Attractors and Inertial Manifolds for Finite Difference Approximations of the Complex Ginzburg–Landau Equation

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Abstract

A semi–discrete spatial finite difference approximation and two fully discrete finite difference approximations to the complex Ginzburg–Landau equation are considered in this paper.

The existence of an inertial manifold is proved inside a discrete H^1 absorbing ball for the semi–discrete approximation by showing certain spectral properties hold for the linear operator and certain Lipschitz properties hold for the non–linear term. Convergence of the true and discrete inertial manifolds is then shown by considering the contraction mappings used to construct the inertial manifolds.

A fully implicit scheme is shown to form a continuous semi–group in a discrete L^2 space and absorbing balls of radii independent of the spatial and temporal mesh sizes in discrete L^2 and H^1 spaces are constructed; existence of a global attractor follows. Furthermore the existence of an inertial manifold is proved by showing certain spectral and Lipschitz properties hold. For a semi–implicit scheme existence is proved of absorbing balls in the discrete space L^2 of radii independent of the spatial and temporal mesh; existence of a global attractor follows.

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1 Introduction

The complex Ginzburg–Landau equation is an important equation in a number of scientific fields – it models the evolution of the amplitude of perturbations to steady state solutions at the onset of instability. In fluid dynamics it is found, for example, in the study of Poiseuille flow, Rayleigh–Bénard convection and Taylor–Couette flow [30, 25, 8]. The equation is also used to model the transition to turbulence in chemical mediums [20]. It derives the name used here from the study of superconductivity where it models the phase transition of the material from a superconducting phase to a non–superconducting phase [5, 11]. As a phase transition equation it is closely related to other phase transition equations such as the Allen–Cahn or the Chafee–Infante equation (see [2] and [4]) or the Cahn–Hilliard equation [3].

Like these other phase transition equations the Ginzburg–Landau equation is an example of a dissipative equation whose long time dynamical behaviour is captured by a global attractor [31]. However, unlike the Allen–Cahn or Cahn–Hilliard

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equations, the Ginzburg–Landau equation does not have a gradient structure except for certain special parameter combinations. In [27] a semi–discrete finite difference approximation to the Ginzburg–Landau found by a spatial discretization was considered. Existence of a global attractor was shown and upper–semicontinuity was proved to the true global attractor. It was also shown that solutions to the semi–discrete problem lie in a discrete Gevrey class, which was exploited to avoid the use of non–smooth error estimates.

In this paper we prove that there exist absorbing balls in discrete L^2 and H^1 spaces of radii independent of the spatial mesh Δx and temporal mesh Δt for the fully implicit finite difference approximation, found by an application of the implicit Euler method to the semi–discrete system. A restriction on Δt is required but this is independent of Δx. We conclude the existence of a global attractor bounded independently of Δx and Δt in the discrete L^2 and H^1 spaces. For a semi–implicit finite difference approximation found from the semi–discrete approximation by discretizing in time we prove the existence of absorbing balls in discrete L^2 of radii independent of Δx and Δt; existence of a global attractor follows.

Another important property that the complex Ginzburg–Landau equation shares with a number of other dissipative equations is that there exists an inertial manifold: that is an exponentially attracting positively invariant manifold which contains the global attractor. Existence of an inertial manifold was first shown for the Ginzburg–Landau equation by Doering et al [10] and was established using a cone condition. Below, we construct an inertial manifold for the Ginzburg–Landau equation using the method outlined by Jones and Stuart in [23]. This is based on contraction mappings in Banach spaces and avoids the explicit use of a cone condition or squeezing property. Using the approach in [23] we prove the existence of an inertial manifold for the semi–discrete and the fully implicit finite difference approximations. For the semi–discrete approximation we consider convergence of the approximate inertial manifold to the true inertial manifold. inside a Gevrey ball of regularity. This avoids the use if a non–smooth error estimate. In [21], Jones applied these same techniques for a semi–discrete spatial finite difference approximation to the Kuramoto–Sivashinsky equation and in addition considered C^1 convergence of the inertial manifolds. In contrast to Jones we consider in the discrete H^1 space and avoid the use of a non–smooth error estimate by using the Gevrey regularity. Inertial manifolds for finite difference approximations to the Kuramoto–Sivashinsky equation were first considered by Foias and Titi in [18]. For spectral and finite element approximations to partial differential equations existence and convergence of inertial manifolds has been considered, for example, in [23] and Jones and Titi [24]. The relation between Gevrey regularity and inertial manifolds is considered in [22], in which it is proposed that for smooth solutions (i.e. in a Gevrey class of regularity) there is no advantage to the non–linear Galerkin method over the standard Galerkin method.

This paper is organized as follows. We commence by reviewing for an evolution equation in a general Hilbert space X existence results for inertial manifolds. In Section 1.2 the complex Ginzburg–Landau equation is introduced and the results of Section 1.1 are applied to conclude the existence of an inertial manifold. In Section

1.3 the semi–discrete and fully discrete finite difference approximations are presented and some discrete Sobolev space results are discussed.

In Section 2 we apply the general theory outlined in Section 1.1 to prove the existence of an inertial manifold for the semi–discrete system. Inside a Gevrey ball of regularity we then show convergence of the semi–discrete to the true inertial manifold.

In Section 3.1 we examine the fully implicit finite difference approximation. We show that this defines a dynamical system and that there exist absorbing balls in the discrete spaces L^2 and H^1; from which we conclude the existence of a global attractor. We then prove the existence of an inertial manifold for this fully discrete system.

We conclude by considering a semi–implicit scheme for which we prove the existence of absorbing balls in the discrete space L^2 of radii independent of Δx and Δt.

1.1 Existence of an Inertial Manifold

Consider the abstract evolution equation

$$U_t + AU = F(U) \tag{1.1}$$

$$U(0) = U^0$$

in a separable Hilbert space X with inner–product $\langle \bullet, \bullet \rangle$ and norm $\| \bullet \|^2 = \langle \bullet, \bullet \rangle$.

We assume that A is a sectorial operator so that A generates an analytic semigroup $L(t)$ and fractional powers of A are defined. These may be used to define new Hilbert spaces X^γ with norm $\| \bullet \|^2_{X^\gamma} = \left\langle A^{\gamma/2}\bullet, A^{\gamma/2}\bullet \right\rangle$, see for example [1] or [29]. Furthermore suppose that F satisfies sufficient conditions to form a Lipschitz continuous semigroup $S(t) : X^\gamma \to X^\gamma$, for some $\gamma \geq 0$, so that the solution $U(t)$ of (1.1) at time t may be written as $U(t) = S(t)U^0$.

Define P_m to be the projection onto the eigenfunctions of A associated with the first m eigenvalues and define $Q_m = I - P_m$. Furthermore define the spaces Y and Z by

$$Y = P_m X \text{ and } Z = Q_m X,$$

so that $X = Y \oplus Z$. We seek an inertial manifold $\mathcal{M}$ which is the graph of a function $\Phi : Y \to Z$,

$$\mathcal{M} = graph(\Phi).$$

The function Φ relates the high wave numbers to the low wave numbers and is often termed the "slaving function".

We now state sufficient conditions to obtain the existence of an inertial manifold. The solution to (1.1) at time t may be written as

$$U(t) = L(t)U^0 + N(U^0, t),$$

where

$$L(t) = e^{-At}, \qquad N(V,t) := \int_0^t L(t-s)F(S(s)V)ds.$$

Let $G : X^\gamma \to X^\gamma$, be a mapping of the form $G(U) = LU + N(U)$, where $L, N(U) : X^\gamma \to X^\gamma$ are given above. So given $U^n \in X^\gamma$ the mapping G gives U^{n+1}:

$$U^{n+1} = G(U^n).$$

Let us make the following assumptions on G:

Assumptions G

There exist positive constants a, b, c, B such that:

$$\|Lz\|_{X^\gamma} \leq a\|z\|_{X^\gamma} \qquad \forall z \in Z; \tag{G1}$$

$$\exists! w \in Y : Lw = p,\ \forall p \in Y \ \ \& \ \ b\|y\|_{X^\gamma} \leq \|Ly\|_{X^\gamma} \leq c\|y\|_{X^\gamma} \quad \forall y \in Y; \tag{G2}$$

$$\|\mathcal{R}(N(u) - N(v))\|_{X^\gamma} \leq B\|u - v\|_{X^\gamma} \ \ \forall u, v \in X^\gamma, \quad \|\mathcal{R}N(u)\|_{X^\gamma} \leq B \ \ \forall u \in X^\gamma, \tag{G3}$$

where $\mathcal{R}$ equals either I, P_m or Q_m.

Under these assumptions Jones and Stuart [23] prove the existence of an inertial manifold for the mapping G whenever the following conditions are fulfilled:

Conditions C

There exist constants $\delta, \epsilon \in (0, \infty)$ and $\mu \in (0, 1)$ such that:

$$b^{-1}B(1 + \delta) \leq \mu. \tag{C1}$$

$$a\epsilon + B \leq \epsilon. \tag{C2}$$

$$\theta := a\delta + B(1 + \delta) \leq \delta\phi, \tag{C3}$$

where $\phi := b - B(1 + \delta) > 0$ by (C1).

$$a + B(1 + \delta) \leq \mu. \tag{C4}$$

Conditions C' There exist $\delta', \epsilon' > 0, K > 1$ and $\mu \in (0, 1)$ such that Conditions (C1)–(C4) hold for all $\delta \in [\delta', K\delta'], \epsilon \in [\epsilon', K\epsilon']$.

Theorem 1.1

Suppose that assumptions G and conditions C' hold for the mapping G. Then there exists a unique $\Phi : Y \to Z$ such that $\mathcal{M} = graph(\Phi)$ is an inertial manifold: an exponentially attracting, positively invariant, finite dimensional manifold.

Proof A complete proof may be found in Section 2 of Jones and Stuart [23] or [16]. We note here that $\Phi : Y \to Z$ is found as the fixed point of a contraction mapping T on the $\Gamma(\epsilon, \delta) \subset C(Y, Z)$, where the space $\Gamma(\epsilon, \delta)$ is endowed with the following properties

$$\forall \psi \in \Gamma, \quad \|\psi\|_\Gamma := \sup_{p \in Y} \|\psi(p)\|_{H^1} \leq \epsilon,$$

$$\forall \psi \in \Gamma,\ p_1,\ p_2 \in Y,\ \|\psi(p_1) - \psi(p_2)\|_{H^1} \leq \delta$$

and T is defined by

$$p = L\zeta + P_m N(\zeta + \Phi(\zeta))$$

$$(T\Phi)p = L\Phi(\zeta) + Q_m N\left(\zeta + \Phi(\zeta)\right).$$

It is found that T is Lipschitz continuous with constant $\mu \in (0,1)$ from which the contraction property follows. $\square$

To obtain global Lipschitz conditions of the form (G3) on the non–linear term it is usually necessary to use a cut off function and obtain the existence of an inertial manifold for (1.1) inside an absorbing ball $\mathcal{B}_\gamma(\rho)$, see [16].

Remark 1.1 *It is shown in [23, Lemmas 4.1 and 4.2] that assumptions G and conditions C are satisfied for a differential equation of the form* (1.1) *inside an absorbing ball $\mathcal{B}_\gamma(\rho)$ whenever the the following assumptions hold :*

Assumptions E

- The operator A is sectorial;
- the equation generates a Lipschitz continuous semigroup $S(t) : X^\gamma \mapsto X^\gamma$ and there exists a $\rho > 0$ such that the ball $\mathcal{B}_\gamma(\rho) \subset X^\gamma$ is absorbing,
- $\exists \gamma \geq 0, \beta \in [0,1)$ and $C = C(\rho) > 0$ such that the nonlinear function $F : X^\gamma \to X^{\gamma-\beta}$ satisfies

$$\begin{aligned} \|F(u)\|_{X^{\gamma-\beta}} &\leq C \quad \forall u \in \mathcal{B}_\gamma(\sigma) \\ \|F(u) - F(v)\|_{X^{\gamma-\beta}} &\leq C\|u - v\|_{X^\gamma} \quad \forall u, v \in \mathcal{B}_\gamma(\sigma); \end{aligned}$$

- the eigenvalues of A, $\{\lambda_i\}$, satisfy

  a) $0 < \lambda_1 \leq \lambda_2 \cdots$,

  b) the *spectral gap* condition, i.e. for any $K_3, K_4 > 0$ there exists an integer q such that
  $$\lambda_{q+1}^{1-\beta} \geq K_3, \qquad \lambda_{q+1} - \lambda_q \geq K_4 \lambda_{q+1}^\beta;$$
  for some $\beta \in [0,1)$.

