PENNY–SHAPED CRACK BRIDGED BY FIBRES

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Abstract

An axi–symmetric problem for a penny–shaped crack bridged by fibres is considered. The study is reduced to the analysis of a hypersingular integral equation with respect to the relative crack–face separation over a circular domain occupied by crack. The special case of a large crack subjected to Mode–I loading is examined. A matched asymptotic expansions technique is used in order to estimate the stress intensity factor at the crack edge and to evaluate the magnitude of applied load which provides the critical opening at the centre of the crack.

Introduction

The simplest model of the crack–bridging phenomenon that allows one to consider a matrix crack propagation problem in the context of classical fracture mechanics, represents the intact composite by an effective homogeneous isotropic medium and replaces the distinct bridging fibres by springs, distributed continuously on the crack surfaces. Application of this model to the prediction of crack growth includes two steps: the determination of an appropriate force–separation law that characterizes the restraining tractions provided by the fibres, and stress analysis performed for a crack with a certain surface traction distribution. In the present paper we deal only with the second step, namely we consider a penny–shaped bridged crack in an elastic infinite body and describe the effect of fibres by introducing restraining tractions that follow either a linear or a quadratic force–separation law.

The problem is formulated in terms of a hypersingular integral equation over a domain occupied by the crack which in the case of a large crack contains a small parameter multiplying the integral term. The main results relating to the asymptotic analysis of the integral equation for a long straight crack, subjected to Mode–I loading in plane strain, were established in [1], where a linear problem was solved explicitly to second order. An example of a nonlinear bridging force–separation law was considered in [2].

The purpose of the present paper is to extend the method [1] to the case of a penny–shaped bridged crack and to evaluate the stress intensity factor at the crack edge and the magnitude of the applied load which provides the critical opening at the centre of the crack. Two critical conditions, for matrix crack growth with fibres remaining intact and for fibre failure preceding crack extension, are analyzed.

Formulation of the problem. Consider a penny–shaped crack occupying the region

$$S = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3) : \bar{x}_3 = 0, \|\bar{\mathbf{x}}\| < a\},$$

where a is a positive constant and $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$. Let a uniform normal tensile load with magnitude $\bar{\sigma}$ be applied to the crack surfaces and let $\bar{\mathbf{b}}$ denote the crack–opening displacement

$$\bar{\mathbf{b}} = (0, 0, \bar{b}(\|\bar{\mathbf{x}}\|)).$$

Assume that the force–separation law has the form

$$\bar{\gamma} = \bar{\gamma}(\bar{b}(\|\bar{\mathbf{x}}\|)).$$

In this case the problem can be formulated as an integral equation with respect to the function $\bar{b}(\|\bar{\mathbf{x}}\|)$

$$\int_S ds\ \bar{b}(\|\bar{\mathbf{x}}\|)\, \Gamma(\bar{\mathbf{x}}, \bar{\mathbf{x}}') + \bar{\gamma}(\bar{b}(\|\bar{\mathbf{x}}'\|)) = \bar{\sigma}, \quad \|\bar{\mathbf{x}}'\| < a, \tag{1}$$

where $\bar{b}$ vanishes on the boundary ∂S of the region S. Here

$$\Gamma(\bar{\mathbf{x}}, \bar{\mathbf{x}}') = -\frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)} \frac{1}{\|\bar{\mathbf{x}} - \bar{\mathbf{x}}'\|^3}$$

and λ, μ are the Lamé moduli.

Introduce the system of polar coordinates $(\bar{r}, \theta)$ with the origin at the centre of the disk S. In these coordinates the equation (1) takes the form

$$-\frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)} \int_0^a \int_0^{2\pi} \frac{\bar{r}\ \bar{b}(\bar{r})\ d\bar{r}\ d\theta}{(\bar{r}^2 + (\bar{r}')^2 - 2\bar{r}'\bar{r}\cos(\theta - \theta'))^{3/2}} \quad + \quad \bar{\gamma}(\bar{b}(\bar{r}')) = \bar{\sigma},$$
$$(\bar{r}', \theta') \in S.$$

Integrating with respect to θ we derive

$$-\frac{\mu(\mu+\lambda)}{2\pi(\lambda+2\mu)} \int_0^a \frac{\bar{b}(\bar{r})}{(\bar{r}-\bar{r}')^2} \frac{2\bar{r}}{\bar{r}+\bar{r}'} E\left(\frac{2\sqrt{\bar{r}\bar{r}'}}{\bar{r}+\bar{r}'}\right) d\bar{r} + \bar{\gamma}(\bar{b}(\bar{r}')) = \bar{\sigma},\ 0 < \bar{r}' < a, \tag{2}$$

where $E(k)$ denotes the complete elliptic integral of the second kind (note that equation (2) agrees with equation (2.9) of [5], if the applied loading is identified with $\bar{\sigma} - \bar{\gamma}(\bar{b}(\bar{r}')))$. The solution of (2) is required to satisfy the conditions

$$\bar{b}(a) = 0, \quad \bar{b}'(0) = 0. \tag{3}$$

The integral on the left–hand side of (2) is specified in the finite–part Hadamard sense [6]. As $\bar{r} \to \bar{r}'$, the kernel

$$\mathcal{K}(\bar{r}, \bar{r}') = \frac{1}{(\bar{r}-\bar{r}')^2} \frac{2\bar{r}}{\bar{r}+\bar{r}'} E\left(\frac{2\sqrt{\bar{r}\bar{r}'}}{\bar{r}+\bar{r}'}\right)$$

of the integral operator (2) has the asymptotic form

$$\mathcal{K}(\bar{r}, \bar{r}') \sim \frac{1}{(\bar{r}-\bar{r}')^2} + \frac{\bar{r}}{(\bar{r}+\bar{r}')^3}[2\ln 2 - \ln|\bar{r}-\bar{r}'| + \ln(\bar{r}+\bar{r}')], \quad \bar{r} \to \bar{r}'.$$

