eta-\overline{\eta})\sqrt{s}}\,,$$

it is easy to see that

$$\lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} \varepsilon(s) = 0\,, \qquad \lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} \varepsilon(s)e^{(\overline{\eta}-\eta)\sqrt{s}} = 0\,. \tag{6.9}$$

Therefore

$$\lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} g_2(s)e^{-\eta\sqrt{s}} = \lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} \left(\frac{\eta + \overline{\eta}}{\eta - \overline{\eta}}e^{-\eta\sqrt{s}} - \frac{2\overline{\eta}}{\eta - \overline{\eta}}e^{-\overline{\eta}\sqrt{s}} + \varepsilon(s)\frac{2\overline{\eta}}{\eta - \overline{\eta}}e^{-\eta\sqrt{s}}\right) = 0\,,$$

and similarly,

$$\lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} g_3(s)e^{-\overline{\eta}\sqrt{s}} = 0\,.$$

Thus, by (6.9)

$$\lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} \tilde{\mathbf{H}}(s) = \lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} \frac{(c\eta + d/\sqrt{s})e^{\eta\sqrt{s}} - (c\overline{\eta} + d/\sqrt{s})(e^{\eta\sqrt{s}} - \varepsilon(s)e^{\overline{\eta}\sqrt{s}})}{\eta^2 e^{\eta\sqrt{s}} - \overline{\eta}^2(e^{\eta\sqrt{s}} - \varepsilon(s)e^{\overline{\eta}\sqrt{s}})} = \frac{c}{\eta + \overline{\eta}}\,.$$

Since $\operatorname{Re}\eta \neq 0$ and $c \neq 0$, this limit is finite and nonzero. Consequently, $\mathbf{H}$ satisfies the conditions in Proposition 3.8, and hence conditions (1) and (2) in Theorem 3.7. Therefore, the conclusions of Corollary 4.6 (ii) hold, so there exist sequences (ε_n) and (s_n) with $\varepsilon_n \downarrow 0$ and $\operatorname{Re} s_n \to \infty$ as $n \to \infty$ and such that the system (6.5)–(6.8) with feedback $u(t) = -y(t-\varepsilon_n)$ has a mode with exponent s_n. $\diamond$

Example 6.3 In this example we will illustrate Proposition 3.9 by considering robustness with respect to delays for the coupled beam example from Rebarber [19]. Let $x_1 \in (0,1)$ and consider the following system

$$w_{tt}(x,t) - w_{xxxx}(x,t) = 0\,, \qquad x \in (0,x_1) \cup (x_1,1)\,, \quad t > 0\,, \tag{6.10}$$

$$w(0,t) = w_{xx}(0,t) = w(1,t) = w_{xx}(1,t) = 0\,, \tag{6.11}$$

$$w(x_1^-,t) = w(x_1^+,t)\,, \quad w_x(x_1^-,t) = w_x(x_1^+,t)\,, \quad w_{xxx}(x_1^-,t) = w_{xxx}(x_1^+,t)\,, \tag{6.12}$$

$$w_{xx}(x_1^-,t) - w_{xx}(x_1^+,t) = u(t)\,, \qquad y(t) = k w_{xt}(x_1,t)\,, \tag{6.13}$$

where $k > 0$. Clearly, (**A 1**) is satisfied and it is shown in [19] that the feedback $u(t) = -y(t)$ renders the system exponentially stable. We will show that this stability is not robust with respect to delays.