1.2 The Complex Ginzburg-Landau Equation

Now consider the complex Ginzburg–Landau equation with cubic non–linearity and periodic boundary conditions on the interval $\Omega = [0,1]$, namely:

$$\begin{gathered} U_t = RU - (1+i\nu)A_0 U - (1+i\mu)|U|^2 U \\ U(0) = U^0. \end{gathered} \tag{1.2}$$

Here R, ν and μ are all real parameters and A_0 is the linear operator given by

$$A_0 := -\Delta \;\text{ with domain } D(A_0) := \left\{ u \in L^2_{\text{per}} : |A_0 u|_{L^2} < \infty \right\}. \tag{1.3}$$

Then $U(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{C}$. In the notation of the previous section $X = L^2_{\text{per}}$. Henceforth we use L^2 to denote L^2_{per}, the standard Lebesgue space of periodic functions on Ω with norm given for $V \in L^2$ by

$$|V|_{L^2} = \left\{ \int_\Omega |V(x)|^2 \, dx \right\}^{\frac{1}{2}}.$$

Since the operator A_0 defined by (1.3) is not sectorial; we introduce the linear operator $\widetilde{A}_0$ defined by

$$\widetilde{A}_0 := I + A_0, \tag{1.4}$$

with domain, $D(\widetilde{A}_0) = D(A_0)$, and reformulate (1.2) as

$$U_t + (1+i\nu)\widetilde{A}_0 U = F_0(U), \tag{1.5}$$

$$U^0 = U(0),$$

where for $\widetilde{R} := R + (1+i\nu)$, F_0 is given by

$$F_0(V) = \widetilde{R}V - (1+i\mu)|V|^2 V. \tag{1.6}$$

It is a standard result to show that the eigenvalues of A_0 and $\widetilde{A}_0$ respectively are given by

$$\Lambda_k = 4k^2\pi^2 \text{ and } \widetilde{\Lambda}_k = 1 + 4k^2\pi^2 \quad k \in \mathbb{Z} \tag{1.7}$$

and eigenvectors $\Psi_k = e^{2\pi ikx}$ $k \in \mathbb{Z}$. Then given $V \in L^2$, V may be written as the Fourier series

$$V = \sum_{k=-\infty}^{\infty} a_k \Psi_k, \qquad \text{where } \; a_k = \langle V, \Psi_k \rangle . \tag{1.8}$$

The *Sobolev space* $\{H^{2s}, \|\bullet\|_{H^{2s}}\}$ may be defined as

$$H^{2s} := D(\widetilde{A}_0^s) = \left\{ V \in L^2 : |\widetilde{A}_0^s V|_{L^2} < \infty \right\}, \tag{1.9}$$

with

$$\|V\|_{H^{2s}} := |\widetilde{A}_0^s V|_{L^2} = \left\{ \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k^{2s} |a_k|^2 \right\}^{\frac{1}{2}}. \tag{1.10}$$

Definition (1.10) is norm equivalent to the standard norm on H^{2s} defined through distributional derivatives. In a similar way (such as in [17, 9] and [12]), we define the *Gevrey space of regularity* $\{G_\tau, \|\bullet\|_{G_\tau}\}$, $\tau > 0$, by

$$G_\tau := D\left(\widetilde{A}_0^{1/2} e^{\tau \widetilde{A}_0^{1/2}}\right) = \left\{ V \in L^2 : |\widetilde{A}_0^{1/2} e^{\tau \widetilde{A}_0^{1/2}} V|_{L^2} < \infty \right\}, \tag{1.11}$$

with

$$\|V\|_{G_\tau} := |\widetilde{A}_0^{1/2} e^{\tau \widetilde{A}_0^{1/2}} V|_{L^2} = \left\{ \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k e^{2\tau \widetilde{\Lambda}_k^{1/2}} |a_k|^2 \right\}^{\frac{1}{2}}.$$

From the definition of the Gevrey spaces it is clear that given $V \in G_\tau$, the Fourier coefficients of V decay exponentially in k and hence V is a smooth analytic function. In fact for all $\alpha, \tau > 0$ we have that $G_\tau \subset H^\alpha$.

We use the following notation to denote to denote the ball centre 0 of radius ρ in L^2 $(= H^0)$, H^1 and G_τ respectively

$$\mathcal{B}_0(\rho) := \left\{ V \in L^2 : |V|_{L^2} \le \rho \right\};$$

$$\mathcal{B}_1(\rho) := \left\{ V \in H^1 : \|V\|_{H^1} \le \rho \right\};$$
$$\mathcal{B}_{G_{\tau,\Delta x}}(\rho) := \{V \in G_\tau : \|V\|_{G_\tau} \le \rho\}.$$

The following theorem summarizes some well known results for the Ginzburg-Landau equation:

Theorem 1.2 *The following properties hold for the Ginzburg–Landau equation* (1.2)*:*
CGL1 *Given $U^0 \in L^2$ there exists a unique solution*

$$U(t) \in C([0,T];L^2) \bigcap L^2((0,T);H^1)\ \forall T < \infty,$$

and so there exists a semigroup $S(\bullet)\bullet \in C\left(I\!R^+ \times L^2, L^2\right)$, defined by $S(t)U^0 = U(t)$.
CGL2 *There exists a constant $\rho_0 = \rho_0(R) > 0$ such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant for the semigroup $\{S(t)\}_{t\ge 0}$.*
CGL3 *There exists a constant $\rho_1 = \rho_1(R) > 0$ such that $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant for the semigroup $\{S(t)\}_{t\ge 0}$.*
CGL4 *The dynamical system given by the complex Ginzburg–Landau equation possesses a global attractor $\mathcal{A}$, $\mathcal{A} = \omega\left(\mathcal{B}_1(\rho_1)\right)$.*
CGL5 *There exists a constant $\rho_G = \rho_G(R)$ such that $\mathcal{B}_{G_{\tau,\Delta x}}(\rho_G)$ is absorbing and positively invariant for the semigroup $\{S(t)\}_{t\ge 0}$.*

Proof CGL1: This may be proved using a Faedo-Galerkin approach, details may be found in Temam [31]. **CGL2,CGL3,CGL4**: These were first shown for the Ginzburg–Landau equation in Ghidaglia and Héron [19] but may also be found in [31] or Doering et al [10]. **CGL5** This was proved by Doelman and Titi [9]. □

We now turn our attention to the existence of an inertial manifold: an exponentially attracting, positively invariant, finite dimensional manifold which contains the global attractor. The first proof of an inertial manifold for the Ginzburg–Landau equation was given by Doering et al [10].

In the framework of Section1.1 let the space $X = L^2$, let P_m denote the projection onto the eigenfunctions Ψ_k of $\widetilde{A}_0$ such that $|k| \le m$ and let $Q_m = I - P_m$. Thus $P_m : L^2 \to \text{Span}\{\Psi_{-m}, \cdots, \Psi_0, \cdots, \Psi_m\}$. Let $Y = P_m L^2$, and $Z = Q_m L^2$, so that $L^2 = Y \oplus Z$.

The following lemmas yield conditions on the spectrum of $\widetilde{A}_0$ and the function non–linear function F to satisfy *assumptions E*. Both proofs are straightforward and maybe found in this form in [26] or a slightly different form in [10].

Lemma 1.1 (Spectral Gap) *For any $K_1, K_2 > 0$ there exists $m \in I\!N$ such that the eigenvalues $\widetilde{\Lambda}_{m+1}$ and $\widetilde{\Lambda}_m$ satisfy*

$$\widetilde{\Lambda}_{m+1} \ge K_1, \quad \textit{and} \quad \widetilde{\Lambda}_{m+1} - \widetilde{\Lambda}_m \ge K_2.$$

Lemma 1.2 *Let $\rho > 0$ be given. Then there exists $C = C(\rho)$ such that the non–linear function F_0 defined by* (1.6) *satisfies in one spatial dimension*

$$\|F_0(U)\|_{H^1} \le C \quad \forall U \in \mathcal{B}_1(\rho)$$

and

$$\|F_0(U) - F_0(V)\|_{H^1} \le C\|U - V\|_{H^1} \quad \forall U, V \in \mathcal{B}_1(\rho).$$

Theorem 1.3 *There exists an inertial manifold $\mathcal{M}$ for the one dimensional Ginzburg–Landau equation* (1.5) *which may be represented as the graph of $\Phi : Y \to Z$ within the absorbing ball $\mathcal{B}_1(\rho_1)$.*

Proof Lemmas 1.2 and 1.1 and Theorem 1.2 ensure that all of *assumptions E* are satisfied with $X = L^2$, $\gamma = 1$, and $\beta = 0$. Hence by Remark 1.1 the assumptions of Theorem 1.1 are satisfied inside the absorbing ball $\mathcal{B}_1(\rho_1)$. $\Box$

1.3 The Discretizations

In the previous section we discussed the complex Ginzburg–Landau equation (1.2) with periodic boundary conditions. We introduce in this section the semi–discrete approximation, some fully discrete approximations and various discrete Sobolev space results. Similar discrete Sobolev results may be found for example in the works of Mokin [28] or Yan, [32, 33].

Consider the interval $[0, 1]$ divided into J uniformly distributed mesh points a distance $\Delta x := 1/J$ apart. We let subscripts indicate the spatial mesh point, so that $u_j(\bullet)$ is our approximation to $U(j\Delta x, \bullet)$. Define δ_+ to be the usual forward difference approximation to the derivative and δ_- to be the backward difference approximation. From this we find the standard approximation for second derivatives

$$\delta^2 u_j = \delta_+\delta_- u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}. \tag{1.12}$$

Let $\tilde{v} \in \ell^2$, $\tilde{v} = (\cdots, v_{-1}, v_0, v_1, \cdots, v_{J-1}, v_J, v_{J+1}, \cdots)^T$, be such that

$$\tilde{v}_k = \tilde{v}_{k+J}, \quad \forall\, k \in \mathbb{Z}. \tag{1.13}$$

Then $\tilde{v}$ is said to satisfy periodic boundary conditions. Hence-forward we let $v = (v_0, \cdots, v_{J-1}) \in \mathbb{C}^J_{\text{per}}$ denote $v \in \ell^2$ satisfying (1.13) and make free use of the periodicity. Thus, for example, $v_{-1} = v_{J-1}$, $v_J = v_0$, $v_{J+1} = v_1$. Let M be the $J \times J$ diagonal matrix with entries Δx down the diagonal:

$$M = \text{diag}(\Delta x, \cdots, \Delta x).$$

In finite element terms M corresponds to a mass matrix constructed through mass lumping. We denote the $J \times J$ matrix arising from the forward difference approximation to the derivative by

$$D = \Delta x^{-1} \begin{pmatrix} -1 & 1 & \cdots & 0 \\ 0 & -1 & 1 & \\ & \ddots & \ddots & \\ 1 & & & -1 \end{pmatrix}, \tag{1.14}$$

where the non–zero corner element arises from the periodicity. If we define the $J \times J$ matrix A by

$$A := \Delta x^{-1} \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & -1 & 2 & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix}, \tag{1.15}$$

then the matrix $M^{-1}A$ is the matrix representation of the standard finite difference approximation to $-\boldsymbol{\Delta}$ for a problem with periodic boundary conditions arising from (1.12).

As in the continuous case we define a linear operator which allows us to define the mathematical setting for the problem. This new linear operator corresponds to shifting the spectrum to make it positive. In this discrete setting define

$$\widetilde{A} = I \ + \ M^{-1}A. \tag{1.16}$$

All that remains is to introduce notation for the non–linear term. For $v \in \mathbb{C}^J_{\mathrm{per}}$ we introduce the diagonal matrix $G(v)$ with entries v_j on the diagonal. The non–linear term is then written using

$$G(|v|^2) := \begin{pmatrix} |v_0|^2 & & 0 \\ & \ddots & \\ 0 & & |v_{J-1}|^2 \end{pmatrix}. \tag{1.17}$$

where it is understood that $|v|^2$ is the element of $\mathbb{C}^J_{\mathrm{per}}$ given by

$$|v|^2 = \left(|v_0|^2, \cdots, |v_{J-1}|^2\right)^T.$$

Thus we are in a position to consider the spatial discretization. This reduces the partial differential equation to a finite system of J ordinary differential equations.