To describe the crack–opening displacement at the point $\bar{r}' = 0$ we shall use the representation of the hypersingular integral (1)

$$\int_S ds\ \bar{b}(\|\bar{\mathbf{x}}\|)\ \Gamma(\bar{\mathbf{x}}, \bar{\mathbf{x}}') = \int_S ds\ \Gamma(\bar{\mathbf{x}}, \bar{\mathbf{x}}')\ \{\bar{b}(\|\bar{\mathbf{x}}\|) - \bar{b}(\|\bar{\mathbf{x}}'\|)$$

$$-(\bar{\mathbf{x}}-\bar{\mathbf{x}}')\cdot\nabla\bar{b}(\|\bar{\mathbf{x}}'\|)\}$$

$$+\int_{\partial S} dl\,\left[\frac{\partial\Xi}{\partial n}(\bar{b}(\|\bar{\mathbf{x}}'\|)+(\bar{\mathbf{x}}-\bar{\mathbf{x}}')\cdot\nabla\bar{b}(\|\bar{\mathbf{x}}'\|))-\Xi\,\mathbf{n}\cdot\nabla\bar{b}(\|\bar{\mathbf{x}}'\|)\right]. \qquad (4)$$

Formula (4) is obtained in the same way as in [4]; it follows from the equalities

$$\Gamma(\bar{\mathbf{x}},\bar{\mathbf{x}}')=\Delta\Xi,\quad \Xi=-\frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)}\frac{1}{\|\bar{\mathbf{x}}-\bar{\mathbf{x}}'\|},$$

$$\bar{b}(\|\bar{\mathbf{x}}\|)=\{\bar{b}(\|\bar{\mathbf{x}}\|)-\bar{b}(\|\bar{\mathbf{x}}'\|)-(\bar{\mathbf{x}}-\bar{\mathbf{x}}')\cdot\nabla\bar{b}(\|\bar{\mathbf{x}}'\|)\}$$
$$+\{\bar{b}(\|\bar{\mathbf{x}}'\|)+(\bar{\mathbf{x}}-\bar{\mathbf{x}}')\cdot\nabla\bar{b}(\|\bar{\mathbf{x}}'\|)\},$$

and a formal application of Green's theorem. Here, $\Delta=\partial^2/\partial\bar{x}_1^2+\partial^2/\partial\bar{x}_2^2$, $\nabla=(\partial/\partial\bar{x}_1,\partial/\partial\bar{x}_2)$, $\mathbf{n}=(n_1,n_2)$ is the unit outward normal vector with respect to ∂S.

As in [2], [3], one can introduce nondimensional variables

$$\sigma=\bar{\sigma}/\bar{\sigma}_0,\quad \gamma=\bar{\gamma}/\bar{\sigma}_0,\quad r=\bar{r}/a, \qquad (5)$$

$$b=\frac{\mu(\mu+\lambda)\bar{\sigma}_0\bar{b}}{2\pi(\lambda+2\mu)\bar{K}_0^2},\quad \varepsilon=\frac{\bar{K}_0^2}{\bar{\sigma}_0^2 a}, \qquad (6)$$

with $\bar{\sigma}_0$ regarded, say, as a single–valued fibre strength and the parameter $\bar{K}_0$ characterizing the fracture toughness of the unreinforced material, and rewrite the problem (2), (3) in the form

$$-\varepsilon\int_0^1\frac{b(r)}{(r-r')^2}\frac{2r}{(r+r')}E\left(\frac{2\sqrt{rr'}}{r+r'}\right)dr+\gamma(b(r'))=\sigma,\quad 0<r'<1, \qquad (7)$$

$$b(1)=0,\quad b'(0)=0. \qquad (8)$$

We shall construct the asymptotic approximation of the solution of (7), (8) under the assumption that the parameter ε is small (the case of a penny–shaped crack of large radius, $a\gg 1$). We shall follow the asymptotic algorithm developed in [1].

Asymptotic procedure. Suppose that $b(r')$ admits the asymptotic representations

$$b(r')\sim b_0(r')+b_1(r')+\ldots+b_n(r')+o(\varepsilon^n)\ \text{ as }\ \varepsilon\to 0,$$

with r' being fixed and different from 1, and

$$b(1-\varepsilon\rho') \sim \tilde{b}_0(\rho') + \tilde{b}_1(\rho') + \ldots + \tilde{b}_m(\rho') + o(\varepsilon^m) \ \text{ as } \ \varepsilon \to 0,$$

with fixed $\rho' = \frac{1-r'}{\varepsilon}$. The functions $b_i(r'), \tilde{b}_i(\rho')$ depend also on ε and $\ln\varepsilon$ ($\ln\varepsilon$ is regarded as $O(1)$). The asymptotic matching principle of Van Dyke [7] gives

$$\sum_{k=0}^{n} S_k^m\left(\frac{1-r'}{\varepsilon}\right) = \sum_{k=0}^{m} \tilde{S}_k^n(r'), \tag{9}$$

where functions $S_k^m(\rho'), \tilde{S}_k^n(r')$ are introduced in the following way

$$b_k(1-\varepsilon\rho') \sim S_k^m(\rho') + o(\varepsilon^m), \ \ \tilde{b}_k\left(\frac{1-r'}{\varepsilon}\right) \sim \tilde{S}_k^n(r') + o(\varepsilon^n) \ \text{ as } \ \varepsilon \to 0.$$

In this case a formal uniform expansion of $b(r')$ is specified by

$$b(r') \sim \sum_{k=0}^{n}(b_k(r') + \tilde{b}_k(\rho') - S_k^n(\rho')) + o(\varepsilon). \tag{10}$$

Solution to lowest order. As $\varepsilon \to 0$ in (7) and r' is fixed, (7) reduces to the algebraic equation

$$\gamma(b_0) = \sigma. \tag{11}$$

The solution of this equation is constant; it violates the boundary condition $b_0(1) = 0$. In order to compensate the discrepancy, we construct a boundary layer approximation $\tilde{b}_0(\rho')$ in the vicinity of the edge of the crack. Since b_0 is constant, it follows immediately that

$$S_0^0(\rho') = b_0. \tag{12}$$

Now we identify r' with $1-\varepsilon\rho'$ and expand (7) to lowest order as $\varepsilon \to 0$, with ρ' fixed. Equation (7) is then written in the form

$$-\varepsilon\int_0^1 \frac{dr}{(r-1+\varepsilon\rho')^2}\,\frac{2r}{r+1-\varepsilon\rho'}\,E\left(\frac{2\sqrt{r(1-\varepsilon\rho')}}{r+1-\varepsilon\rho'}\right)$$
$$\times\,[b_0 + \tilde{b}_0(\rho) - S_0^0(\rho)] + \gamma(\tilde{b}_0(\rho')) = \sigma. \tag{13}$$

