# CRITICAL LOAD FOR A MODE–1 CRACK REINFORCED BY BRIDGING FIBRES

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**Abstract**

A long matrix crack in a unidirectional fibre–reinforced composite, perpendicular to the fibre direction and subjected to Mode–I loading, is considered. The intact composite material is modelled as homogeneous linearly elastic and the bridging fibres are replaced by restraining tractions acting on the crack surfaces. The problem is formulated in terms of a hypersingular integral equation with respect to the relative crack–face separation and contains a small parameter multiplying the integral term. Two limiting cases, related to the maximal opening in the middle of the crack before fibres break and to the maximal opening at the ends of the region where fibres are broken, are analyzed. The asymptotic solution of the problem is constructed with the use of a matched expansions technique. The values of applied load, corresponding to these two limiting situations, are estimated.

## 1 Introduction

A typical failure process in fibre–reinforced ceramics is characterized by formation of a matrix crack which extends through the composite leaving unbroken fibres behind its front. Intact fibres provide bridging restraint across the crack and increase the fracture toughness of the composite. Further loading may cause fibre breakage behind the crack front which occurs initially without matrix crack growth and is accompanied by stable or unstable extension of unbridged zone. This leads to reduction of the stress required for matrix cracking and, subsequently, causes complete failure of the composite.

The value of applied load at which fibres begin to fail depends on the fibre strength, the initial crack configuration and on the nature of the fibre–matrix interface.

The simplest model that allows analysis of the crack reinforcement effect is based on the stress intensity approach in which the effect of fibres is modelled by restraining tractions $\gamma$ acting on the crack surfaces and the intact material is regarded as homogeneous linearly elastic with moduli appropriate to the composite. The tractions $\gamma$ are taken to be proportional to the fibre axial stress and dependent on the relative crack–face separation $\phi$. The maximum value, $c_m$ say, of the function $\gamma$ corresponds to a single–valued fibre strength, and the fibre failure is assumed to occur when the stress in a bridging fibre reaches this value.

With the use of this model the problem for a crack in a unidirectional fibre–reinforced composite, which extends along the segment $(-a, a)$ of the $x$–axis perpendicular to the fibre direction and is subjected to Mode–I loading, can be formulated as a hypersingular integral equation with respect to the relative displacement $\phi$ of the crack faces. In the case of a long crack, $a \gg 1$, this equation contains a small parameter $\varepsilon$ (proportional to $1/a$) multiplying the integral term

$$-\varepsilon \int_{-1}^{1} \frac{dy}{(y-x)^2}\phi(y) + \gamma(\phi(x)) = \sigma, \ |x| < 1. \tag{1}$$

Here we use nondimensional variables which are introduced in the same way as in [2] (p.94, formulae (2), (4)). The quantity $\sigma$ on the right–hand side of (1) characterizes the tensile stress that the applied load would generate across the $x$–axis in uncracked composite; in general, it depends on $x$. The displacement $\phi$ is required to satisfy zero opening conditions at the ends of the crack

$$\phi(-1) = \phi(1) = 0. \tag{2}$$

The integral on the left–hand side of (1) is interpreted in the finite–part Hadamard sense [4] and is written in the form

$$\begin{aligned}\int_{-1}^{1} \frac{\phi(y)dy}{(y-x)^2} &= \int_{-1}^{1} \frac{dy}{(y-x)^2}[\phi(y) - \phi(x) - (y-x)\phi'(x)] \\ &\quad - \frac{2\phi(x)}{1-x^2} + \phi'(x)\ln\left|\frac{1-x}{1+x}\right|, \end{aligned} \tag{3}$$

for any fixed point $x$ of the segment $(-1, 1)$ at which $\phi$ is differentiable.

A method for analysis of such class of singular integral equations was developed by Willis and Nemat–Nasser [1], based on the matched asymptotic expansions technique. The linearized problem (1)–(2), with $\gamma(\phi) = \phi$ and $\sigma = \text{const}$, was solved explicitly in [1], a nonlinear example, with $\gamma(\phi) = k\phi(\phi_* - \phi)$, $\sigma = \text{const}$, was considered in [3].

In the present paper we estimate the value of applied load at which fibres begin to fail in the middle zone of the crack, assuming that the crack does not extend. The function $\gamma$ is taken to be such that

$$\gamma(\phi_m) = c_m, \quad \gamma'(\phi_m) = 0. \tag{4}$$

The remotely applied stress $\sigma$ on the right–hand side of (1) is assumed to be constant and specified by

$$\sigma = c_m + c_1 + c_2, \; c_1 = O(\varepsilon), \; c_2 = o(\varepsilon). \tag{5}$$

We define the critical value $c_{crit}$ of applied load as a quantity corresponding to the maximal opening $\phi_m$ in the middle of the crack and estimate it to order $O(\varepsilon^{3/2})$.

The formal procedure for finding the asymptotic solution is as follows. The opening displacement outside neighbourhoods of the ends of the crack ($x$ is fixed) is represented in the form of the asymptotic expansion with respect to the half–integer powers of a small parameter $\varepsilon$

$$\phi(x) \sim \phi_0(x) + \phi_1(x) + \cdots + \phi_n(x) + o(\varepsilon^{n/2}) \quad \text{as} \;\; \varepsilon \to 0. \tag{6}$$

The functions $\phi_k(x)$ depend algebraically on $\varepsilon^{1/2}$ and $\ln\varepsilon$, and $\ln\varepsilon$ is regarded as $O(1)$. Near the ends of the crack stretched variables are introduced

$$x' = \frac{1+x}{\varepsilon}, \quad x'' = \frac{1-x}{\varepsilon}, \tag{7}$$

and the crack opening is described by solutions of the boundary layer type

$$\phi(-1+\varepsilon x') \sim \tilde{\phi}_0(x') + \tilde{\phi}_1(x') + \cdots + \tilde{\phi}_m(x') + o(\varepsilon^{m/2}), \tag{8}$$

$$\phi(1-\varepsilon x'') \sim \tilde{\tilde{\phi}}_0(x'') + \tilde{\tilde{\phi}}_1(x'') + \cdots + \tilde{\tilde{\phi}}_m(x'') + o(\varepsilon^{m/2}) \quad \text{as} \;\; \varepsilon \to 0, \tag{9}$$

with fixed $x'$ and $x''$. An approximation to the solution which is uniform over the length of the crack is generated with the use of the matched expansions technique. Namely, the coefficients $\phi_k(x), \;\; \tilde{\phi}_k(x'), \;\; \tilde{\tilde{\phi}}_k(x'')$ of the expansions (6), (8)–(9) are required to satisfy the matching principle of Van Dyke [5] that is formulated as follows: if the representations

$$\phi_k(-1+\varepsilon x') \sim \bar{S}_k^m(x') + o(\varepsilon^{m/2}), \quad \phi_k(1-\varepsilon x'') \sim \bar{\bar{S}}_k^m(x'') + o(\varepsilon^{m/2}), \tag{10}$$

$$\tilde{\phi}_k\left(\frac{1+x}{\varepsilon}\right) \sim \tilde{S}_k^n(x) + o(\varepsilon^{n/2}), \quad \tilde{\tilde{\phi}}_k\left(\frac{1-x}{\varepsilon}\right) \sim \tilde{\tilde{S}}_k^n(x) + o(\varepsilon^{n/2}) \quad \text{as} \;\; \varepsilon \to 0, \tag{11}$$

hold, then

$$\sum_{k=0}^{n} \bar{S}_k^m\left(\frac{1+x}{\varepsilon}\right) = \sum_{k=0}^{m} \tilde{S}_k^n(x), \qquad \sum_{k=0}^{n} \bar{\bar{S}}_k^m\left(\frac{1-x}{\varepsilon}\right) = \sum_{k=0}^{m} \tilde{\tilde{S}}_k^n(x) \tag{12}$$

exactly. The formal uniform asymptotic expansion of the solution $\phi(x)$ of the problem (1)–(2) is determined by

$$\phi(x) \sim \sum_{k=0}^{n} [\phi_k(x) + \tilde{\phi}_k(x') + \tilde{\tilde{\phi}}_k(x'') - \bar{S}_k^n(x') - \bar{\bar{S}}_k^n(x'')] + o(\varepsilon^{n/2}), \qquad (13)$$

for $-1 \le x \le 1$, $x'$, $x''$ related by (7). We substitute the expansion (13) into equation (1) and take $\varepsilon \to 0$ with one of $x$, $x'$ or $x''$ fixed. The formulation (1)–(2) reduces then to a set of the problems for the evaluation of the functions $\phi_k(x)$, $\tilde{\phi}_k(x')$, $\tilde{\tilde{\phi}}_k(x'')$. The condition of the maximal opening in the middle of the crack

$$\phi(0) = \phi_m$$

is used to estimate the critical value $c_{crit}$.