Semi–Discrete Problem (SD) :

Find $u(t) = (u_0(t), \cdots, u_{J-1}(t))^T \ \in \ \mathbb{C}^J_{\mathrm{per}}$ satisfying

$$\left\{ \begin{array}{c} u_t = Ru - (1+i\nu)M^{-1}Au - (1+i\mu)G(|u|^2)u \\ u(0) = u^0 \in \mathbb{C}^J_{\mathrm{per}} \end{array} \right. . \tag{1.18}$$

Equivalently we may write **SD** to look more like (1.5) :

$$u_t + (1+i\nu)\widetilde{A}u = F(u), \quad u(0) = u^0 \tag{1.19}$$

where

$$F(v) := \widetilde{R}v - (1+i\mu)G(|v|^2)v \tag{1.20}$$

and $\widetilde{R} = (R + (1+i\nu))$.

Although the semi–discrete problem (1.18) is of interest, a further discretization in time is required before we get a numerical scheme that we can implement on a computer. Let $\Delta t \in (0, \infty)$ be the time step and u^n denote the approximation to $U(n\Delta t)$. The time derivative is then approximated by a straightforward one–step finite difference approximation.

From an analytical point of view the easiest fully discrete problem to consider is a fully implicit scheme. This is constructed by applying the backward Euler's method to (1.18).

Discrete Fully Implicit Problem (DI) :

Find $u^n = (u_0^n, \cdots, u_{J-1}^n)^T \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\left\{ \begin{array}{l} \frac{u^{n+1}-u^n}{\Delta t} = Ru^{n+1} - (1+i\nu)M^{-1}Au^{n+1} - (1+i\mu)G(|u^{n+1}|^2)u^{n+1} \\ \qquad\qquad u(0) = u^0 \in \mathbb{C}^J_{\text{per}} \end{array} \right. \tag{1.21}$$

or equivalently

$$\frac{u^{n+1}-u^n}{\Delta t} + (1+i\nu)\widetilde{A}u^{n+1} = F(u^{n+1}). \tag{1.22}$$

$$u(0) = u^0 \in \mathbb{C}^J_{\text{per}}$$

In order to implement this scheme a non–linear solver such as Newton's method is required. However, based on linear intuition, we can expect this scheme to have good stability properties and to have dynamics close to those of the continuous time systems. The following scheme has a combination of explicit and implicit terms on the right hand side and is an example of a semi–implicit scheme.

Discrete Explicit-Implicit Non–linear Term (DEI) :

Find $u^n = (u_0^n, \cdots, u_{J-1}^n)^T \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\left\{ \begin{array}{l} \frac{u^{n+1}-u^n}{\Delta t} = Ru^{n+1} - (1+i\nu)M^{-1}Au^{n+1} - (1+i\mu)G(|u^{n}|^2)u^{n+1} \\ \qquad\qquad u(0) = u^0 \in \mathbb{C}^J_{\text{per}} \end{array} \right. \tag{1.23}$$

This scheme is relatively easy to implement as all that is required to find the solution u^{n+1} at the $(n+1)$th time level from the solution u^n at the nth time level is a tridiagonal solver adapted to handle the two corner elements arising from the periodic boundary conditions.

We now introduce the mathematical setting for the discrete equations and prove some discrete Sobolev inequalities. For $s \in \mathbb{R}$, we define

$$s\rceil := \text{ smallest integer} > s\ , \quad s\rfloor := \text{ greatest integer} \leq s.$$

The first lemma gives the eigenvalues and eigenvectors of the finite difference approximation to $-\Delta$ from, for example, Conte and de Boor [7].

Lemma 1.3 *The linear operator $M^{-1}A$ has eigenvalues*

$$\lambda_k = \frac{4}{\Delta x^2}\sin^2(k\pi\Delta x) \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor \tag{1.24}$$

and the linear operator $\widetilde{A} = I + M^{-1}A$ has eigenvalues

$$\widetilde{\lambda_k} = 1 + \lambda_k \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor. \tag{1.25}$$

The corresponding orthonormal eigenvectors ψ_k for both $M^{-1}A$ and $\widetilde{A}$ are given by

$$\psi_k = \left(1, e^{2\pi ik\Delta x}, \ldots, e^{2\pi ik(J-1)\Delta x}\right) \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor. \tag{1.26}$$

The operators $\widetilde{A}$ and $(1+i\nu)\widetilde{A}$ are sectorial operators. This follows immediately from the finite dimensionality and the eigenvalues being bounded away from 0.

The following lemma, proved in [27], bounds the ratio of the continuous eigenvalues (1.7) to the discrete eigenvalues (1.24) and (1.25).

Lemma 1.4 *The ratio of discrete to continuous eigenvalues* $r(k) = \lambda_k/\Lambda_k$, *where* $k = -J/2\rceil, \cdots, -1, 1, \cdots, J/2\rfloor$, *satisfies*

$$\frac{4}{\pi^2} \leq r(k) \leq 1 \quad k = -J/2\rceil, \cdots, -1, 1, \cdots, J/2\rfloor \tag{1.27}$$

and the ratio $\widetilde{r}(k) = \widetilde{\lambda_k}/\widetilde{\Lambda}_k$, $k = -J/2\rceil, \cdots, J/2\rfloor$

$$\frac{4}{\pi^2} \leq \widetilde{r}(k) \leq 1 \quad k = -J/2\rceil, \cdots, J/2\rfloor. \tag{1.28}$$

We now define discrete analogues of the complexified Sobolev spaces of Section 1.2, [29] or [31, page 273]. First we define the discrete L^p spaces : $L^p_{\Delta x}$. Let $u = (u_0, u_1, \ldots, u_{J-1})^T \in \mathbb{C}^J_{\text{per}}$ and $v = (v_0, v_1, \ldots, v_{J-1})^T \in \mathbb{C}^J_{\text{per}}$. Then the **discrete** $L^p_{\Delta x}$ **space** on $[0,1]$ is defined to be the normed linear space $\{\mathbb{C}^J_{\text{per}}, |\bullet|_{L^p_{\Delta x}}\}$ where

$$|v|_{L^p_{\Delta x}} = \begin{cases} \left\{\sum_{j=0}^{J-1} \Delta x |v_j|^p\right\}^{1/p} & \text{for } 1 \leq p < \infty \\ \sup_{0\leq j\leq J-1} |v_j| & \text{for } p = \infty. \end{cases} \tag{1.29}$$

The **discrete** $L^2_{\Delta x}$ **inner product** is defined by

$$\langle u, v\rangle = u^T M \overline{v} = \sum_{j=0}^{J-1} \Delta x u_j \overline{v_j} \tag{1.30}$$

Consider the expansion of $v = (v_0, \ldots, v_{J-1})^T \in L^2_{\Delta x}$ as a Fourier series based on the eigenvectors of $\widetilde{A}$ given in Lemma 1.3, so that

$$v = \sum_{k=-J/2\rceil}^{J/2\rfloor} a_k \psi_k. \tag{1.31}$$

For $s > 0$, we define the **discrete Sobolev space** $H^{2s}_{\Delta x}$ as the normed linear space $\left\{\mathbb{C}^J_{\text{per}}, \|\bullet\|_{H^{2s}_{\Delta x}}\right\}$ where

$$\|v\|_{H^{2s}_{\Delta x}} = |\widetilde{A}^s v|_{L^2_{\Delta x}} = \left\{\sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_k^{2s} |a_k|^2\right\}^{\frac{1}{2}}. \tag{1.32}$$

Note that for $s = 0$ we recover the $L^2_{\Delta x}$ norm, and for $s = \frac{1}{2}$ we find the space $H^1_{\Delta x}$ which approximates H^1.

We define the **discrete Dirichlet inner–product** by

$$\langle u, v\rangle_A := u^T A \overline{v} = -\sum_{j=0}^{J-1} \Delta x u_j \delta^2 \overline{v}_j,$$

where $j = 0$ and $j = J-1$ are dealt with by the periodicity. The inner–product allows us to define the semi-norm $\|\bullet\|_1$ by

$$\|v\|_1^2 = \langle v, v\rangle_A = v^T A\overline{v} \geq 0. \tag{1.33}$$

Thus an alternative definition for the **discrete $H^1_{\Delta x}$ space** is the normed vector space $\{\mathbb{C}^J, \|\bullet\|_{H^1_{\Delta x}}\}$, where

$$\|v\|_{H^1_{\Delta x}} = \left\{|v|^2_{L^2_{\Delta x}} + \|v\|_1^2\right\}^{1/2}. \tag{1.34}$$

The definitions of the space $H^1_{\Delta x}$ through the operator $\widetilde{A}$ in (1.32) and through discrete derivatives in (1.34) are equivalent: see for example [26]. It is a straightforward application of the Cauchy–Schwarz inequality to deduce the following interpolation inequality

$$\|v\|_1^2 \leq |v|_{L^2_{\Delta x}} |M^{-1}Av|_{L^2_{\Delta x}}. \tag{1.35}$$

In a similar way we define the **discrete Gevrey class of regularity** τ, $G_{\tau,\Delta x}$, to be the normed linear space $\left\{\mathbb{C}^J_{\text{per}}, \|\bullet\|_{G_{\tau,\Delta x}}\right\}$ where

$$\|v\|_{G_{\tau,\Delta x}} = |\widetilde{A}e^{\tau\widetilde{A}^{1/2}} v|_{L^2_{\Delta x}} = \left\{\sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \widetilde{\lambda}_k e^{2\tau\widetilde{\lambda}_k^{1/2}} |a_k|^2\right\}^{\frac{1}{2}} \tag{1.36}$$

and $\widetilde{\lambda}_k$ is the k^{th} eigenvalue of $\widetilde{A}$ given in Lemma 1.3.

We define the ball centre 0, radius ρ in $L^2_{\Delta x}(= H^0_{\Delta x})$, $H^1_{\Delta x}$ and $G_{\tau,\Delta x}$ respectively by

$$\mathcal{B}_0(\rho) := \left\{v \in \mathbb{C}^J_{\text{per}} \;:\; |v|_{L^2_{\Delta x}} \leq \rho\right\},$$

$$\mathcal{B}_1(\rho) := \left\{v \in \mathbb{C}^J_{\text{per}} \;:\; \|v\|_{H^1_{\Delta x}} \leq \rho\right\},$$

$$\mathcal{B}_{G_\tau}(\rho) := \left\{v \in \mathbb{C}^J_{\text{per}} \;:\; \|v\|_{G_{\tau,\Delta x}} \leq \rho\right\}.$$

We relate the discrete space $H^1_{\Delta x}$ and the conventional Sobolev space H^1 in the theorem, following this definition.

Definition 1.1 Let $\mathbf{V} \subset H^1$ be the space of piecewise–linear functions and let $\{\Phi_j\}$ be the standard basis of "hat" functions given by

$$\Phi_k(x) = \begin{cases} (x - x_{k-1})/\Delta x & \text{for } x_{k-1} \leq x < x_k \\ (x_{k+1} - x)/\Delta x & \text{for } x_k \leq x < x_{k+1} \qquad k = 1, \cdots, J-2 \\ 0 & \text{otherwise;} \end{cases}$$

with

$$\Phi_0(x) = \begin{cases} (x_1 - x)/\Delta x & \text{for } x_0 \leq x < x_1 \\ 0 & \text{otherwise;} \end{cases}$$

$$\Phi_{J-1}(x)) = \begin{cases} (x - x_{J-2})/\Delta x & \text{for } x_{J-2} \leq x < x_{J-1} \\ 0 & \text{otherwise;} \end{cases}$$

and $x_j = j\Delta x$. Then we define the *prolongation of* $H^1_{\Delta x}$ *into* H^1, $P_L : H^1_{\Delta x} \to \mathbf{V} \subset H^1$, for $v \in \mathbb{C}^{J}_{\text{per}}$ by

$$P_L v = \sum_{j=0}^{J-1} v_j \Phi_j. \quad \square$$

Theorem 1.4 *Let* $\mathbf{V} \subset H^1$ *be the space of piecewise–linear functions, let* $v \in H^1_{\Delta x}$ *and define* $V \in \mathbf{V}$ *by*

$$V := P_L v,$$

where P_L *is the prolongation defined in Definition 1.1. Then* $\exists$ *a constant* $\kappa > 1$ *such that*

$$\kappa^{-1}\|V\|_{H^1} \;\leq\; \|v\|_{H^1_{\Delta x}} \;\leq \kappa\|V\|_{H^1} \quad \forall \Delta x > 0.$$

Proof This proof relies on the one–dimensional setting of the problem, and maybe found in [27]. $\square$

The following definition enables us to relate the discrete spaces $H^{2s}_{\Delta x}$ and the conventional Sobolev spaces H^{2s}.