Since $b_0 - S_0^0(\rho') = 0$, the integral on the left–hand side of (13) contains only $\tilde{b}_0(\rho)$. The transformation $r = 1 - \varepsilon\rho$ gives

$$-\varepsilon \int_0^1 \frac{dr}{(r-1+\varepsilon\rho')^2} \, \frac{2r}{r+1-\varepsilon\rho'} \, E\left(\frac{2\sqrt{r(1-\varepsilon\rho')}}{r+1-\varepsilon\rho'}\right) \tilde{b}_0(\rho)$$

$$= -\int_0^{1/\varepsilon} \frac{dr}{(\rho-\rho')^2} \, \frac{2(1-\varepsilon\rho)}{2-\varepsilon\rho-\varepsilon\rho'} \, E\left(\frac{2\sqrt{(1-\varepsilon\rho)(1-\varepsilon\rho')}}{2-\varepsilon\rho-\varepsilon\rho'}\right) \tilde{b}_0(\rho). \tag{14}$$

The integral on the right–hand side of (14) can be expressed as follows:

$$-\int_0^{1/\varepsilon} \frac{d\rho}{(\rho-\rho')^2} \, \frac{2(1-\varepsilon\rho)}{2-\varepsilon\rho-\varepsilon\rho'} \, E\left(\frac{2\sqrt{(1-\varepsilon\rho)(1-\varepsilon\rho')}}{2-\varepsilon\rho-\varepsilon\rho'}\right) \tilde{b}_0(\rho)$$

$$= -\int_0^{\infty} \frac{d\rho}{(\rho-\rho')^2} \, \tilde{b}_0(\rho) + \int_{1/\varepsilon}^{\infty} \frac{d\rho}{(\rho-\rho')^2} \, \tilde{b}_0(\rho) + O(\varepsilon). \tag{15}$$

When $\rho > 1/\varepsilon$, $\tilde{b}_0(\rho)$ can be replaced by $\tilde{S}_0^0(1-\varepsilon\rho')$, and due to the asymptotic matching principle (9) it equals b_0. The evaluation of the second integral on the right–hand side of (15) gives

$$-\varepsilon \int_0^1 \frac{dr}{(r-1+\varepsilon\rho')^2} \, \frac{2r}{r+1-\varepsilon\rho'} \, E\left(\frac{2\sqrt{r(1-\varepsilon\rho')}}{r+1-\varepsilon\rho'}\right) \tilde{b}_0(\rho)$$

$$= -\int_0^{\infty} \frac{d\rho}{(\rho-\rho')^2} \, \tilde{b}_0(\rho) + O(\varepsilon).$$

Thus, the equation for the boundary layer approximation $\tilde{b}_0(\rho)$ takes the form

$$-\int_0^{\infty} \frac{d\rho}{(\rho-\rho')^2} \tilde{b}_0(\rho) + \gamma(\tilde{b}_0(\rho')) = \sigma. \tag{16}$$

The solution of this equation is required to satisfy the conditions

$$\tilde{b}_0(0) = 0, \quad \tilde{b}_0(\rho') \sim b_0 \;\text{ as }\; \rho' \to \infty. \tag{17}$$

The problem (16)–(17) is the same as for the case of a two–dimensional plane strain crack. The explicit form of the solution in the case of a linear function γ was derived in [1]. In the case of quadratic force law this problem was studied in [2].

The first order approximation of $\tilde{b}_0(\frac{1-r'}{\varepsilon})$ is

$$\tilde{b}_0\left(\frac{1-r'}{\varepsilon}\right) \sim b_0 - \frac{\varepsilon b_0}{\gamma'(b_0)}\frac{1}{1-r'} + o(\varepsilon) \equiv \tilde{S}_0^1(r') + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0. \tag{18}$$

Solution to first order. The term $b_1(r')$ of the asymptotic expansion of the crack–opening displacement satisfies the equation

$$-\varepsilon\int_0^1 \frac{b_0}{(r-r')^2}\frac{2r}{r+r'}E\left(\frac{2\sqrt{rr'}}{r+r'}\right)dr + b_1(r')\gamma'(b_0) = 0, \tag{19}$$

with fixed r' in the interval $(0,1)$.

With the use of the representation (4) one can show that the solution b_1 of equation (19) has the form

$$b_1(r') \sim -\frac{\varepsilon b_0}{\gamma'(b_0)}\left[\frac{1}{1-r'}E(\frac{2\sqrt{r'}}{1+r'}) + \frac{1}{1+r'}K(\frac{2\sqrt{r'}}{1+r'})\right], \tag{20}$$

$$b_1'(r') \sim -\frac{3}{2}\pi r'\frac{\varepsilon b_0}{\gamma'(b_0)} \quad \text{as} \quad r' \to 0,$$

and, therefore, it satisfies the condition $b_1'(0) = 0$. In (20) $E(k)$ and $K(k)$ are the complete elliptic integrals of the second and first kind, respectively.

Near the edge of the crack b_1 admits the asymptotic representation

$$b_1(1-\varepsilon\rho') \sim \frac{b_0}{\gamma'(b_0)}\left[-\frac{1}{\rho'} + \frac{\varepsilon}{2}\ln\varepsilon + \frac{\varepsilon}{2}\ln\rho' - \varepsilon + o(\varepsilon)\right] \quad \text{as} \quad \varepsilon \to 0.$$

It violates the boundary condition $b_1(1) = 0$, and the discrepancy is compensated by the solution of the boundary layer type $\tilde{b}_1(\rho') - S_1^1(\rho')$, with

$$S_1^1(\rho') = \frac{b_0}{\gamma'(b_0)}\left[-\frac{1}{\rho'} + \frac{\varepsilon}{2}\ln\varepsilon + \frac{\varepsilon}{2}\ln\rho' - \varepsilon\right]. \tag{21}$$

Due to the asymptotic matching principle (9)

$$\tilde{b}_1(\frac{1-r'}{\varepsilon}) \sim \frac{b_0}{\gamma'(b_0)}\left[\frac{\varepsilon}{2}\ln(1-r') - \varepsilon\right] \quad \text{as} \quad \varepsilon \to 0. \tag{22}$$