In Section 3 a long crack, partially bridged by fibres, is considered. It is assumed that fibre failure occurs in the middle zone of the crack and broken fibres do not contribute to crack bridging ($\gamma = 0$). In this case the critical value of applied load is determined as a quantity which corresponds to the critical opening at the end points of the broken region. This value depends on the relation between the length of the crack and the length of the region where fibres are broken.

# 2 Critical opening of a bridged crack in an elastic plane

**Solution to lowest order.** It follows from the representation (3) that the integral on the left–hand side (1) outside neighbourhoods of the ends of the crack is bounded. After substitution of the expansion (13) in (1), with $\varepsilon \to 0$ and $x$ fixed, equation (1) reduces to an algebraic equation for the function $\phi_0$

$$\gamma(\phi_0(x)) = c_m. \qquad (14)$$

Since the function $\gamma$ satisfies the conditions (4), it gives immediately that

$$\phi_0(x) = \phi_m = \text{const},$$

and, therefore,

$$\bar{S}_0^0(x') = \bar{\bar{S}}_0^0(x'') = \phi_m.$$

Near the ends of the crack the contribution of the integral term in equation (1) becomes important. Taking $x = -1 + \varepsilon x'$ and expanding (1) to lowest order as $\varepsilon \to 0$, with $x'$ fixed, one can write equation (1) in the form

$$-\varepsilon \int_{-1}^{1} \frac{dy}{(y-x)^2} [\phi_m + \tilde{\phi}_0(y') - \bar{S}_0^0(y') + \tilde{\tilde{\phi}}_0(y'') - \bar{\bar{S}}_0^0(y'')] + \gamma(\phi(-1+\varepsilon x')) \sim c_m, \qquad (15)$$

where $y' = \varepsilon^{-1}(1+y)$.

In view of the asymptotic estimates

$$\varepsilon \int_{-1}^{1} \frac{dy}{(y+1-\varepsilon x')^2}[\phi_m - \bar{S}_0^0(y') + \tilde{\tilde{\phi}}_0(y'') - \bar{\bar{S}}_0^0(y'')] = O(\varepsilon),$$

$$\int_{-1}^{1} \frac{dy}{(y-x)^2}\tilde{\phi}_0(y') = \int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\phi}_0 - \int_{2/\varepsilon}^\infty \frac{dy'}{(y'-x')^2}\tilde{\phi}_0(y')$$
$$\sim \int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\phi}_0(y'),$$

the relation (15) gives the equation for the lowest order term $\tilde{\phi}_0(x')$ of the boundary layer near the left end of the crack

$$-\int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\phi}_0(y') + \gamma(\tilde{\phi}_0(x')) = c_m. \tag{16}$$

This has exactly the same form as in [1]. The function $\tilde{\phi}_0(x')$ is required to satisfy the conditions

$$\tilde{\phi}_0(0) = 0,\ \tilde{\phi}_0(x') \to \phi_m \text{ as } x' \to \infty. \tag{17}$$

Due to the symmetry of the original problem, the problem for $\tilde{\tilde{\phi}}_0(x'')$ looks exactly the same.

The following asymptotic relations hold

$$\tilde{\phi}_0\left(\frac{1+x}{\varepsilon}\right) \sim \phi_m - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1+x}} + o(\varepsilon^{1/2}), \tag{18}$$

$$\tilde{\tilde{\phi}}_0\left(\frac{1-x}{\varepsilon}\right) \sim \phi_m - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1-x}} + o(\varepsilon^{1/2}) \text{ as } \varepsilon \to 0. \tag{19}$$

They can be proved as follows. Equation (16) with $x' = (1+x)/\varepsilon$ yields the asymptotic representation

$$\varepsilon \int_{-1}^\infty \frac{dy}{(y-x)^2}\phi_m - \varepsilon \int_{-1}^\infty \frac{dy}{(y-x)^2}\left[\tilde{\phi}_0\left(\frac{1+y}{\varepsilon}\right) - \phi_m\right]$$
$$+\gamma(\phi_m) + \frac{1}{2}\left[\tilde{\phi}_0\left(\frac{1+x}{\varepsilon}\right) - \phi_m\right]^2 \gamma''(\phi_m) \sim c_m.$$

The integrand in the second term is of order $O(1)$ only in an $\varepsilon$–neighbourhood of $y=-1$. Thus, this term can be neglected in comparison with the first one. Evaluating the remaining integral and noting (14) then yields

$$\frac{1}{2}\left[\tilde{\phi}_0\left(\frac{1+x}{\varepsilon}\right) - \phi_m\right]^2 \gamma''(\phi_m) + \frac{\varepsilon\phi_m}{1+x} = 0.$$

Therefore,

$$\tilde{\phi}_0\left(\frac{1+x}{\varepsilon}\right) - \phi_m \sim -\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1+x}} \quad \text{as } \varepsilon \to 0. \tag{20}$$

Sign "$-$" is chosen due to the inequality $\phi(x) \le \phi_m$. Relation (20) yields (18) and so, by the definition (11) of $\tilde{S}_k^n$,

$$\tilde{S}_0^1(x) = \phi_m - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1+x}}.$$

Similarly, one can obtain (19) and

$$\tilde{\tilde{S}}{}_0^1(x) = \phi_m - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1-x}}.$$

Now reverting to equation (16) the integral over $(0,\infty)$ is represented by

$$\int_0^\infty \frac{\tilde{\phi}_0(y')dy'}{(y'-x')^2} = \int_0^A \frac{dy'}{(y'-x')^2}[\tilde{\phi}_0(y') - \tilde{\phi}_0(x') - (y'-x')\tilde{\phi}_0(x')]$$

$$-\frac{A\tilde{\phi}_0(x')}{x'(A-x')} + \tilde{\phi}_0'(x')\ln\left(\frac{A-x'}{x'}\right) + \int_A^\infty \frac{\tilde{\phi}_0(y')dy'}{(y'-x')^2}, \tag{21}$$

where $0 < x' < A,\ \ A \gg 1$. The first integral on the right–hand side (21) is regular, and the last term can be evaluated as follows

$$\int_A^\infty \frac{\tilde{\phi}_0(y')dy'}{(y'-x')^2} \sim \frac{\phi_m}{A-x'} - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\left[\frac{\sqrt{A}}{x'(A-x')}\right.$$

$$\left. + \frac{1}{2(x')^{3/2}}\ln\left|\frac{\sqrt{A}-\sqrt{x'}}{\sqrt{A}+\sqrt{x'}}\right|\right].$$