Definition 1.2 Let $s \geq \frac{1}{2}$ and define $P_{\Delta x} : H^{2s} \to H^{2s}_{\Delta x}$ to be the operator which evaluates the continuous function $V(x) \in H^{2s}$ at the grid points. Thus

$$P_{\Delta x} V(x) = \left(V(0), V(\Delta x), \ldots, V((J-1)\Delta x)\right)^T.$$

Let $\mathcal{W} \subset H^{2s}$ be a bounded set , then

$$P_{\Delta x}\mathcal{W} = \bigcup_{W \in \mathcal{W}} P_{\Delta x} W. \quad \square$$

We now present some results we require later about the discrete spaces. The proofs of these Lemmas may be found in [27].

Lemma 1.5 (Bounds on $L^\infty_{\Delta x}$)
The discrete space $H^1_{\Delta x}$ *is embedded in the discrete space* $L^\infty_{\Delta x}$ *independently of* Δx. *Precisely*

$$i) \;\; |v|^2_{L^\infty_{\Delta x}} \leq 3\|v\|^2_{H^1_{\Delta x}}, \quad \forall v \in H^1_{\Delta x}.$$

Also, for any $\epsilon > 0$, *we have*

$$ii) \;\; |v|^2_{L^\infty_{\Delta x}} \leq (1 + \frac{2}{\epsilon^2})|v|^2_{L^2_{\Delta x}} + \frac{1}{2}\epsilon^2\|v\|^2_1 \quad \forall v \in H^1_{\Delta x}.$$

Finally,

$$iii) \;\; |v|^2_{L^\infty_{\Delta x}} \leq |v|^2_{L^2_{\Delta x}} + 2|v|_{L^2_{\Delta x}}\|v\|_1 \quad \forall v \in H^1_{\Delta x}.$$

Lemma 1.6 *Given $v \in \mathbb{C}^J_{\rm per}$, for $1 \le q \le p < \infty$ we have the following bound on the $L^p_{\Delta x}$ norm*

$$|v|^p_{L^p_{\Delta x}} \le \left\{ |v|^2_{L^2_{\Delta x}} + 2|v|_{L^2_{\Delta x}} \|v\|_1 \right\}^{(p-q)/2} |v|^q_{L^q_{\Delta x}}. \tag{1.37}$$

Furthermore for Dv we find that the following two equalities hold :

$$|Dv|^2_{L^2_{\Delta x}} = \|v\|^2_1 \qquad \textit{and} \qquad \|Dv\|^2_1 = |M^{-1}Av|^2_{L^2_{\Delta x}}, \tag{1.38}$$

and that

$$|Dv|^4_{L^4_{\Delta x}} \le 6 \left\{ |v|^2_{L^2_{\Delta x}} + |M^{-1}Av|^2_{L^2_{\Delta x}} \right\} \|v\|^2_1. \tag{1.39}$$

The next lemma states a Lipschitz inequality for the non–linear term.

Lemma 1.7

With $G(\cdot)$ defined in (1.17), for all $u, v \in L^2_{\Delta x}$, $\mu \in \mathbb{R}$, we have that

$$Re\left\{ (1+i\mu) \left\langle G(|u|^2)u - G(|v|^2)v, u - v \right\rangle \right\} \le (1+\mu^2)^{1/2} (|u|^2_{L^\infty_{\Delta x}} + |v|^2_{L^\infty_{\Delta x}}) |u-v|^2_{L^2_{\Delta x}}.$$

The following two lemmas are discrete versions of standard Gronwall inequalities. The proof of the first may be found in Lemma 7.1 of [13] and the proof of the second in [14, Appendix 2].

Lemma 1.8 (Discrete Gronwall [13])

Let $0 \le y^n \le R$ for $0 \le n\Delta t \le T$. If for some constants $A_1, A_2, B, \alpha_1, \alpha_2, \beta > 0$ we have that

$$y^n \le A_1 (n\Delta t)^{-1+\alpha_1} + A_2 (n\Delta t)^{-1+\alpha_2} + B\Delta t \sum_{k=1}^{n} ((n-k+1)\Delta t)^{-1+\beta} y^k,$$

for $0 < n\Delta t \le T$, then there exists constants $\Delta t_0 = \Delta t_0(R, B, \beta)$ and $C = C(B, T, \alpha_1, \alpha_2, \beta)$ such that for $\Delta t < \Delta t_0$

$$y^n \le C \left(A_1 (n\Delta t)^{-1+\alpha_1} + A_2 (n\Delta t)^{-1+\alpha_2} \right), \quad 0 < n\Delta t \le T.$$

Lemma 1.9 (Discrete Uniform Gronwall [14])

Let q^n, p^n, y^n be three positive sequences satisfying:

$$\frac{y^{n+1} - y^n}{\Delta t} \le q^n y^{n+1} + p^n \quad \forall n > n_0$$

$$\sum_{n=k_0}^{N+k_0} \Delta t q^n \le a_1(r), \quad \sum_{n=k_0}^{N+k_0} \Delta t p^n \le a_2(r), \quad \sum_{n=k_0}^{N+k_0} \Delta t y^n \le a_3(r)$$

where

$$\sum_{n=k_0}^{N+k_0} \Delta t = r.$$

Then provided

$$\Delta t q^n \le 1 - \delta \quad \forall n > n_0$$

we have

$$y^{N+k_0+1} \le \left[\frac{a_3}{r} + a_2 \right] \exp(a_1/\delta).$$

2 A Semi–Discrete Inertial Manifold

In [27] discrete analogues **SD1–SD4** were proved of the continuous properties **CGL1–CGL4** :

Theorem 2.1 *The following properties hold for the semi–discrete Ginzburg–Landau equation:*
SD1 *Given $U^0 \in L^2_{\Delta x}$ there exists a unique solution $U \in C^1(0,T; L^2_{\Delta x})$ $\forall T < \infty$, and the mapping $U^0 \to U(t)$ is continuous and bounded from $L^2_{\Delta x}$ into itself $\forall t > 0$ so that a semigroup $S(t)$ may be defined by $S_{\Delta x}(t)U^0 = U(t)$ $\forall t \geq 0$.*
SD2 *There exists a constant $\rho_0 = \rho_0(R) > 0$ such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant for the semigroup $\{S_{\Delta x}(t)\}_{t \geq 0}$.*
SD3 *There exists a constant $\rho_1 = \rho_1(R) > 0$ such that $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant for the semigroup $\{S_{\Delta x}(t)\}_{t \geq 0}$.*
SD4 *The dynamical system given by the semi–discrete Ginzburg–Landau equation* (1.18) *possesses a global attractor $\mathcal{A}$, $\mathcal{A} = \omega(\mathcal{B}_1(\rho_1))$.*
SD5 *There exists a constant $\rho_G = \rho_G(R) > 0$ such that $\mathcal{B}_{G_{\tau,\Delta x}}(\rho_G)$ is absorbing and positively invariant for the semigroup $\{S_{\Delta x}(t)\}_{t \geq 0}$.*

We now define the discrete analogues of the projections P_m and Q_m that is the projection $\mathcal{P}_m$ onto low Fourier modes $\mathcal{P}_m : L^2_{\Delta x} \to \mathrm{Span}\{\psi_{-m}, \cdots, \psi_0, \cdots, \psi_m\}$; the projection $\mathcal{Q}_m$ onto high modes $\mathcal{Q}_m = I - \mathcal{P}_m$. Furtermore we define the spaces

$$Y_{\Delta x} := \mathcal{P}_m L^2_{\Delta x} \quad \text{and} \quad Z_{\Delta x} := \mathcal{Q}_m L^2_{\Delta x},$$

so that $L^2_{\Delta x} = Y_{\Delta x} \oplus Z_{\Delta x}$.

We prove the existence of an inertial manifold for the semi–discrete system by proving assumptions E hold and using Remark 1.1 to invoke Theorem 1.1. To achieve this we make use of the following lemma.

Lemma 2.1 *The eigenvalues $\{\lambda_k\}$ of $M^{-1}A$ satisfy:*

$$\lambda_k^{1/2} \leq 2k\pi, \quad \textit{and } \lambda_k^{1/2} \geq 2\left(k\pi - \frac{k^3\pi^3\Delta x^2}{6}\right). \tag{2.1}$$

Proof The inequalities follow from Taylor's theorem with a Lagrange form of remainder and standard trigonometric identities. □

We now prove the spectral gap condition holds for the linear operator $\widetilde{A}$.

Lemma 2.2 (Spectral Gap) *Given K_1, $K_2 > 0$ there exists $m \in \mathbb{Z}$ and $\Delta x_0 > 0$, such that for all $\Delta x < \Delta x_0$, the eigenvalues of $\widetilde{A}$ satisfy*

$$\widetilde{\lambda}_{m+1} \geq K_1 \quad \textit{and} \quad \widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m \geq K_2.$$

Proof First we fix m and Δx_0 and note that the result is straightforward for $K_1 \leq 1$. Let

$$m > \max\left\{\frac{\sqrt{K_1 - 1}}{2\pi} - 1, \frac{K_2}{8\pi^2} - \frac{1}{2}\right\} \tag{2.2}$$

and

$$\Delta x_0 = \min\left\{\frac{1 - K_1 + 4\pi^2(m+1)^2}{4\pi^4}, \frac{8\pi^2 m + 4\pi^2 - K_2}{4\pi^4}, \frac{3}{(m+1)^4}\right\}. \tag{2.3}$$

By the choice of m we have that $1 - K_1 + 4\pi^2(m+1)^2 > 0$ and $8\pi^2 m + 4\pi^2 - K_2 > 0$ whence Δx_0 is well defined. By the choice of Δx_0 we have that $m > \frac{\sqrt{K_1 + 4\pi^4 \Delta x_0 - 1}}{2\pi} - 1$ and $m > \frac{1}{8\pi^2}\left(K_2 + 4\pi^4 \Delta x_0 - 4\pi^2\right)$, from which

$$\widetilde{\Lambda}_{m+1} > K_1 + 4\pi^4 \Delta x_0 \tag{2.4}$$

and similarly

$$\widetilde{\Lambda}_{m+1} - \widetilde{\Lambda}_m > K_2 + 4\pi^4 \Delta x_0. \tag{2.5}$$

Clearly we have

$$\widetilde{\lambda}_{m+1} = \widetilde{\Lambda}_{m+1} - (\widetilde{\Lambda}_{m+1} - \widetilde{\lambda}_{m+1}),$$

so that by inequalities (2.4) and (2.1) we see that

$$\begin{aligned}
\widetilde{\lambda}_{m+1} &\geq K_1 + 4\pi^4 \Delta x_0 - (\widetilde{\Lambda}_{m+1} - \widetilde{\lambda}_{m+1}) \\
&\geq K_1 + 4\pi^4 \Delta x_0 - \frac{4\pi^4 m^4 \Delta x^2}{3} + \frac{m^6 \pi^6 \Delta x^4}{9} \\
&\geq K_1 + 4\pi^4 \Delta x_0 - 4\pi^4 \Delta x_0 \quad = \quad K_1.
\end{aligned}$$

Also,

$$\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m = \widetilde{\Lambda}_{m+1} - \widetilde{\Lambda}_m + (\widetilde{\lambda}_{m+1} - \widetilde{\Lambda}_{m+1}) - (\widetilde{\lambda}_m - \widetilde{\Lambda}_m),$$

so by (2.1) and the fact that $\widetilde{\lambda}_m - \widetilde{\Lambda}_m \leq 0$ for all $m \in \mathbb{Z}$:

$$\begin{aligned}
\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m &= K_2 + 4\pi^4 \Delta x_0 - \frac{4\pi^4 (m+1)^4 \Delta x^2}{3} + \frac{(m+1)^6 \pi^6 \Delta x^4}{9} \\
&\geq K_2 + 4\pi^4 \Delta x_0 - 4\pi^4 \Delta x_0 \quad = \quad K_2. \qquad \square
\end{aligned}$$

By the choice of Δx_0 we have ensured that the break in the spectrum is the same for the finite difference approximation and the partial differential equation. We now turn our attention to proving the relevant Lipschitz conditions for the nonlinearity.