To obtain an equation for $\tilde{b}_1(\rho')$, we put $r' = 1 - \varepsilon\rho'$ in (7) and collect terms up to order ε. Thus,

$$-\varepsilon \int_0^1 \frac{dr}{(r-1+\varepsilon\rho')^2}\ \frac{2r}{r+1-\varepsilon\rho'}\ E\left(\frac{2\sqrt{r(1-\varepsilon\rho')}}{r+1-\varepsilon\rho'}\right)\Big[b_0 + \tilde{b}_0(\rho)$$

$$-S_0^0(\rho) + b_1(r) + \tilde{b}_1(\rho) - S_1^1(\rho)\Big] + \gamma(\tilde{b}_0(\rho') + \tilde{b}_1(\rho')) = 0.$$

Note that

$$-\varepsilon \int_0^1 \frac{dr}{(r-1+\varepsilon\rho')^2}\ \frac{2r}{r+1-\varepsilon\rho'}\ E\left(\frac{2\sqrt{r(1-\varepsilon\rho')}}{r+1-\varepsilon\rho'}\right)$$

$$\times[b_0 - S_0^0(\rho) + b_1(r) - S_1^1(\rho)] = o(\varepsilon),$$

and the integral containing $\tilde{b}_0(\rho)$ and $\tilde{b}_1(\rho)$ can be written in the form

$$-\varepsilon \int_0^1 \frac{dr}{(r-1+\varepsilon\rho')^2}\ \frac{2r}{r+1-\varepsilon\rho'}\ E\left(\frac{2\sqrt{r(1-\varepsilon\rho')}}{r+1-\varepsilon\rho'}\right)\ [\tilde{b}_0(\rho) + \tilde{b}_1(\rho)]$$

$$= -\int_0^\infty \frac{d\rho}{(\rho-\rho')^2}\ \tilde{b}_1(\rho) + \int_{1/\varepsilon}^\infty \frac{d\rho}{(\rho-\rho')^2}\ \tilde{b}_1(\rho) - \int_0^\infty \frac{d\rho}{(\rho-\rho')^2}\ \tilde{b}_0(\rho)$$

$$+ \int_{1/\varepsilon}^\infty \frac{d\rho}{(\rho-\rho')^2}\ \tilde{b}_0(\rho) - \frac{\varepsilon}{2} b_0 \ln\varepsilon - \frac{\varepsilon}{2} b_0 \ln\rho' - \frac{\varepsilon}{2}\int_0^\infty \frac{b_0 - \tilde{b}_0(\rho)}{\rho-\rho'} d\rho + o(\varepsilon). \quad (23)$$

The second and fourth integrals in the right–hand side of (23) are calculated with the use of the asymptotic expansions (18), (22). The results are

$$\int_{1/\varepsilon}^\infty \frac{d\rho}{(\rho-\rho')^2}\tilde{b}_0(\rho) = \varepsilon b_0 + o(\varepsilon), \quad \int_{1/\varepsilon}^\infty \frac{d\rho}{(\rho-\rho')^2}\tilde{b}_1(\rho) = o(\varepsilon).$$

Thus, the problem for the boundary layer function $\tilde{b}_1(\rho')$ is

$$-\int_0^\infty \frac{d\rho}{(\rho-\rho')^2}\tilde{b}_1(\rho) + \tilde{b}_1(\rho')\gamma'(\tilde{b}_0(\rho')) = -\varepsilon b_0 + \frac{\varepsilon b_0}{2}\Big[\ln\varepsilon + \ln\rho'\Big]$$

$$+\frac{\varepsilon}{2}\int_0^\infty \frac{b_0 - \tilde{b}_0(\rho)}{\rho-\rho'} d\rho, \quad (24)$$

$$\tilde{b}_1(0) = 0, \ \ \tilde{b}_1(\rho') \sim \frac{b_0\varepsilon}{\gamma'(b_0)}\left[\frac{1}{2}\ln\varepsilon + \frac{1}{2}\ln\rho' - 1\right] \ \text{ as } \ \rho' \to \infty. \quad (25)$$

Since equation (24) is linear, the solution $\tilde{b}_1(\rho')$ can be taken in the form

$$\tilde{b}_1(\rho') = \varepsilon \ln \varepsilon \tilde{\phi}_1(\rho') + \varepsilon \tilde{\phi}_2(\rho'). \tag{26}$$

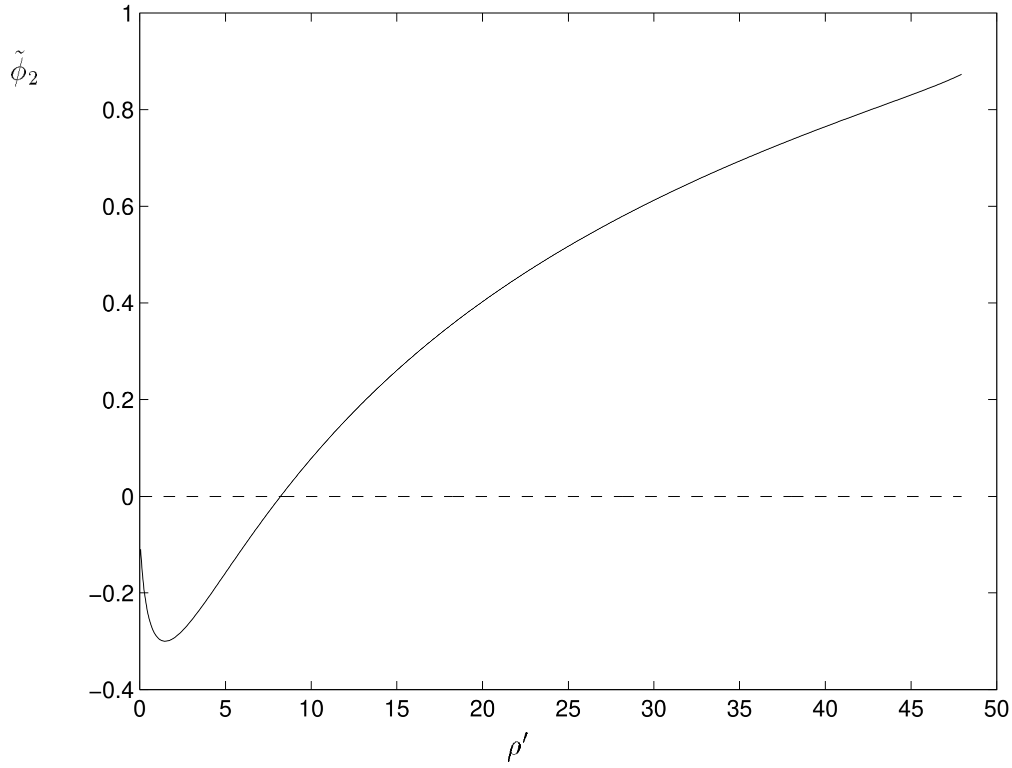

Fig. 1. Plot of the function $\tilde{\phi}_2$ for the linear force law $\gamma = b$ and $\sigma = 1$.