The problem (16)–(17) has been solved numerically for the quadratic function $\gamma$

$$\gamma(\phi) = \phi(\phi_* - \phi). \tag{22}$$

The function $\tilde{\phi}_0$ was calculated as the solution of the equation

$$-\int_0^A \frac{dy'}{(y'-x')^2}[\tilde{\phi}_0(y') - \tilde{\phi}_0(x') - (y'-x')\tilde{\phi}_0'(x')]$$

$$+\frac{A\tilde{\phi}_0(x')}{x'(A-x')} - \tilde{\phi}_0'(x')\ln\left(\frac{A-x'}{x'}\right) - \frac{\phi_m}{A-x'}$$

$$+\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\left[\frac{\sqrt{A}}{x'(A-x')} + \frac{1}{2(x')^{3/2}}\ln\left|\frac{\sqrt{A}-\sqrt{x'}}{\sqrt{A}+\sqrt{x'}}\right|\right]$$

$$+\gamma(\tilde{\phi}_0(x')) = c_m,\quad 0 < x' < A. \tag{23}$$

The numerical calculations have been done for the values $A = 200, \phi_* = 5$. The plot of $\tilde{\phi}_0$ versus $x'$ and comparison with the asymptotic approximation (18) (dashed line) is presented in Fig. 1. Although the asymptotic approximation (18) is valid formally as $x' \to \infty$, the figure shows that, in practice, it provides a remarkable approximation for $x' \geq \text{const} \simeq 5$.

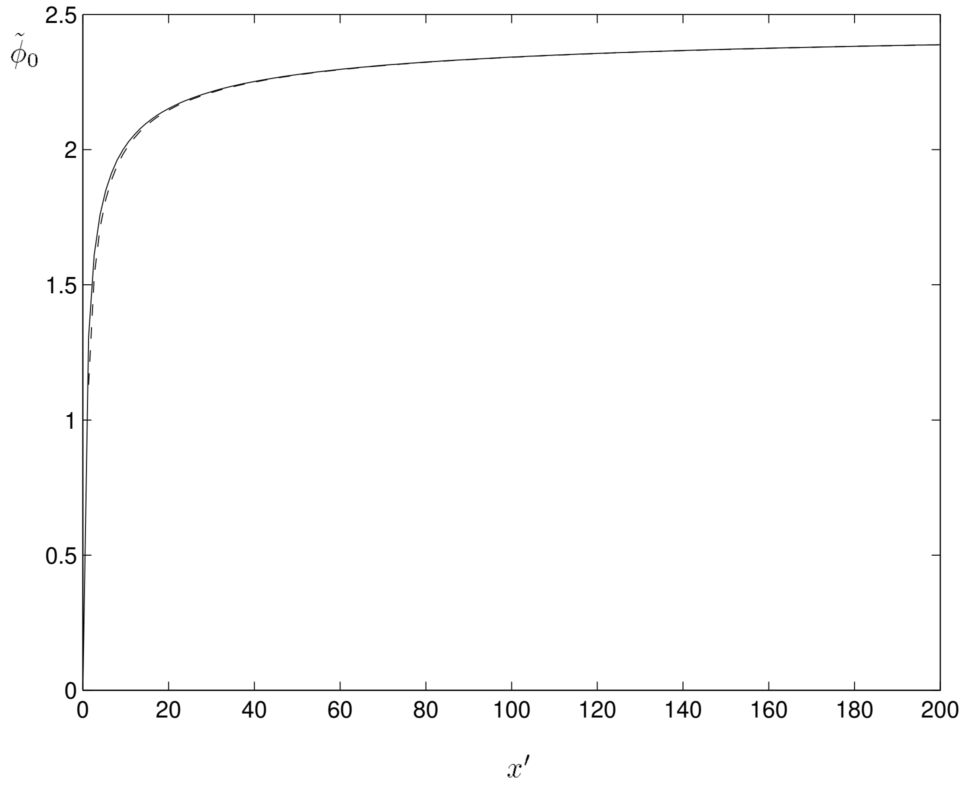


**Fig. 1**. Solution $\tilde{\phi}_0(x')$ of equation (16) (solid line) and its asymptotic approximation (18) (dashed line), for $\gamma = \phi(\phi_* - \phi)$, $\phi_* = 5$.

**Solution to first order.** The results of this section are based on the work [2], where the first two terms of the asymptotic expansion of the critical load were obtained. The critical crack opening is first attained in the middle of the crack, $x = 0$, and then $\phi(0) = \phi_m$. When this occurs, the equation with respect to $\phi_1$ (next order term in the asymptotic approximation (13)) has the form

$$-\varepsilon \int_{-1}^{1} \frac{dy}{(y-x)^2} \phi_m + \frac{1}{2}\gamma''(\phi_m)[\phi_1(x)]^2 = c_1, \tag{24}$$

and, therefore,

$$\phi_1(x) = -\left[ -\frac{2}{\gamma''(\phi_m)} \left( \frac{2\phi_m \varepsilon}{1-x^2} - c_1 \right) \right]^{1/2},$$

with the sign "$-$" in front of the expression on the right–hand side chosen due to the inequality $\phi(x) \leq \phi_m$. Clearly, the relation

$$\phi(0) - \phi_m = o(\varepsilon)$$

holds if and only if

$$c_1 = 2\phi_m \varepsilon.$$

Thus, it yields the estimate for the critical external load

$$c_{crit} \sim c_m + 2\varepsilon\phi_m \tag{25}$$

and the representation

$$\phi_1(x) = -\left[ -\frac{4\phi_m \varepsilon}{\gamma''(\phi_m)} \right]^{1/2} \frac{|x|}{\sqrt{1-x^2}}. \tag{26}$$

In the vicinity of the end point $x = -1$

$$\phi_1(-1 + \varepsilon x') \sim -\left[ -\frac{2\phi_m}{\gamma''(\phi_m)} \right]^{1/2} \frac{1}{\sqrt{x'}} + o(1) \quad \text{as} \ \ \varepsilon \to 0,$$

and, therefore, the asymptotic matching principle is satisfied

$$\overline{S}_1^0\left(\frac{1+x}{\varepsilon}\right) + \overline{S}_0^0\left(\frac{1+x}{\varepsilon}\right) = \tilde{S}_0^1(x).$$

Here

$$\overline{S}_1^0(x') = -\left[ -\frac{2\phi_m}{\gamma''(\phi_m)} \right]^{1/2} \frac{1}{\sqrt{x'}}.$$

However, the first order term $\phi_1$ is not smooth in the middle of the crack, and it should be treated by constructing an additional central layer concentrated in a neighbourhood of the point $x = 0$.

In the vicinity of the origin we introduce a new scaled variable $X$

$$x = \alpha X, \quad 0 < \alpha \ll 1.$$

Due to the asymptotic relation

$$\phi_1(\alpha X) \sim -\left[ -\frac{4\phi_m \varepsilon}{\gamma''(\phi_m)} \right]^{1/2} \alpha |X| + o(\sqrt{\varepsilon}\alpha) \ \ \text{as} \ \ \alpha \to 0,$$

the opening displacement in the middle zone of the crack is represented by the sum

$$\phi_m + \phi_1(x) + \Phi_1(X) - S_1(X) + \dots,$$

where

$$S_1(X) = -\left[ -\frac{4\phi_m \varepsilon}{\gamma''(\phi_m)} \right]^{1/2} \alpha |X|, \quad \Phi_1(X) \sim S_1(X) \ \text{ as } \ |X| \to \infty.$$

Let the function $\Psi_1(X)$ be such that

$$\Phi_1(X) = \Psi_1(X) - \left[ -\frac{4\phi_m \varepsilon}{\gamma''(\phi_m)} \right]^{1/2} \alpha (1 + X^2)^{1/2}. \tag{27}$$