Lemma 2.3 *Let $\rho > 0$ be given. Then there exists $C = C(\rho) > 0$ such that the non–linear function F given by* (1.20) *satisfies*

$$\|F(u)\|_{H^1_{\Delta x}} \leq C \quad \forall u \in \mathcal{B}_1(\rho)$$

and

$$\|F(u) - F(v)\|_{H^1_{\Delta x}} \leq C \|u - v\|_{H^1_{\Delta x}} \quad \forall u, v \in \mathcal{B}_1(\rho).$$

Proof Since by the definition of F we have that $F(0)=0$, it is sufficient to prove the second inequality alone. Recall that

$$\|F(u)-F(v)\|^2_{H^1_{\Delta x}} = |F(u)-F(v)|^2_{L^2_{\Delta x}} + \|F(u)-F(v)\|^2_1.$$

By the straightforward inequality

$$(a+b)^2 \le 2a^2+2b^2, \quad \forall a,b\in \mathbb{R} \tag{2.6}$$

and Lemma 1.5 there exists a constant $C_1=C_1(\rho)$ such that

$$\begin{aligned}
|F(u)-F(v)|^2_{L^2_{\Delta x}} &= \sum_{j=0}^{J-1}\Delta x\left|\widetilde{R}(u_j-v_j)-(1+i\mu)\left(|u_j|^2u_j-|v_j|^2v_j\right)\right|^2 \\
&\le 2|\widetilde{R}|^2|u-v|^2_{L^2_{\Delta x}} + 2(1+\mu^2)^2C_1|u-v|^2_{L^2_{\Delta x}} \\
&= C_1|u-v|^2_{L^2_{\Delta x}}.
\end{aligned}$$

Similarly by (2.6), Lemma 1.5 and noting that for all $u,v\in\mathbb{C}^J_{\text{per}}$ the following equality holds

$$\begin{aligned}
&\delta_+(|u_j|^2u_j)-\delta_+(|v_j|^2v_j)\\
&= \left(u^2_{j+1}+\overline{u}_{j+1}u_j+|u_j|^2\right)\delta_+(u_j-v_j)+\delta_+v_j(u_{j+1}+v_{j+1}+\overline{v}_j)(u_{j+1}-v_{j+1})\\
&\quad +\delta_+v_j(u_{j+1}+u_j)(\overline{u}_j-\overline{v}_j)+\delta_+v_j(u_j-v_j)\overline{v}_j,
\end{aligned}$$

we find there exists $C_2=C(\rho_1)$ such that

$$\begin{aligned}
\|F(u)-F(v)\|^2_1 &= \sum_{j=0}^{J-1}\Delta x\left|\widetilde{R}\delta_+(u_j-v_j)-(1+i\mu)\delta_+(|u_j|^2u_j-|v_j|^2v_j)\right|^2\\
&\le 2|\widetilde{R}|^2\|u-v\|^2_1+2(1+\mu^2)\|G(|u|^2)u-G(|v|^2)v\|^2_1\\
&\le C_2\|u-v\|^2_1. \quad \square
\end{aligned}$$

We now invoke Theorem 1.1 to prove the existence of an inertial manifold.

Theorem 2.2 *There exists an inertial manifold $\mathcal{M}_{\Delta x}$ for the semi–discrete Ginzburg–Landau equation* (1.2) *which may be represented as a graph of $\Phi: Y_{\Delta x}\to Z_{\Delta x}$ within the absorbing ball $\mathcal{B}_1(\rho_1)$.*

Proof Lemmas 2.2 and 2.3 and Theorem 2.1 ensure that all of *assumptions E* are satisfied with $X=L^2_{\Delta x}$, $A=\widetilde{A}$, $\gamma=1$, and $\beta=0$. Hence by Remark 1.1 the assumptions of Theorem 1.1 are satisfied inside the absorbing ball $\mathcal{B}_1(\rho_1)$. $\square$

Although not required for the proof of an inertial manifold, it was shown in [26] that the semi–discrete Ginzburg–Landau equation given by (1.2) satisfies a cone condition.

2.1 Convergence of the Semi–Discrete Inertial Manifold

We show in this section that inside a Gevrey absorbing ball the inertial manifold for (1.5) described as the graph of Φ and the inertial manifold for (1.18) described as the graph of $\Phi_{\Delta x}$ may be made arbitrarily close in H^1 over a finite time interval. To achieve this we use a similar analysis to that used by Jones in [21] in which he proves the existence and convergence of a semi–discrete inertial manifold of a finite difference approximation to the Kuramoto–Sivashinsky equation. In contrast to the analysis in [21] we exploit the Gevrey regularity of the solutions to avoid a non–smooth error estimate.

For the purposes of this section we introduce the following projection operator from the space H^1 to $H^1_{\Delta x}$. Define $\Theta : H^1 \to H^1_{\Delta x}$ for V given by (1.8) by

$$\Theta V(x) := \mathcal{P}_{J/2} P_{\Delta x} V(x) = \sum_{k=-\lceil J/2 \rceil}^{-\lfloor J/2 \rfloor} a_k \psi_k.$$

Furthermore define $\Theta^{-1} : H^1_{\Delta x} \to P_{J/2} H^1 :=\text{Span}\Big\{\Psi_{-\lceil J/2\rceil}, \cdots, \Psi_{\lfloor J/2 \rfloor}\Big\}$ for v given by (1.31) by

$$\Theta^{-1} v(x) = \sum_{k=-\lceil J/2 \rceil}^{-\lfloor J/2 \rfloor} a_k \Psi_k.$$

Note that the projection $\mathcal{P}_{J/2}$ acting on $v \in \mathbb{C}^{J}_{\text{per}}$ is simply the identity, i.e. $\mathcal{P}_{J/2} v = v$.

Recall that by Theorem 1.3 the continuous inertial manifold is given by the graph of $\Phi : Y \to Z$ which is the fixed point of the mapping T on $\Gamma(\epsilon, \delta) \subset C(Y, Z)$, where the space Γ is endowed with the following properties:

$$\forall\ \Psi \in \Gamma, \quad \|\Psi\|_\Gamma := \sup_{p \in Y} \|\Psi(p)\|_{H^1}\ \le \epsilon; \tag{2.7}$$

$$\forall\ \Psi \in \Gamma, \quad p_1, p_2 \in Y, \quad \|\Psi(p_1) - \Psi(p_2)\|_{H^1} \le \delta. \tag{2.8}$$

By Theorem 2.2 the semi–discrete inertial manifold is given by the graph of $\Phi_{\Delta x} : Y_{\Delta x} \to Z_{\Delta x}$ which is the fixed point of the mapping T on $\Gamma_{\Delta x}(\tilde{\epsilon}, \tilde{\delta}) \subset C(Y_{\Delta x}, Z_{\Delta x})$, where the space $\Gamma_{\Delta x}$ is endowed with the following properties:

$$\forall\ \psi \in \Gamma_{\Delta x}, \quad \|\psi\|_{\Gamma_{\Delta x}} := \sup_{p \in Y_{\Delta x}} \|\psi(p)\|_{H^1_{\Delta x}}\ \le \tilde{\epsilon}; \tag{2.9}$$

$$\forall\ \psi \in \Gamma_{\Delta x}, p_1, p_2 \in Y_{\Delta x}, \quad \|\psi(p_1) - \psi(p_2)\|_{H^1_{\Delta x}} \le \tilde{\delta}. \tag{2.10}$$

By construction we have that dim(Y)=dim($Y_{\Delta x}$) and (see [23]) may choose $\tilde{\epsilon}$ and $\tilde{\delta}$ so that

$$\tilde{\epsilon} \ge \epsilon \quad \& \quad \tilde{\delta} \ge \delta. \tag{2.11}$$

Lemma 2.4 *Given any $V \in H^1$ and any $W \in P_{J/2} H^1$, $J \in \mathbb{N}$ we have that*

$$\|\Theta V\|_{H^1_{\Delta x}} \le \|V\|_{H^1} \qquad \textit{and} \qquad \frac{\pi}{2}\|W\|_{H^1} \le \|\Theta W\|_{H^1_{\Delta x}}.$$

Proof Let V be given by 1.8. Since $W \in P_{J/2}H^1$ we may write W as $W = \sum_{k=-J/2\rceil}^{J/2\rfloor} b_k\Psi_k$ and so by the definition of Θ and the $H^1_{\Delta x}$ norm

$$\|\Theta V\|^2_{H^1_{\Delta x}} = \sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_k|a_k|^2 \quad \text{and} \qquad \|\Theta W\|^2_{H^1_{\Delta x}} = \sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}|b_k|^2.$$

The result follows by applying Lemma 1.4. $\square$

We now relate the continuous space $\Gamma(\epsilon,\delta)$ and the discrete space $\Gamma(\widetilde{\epsilon},\widetilde{\delta})$.

Lemma 2.5

Given any $\Psi \in \Gamma(\epsilon,\delta) \subset C(Y,Z)$, define $\xi : Y_{\Delta x} \to Z_{\Delta x}$ by $\xi\bullet := \Theta\Psi(\Theta^{-1}\bullet)$. Then $\xi \in \Gamma_{\Delta x}(\widetilde{\epsilon},\widetilde{\delta})$.

Proof The proof is straightforward. By the definition of the spaces Γ and $\Gamma_{\Delta x}$, Lemma (2.5) and inequalities (2.11)

$$\begin{aligned}
\|\xi\|_{\Gamma_{\Delta x}} &= \sup_{p\in Y_{\Delta x}} \|\Theta\Psi(\Theta^{-1}p)\|_{H^1_{\Delta x}} &= \sup_{\Theta^{-1}p\in Y} \|\Theta\Psi(\Theta^{-1}p)\|_{H^1_{\Delta x}} \\
&\le \sup_{\Theta^{-1}p\in Y} \|\Psi(\Theta^{-1}p)\|_{H^1} \le \epsilon \le \widetilde{\epsilon}.
\end{aligned}$$

Similarly we find that for all $p_1, p_2 \in Y_{\Delta x}$, $\|\xi(p_1) - \xi(p_2)\|_{H^1_{\Delta x}} \le \widetilde{\delta}$. $\square$

Lemma 2.6 *Given $u^0 \in \mathcal{B}_{G_{\tau,\Delta x}}(\rho_G)$ there exists $U^0 \in G_\sigma$, for $\sigma \in (0, 2\tau/\pi)$ such that $u(0) = \Theta U(0)$. Furthermore there exists $C = C(T) > 0$ such that*

$$\|u(t) - \Theta U(t)\|_{H^1_{\Delta x}} \le C(T)\Delta x^2, \qquad \forall t \in (0,T].$$

Proof In [27] it was proved that given $u^0 \in \mathcal{B}_{G_{\tau,\Delta x}}$ then $U^0 := \Theta^{-1}u^0 \in G_\sigma$ for $\sigma \in (0, 2\tau/\pi)$ and that there exists $C = C(T) > 0$ such that for all $t \in (0,T]$

$$\|u(t) - P_{\Delta x}U(t)\|_{H^1_{\Delta x}} \le C(T)\Delta x^2.$$

Using the above and the definitions of the projections Θ, $\mathcal{P}_{J/2}$ and $P_{\Delta x}$ we get

$$\begin{aligned}
\|u(t) - \Theta U(t)\|_{H^1_{\Delta x}} &= \|u(t) - \mathcal{P}_{J/2}P_{\Delta x}U(t)\|_{H^1_{\Delta x}} \\
&= \|\mathcal{P}_{J/2}\left(u(t) - P_{\Delta x}U(t)\right)\|_{H^1_{\Delta x}} \\
&= \|\mathcal{P}_{J/2}\|_{H^1_{\Delta x}}\|u(t) - P_{\Delta x}U(t)\|_{H^1_{\Delta x}} \\
&\le C(T)\Delta x^2. \quad \square
\end{aligned}$$

Theorem 2.3 *Given any $p \in Y_{\Delta x}$ there exists a constant $C = C(p) > 0$ such that inside the absorbing ball $\mathcal{B}_{G_{\tau,\Delta x}}(\rho_G)$ the continuous inertial manifold described by Φ and the semi–discrete inertial manifold described by $\Phi_{\Delta x}$ satisfy*

$$\sup_{p\in Y_{\Delta x}} \|\Phi_{\Delta x}(p) - \Theta\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}} \le C\Delta x^2.$$

Proof Recall from Theorem 1.1 that Φ and $\Phi_{\Delta x}$ are fixed points of mappings T and $T_{\Delta x}$ respectively and that both T and $T_{\Delta x}$ are Lipschitz continuous with constants μ and $\mu_{\Delta x}$. Thus we find

$$\begin{aligned}
\|\Phi_{\Delta x}(p)-\Theta\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}} &= \|T_{\Delta x}\Phi_{\Delta x}(p)-\Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}\\
\leq\ & \|T_{\Delta x}\Phi_{\Delta x}(p)-T_{\Delta x}\Theta\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}+\|T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}\\
\leq\ & \mu_{\Delta x}\|\Phi_{\Delta x}(p)-\Theta\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}+\|T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}.
\end{aligned}$$

We can take the first term on the right hand side across to the left hand side. Thus we need only consider the second term.