The functions $\tilde{\phi}_1(\rho'), \tilde{\phi}_2(\rho')$ are required to satisfy the problems

$$-\int_0^\infty \frac{d\rho}{(\rho-\rho')^2} \tilde{\phi}_1(\rho) + \tilde{\phi}_1(\rho')\gamma'(\tilde{b}_0(\rho')) = \frac{b_0}{2}, \tag{27}$$

$$\tilde{\phi}_1(0) = 0, \tilde{\phi}_1(\rho') \sim \frac{b_0}{2\gamma'(b_0)} \text{ as } \rho' \to \infty;$$

$$-\int_0^\infty \frac{d\rho}{(\rho-\rho')^2}\tilde{\phi}_2(\rho) + \tilde{\phi}_2(\rho')\gamma'(\tilde{b}_0(\rho')) = -b_0 + \frac{b_0}{2}\ln\rho' + \frac{1}{2}\int_0^\infty \frac{b_0 - \tilde{b}_0(\rho)}{\rho-\rho'}d\rho, \tag{28}$$

$$\tilde{\phi}_2(0) = 0, \tilde{\phi}_2(\rho') \sim \frac{b_0}{2\gamma'(b_0)}\ln\rho' - \frac{b_0}{\gamma'(b_0)} \text{ as } \rho' \to \infty.$$

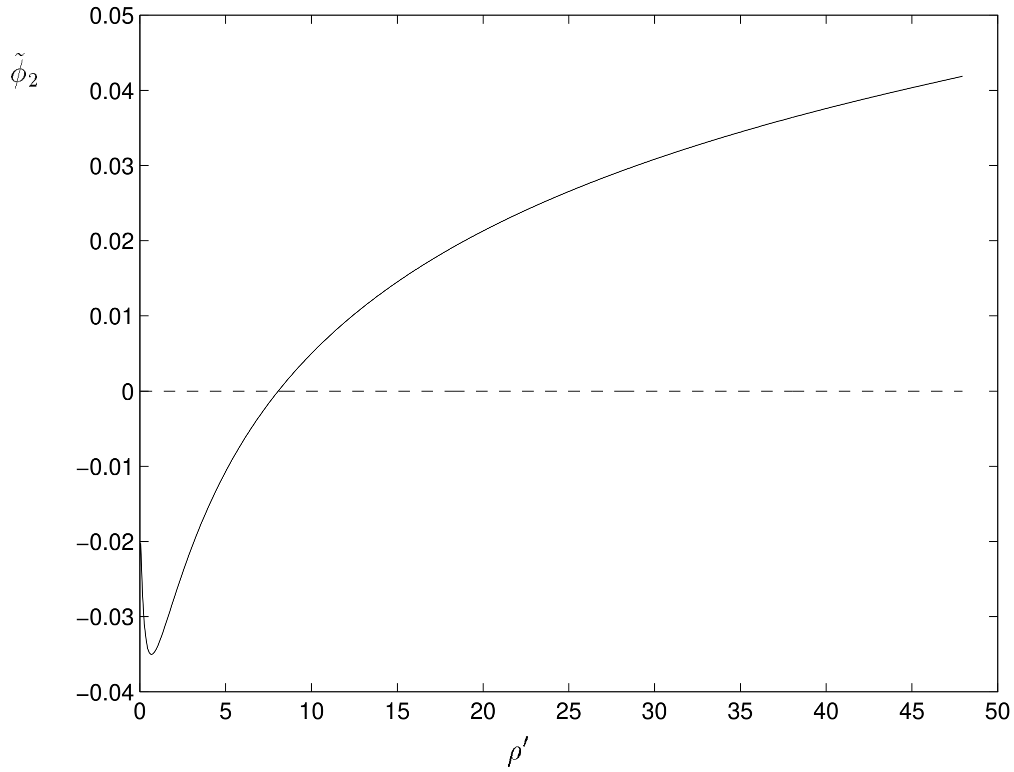

Fig. 2. Plot of the function $\tilde{\phi}_2$ for the quadratic force law $\gamma = b(b_* - b)$ and $\sigma = 1,\ b_* = 5$.

It should be observed that, in contrast with (26), the boundary layer solution in the case of a two–dimensional crack does not depend on $\ln\varepsilon$; it corresponds to the problem (27). This problem has been solved explicitly in [1] for the case of linear function γ and numerically in [2] for the case of quadratic force law. (Here we do not present the results of those calculations and refer the reader to the papers [1], [2].)

Due to the presence of the function $\tilde{b}_0(\rho')$ on the right–hand side of equation (28), we solve the problem (28) numerically. The plots of $\tilde{\phi}_2$ for linear and quadratic functions γ are shown, with $\sigma = 1$, in Fig. 1 and Fig. 2, respectively.