Then the equation with respect to $\Psi_1$ takes the form

$$-\frac{\varepsilon}{\alpha} \int_{-\infty}^{\infty} \frac{dY}{(Y - X)^2} \Psi_1(Y) + \frac{1}{2}\gamma''(\phi_m) \Bigg[ (\Psi_1(X))^2$$

$$-2\Psi_1(X) \left[ -\frac{4\phi_m \varepsilon}{\gamma''(\phi_m)} \right]^{1/2} \alpha (1 + X^2)^{1/2} \Bigg]$$

$$= 2\varepsilon \phi_m \alpha^2 - \varepsilon^{3/2} \left[ -\frac{4\phi_m}{\gamma''(\phi_m)} \right]^{1/2} F_1(X), \tag{28}$$

where

$$F_1(X) = \int_{-1}^{1} \frac{dy}{(y - \alpha X)^2} \left[ \frac{|y|}{\sqrt{1 - y^2}} - |y| + (\alpha^2 + y^2)^{1/2} \right].$$

At infinity $\Psi_1$ satisfies the condition

$$\Psi_1(X) \sim \frac{1}{2} \left[ -\frac{4\phi_m \varepsilon}{\gamma''(\phi_m)} \right]^{1/2} \frac{\alpha}{|X|} \ \text{ as } \ |X| \to \infty.$$

The correct choice of the small parameter $\alpha$ should provide the same order terms on the right–hand side (28). In the present case

$$\alpha = \varepsilon^{1/4}.$$

The function $\Psi_1$ has been calculated numerically with the use of the representation

$$F_1(X) = 2 \ln 2 + \int_{-A}^{A} \frac{dY}{(Y - X)^2} \left[ (1 + Y^2)^{1/2} - (1 + X^2)^{1/2} - \frac{(Y - X)X}{(1 + X^2)^{1/2}} \right]$$

$$+ \frac{2A\sqrt{1 + A^2}}{A^2 - X^2} - \frac{2A(1 + X^2)^{1/2}}{A^2 - X^2} - \frac{X}{\sqrt{1 + X^2}} \ln \left| \frac{1 - AX + \sqrt{(1 + X^2)(1 + A^2)}}{1 + AX + \sqrt{(1 + X^2)(1 + A^2)}} \right|.$$

The finite–part integral from (28) was defined by

$$\int_{-\infty}^{\infty}\frac{dY}{(Y-X)^2}\Psi_1(Y)\sim\int_{-A}^{A}\frac{dY}{(Y-X)^2}[\Psi_1(Y)-\Psi_1(X)-(Y-X)\Psi_1'(X)]$$

$$-\frac{A\Psi_1(X)}{A^2-X^2}+\Psi_1'(X)\ln\left|\frac{A-X}{A+X}\right|$$

$$+\frac{1}{2}\left[-\frac{4\phi_m\varepsilon}{\gamma''(\phi_m)}\right]^{1/2}\alpha\left[\frac{2}{A^2-X^2}+\frac{1}{X^2}\ln\left|\frac{A^2-X^2}{A^2}\right|\right],$$

$$|X|<A,\quad A\gg 1,$$

and the equation was taken in the form

$$-\frac{\varepsilon}{\alpha}\left[\int_{-A}^{A}\frac{dY}{(Y-X)^2}[\Psi_1(Y)-\Psi_1(X)-(Y-X)\Psi_1'(X)]\right.$$

$$\left.-\frac{A\Psi_1(X)}{A^2-X^2}+\Psi_1'(X)\ln\left|\frac{A-X}{A+X}\right|\right]$$

$$+\frac{1}{2}\gamma''(\phi_m)\left[(\Psi_1(X))^2-2\Psi_1(X)\left[-\frac{4\phi_m\varepsilon}{\gamma''(\phi_m)}\right]^{1/2}\alpha(1+X^2)^{1/2}\right]$$

$$=2\varepsilon\phi_m\alpha^2+\varepsilon^{3/2}\left[-\frac{4\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\left[\frac{1}{A^2-X^2}+\frac{1}{2X^2}\ln\left|\frac{A^2-X^2}{A^2}\right|\right]$$

$$-\varepsilon^{3/2}\left[-\frac{4\phi_m}{\gamma''(\phi_m)}\right]^{1/2}F_1(X),\ |X|<A. \tag{29}$$

In Fig. 2 plots of $\varepsilon^{-3/4}\Phi_1$ and $\varepsilon^{-3/4}\Psi_1$ versus $X$ are represented for $A=60, \phi_*=5$ and quadratic function $\gamma$ given by equation (22).

The function $\Phi_1$ related to $\Psi_1$ by (27) gives a smooth first order approximation of the opening displacement in the middle zone of the crack. Due to the fact that $\Phi_1-S_1$ rapidly decays at infinity, it does not contribute to the first order boundary layer terms near the edges of the crack.

It follows from the representation (26) of the function $\phi_1$ that

$$\overline{S}_1^2(x')=-\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\frac{1}{\sqrt{x'}}+\frac{3}{4}\varepsilon\sqrt{x'}\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2},$$

and, therefore, the asymptotic matching principle (12) yields

$$\tilde{S}_1^1=\frac{3}{4}\varepsilon^{1/2}\sqrt{1+x}\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}.$$

The integral equation for the function $\tilde{\phi}_1(x')$ takes the form

$$-\int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\phi}_1(y') + \tilde{\phi}_1(x')\gamma'(\tilde{\phi}_0(x')) = \frac{3}{2}\varepsilon\phi_m. \quad (30)$$

Also, $\tilde{\phi}_1$ is subjected to the conditions

$$\tilde{\phi}_1(0) = 0, \quad \tilde{\phi}_1(x') \sim \frac{3}{4}\varepsilon\sqrt{x'}\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2} \text{ as } x' \to \infty. \quad (31)$$

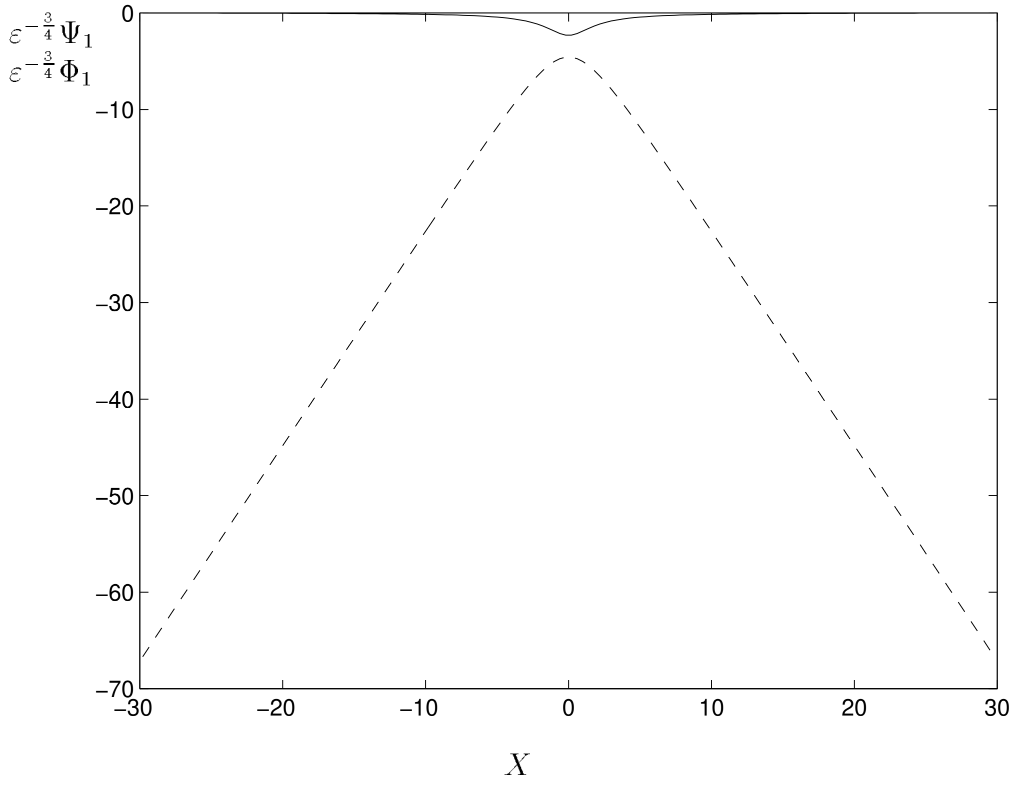


**Fig. 2**. Plots of the functions $\varepsilon^{-\frac{3}{4}}\Phi_1$ (dashed line) and $\varepsilon^{-\frac{3}{4}}\Psi_1$ (solid line) versus $X$, for $\gamma = \phi(\phi_* - \phi)$, $\phi_* = 5$.