Given any $p\in Y_{\Delta x}$ we can find a unique ξ such that

$$p=\mathcal{P}_m S_{\Delta x}(t)\left(\xi+\Theta\Phi(\Theta^{-1}\xi)\right) \tag{2.12}$$

$$T_{\Delta x}\Phi_{\Delta x}(p)=\mathcal{Q}_m S_{\Delta x}(t)\left(\xi+\Theta\Phi(\Theta^{-1}\xi)\right). \tag{2.13}$$

Furthermore we set

$$\eta=\mathcal{P}_m\Theta S(t)\left(\Theta^{-1}\xi+\Phi(\Theta^{-1}\xi)\right) \tag{2.14}$$

$$\Theta T\Phi(\Theta^{-1}\eta)=\Theta\Phi(\Theta^{-1}\eta)=\mathcal{Q}_m\Theta S(t)\left(\Theta^{-1}\xi+\Phi(\Theta^{-1}\xi)\right). \tag{2.15}$$

Now we find that by Lemma 2.4 and the properties of T and $T_{\Delta x}$ that

$$\begin{aligned}
&\|T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}\\
&\leq\ \|T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}\eta)\|_{H^1_{\Delta x}}+\|\Theta T\Phi(\Theta^{-1}\eta)-\Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}\\
&\leq\ \|\mathcal{Q}_m\left(T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}\eta)\right)\|_{H^1_{\Delta x}}+\|\Theta\left(\Phi(\Theta^{-1}\eta)-\Phi(\Theta^{-1}p)\right)\|_{H^1_{\Delta x}}\\
&\leq\ \|\mathcal{Q}_m\left(T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}\eta)\right)\|_{H^1_{\Delta x}}+\|\Phi(\Theta^{-1}\eta)-\Phi(\Theta^{-1}p)\|_{H^1}\\
&\leq\ \|\mathcal{Q}_m\left(T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}\eta)\right)\|_{H^1_{\Delta x}}+\delta\|\Theta^{-1}\eta-\Theta^{-1}p\|_{H^1}.
\end{aligned}$$

Since $\mathcal{Q}_m p=0=\mathcal{Q}_m\eta$ and $\mathcal{P}_m(\Theta T\Phi(\Theta^{-1}\eta))=0=\mathcal{P}_m(T_{\Delta x}\Theta\Phi(\Theta^{-1}p))$ and by the norm equivalence of Lemma 2.4

$$\begin{aligned}
&\|T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}\\
&\quad\leq\ \|\mathcal{Q}_m\left(p+T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\left(\eta-\Theta T\Phi(\Theta^{-1}\eta)\right)\right)\|_{H^1_{\Delta x}}\\
&\qquad+\frac{2\delta}{\pi}\|\mathcal{P}_m\left(\eta+\Theta T\Phi(\Theta^{-1}\eta)-\left(p+T_{\Delta x}\Theta\Phi(\Theta^{-1}p)\right)\right)\|_{H^1_{\Delta x}}.
\end{aligned}$$

Note that

$$p+T_{\Delta x}\Theta\Phi(\Theta^{-1}p)=S_{\Delta x}(t)\left(\xi+\Theta\Phi(\Theta^{-1}\xi)\right)$$

and

$$\eta+\Theta T\Phi(\Theta^{-1}\eta)=\Theta S(t)\left(\Theta^{-1}\xi+\Phi(\Theta^{-1}\xi)\right)$$

so that

$$\begin{aligned}
&\|T_{\Delta x}\Theta\Phi(\Theta^{-1}p)-\Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}}\\
&\leq\ \|\mathcal{Q}_m\left(S_{\Delta x}(t)\left(\xi+\Theta\Phi(\Theta^{-1}\xi)\right)-\Theta S(t)\left(\Theta^{-1}\xi+\Phi(\Theta^{-1}\xi)\right)\right)\|_{H^1_{\Delta x}}\\
&\quad+\frac{2\delta}{\pi}\|\mathcal{P}_m\left(S_{\Delta x}(t)\left(\xi+\Theta\Phi(\Theta^{-1}\xi)\right)-\Theta S(t)\left(\Theta^{-1}\xi+\Phi(\Theta^{-1}\xi)\right)\right)\|_{H^1_{\Delta x}}.
\end{aligned}$$

By applying Lemma 2.6 with $u^0 = \Theta\left\{\Theta^{-1}\xi + \Phi(\Theta^{-1}\xi)\right\}$ we find that

$$\|T_{\Delta x}\Theta\Phi(\Theta^{-1}p) - \Theta T\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}} \leq \left(1 + \frac{2\delta}{\pi}\right) C\Delta x^2;$$

and so

$$\|\Phi_{\Delta x}(p) - \Theta\Phi(\Theta^{-1}p)\|_{H^1_{\Delta x}} \leq \frac{\pi + 2\delta}{\pi(1-\mu_{\Delta x})} C\Delta x^2. \ \square$$

3 The Fully Discrete Finite Difference Schemes

We now consider fully discrete approximations to the continuous complex Ginzburg–Landau equation **CGL**, in particular the fully implicit scheme **DI** and the mixed scheme **DEI**.

To ensure that the fully discrete dynamical systems have the desired dynamical properties it is necessary to enforce restrictions on the time step Δt. Ideally we seek restrictions on Δt independent of the spatial step size Δx and the initial condition u^0. For the scheme **DI** this was achieved, however for the mixed scheme **DEI** we present results with Δt dependent on Δx.

We also considered the θ type method given by

$$\frac{u^{n+1} - u^n}{\Delta t} = Rv^n - (1 + i\nu)M^{-1}Av^n - (1 + i\mu)G\left(|v^n|^2\right) v^n; \tag{3.1}$$

where $v^n := \theta u^{n+1} + (1 - \theta u^n)$ and $\theta \in [0, 1]$. For $\theta > 1/2$ we are able to prove the existence of $L^2_{\Delta x}$ and $H^1_{\Delta x}$ absorbing balls with radius ρ_0^θ and ρ_1^θ respectively; with both ρ_0^θ and ρ_1^θ independent of Δx, Δt and the initial condition u^0. We do not present the analysis here since it is analogous to the analysis for fully implicit scheme (1.21) which we present below.

In [26] two other schemes were investigated: a fully explicit approximation and a mixed explicit/implicit scheme where all terms are implicit except the non–linear term which is fully implicit. Numerical evidence suggests that neither of these schemes has an absorbing property that is independent of the initial data, hence they are not considered below.

3.1 Fully Implicit Scheme DI

For convenience we start by assuming that we are given a semigroup $S^n : L^2_{\Delta x} \to L^2_{\Delta x}$ for (1.21) and establish **DI2** and **DI3** - and then return to **DI1** and **DI4**.

Theorem 3.1 (DI2) *Given $u^0 \in L^2_{\Delta x}$ suppose that $\forall n > 0$ there exists a solution $u^n \in L^2_{\Delta x}$ of (1.21). Then there exists $\rho_0 = \rho_0(R) > 0$, independent of Δx, such that $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant.*

Proof Take the $L^2_{\Delta x}$ inner–product of (1.21) with u^{n+1}, take the real part, complete the square and rearrange to get

$$|u^{n+1}|^2_{L^2_{\Delta x}} \leq |u^n|^2_{L^2_{\Delta x}} + 2R\Delta t|u^{n+1}|^2_{L^2_{\Delta x}} - 2\Delta t\|u^{n+1}\|_1^2 - 2\Delta t|u^{n+1}|^4_{L^4_{\Delta x}}. \tag{3.2}$$

Let us consider the three possible cases for R: $R<0$, $R=0$ and $R>0$.

If $R<0$ then (3.2) becomes

$$|u^{n+1}|^2_{L^2_{\Delta x}} \leq |u^n|^2_{L^2_{\Delta x}} + 2R\Delta t|u^{n+1}|^2_{L^2_{\Delta x}};$$

and since $(1-2R\Delta t)>1$, we see that $|u^n|^2_{L^2_{\Delta x}} \to 0$ as $n\to\infty$. Therefore, for any $\epsilon>0$, we may take $\rho_0=\epsilon$.

If $R=0$ then using Hölder's inequality on the non–linear term in (3.2) yields

$$|u^{n+1}|^2_{L^2_{\Delta x}} \leq |u^n|^2_{L^2_{\Delta x}} - 2\Delta t u^{n+1}|^4_{L^2_{\Delta x}}$$

and so $|u^{n+1}|_{L^2_{\Delta x}} \to 0$ as $n\to\infty$. Hence we may take $\rho_0=\epsilon$ for any $\epsilon>0$.

If $R>0$ then we use that $\forall s\in \mathbb{R}$

$$\frac{1}{2}s^4 - 2Rs^2 \geq -2R^2 \tag{3.3}$$

to bound half the $L^4_{\Delta x}$ term in (3.2) :

$$\begin{aligned}(1-2R\Delta t)|u^{n+1}|^2_{L^2_{\Delta x}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.4)\\ \leq \quad |u^n|^2_{L^2_{\Delta x}} - 2\Delta t\|u^{n+1}\|^2_1 - \Delta t|u^{n+1}|^4_{L^4_{\Delta x}} - 4R\Delta t|u^{n+1}|^2_{L^2_{\Delta x}} + 4R^2\Delta t.\end{aligned}$$

Thus,

$$|u^{n+1}|^2_{L^2_{\Delta x}}(1+2R\Delta t) \leq |u^n|^2_{L^2_{\Delta x}} + 4R^2\Delta t$$

to which we can apply the standard discrete Gronwall argument to get

$$\begin{aligned}|u^{n+1}|^2_{L^2_{\Delta x}} &\leq \left(\frac{1}{1+2R\Delta t}\right)^n |u^0|^2_{L^2_{\Delta x}} + 4R^2\Delta t\sum_{k=1}^{n}\frac{1}{(1+2R\Delta t)^k}\\ &= \frac{1}{(1+2R\Delta t)^n}|u^0|^2_{L^2_{\Delta x}} + 2R + \frac{2R}{(1+2R\Delta t)^n}. \qquad (3.5)\end{aligned}$$

Since $(1+2R\Delta t)>1$ we find

$$\lim_{N\to\infty}|u^{N+1}|^2_{L^2_{\Delta x}} = 2R.$$

Thus for

$$\rho_0^2 > 2R, \tag{3.6}$$

there exists $n_0(|u^0|_{L^2_{\Delta x}})$ such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and ρ_0 is independent of Δx.□

Note that we can sum (3.5) over n from k_0 to $N+k_0$ for $k_0\geq n_0$ to obtain

$$\sum_{n=k_0}^{N+k_0}(|u^{n+1}|^2_{L^2_{\Delta x}} - |u^n|^2_{L^2_{\Delta x}} + 2R\Delta t|u^{n+1}|^2_{L^2_{\Delta x}} + 2\Delta t\|u^{n+1}\|^2_1 + \Delta t|u^{n+1}|^4_{L^4_{\Delta x}}) \leq 4\Delta t R^2 N.$$

Thus by the previous Theorem we find for $k_0>n_0$ that

$$\sum_{n=k_0}^{N+k_0}(2R\Delta t|u^{n+1}|^2_{L^2_{\Delta x}} + 2\Delta t\|u^{n+1}\|^2_1 + \Delta t|u^{n+1}|^4_{L^4_{\Delta x}}) \leq 4rR^2 + \rho_0^2 \tag{3.7}$$

where $r = N\Delta t$.

Having established the existence of the $L^2_{\Delta x}$ absorbing sets we could apply an inverse inequality to find absorbing sets in $H^1_{\Delta x}$. However the bound on the $H^1_{\Delta x}$ norm or the restriction on Δt obtained from this method would be a function of u^0 or Δx. We seek bounds which hold uniformly as $\Delta x \to 0$.