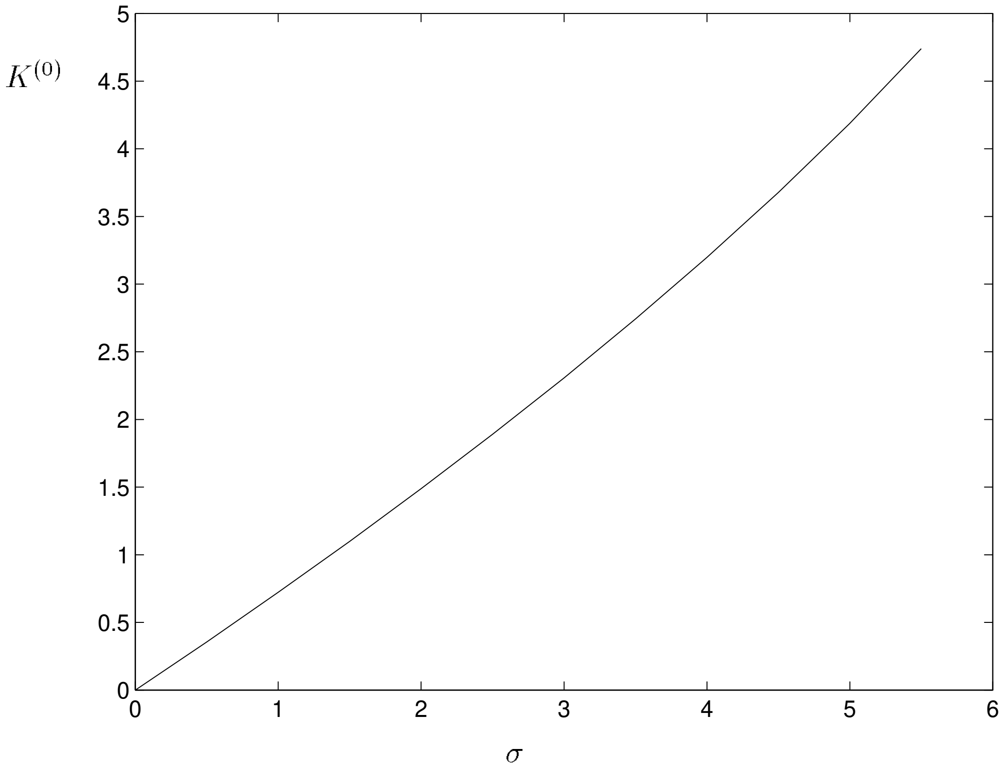

Fig. 3. $K^{(0)}$ versus σ, with $\gamma = b(b_* - b)$ and $b_* = 5$.

Stress intensity factor. Since only the boundary layer terms have the square root behaviour near the edge of the crack, the stress intensity factor is specified by the boundary layers. It has the form

$$K_I \sim (2/\pi)^{1/2}[K^{(0)} + \varepsilon \ln \varepsilon K^{(1)} + \varepsilon K^{(2)} + o(\varepsilon)]. \tag{29}$$

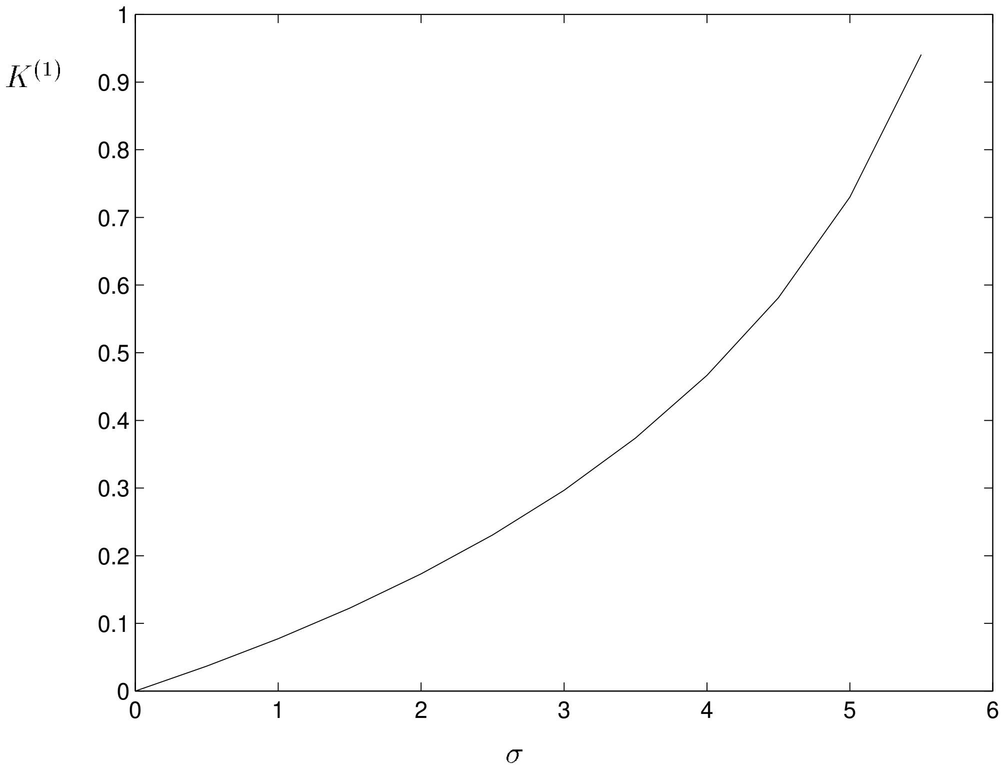

Fig. 4. $K^{(1)}$ versus σ, with $\gamma = b(b_* - b)$ and $b_* = 5$.

Here,

$$K^{(0)} = \int_0^\infty \frac{d\rho'}{\sqrt{\rho'}}\ [\sigma - \gamma(\tilde{b}_0(\rho'))], \tag{30}$$

$$K^{(1)} = \int_0^\infty \frac{d\rho'}{\sqrt{\rho'}} \left[\frac{b_0}{2} - \tilde{\phi}_1(\rho')\gamma'(\tilde{b}_0(\rho'))\right], \tag{31}$$

$$K^{(2)} = \int_0^\infty \frac{d\rho'}{\sqrt{\rho'}} \Big[-b_0 + \frac{b_0}{2} \ln \rho' + \frac{1}{2} \int_0^\infty \frac{b_0 - \tilde{b}_0(\rho)}{\rho - \rho'} d\rho - \tilde{\phi}_2(\rho')\gamma'(\tilde{b}_0(\rho')) \Big]. \quad (32)$$