Equation (30) has been solved numerically for the case of the quadratic function $\gamma$. The plot of $\varepsilon^{-1}\tilde{\phi}_1$ versus $x'$ and comparison with the asymptotic approximation (31) are presented in Fig. 3. The boundary layer follows the square root law, as expected.

**Solution to second order.** Let $x$ be fixed outside neighbourhoods of the crack tips. Substituting expansion (13) into (1), we obtain the linear equation with respect to $\phi_2$

$$\begin{aligned}
&\phi_1(x)\phi_2(x)\gamma''(\phi_m)+\\
&+\varepsilon^{3/2}\Big[-\frac{2\phi_m}{\gamma''(\phi_m)}\Big]^{1/2}\Big[\frac{\gamma''(\phi_m)}{\sqrt{1+x}}\Big(\tilde{S}_0^*(x,\ln\varepsilon)+\tilde{S}_1^*(x,\ln\varepsilon)\Big)\\
&+\frac{\gamma''}{\sqrt{1-x}}\Big(\tilde{\tilde{S}}_0^*(x,\ln\varepsilon)+\tilde{\tilde{S}}_1^*(x,\ln\varepsilon)\Big)\\
&-\frac{3}{4}\sqrt{1+x}\gamma''(\phi_m)\tilde{S}_0^*(x,\ln\varepsilon)-\frac{3}{4}\sqrt{1-x}\gamma''(\phi_m)\tilde{\tilde{S}}_0^*(x,\ln\varepsilon)\\
&+\frac{3}{4}\Big(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1-x}}\Big)\frac{\phi_m}{\gamma''(\phi_m)}\gamma'''(\phi_m)\Big]\\
&-\varepsilon^{3/2}\Big[-\frac{4\phi_m}{\gamma''(\phi_m)}\Big]^{1/2}2\ln\alpha=c_2. \qquad (32)
\end{aligned}$$

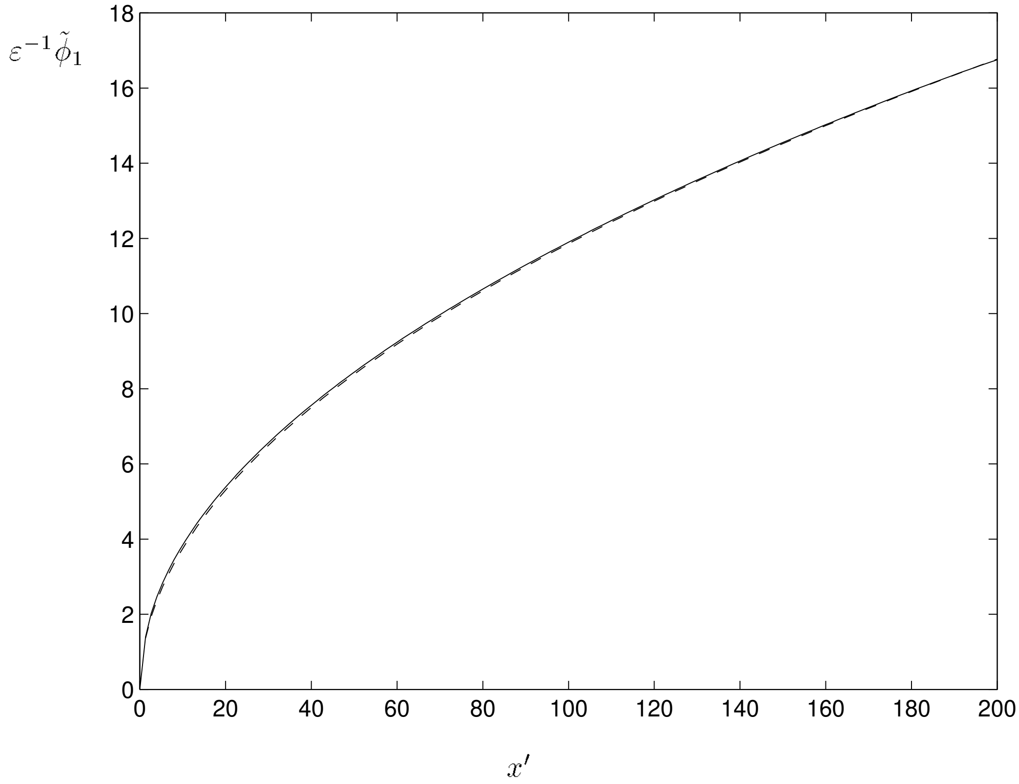


**Fig. 3**. Plots of the function $\varepsilon^{-1}\tilde{\phi}_1(x')$ (solid line) and its asymptotic approximation (31) (dashed line), for $\gamma=\phi(\phi_*-\phi)$, $\phi_*=5$.

Here $\tilde{S}_0^*$, $\tilde{S}_1^*$ denote terms of order $O(\varepsilon)$ in the asymptotic expansions

$$\tilde{\phi}_0\left(\frac{1+x}{\varepsilon}\right) \sim \phi_m - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1+x}} + \varepsilon\tilde{S}_0^*(x,\ln\varepsilon) + o(\varepsilon),$$

$$\tilde{\phi}_1\left(\frac{1+x}{\varepsilon}\right) \sim \frac{3}{4}\varepsilon^{1/2}\sqrt{1+x}\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2} + \varepsilon\tilde{S}_1^*(x,\ln\varepsilon) + o(\varepsilon) \text{ as } \varepsilon\to 0.$$

Similar asymptotic representations hold for the boundary layer in the vicinity of the right end of the crack.