For the remainder of this section we shall assume $R > 0$ since the dynamics of both the fully discrete system **DI** and continuous system are trivial for $R \leq 0$.

Theorem 3.2 (DI3) *Given $u^0 \in L^2_{\Delta x}$ suppose that u^n is a solution of (1.21) for all $n > 0$. Then there exists constants $\Delta t_1 > 0$ and $\rho_1 = \rho_1(R) > 0$, ρ_1 independent of Δx, such that $\forall \Delta t < \Delta t_1$ the ball $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant.*

Proof We seek to bound the $H^1_{\Delta x}$ semi–norm independently of Δx. Take the Dirichlet inner–product of (1.21) with u^{n+1} and take the real part to get

$$\begin{aligned}
&\|u^{n+1}\|_1^2 \\
&\quad\leq \frac{1}{2}(\|u^n\|_1^2 + \|u^{n+1}\|_1^2) + R\Delta t\|u^{n+1}\|_1^2 - \Delta t|M^{-1}Au^{n+1}|^2_{L^2_{\Delta x}} \\
&\quad\quad + \frac{3}{2}(1+\mu^2)^{1/2}\Delta t \sum_{j=0}^{J-1} \Delta x \left|\delta_+ u_j^{n+1}\right|^2 (|u_{j+1}^{n+1}|^2 + |u_j^{n+1}|^2).
\end{aligned}$$

Apply Schwarz's inequality to last term and use the periodic boundary conditions to get

$$\begin{aligned}
\frac{1}{2}\|u^{n+1}\|_1^2 \;\leq\; & \frac{1}{2}\|u^n\|_1^2 + R\Delta t\|u^{n+1}\|_1^2 - \Delta t|M^{-1}Au^{n+1}|^2_{L^2_{\Delta x}} \\
& + 3(1+\mu^2)^{1/2}\Delta t|u^{n+1}|^2_{L^4_{\Delta x}}|Du^{n+1}|^2_{L^4_{\Delta x}}.
\end{aligned} \tag{3.8}$$

By Lemma 1.39 and completing the square on the last term we get

$$\begin{aligned}
\frac{\|u^{n+1}\|_1^2 - \|u^n\|_1^2}{\Delta t} \;\leq\; & \left(2R + 54(1+\mu^2)|u^{n+1}|^4_{L^4_{\Delta x}}\right)\|u^{n+1}\|_1^2 + |u^{n+1}|^2_{L^2_{\Delta x}} \\
& - |M^{-1}Au^{n+1}|^2_{L^2_{\Delta x}}
\end{aligned} \tag{3.9}$$

We now wish to apply the discrete uniform Gronwall inequality (Lemma 1.9) to (3.9). This has an extra condition to the continuous uniform Gronwall inequality: namely we require that $\forall \delta \in (0,1)$,

$$\Delta t\left(2R + 54(1+\mu^2)|u^{n+1}|^4_{L^4_{\Delta x}}\right) < 1 - \delta.$$

However we do not have any bound on $|u^{n+1}|_{L^4_{\Delta x}}$ only on the sum from k_0 to $k_0 + N$. We are able to control the $L^4_{\Delta x}$ term by an $L^2_{\Delta x}$ term by using the "good" term from the discrete Laplacian (thrown away in the semi–discrete and continuous analysis). First note that if $|u^{n^*}|^2_{L^2_{\Delta x}} = 0$ for some $n^* \geq 0$ then the desired result is immediate since $\|u^n\|_1^2 = 0 \ \ \forall n \geq n^*$. Without loss of generality we consider $|u^{n+1}|^2_{L^2_{\Delta x}} \neq 0$. By the interploation inequality (1.35)

$$-|M^{-1}Au^{n+1}|^2_{L^2_{\Delta x}} \leq -\frac{\|u^{n+1}\|_1^4}{|u^{n+1}|^2_{L^2_{\Delta x}}},$$

and by the discrete Gagliardo–Nirenberg inequality (1.37) with $p = 4$ and $q = 2$ we find

$$|u^{n+1}|^4_{L^4_{\Delta x}} \leq \left\{|u^{n+1}|^4_{L^2_{\Delta x}} + 2|u^{n+1}|^3_{L^2_{\Delta x}}\|u^{n+1}\|_1\right\}.$$

If we now apply the generalized Cauchy–Schwarz inequality we get

$$|u^{n+1}|^4_{L^4_{\Delta x}} \leq \left(1+\epsilon^{-1}\right)|u^{n+1}|^4_{L^2_{\Delta x}} + \epsilon|u^{n+1}|^2_{L^2_{\Delta x}}\|u^{n+1}\|_1^2. \tag{3.10}$$

Hence (3.9) becomes :

$$\begin{aligned}
&\frac{\|u^{n+1}\|_1^2 - \|u^n\|_1^2}{\Delta t} \\
&\leq \ \left\{2R + 54(1+\mu^2)\left(\left(1+\epsilon^{-1}\right)|u^{n+1}|^4_{L^2_{\Delta x}} + \epsilon|u^{n+1}|^2_{L^2_{\Delta x}}\|u^{n+1}\|_1^2\right)\right. \\
&\quad \left. - \frac{\|u^{n+1}\|_1^2}{|u^{n+1}|^2_{L^2_{\Delta x}}}\right\}\|u^{n+1}\|_1^2 + |u^{n+1}|^2_{L^2_{\Delta x}}.
\end{aligned} \tag{3.11}$$

Define ϵ by

$$\epsilon := \frac{1}{54(1+\mu^2)\rho_0^4}.$$

By the choice of ϵ and Theorem 3.1 we find from

$$\frac{\|u^{n+1}\|_1^2 - \|u^n\|_1^2}{\Delta t} \ \leq \ \left\{2R + 54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)\rho_0^4\right\}\|u^{n+1}\|_1^2 + |u^{n+1}|^2_{L^2_{\Delta x}}.$$

Define Δt_1 such that for fixed $\delta \ \in (0,1)$

$$\Delta t_1 \ := \ \frac{1-\delta}{2R + 54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)\rho_0^4}. \tag{3.12}$$

We may now apply the discrete uniform Gronwall Lemma 1.9 with constants a_1, a_2 and a_3 found from (3.7) with $r = N\Delta t$ and noting that the condition on Δt holds since $\Delta t < \Delta t_1$:

$$\|u^{k_0+1}\|_1^2 \leq \left(a_2 + \frac{a_3}{r}\right)\exp\left(a_1/\delta\right) \qquad \forall \ \ k_0 \geq n_0 + N.$$

Hence we have a uniform bound on the $H^1_{\Delta x}$ semi–norm which is independent of the initial condition u^0, the spatial step size Δx and the temporal step size Δt provided $\Delta t < \Delta t_1$ where Δt_1 satisfies relation (3.12).
Therefore there exist absorbing balls $\mathcal{B}_1(\rho_1)$ in $H^1_{\Delta x}$ with $n_1 = k_0$, and

$$\rho_1^2 = \rho_0^2 + \left(a_2 + \frac{a_3}{r}\right)\exp\left(a_1/\delta\right),$$

ρ_1 independent of Δx. $\square$

Thus far we have not considered the existence of a continuous semigroup $S^1_{\Delta x} : L^2_{\Delta x} \to L^2_{\Delta x}$ for (1.21). Below we show the existence and uniqueness of a solution to our discrete system which is continuously dependent on the initial condition, which yields existence of the continuous semigroup.

The following theorem, stated in the real case in [6, p.58] or [15], gives an approach for proving existence. The proof was adapted in [26] to the complex case.

Theorem 3.3 *Let B be a closed ball in $\mathbb{C}^n$. Suppose $\Phi : B \to \mathbb{C}^n$ is continuous and satisfies*

$$Re\,\{\langle \Phi(v), v\rangle\} < 0 \qquad \forall v \in \partial B.$$

Then there exists $v \in B$ such that $\Phi(v) = 0$.

The following Lemma gives us the existence of a solution for the system (1.21).

Lemma 3.1 (Existence) *There exists a solution u of the equation*

$$u - R\Delta t u - \Delta t(1 + i\nu)M^{-1}Au - \Delta t(1 + i\mu)G(|u|^2)u = v$$

for all $\Delta t \geq 0$ and all $v \in \mathbb{C}^J_{\text{per}}$.

Proof Let v be given. Our task is to show that we can find u. To do this we apply Theorem 3.3. Let B be the ball of radius $\sqrt{2}\rho$ in $\mathbb{C}^J_{\text{per}}$, where

$$\rho^2 = \frac{1}{1 + 2R\Delta t}(|v|^2_{L^2_{\Delta x}} + 4\Delta t R^2)$$

and define $\Phi : B \to \mathbb{C}^J_{\text{per}}$ by

$$-\Phi(u) = 2(u - v) - 2R\Delta t u - \Delta t(1 + i\nu)M^{-1}Au - \Delta t(1 + i\mu)G(|u|^2)u.$$

Take the $L^2_{\Delta x}$ inner–product of $\Phi(u)$ with u to get

$$\begin{aligned} -\langle \Phi(u), u\rangle &= 2|u|^2_{L^2_{\Delta x}} - 2\langle v, u\rangle - 2\Delta t \Big(R|u|^2_{L^2_{\Delta x}} - (1 + i\mu)\Big\langle M^{-1}Au, u\Big\rangle \\ &\quad -(1 + i\mu)\Big\langle G(|u|^2)u, u\Big\rangle\Big) \\ &= 2|u|^2_{L^2_{\Delta x}} - (\langle v, u\rangle + \langle u, v\rangle) - 2\Delta t(R|u|^2_{L^2_{\Delta x}} - \|u\|^2_1 - |u|^4_{L^4_{\Delta x}}). \end{aligned}$$

Take the real part and use (3.3) to bound half the $L^4_{\Delta x}$ norm

$$\begin{aligned} -\text{Re}\{\langle \Phi(u), u\rangle\} &\geq 2|u|^2_{L^2_{\Delta x}} - |u|^2_{L^2_{\Delta x}} - |v|^2_{L^2_{\Delta x}} + 2R^2\Delta t\|u\|^2_1 + 2R\Delta t|u|^2_{L^2_{\Delta x}} \\ &\quad - 2R^2\Delta t + \Delta t|u|^4_{L^4_{\Delta x}} \\ &\geq (1 + 2R\Delta t)|u|^2_{L^2_{\Delta x}} - (|v|^2_{L^2_{\Delta x}} + 4\Delta t R^2). \end{aligned}$$

Now for $u \in \partial B$ we see

$$-\text{Re}\,\langle \Phi(u), u\rangle \geq (1 + 2R\Delta t)(2\rho_1^2 - \rho_1^2) > 0.$$

Therefore, by Theorem 3.3, a solution exists. $\square$

Lemma 3.2 (uniqueness) *Given $u^0 \in L^2_{\Delta x}$ suppose that Δt satisfies*

$$\Delta t < \Delta t_2 := \frac{1}{2(R + 6(1+\mu^2)^{1/2})\rho_1^2}, \tag{3.13}$$

where ρ_1 is the radius of the $H^1_{\Delta x}$ absorbing ball of Theorem 3.2. Then $\forall n \geq n_1$ there exists a unique solution u^n to (1.21) and u^n is continuously dependent on the initial data u^0.

Proof Suppose for a given u^n we had two solutions u^{n+1}, v^{n+1} satisfying,

$$u^{n+1} = u^n - \Delta t \left\{ Ru^{n+1} - (1+i\nu)M^{-1}Au^{n+1} - (1+i\mu)G(|u^{n+1}|^2)u^{n+1} \right\}$$

and

$$v^{n+1} = u^n - \Delta t \left\{ Rv^{n+1} - (1+i\nu)M^{-1}Av^{n+1} - (1+i\mu)G(|v^{n+1}|^2)v^{n+1} \right\}.$$

Then subtracting, taking the inner–product with $u^{n+1} - v^{n+1}$, and taking the real part gives

$$\begin{aligned} |u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} &= R\Delta t |u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} - \Delta t \|M^{-1}A(u^{n+1} - v^{n+1})\|_1 \\ &\quad - \mathrm{Re}\left\{ \Delta t(1+i\mu) \left\langle G(|u^{n+1}|^2)u^{n+1} - G(|v^{n+1}|^2)v^{n+1}, u^{n+1} - v^{n+1} \right\rangle \right\}. \end{aligned}$$

By Lemma 1.7 we find

$$\begin{aligned} |u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} &\leq R\Delta t |u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} - \Delta t \|M^{-1}A(u^{n+1} - v^{n+1})\|_1 \\ &\quad + \Delta t (1+\mu^2)^{1/2} \left(3|u^{n+1}|^2_{L^\infty_{\Delta x}} + 3|v^{n+1}|^2_{L^\infty_{\Delta x}} \right) |u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}}. \end{aligned}$$

Hence for $n > n_1$, we find

$$1 \leq \Delta t R + 6\Delta t (1+\mu^2)^{1/2} \rho_1^2,$$

and provided the restriction (3.13) holds, we have a unique solution .