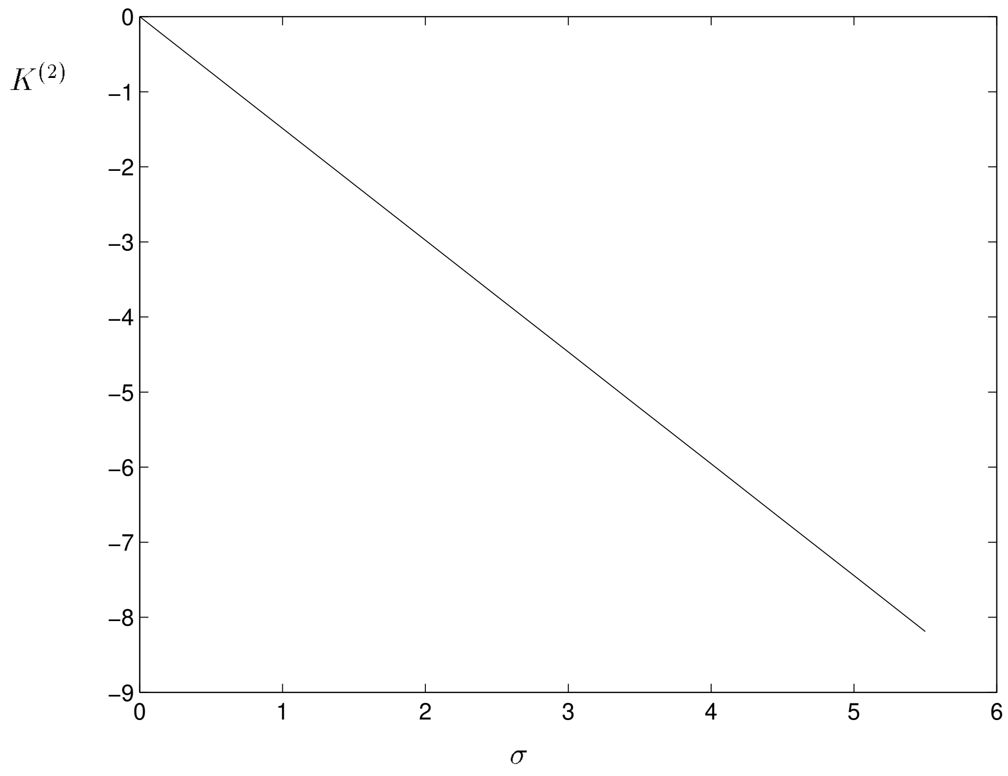

Fig. 5. $K^{(2)}$ versus σ, with $\gamma = b$.

Note that in comparison with the case of a plane strain semi–infinite crack the asymptotic formula (29) gives the same principal term $K^{(0)}$, but it differs in the next terms. A plot of the factor $K^{(0)}$ is shown in Fig.3; this is the same as in [2]. The calculation of the coefficient $K^{(1)}$ follows the same pattern as in [1], [2]. For the linear case ($\gamma = kb$, k is a positive constant) the factor $K^{(1)}$ is specified by

$$K^{(1)} = \frac{\pi}{4}\, k^{-3/2}\sigma. \quad (33)$$

The equality (33) follows from the representation of the solution $\tilde{\phi}_1(\rho')$ of the problem (27) obtained in [8]

$$\tilde{\phi}_1(\rho') = \frac{b_0}{2k} V\left(\frac{k\rho'}{\pi}\right),$$

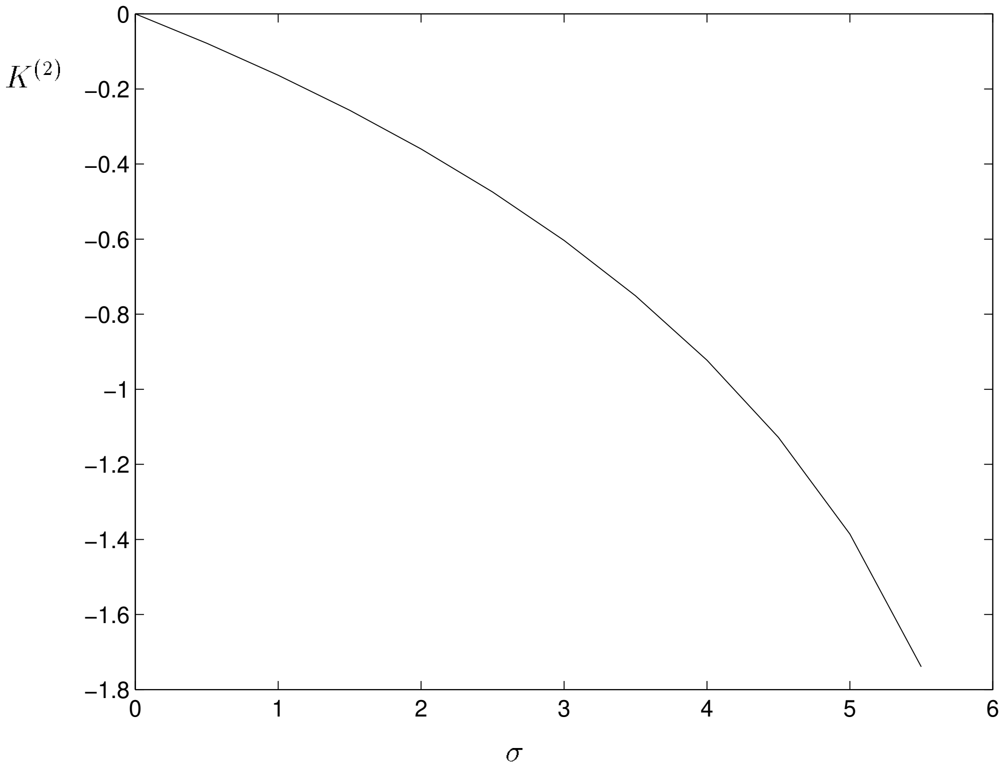

Fig. 6. $K^{(2)}$ versus σ, with $\gamma = b(b_* - b)$ and $b_* = 5$.

$$V(\eta) = \int_0^\eta v(\xi)d\xi = 1 - \int_\eta^\infty v(\xi)d\xi,$$

$$v(\eta) = \frac{1}{\pi}\int_0^\infty \frac{t\,dt}{(1+t^2)^{3/4}} \exp\left[-\eta t - \frac{1}{\pi}\int_0^t \frac{\ln r\,dr}{1+r^2}\right],$$

and from the asymptotic expansion (see [8])

$$V(\eta) \sim 2\pi^{-1/2}\eta^{1/2} \text{ as } \eta \to 0.$$

For the quadratic case ($\gamma = kb(b_* - b)$, k and b_* are positive constants) the factor $K^{(1)}$ is evaluated from (31). The coefficient $K^{(1)}$ versus the load σ is presented in Fig. 4; one can see that $\varepsilon \ln \varepsilon K^{(1)} < 0$. The quantity $K^{(2)}$ is plotted in Figs 5 and 6 for the linear and quadratic function γ, respectively. They show that the coefficient $K^{(2)}$ is negative.

Evaluation of the critical load. Let $b = b_m$ be the maximum of the smooth function γ. Then, $\gamma'(b_m) = 0$. Suppose, also, that the crack will extend if the stress intensity factor K_I reaches a critical value K_c. The load σ can become critical in one of two ways. The stress intensity factor at the crack edge may reach its critical value K_c before the maximum value b_m of b is reached. If this happens, the crack will extend and, as it does so, K_I will increase further. The crack opening displacement will also increase until, at the crack centre, $b = b_m$. Thereafter, a region of fibre breakage will expand from $r' = 0$, including further increase of K_I. Alternatively, the crack opening displacement b may reach the critical value b_m at $r' = 0$, while the stress intensity factor K_I is below K_c. Fibres near $r' = 0$ will break, thereby allowing further crack opening as well as increase of K_I. Either event will precipitate instability and therefore rapid crack growth, so long as the load σ is maintained.