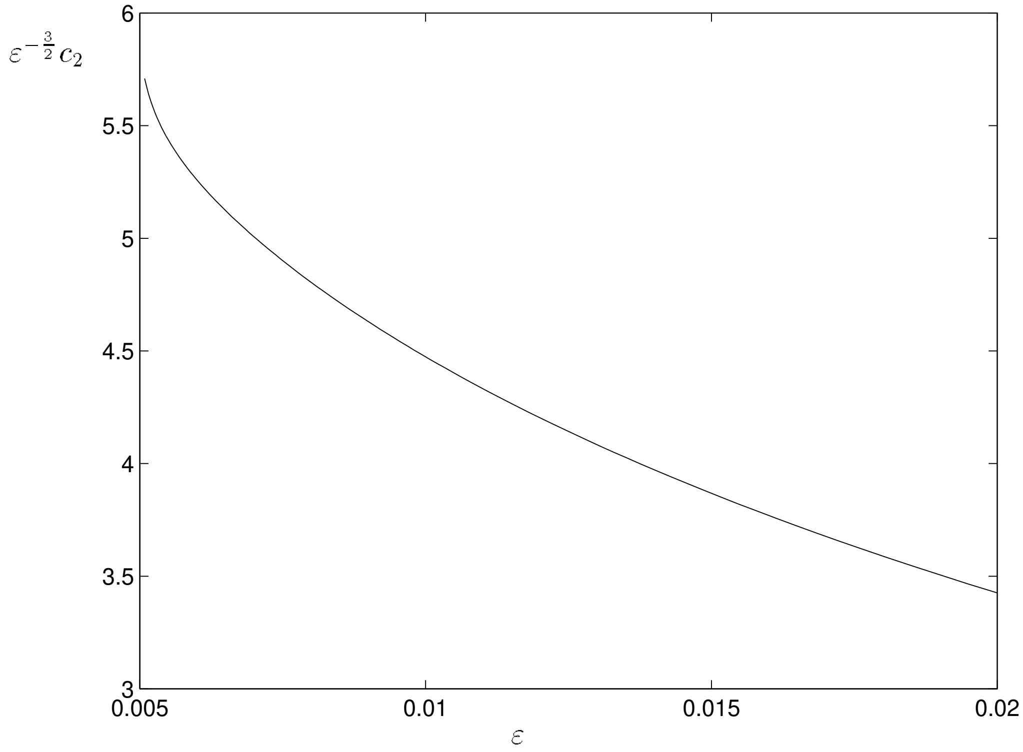


**Fig. 4**. Correction term $\varepsilon^{-3/2}c_2$ of the critical load $c_{crit}$ versus $\varepsilon$.

The functions $\tilde{S}_i^*$, $\tilde{\tilde{S}}_i^*$, $i = 0,1$, can be evaluated from the numerical calculations for first two terms in the asymptotic expansion of the boundary layer.

The function $\phi_1(x)$ vanishes at the origin. Therefore, equation (32) at $x = 0$ yields

$$c_2 = -\varepsilon^{3/2}\left[-\frac{4\phi_m}{\gamma''(\phi_m)}\right]^{1/2} 2\ln\alpha$$

$$+\varepsilon^{3/2}\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\gamma''(\phi_m)\left[\frac{1}{4}\tilde{S}_0^*(0,\ln\varepsilon)+\frac{1}{4}\tilde{\tilde{S}}_0^*(0,\ln\varepsilon)\right.$$

$$\left.+\tilde{S}_1^*(0,\ln\varepsilon)+\tilde{\tilde{S}}_1^*(0,\ln\varepsilon)\right]+\frac{3}{4}\varepsilon^{3/2}\left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\frac{\phi_m}{\gamma''(\phi_m)}\gamma'''(\phi_m). \tag{33}$$

Thus, the correction term $c_2$ to the critical load has the order $O(\varepsilon^{3/2}\ln\varepsilon)$.

In Fig. 4 the plot $\varepsilon^{-3/2}c_2$ versus $\varepsilon$ is presented; clearly, the shape of the curve is determined by the logarithmic term in (33). Fig. 5 includes $c_{crit}$ versus $\varepsilon$ in comparison with the value $c_m$. One can see that the presence of bridging fibres allows the critical load to take the value greater than $c_m$.

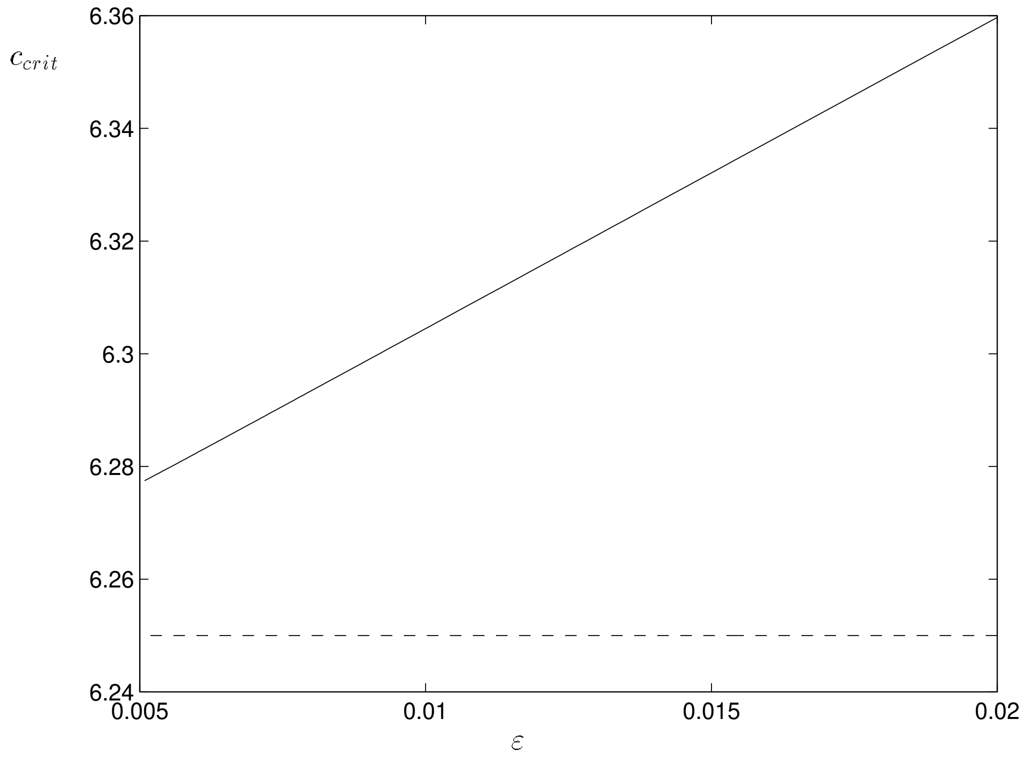


**Fig. 5**. Plot of $c_{crit}$ (solid line) versus $\varepsilon$ in comparison with the value $c_m$ (dashed line).

# 3 Partially bridged crack

In this section we consider a long crack, partially bridged by fibres. We suppose that fibres are broken in the middle zone of the crack for $|x| < \eta$, where $\eta$ is a small nondimensional positive parameter. The function $\gamma(\phi(x))$ is taken to be such that

$$\gamma(\phi_m) = c_m,\ \gamma'(\phi_m) = 0,$$

$$\gamma(\phi) = 0 \ \text{ for } \ \phi > \phi_m.$$

The crack opening displacement $\phi$ satisfies the integral equation

$$-\varepsilon \int_{-1}^{1} \frac{\phi(y)dy}{(y-x)^2} dy = \begin{cases} c, & |x| < \eta, \\ c - \gamma(\phi(x)), & \eta < |x| < 1 \end{cases} \tag{34}$$

and the homogeneous boundary conditions at the ends of the crack

$$\phi(-1) = \phi(1) = 0.$$

Here $c$ is constant, $c \leq c_m + o(1)$.

Suppose that at the points $x = \pm\eta$ the opening displacement $\phi$ is equal to $\phi_m$. The critical value of the applied load $c$ is determined as a quantity which corresponds to the opening $\phi = \phi_m$ at the end points $x = \pm\eta$ of the broken zone.