Let $s = \imath\omega^2$, where we choose $\omega \in \{re^{\imath\theta} \mid r \geq 0,\ \theta \in [-\pi/2, 0]\}$ for $\operatorname{Re} s \geq 0$, and let

$$h(x_1,\omega) = \frac{\sinh(\omega x_1) - \cosh(\omega x_1)\tanh\omega(x_1-1)}{\cos(\omega x_1)\tan\omega(x_1-1) - \sin(\omega x_1)}\,.$$

Computing the transfer function for (6.10)–(6.13) we find that (**A 2**) holds and we obtain

$$\mathbf{H}(s) = \frac{\omega[\cosh(\omega x_1) + h(x_1,\omega)\cos(\omega x_1)]}{(-2\imath)[\sinh(\omega x_1) - \cosh(\omega x_1)\tanh\omega(x_1-1)]}\,.$$

From Section 4, we see that the poles of $\mathbf{H}(s)$ are contained in the set of exponents of the modes of (6.10)–(6.13) with $u(t) = 0$. This set of exponents is the set of eigenvalues of the operator

$$A := \begin{pmatrix} 0 & I \\ -\mathfrak{D}^4 & 0 \end{pmatrix}, \tag{6.14}$$

where as in Section 4, $\mathfrak{D}$ denotes spatial differentiation. Here the state space is

$$X = \{(w_1,w_2)^T \in H^2[0,1] \oplus L^2[0,1] \mid w_1(0) = w_1(1) = 0\}\,,$$

and the domain of A is given by

$$\begin{aligned}\operatorname{dom}(A) = \{(w_1,w_2)^T \in H^4[0,1] \oplus H^2[0,1] \mid w_1(0) &= w_1(1) \\ = \mathfrak{D}^2 w_1(0) = \mathfrak{D}^2 w_1(1) &= w_2(0) = w_2(1) = 0\}\,.\end{aligned}$$

It is well-known that A is skew-adjoint and has compact resolvent, hence the spectrum of A consists of purely imaginary eigenvalues. Moreover, there are infinitely many eigenvalues, with ∞ being the only accumulation point. As already mentioned, an application of the feedback law $u(t) = -y(t)$ to (6.10)–(6.13) results in a closed-loop system which is exponentially stable, and hence modally stable. A combination of Lemma 2.4 and Proposition 2.3 then shows that each eigenvalue of A is a pole of $\mathbf{H}$, and hence $\mathbf{H}$ has infinitely many poles on the imaginary axis.

The zeros of $\mathbf{H}(s)$ are contained in the set of exponents of the modes of the zero dynamics of (6.10)–(6.13). The zero dynamics are given by the uncontrolled inverse system of (6.10)–(6.13), i.e. the system which is obtained by interchanging the roles of u and y. Clearly, the transfer function of the inverse system is given by $1/\mathbf{H}$. The exponents of the modes of the zero dynamics are identical to the eigenvalues of the operator A^{inv}, where A^{inv} is given by the right-hand side of (6.14) with

$$\begin{aligned}\mathrm{dom}(A^{inv}) = \{(w_1, w_2)^T \in X \mid\ & w_1|_{[0,x_1)} \in H^4[0, x_1),\ w_1|_{(x_1,1]} \in H^4(x_1, 1],\\ & w_2 \in H^2[0, 1],\ \mathcal{D}^2 w_1(0) = \mathcal{D}^2 w_1(1) = w_2(0) = w_2(1) = 0,\\ & \mathcal{D} w_2(x_1) = 0,\ \mathcal{D}^3 w_1(x_1^-) = \mathcal{D}^3 w_1(x_1^+)\}\,.\end{aligned}$$

It is shown in [19] that this operator is skew-adjoint and has compact resolvent, and so the spectrum of A^{inv} consists of purely imaginary eigenvalues. Moreover, there are infinitely many eigenvalues, with ∞ being the only accumulation point. Since the feedback $y(\cdot) = -u(\cdot)$ applied to the inverse system leads to an exponentially stable closed-loop system, we can argue as above to show that $\mathbf{H}$ has infinitely many zeros on the imaginary axis.

Combining our findings, it follows that $\mathbf{H}$ satisfies (3.22). So there exist sequences (ε_n) and (ω_n) in $(0, \infty)$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} \omega_n = \infty$ and such that the system (6.10)–(6.13) with the feedback $u(t) = -y(t - \varepsilon_n)$ has a mode with exponent $\imath\omega_n$. $\diamond$

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