Here, the critical load for the attainment of b_m is estimated, assuming that condition $K_I < K_c$ is maintained. We suppose that the load

$$\sigma = \sigma_m + \sigma_1, \tag{34}$$

with $\sigma_m = \gamma(b_m)$ and σ_1 being small, is applied on the crack faces, and the relative displacement at the centre of the crack is

$$b(0) = b_m. \tag{35}$$

The objective is to estimate the value of the small correction term σ_1 which achieves the condition (35).

The solution to lowest order of the problem (7), (8) with σ given by (34) is specified by

$$b(r') \sim b_m + \tilde{b}_0(\rho') - S_0^0(\rho') + o(1),$$

where $S_0^0(\rho') = b_m$. The boundary layer function $\tilde{b}_0(\rho')$ satisfies the problem

$$-\int_0^\infty \frac{d\rho}{(\rho-\rho')^2}\tilde{b}_0(\rho) + \gamma(\tilde{b}_0(\rho')) = \sigma_m, \tag{36}$$

$$\tilde{b}_0(0) = 0, \quad \tilde{b}_0(\rho') \sim b_m \ \text{ as } \ \rho' \to \infty. \tag{37}$$

As $\varepsilon \to 0$, the asymptotic expansion of $\tilde{b}_0(\frac{1-r'}{\varepsilon})$ has the form

$$\tilde{b}_0\left(\frac{1-r'}{\varepsilon}\right) \sim b_m - \left[-\frac{2b_m}{\gamma''(b_m)}\right]^{1/2}\sqrt{\frac{\varepsilon}{1-r'}} + o(\sqrt{\varepsilon}) \ \text{ as } \ \varepsilon \to 0. \tag{38}$$

The function $b_1(r')$ solves the equation

$$-\varepsilon\int_0^1 \frac{dr}{(r-r')^2}\ \frac{2r}{r+r'}\ E\left(\frac{2\sqrt{rr'}}{r+r'}\right)\ b_m + \frac{1}{2}\gamma''(b_m)b_1(r')^2 = \sigma_1, \tag{39}$$

with fixed $r' \in (0,1)$. Outside a neighbourhood of the centre of the disk S we have

$$b_1(r') = -\left[-\frac{2}{\gamma''(b_m)}\right]^{1/2}\left[-\sigma_1 - \varepsilon b_m \int_0^1 \frac{dr}{(r-r')^2}\frac{2r}{r+r'}E\left(\frac{2\sqrt{rr'}}{r+r'}\right)\right]^{1/2}$$

and, therefore, $b_1(1) = 0$. In the middle zone of the crack ($r' \to 0$) one can use (4) and obtain the asymptotic formula

$$b_1(r') \sim -\left[-\frac{2}{\gamma''(b_m)}\right]^{1/2}\left[-\sigma_1+\varepsilon b_m\left\{\frac{1}{1-r'}E\left(\frac{2\sqrt{r'}}{1+r'}\right)+\frac{1}{1+r'}K\left(\frac{2\sqrt{r'}}{1+r'}\right)\right\}\right]^{1/2}$$

$$\sim -\left[-\frac{2}{\gamma''(b_m)}\right]^{1/2}\left[-\sigma_1 + \varepsilon\pi b_m\right]^{1/2} \ \text{ as } \ r' \to 0.$$

The condition (35) gives then

$$\sigma_1 = \varepsilon\pi b_m.$$

Thus, the critical value of applied load σ is specified by

$$\sigma_{crit} \sim \sigma_m + \varepsilon\pi b_m + o(\varepsilon) \quad \text{as } \ \varepsilon \to 0. \tag{40}$$

In dimensional variables (5)–(6) the relation (40) has the form

$$\bar{\sigma}_{crit} \sim \bar{\sigma}_m + \frac{\mu(\mu+\lambda)}{2(\lambda+2\mu)} \frac{\bar{b}_m}{a} + o(1/a) \quad \text{as} \quad a \to \infty.$$

Note that the asymptotic representation for σ_{crit} in the case of a two–dimensional crack was obtained in [3] (p.96, formula (19)). The relations (40) and (19) of [3] give the same principal term σ_m, but they are different in the terms of order $O(\varepsilon)$.

The principal term $K^{(0)}$ of the stress intensity factor K_I at the edge of the crack is given by (30), with $\sigma = \sigma_{crit}$ and $\tilde{b}_0$ specified by (36)–(38). The results of numerical calculation presented in Fig. 3 show that for the quadratic function γ the factor $K^{(0)}$ monotonically increases with increasing σ, and therefore, the condition $K^{(0)} < K_c$ implies the relation $\sigma_{crit} < \sigma_k$, where σ_k denotes the value of applied load which provides the condition $K_I = K_c$.

References

[1] **Willis, J.R. and Nemat-Nasser, S.** "Singular perturbation solution of a class of singular integral equations" *Quart Appl Math* **48** (1990) pp 741-753

[2] **Movchan, A.B. and Willis, J.R.** "Asymptotic analysis of the reinforcement of a brittle crack by bridging fibres" *Quart J Mech Appl Math* **46** (1993) pp 331-350

[3] **Willis, J.R.** "Asymptotic analysis of crack bridging by ductile fibres" *Composites* **24** (1993) pp 93-97

[4] **Movchan, A.B. and Willis, J.R.** "An asymptotic procedure and numerical study for the analysis of an elastic body with a thin subsurface crack" *Euro J Appl Math* bf 6 (1995) pp 1-23

[5] **Guidera, J.T. and Lardner, R.W.** "Penny–shaped cracks" *Journal of Elasticity* **5** (1975) pp 59-73

[6] **Hadamard, J.** *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Dover, New York, 1952)

[7] **Van Dyke, M.D.** *Perturbation Methods in Fluid Mechanics* (Academic Press, New York, 1964)

[8] **Olmstead, W.E. and Gautesen, A.K.** "Asymptotic solution of some singularly perturbed Fredholm equations" *Z Angew Math Phys* **40** (1989) pp 230-244