**Solution to lowest order.** Outside a neighbourhood of the region where fibres are broken, the leading part of the displacement $\phi$ is determined by

$$\phi(x) \sim \phi_0, \quad \phi_0 = \text{const},$$

$$\gamma(\phi_0) = c.$$

The constant $\phi_0$ leaves a discrepancy in the boundary conditions at the crack ends and in the equation in the middle zone $|x| < \eta$ of the crack. In order to compensate the discrepancy in the middle zone, we introduce a new stretched coordinate $\tilde{X} = \frac{x}{\eta}$ and construct the function $\tilde{\Phi}(\tilde{X})$ such that the following representation is valid in the middle zone of the crack

$$\phi \sim \phi_0 + \tilde{\Phi}(\tilde{X}). \tag{35}$$

Substituting (35) into the original integral equation (34), one can obtain

$$-\frac{\varepsilon}{\eta} \int_{-1/\eta}^{1/\eta} \frac{\tilde{\Phi}(\tilde{Y})}{(\tilde{Y}-\tilde{X})^2} d\tilde{Y} = \begin{cases} c, & |\tilde{X}| < 1, \\ c - \gamma(\phi_0 + \tilde{\Phi}(\tilde{X})), & 1 < |\tilde{X}| < 1/\eta, \end{cases}$$

$$\tilde{\Phi}(-1) = \tilde{\Phi}(1) = \phi_m - \phi_0.$$

As $\eta \to 0$, to lowest order approximation

$$-\frac{\varepsilon}{\eta} \int_{-\infty}^{\infty} \frac{\tilde{\Phi}(\tilde{Y})}{(\tilde{Y}-\tilde{X})^2} d\tilde{Y} = \begin{cases} c, & |\tilde{X}| < 1, \\ c - \gamma(\phi_0 + \tilde{\Phi}(\tilde{X})), & |\tilde{X}| > 1, \end{cases}$$

$$\tilde{\Phi}(-1) = \tilde{\Phi}(1) = \phi_m - \phi_0.$$

To provide the balance, the term $\tilde{\Phi}$ should have the order $O(\frac{\eta}{\varepsilon})$.

Let $k = \eta\varepsilon^{-1}$ be a positive constant. Introduce the notation $\tilde{\Phi}(\tilde{X}) = k\tilde{\Psi}$, where the function $\tilde{\Psi}$ satisfies the nonlinear equation

$$-\int_{-\infty}^{\infty} \frac{\tilde{\Psi}(\tilde{Y})}{(\tilde{Y}-\tilde{X})^2} d\tilde{Y} = \begin{cases} c, & |\tilde{X}| < 1, \\ c - \gamma(\phi_0 + k\tilde{\Psi}(\tilde{X})), & |\tilde{X}| > 1, \end{cases} \tag{36}$$

together with the conditions

$$\tilde{\Psi}(-1) = \tilde{\Psi}(1) = k^{-1}(\phi_m - \phi_0).$$

**The case of small values of the parameter $k$.** Assume that $k = o(1)$. Then

$$c = c_m + \tilde{c}, \quad \tilde{c} = o(1), \quad \phi_0 = \phi_m,$$

and the term $\gamma(\phi_m + k\tilde{\Psi}(\tilde{X}))$ can be written in the form

$$\gamma(\phi_m + k\tilde{\Psi}(\tilde{X})) \sim \gamma(\phi_m) + \frac{1}{2}(k\tilde{\Psi}(\tilde{X}))^2\gamma''(\phi_m).$$

The function $\tilde{\Psi}$ satisfies the equation

$$-\int_{-\infty}^{\infty} \frac{\tilde{\Psi}(\tilde{Y})}{(\tilde{Y}-\tilde{X})^2} d\tilde{Y} = c_m\chi(\tilde{X}),$$

and the conditions

$$\tilde{\Psi}(-1) = \tilde{\Psi}(1) = 0.$$

Here $\chi(\tilde{X})$ is defined by

$$\chi(\tilde{X}) = \begin{cases} 1, & |\tilde{X}| < 1, \\ 0, & |\tilde{X}| \geq 1. \end{cases}$$

Thus,

$$\tilde{\Psi}(\tilde{X}) = \frac{c_m}{\pi^2}[-(1-\tilde{X})\ln|1-\tilde{X}| - (1+\tilde{X})\ln|1+\tilde{X}| + 2\ln 2],$$

and

$$\tilde{\Psi}(\tilde{X}) \sim \frac{c_m}{\pi^2}[-2\ln|\tilde{X}| - 2 + 2\ln 2] + o(1) \quad \text{as} \quad |\tilde{X}| \to \infty. \tag{37}$$

Solutions of the boundary layer type. In the vicinity of the end points $x = \pm 1$ scaled coordinates $x'$, $x''$ (7) are used. The boundary layers $\tilde{\phi}_0(x')$ and $\tilde{\tilde{\phi}}_0(x'')$, which describe the crack opening in neighbourhoods of the crack ends, are introduced. The leading part of the crack opening displacement takes the form

$$\phi(x) \sim \phi_m + \tilde{\Phi}(\tilde{X}) + \tilde{\phi}_0(x') - \phi_m + \tilde{\tilde{\phi}}_0(x'') - \phi_m.$$

The term $\tilde{\Phi}$ leaves a discrepancy of order $\varepsilon^{-1}\eta\ln\eta$ which should be compensated on the next step of the asymptotic algorithm.

The boundary layer $\tilde{\phi}_0$ satisfies the problem

$$-\int_0^\infty \frac{\tilde{\phi}_0(y')}{(y'-x')^2}dy' + \gamma(\tilde{\phi}_0(x')) = c_m,$$

$$\tilde{\phi}_0(0)=0, \quad \tilde{\phi}_0(x') \to \phi_m \quad \text{as} \quad x' \to \infty,$$

c.f. (16), (17). Due to the symmetry, the same formulation is valid for the function $\tilde{\tilde{\phi}}_0(x'')$.

The solutions of the boundary layer type admit the asymptotic representations (c.f. (18), (19))

$$\tilde{\phi}_0\left(\frac{1+x}{\varepsilon}\right) \sim \phi_m - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1+x}} + o(\varepsilon^{1/2}),$$

$$\tilde{\tilde{\phi}}_0\left(\frac{1-x}{\varepsilon}\right) \sim \phi_m - \left[-\frac{2\phi_m}{\gamma''(\phi_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1-x}} + o(\varepsilon^{1/2}) \ \text{ as } \ \varepsilon \to 0.$$

Estimate of the critical load. To obtain the correction term for the critical load, the first order approximation for the crack opening in the middle zone of the crack (outside neighbourhoods of $x=\pm 1$) is considered.

The relation (37) yields (for fixed value of $x$)

$$\tilde{\Phi}(x/\eta) \sim \tilde{\Sigma}(x) + o(\eta/\varepsilon) \quad \text{as} \quad \eta \to 0,$$

where

$$\tilde{\Sigma}(x) = \frac{\eta}{\varepsilon}\,\frac{c_m}{\pi^2}\left[2\ln\eta + 2\ln 2 - 2\ln|x| - 2\right].$$

Outside neighbourhoods of the crack ends

$$\phi(x) \sim \phi_m + \tilde{\Phi}(\tilde{X}) + \phi_1(x) + o(\max\{\varepsilon^{1/2}, \eta/\varepsilon\}).$$

The function $\phi_1$ satisfies an algebraic equation whose form depends on the relation between $\varepsilon$ and $\eta$.

Consider the following range of values of parameters $\varepsilon$ and $\eta$.

a) $\varepsilon^{3/2} = o(\eta)$. The equation with respect to $\phi_1$ takes the form

$$\frac{1}{2}\,[\tilde{\Sigma}(x) + \phi_1(x)]^2\,\gamma''(\phi_m) = \tilde{c}.$$

The conditions $\phi(\eta) = \phi(-\eta) = \phi_m$ yield

$$\phi_1(-\eta) = \phi_1(\eta) = 0.$$

Thus,

$$\tilde{c} = 2\left(\frac{\eta}{\varepsilon}\right)^2 \left[\frac{c_m}{\pi^2}(\ln 2 - 1)\right]^2 \gamma''(\phi_m). \tag{38}$$

This quantity is negative, and, therefore, the critical value of the applied load $c_m + \tilde{c}$ is less than $c_m$.

b) $\eta = O(\varepsilon^{3/2})$. In this case the function $\phi_1(x)$ satisfies the equation

$$\frac{2\varepsilon\phi_m}{1-x^2} + \frac{1}{2}\left[\tilde{\Sigma}(x) + \phi_1(x)\right]^2 \gamma''(\phi_m) = \tilde{c}$$

and the constant $\tilde{c}$ is estimated by

$$\tilde{c} = 2\varepsilon\phi_m + 2\left(\frac{\eta}{\varepsilon}\right)^2 \left[\frac{c_m}{\pi^2}(\ln 2 - 1)\right]^2 \gamma''(\phi_m) + o(\varepsilon). \tag{39}$$

The second term in the right–hand side (39) is negative, and, therefore, $\tilde{c} < 2\varepsilon\phi_m$.

c) $\eta = o(\varepsilon^{3/2})$. For this range of small values of the length of the zone of broken fibres one has the following equation with respect to $\phi_1$

$$\frac{2\varepsilon\phi_m}{1-x^2} + \frac{1}{2}(\phi_1(x))^2\gamma''(\phi_m) = \tilde{c}.$$

The correction term for the critical load is

$$\tilde{c} = 2\varepsilon\phi_m + o(\varepsilon), \tag{40}$$

Thus, the correction terms which characterize the critical load in comparison with the case of unbroken fibres [2] have the high order $o(\varepsilon)$ contribution.

**The case** $k = O(1)$**.** The nonlinear problem (36) has been solved numerically for $\gamma$ specified by

$$\gamma(\phi) = \begin{cases} \phi(\phi_* - \phi), & \phi \le \phi_m, \\ 0, \phi > \phi_m, \end{cases}$$

with constant $\phi_*$ (for numerical calculations $\phi_* = 5$ was used) and $\phi_m = \phi_*/2$. The critical value of the external load $c$ was estimated with the use of the conditions

$$\phi_0 + k\tilde{\Psi}(-1) = \phi_0 + k\tilde{\Psi}(1) = \phi_m.$$

In Figs. 6(a)-(d) plots of the correction terms in the middle zone of the crack are presented for different values of the parameter $k$. The contribution from the middle zone is smaller as $k$ decreases. The plot of the critical load versus $k$ is presented in Fig. 7. The presence of broken fibres in the middle zone of the crack causes decrease of the critical value of the applied load. It follows from the analysis above that unstable fibre failure occurs in the middle zone of the crack, when the length $\eta$ of the region where fibres are broken and the parameter $\varepsilon$ (reciprocal to the length of the crack) have the same order of magnitude. This result is similar to one obtained by Marshall and Cox [6].

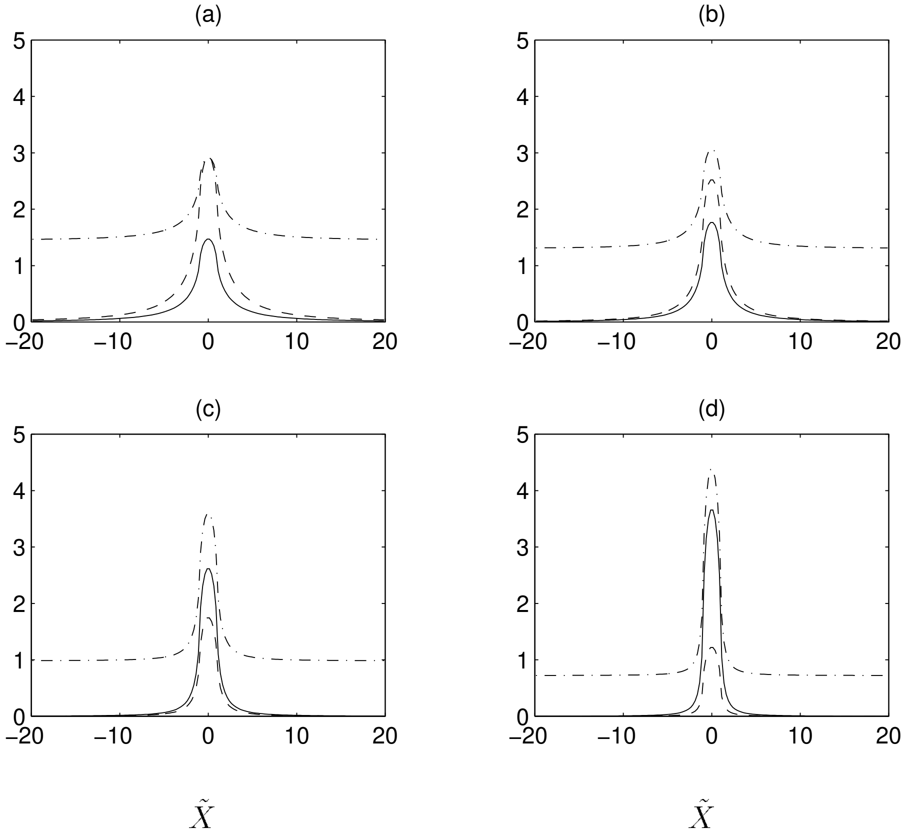


**Fig. 6**. Correction terms $\tilde{\Phi}(\tilde{X})$ (solid line), $\tilde{\Psi}(\tilde{X})$ (dashed line) and the function $\phi_0 + \tilde{\Phi}$ (dotted line): (a) $k = 0.5$, (b) $k = 0.7$, (c) $k = 1.5$, (d) $k = 3.0$.

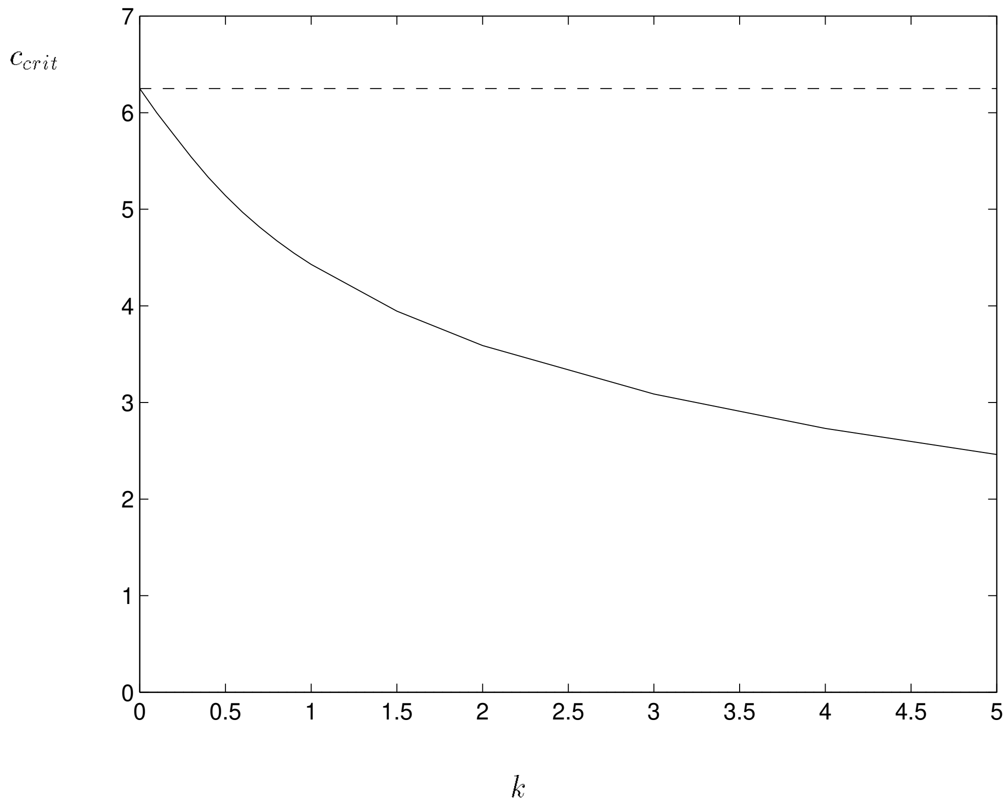


**Fig. 7**. Plot of $c_{crit}$ (solid line) versus $k$ for the partially bridged crack in comparison with the value $c_m$ (dashed line).

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