Continuity with respect to initial data is established by proving Lipschitz continuity. Let u^{n+1}, v^{n+1} be two solutions to (1.21) at step $n+1$ with initial conditions u^0 and v^0 respectively. Subtracting the equations for u^{n+1} and v^{n+1}, taking the $L^2_{\Delta x}$ innerproduct with $u^{n+1} - v^{n+1}$, taking the real part and using Lemma 1.7 we find

$$\begin{aligned} &\frac{|u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} - |u^n - v^n|^2_{L^2_{\Delta x}}}{2\Delta t} \\ &\quad \leq R|u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} - \|M^{-1}A(u^{n+1} - v^{n+1})\|^2 \\ &\qquad + (1+\mu^2)^{1/2} \left(|u^{n+1}|^2_{L^\infty_{\Delta x}} + |v^{n+1}|^2_{L^\infty_{\Delta x}} \right) |u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} \\ &\quad \leq R|u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} + 6(1+\mu^2)^{1/2}\rho_1^2 |u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}}. \end{aligned}$$

Thus we find that

$$|u^{n+1} - v^{n+1}|^2_{L^2_{\Delta x}} \leq \frac{1}{1 - 2R\Delta t - 12\Delta t(1+\mu^2)^{1/2}\rho_1^2} |u^n - v^n|^2_{L^2_{\Delta x}},$$

and by the choice of Δt the lemma is proved. □

Together, Theorems 3.1 and 3.2 establish the existence of a continuous semigroup $S^1_{\Delta x} : L^2_{\Delta x} \to L^2_{\Delta x}$. We now conclude the existence of a global attractor.

Theorem 3.4 (DI4) *Suppose that Δt is restricted so that*

$$\Delta t < \Delta t_3 := \min\{1, \Delta t_1, \Delta t_2\} \tag{3.14}$$

where Δt_1, Δt_2 are given by (3.12) *and* (3.13). *Then dynamical system given by the fully implicit discrete Ginzburg–Landau equation* (1.21) *possesses a global attractor $\mathcal{A}_{\Delta x}$,*

$$\mathcal{A}_{\Delta x} = \omega(\mathcal{B}_1(\rho_1))$$

which is bounded independently of Δx in $H^1_{\Delta x}$.

3.2 A Fully Discrete Inertial Manifold

We employ the same notation as in Section 2 and let $\mathcal{P}_m$ denote the projection onto low Fourier modes, $\mathcal{Q}_m$ denote the projection onto high Fourier modes, and $Y_{\Delta x} := \mathcal{P}_m L^2_{\Delta x}$ and $Z_{\Delta x} := \mathcal{Q}_m H^1_{\Delta x}$. For the purposes of this section we assume that $u^0 \in \mathcal{B}_1(\rho_1)$, where ρ_1 was defined in Theorem 3.2.

We prove the existence of an inertial manifold for (1.21) inside the $H^1_{\Delta x}$ absorbing ball $\mathcal{B}_1(\rho_1)$ by proving existence of a global inertial manifold for the prepared equation. The prepared equation has the same dynamics as (1.21) for solutions in $\mathcal{B}_1(\rho_1)$ but satisfies a global Lipschitz property. The use of a prepared equation and our proof of an inertial manifold for it is an application of the analysis described in [23] and [16] which is based on mappings in Banach spaces and hence is readily applicable to this fully discrete setting.

We start by defining a suitable cut–off function in order to define the prepared equation. Let $\theta : \mathbb{R}^+ \to [0,1]$ be a C^∞ function such that

$$\theta(s) = \begin{cases} 1 & 0 \le s \le \rho_1 \\ 0 & s \ge 2\rho_1 \end{cases} \tag{3.15}$$

and

$$|\theta'(s)| \le 2 \ \forall s \ge 0. \tag{3.16}$$

The following Lemma is proved in a standard way, such as in [15] for example. It is proved in this particular context in [26].

Lemma 3.3 *Define the function $F_\theta : \mathbb{C}^J_{\mathrm{per}} \to \mathbb{C}$ by $F_\theta(u) := \theta(\|u\|_{H^1_{\Delta x}})F(u)$, where θ is defined by* (3.15) *and* (3.16). *Then $\exists\ K_3,\ K_4 > 0$ such that $F_\theta(u)$ satisfies the estimates*

$$\begin{aligned} \|F_\theta(u)\|_{H^1_{\Delta x}} &\le K_3 \ \ \forall\ u \in \mathbb{C}^J_{\mathrm{per}} \\ \|F_\theta(u) - F_\theta(v)\|_{H^1_{\Delta x}} &\le K_4 \|u - v\|_{H^1_{\Delta x}} \ \ \forall\ u, v \in C^J. \quad \Box \end{aligned}$$

We now define the prepared equation:

$$\frac{u^{n+1} - u^n}{\Delta t} + (1 + i\nu)\tilde{A}u^{n+1} = F_\theta(u^{n+1}). \tag{3.17}$$

By the previous lemma, all solutions to (3.17) are the same as for (1.21) inside the ball $\mathcal{B}_1(\rho_1)$. Hence to comprehend the long time dynamics of (1.21) it is sufficient to examine (3.17). We prove the existence of a global invariant manifold for (3.17) which yields the existence of an invariant manifold for (1.21) inside the ball. For the discrete problem (1.21) we simply find the mapping $G_{\Delta x,\Delta t}$ from (3.17). Rearranging (3.17) we get

$$\left[I + \Delta t(1+i\nu)\widetilde{A}\right] u^{n+1} = u^n + \Delta t F_\theta(u^{n+1}),$$

and so

$$u^{n+1} = \left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} u^n + \Delta t \left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} F_\theta(u^{n+1}) \tag{3.18}$$

or

$$u^{n+1} = G_{\Delta x,\Delta t}(u^n). \tag{3.19}$$

Thus we find the mapping $G_{\Delta x,\Delta t}$ for (3.17) is given for $u \in \mathbb{C}^J_{\text{per}}$ by

$$G_{\Delta x,\Delta t} u := Lu + N(u) \tag{3.20}$$

where

$$L_{\Delta x,\Delta t} = \left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} \tag{3.21}$$

and

$$N_{\Delta x,\Delta t}(u^n) := \Delta t \left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} F_\theta(S^1_{\Delta x} u^n). \tag{3.22}$$

We now show that the map $G_{\Delta x,\Delta t}$ satisfies assumptions G, with $A = \widetilde{A}$, $G = G_{\Delta x,\Delta t}$ and with the projections $P_m = \mathcal{P}_m$ and $Q_m = \mathcal{Q}_m$.

Lemma 3.4

For all $\Delta x < 1/(m+1)$ and $\Delta t > 0$ $\exists K_5 \in \mathbb{R}$ such that the mapping $G_{\Delta x,\Delta t}$ defined by (3.20) satisfies assumptions G with $\gamma = 1$ and

$$\begin{aligned} a &:= |1 + \Delta t(1+i\nu)\widetilde{\lambda}_{m+1}|^{-1} && (3.23)\\ b &:= |1 + \Delta t(1+i\nu)\widetilde{\lambda}_{m}|^{-1} && (3.24)\\ c &:= |1 + \Delta t(1+i\nu)\widetilde{\lambda}_{0}|^{-1} && (3.25)\\ B &:= \Delta t K_5 |1 + \Delta t(1+i\nu)\widetilde{\lambda}_{0}|^{-1}. && (3.26) \end{aligned}$$

Proof First note that given any $v \in H^1_{\Delta x}$ with Fourier coefficients v_k and any $\alpha \in \mathbb{R}$ we have,

$$\|(I + \Delta t(1+\imath\nu)\widetilde{A})^\alpha v\|^2_{H^1_{\Delta x}} = \sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_k \left|1 + \Delta t(1+i\nu)\widetilde{\lambda}_k\right|^{2\alpha} |v_k|^2. \tag{3.27}$$

G1) Let $z \in Z_{\Delta x}$ have Fourier coefficients a_k then by (3.27)

$$\begin{aligned} \|L_{\Delta x,\Delta t} z\|^2_{H^1_{\Delta x}} &= \sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_k |1 + \Delta t(1+i\nu)\widetilde{\lambda}_k|^{-2} |a_k| \\ &\le |1 + \Delta t(1+i\nu)\widetilde{\lambda}_{m+1}|^{-2} \|z\|^2_{H^1_{\Delta x}}, \end{aligned}$$

and so G1 is satisfied with a defined by (3.23).

G2) First note that given any $p \in Y_{\Delta x}$ $\exists ! w \in Y_{\Delta x}$, $w = (I + \Delta t(1+i\nu)\widetilde{A})p$ such that $L_{\Delta x} w = p$. Let $y \in Y_{\Delta x}$ have Fourier coefficients a_k then by (3.27)

$$\|L_{\Delta x,\Delta t} y\|^2_{H^1_{\Delta x}} \geq |1 + \Delta t(1+i\nu)\widetilde{\lambda}_m|^{-2} \|y\|^2_{H^1_{\Delta x}},$$

and

$$\|L_{\Delta x,\Delta t} y\|^2_{H^1_{\Delta x}} \leq |1 + \Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2} \|y\|^2_{H^1_{\Delta x}}.$$

Thus, G2 is satisfied with b and c given by (3.24) and (3.25) respectively.

G3) Note that since $F_\theta(0) = 0$ it is sufficient to prove the second inequality alone. Let $F_\theta(S^1_{\Delta x} u^n) - F_\theta(S^1_{\Delta x} v^n)$ have Fourier coefficients b_k and consider $\|N(u^n) - N(v^n)\|_{H^1_{\Delta x}}$:

$$\begin{aligned}
&\|N(u^n) - N(v^n)\|^2_{H^1_{\Delta x}} \\
&= \Delta t \sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_k |1 + (1+i\nu)\widetilde{\lambda}_k|^2 |b_k|^2 \\
&\leq \Delta t |1 + \Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2} \left\| F_\theta(S^1_{\Delta x} u^n) - F_\theta(S^1_{\Delta x} v^n) \right\|^2_{H^1_{\Delta x}}.
\end{aligned}$$

Since F_θ is Lipschitz with constant K_4, we get

$$\|N(u) - N(v)\|^2_{H^1_{\Delta x}} \leq \Delta t |1 + \Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2} K_4 \|S^1_{\Delta x} u^n - S^1_{\Delta x} v^n\|^2_{H^1_{\Delta x}}.$$

By the continuity of the discrete semigroup $S^1_{\Delta x}$ we have that

$$\|N(u) - N(v)\|^2_{H^1_{\Delta x}} \leq \Delta t |1 + \Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2} K_5 \|u^n - v^n\|^2_{H^1_{\Delta x}}.$$

The other inequalities may be established in a similar manner. Thus G3 is satisfied with B defined by (3.26). □

We now prove that conditions C' hold provided enough Fourier modes are taken.

Lemma 3.5 *Let $\epsilon, \delta, \alpha \in (0,1)$, $\mu \in (e^{-\alpha}, 1)$ be given and let*

$$\Delta t := \frac{\alpha}{\widetilde{\lambda}_{q+1}}.$$

Then there exists $m \in \mathbb{Z}$ and $\Delta x_0 > 0$ such that $\forall \Delta x < \Delta x_0$ the constants a, b, c, B as defined by (3.23–3.26) satisfy conditions C'.

Proof First we make our choice of $m \in \mathbb{Z}$ and $\Delta x_0 \in \mathbb{R}$ from Lemma 2.2 in order to satisfy the spectral gap property of Lemma 2.2 with

$$\begin{aligned}
&K_1 \\
&:= \max\left\{ \frac{\alpha(1+\delta)}{\mu} K_5 \left|1 + (1+i\nu)\alpha\right|, \frac{(1 + |1 + (1+i\nu)\alpha|)\, |1 + (1+i\nu)\alpha| K_5}{2|1 + (1+i\nu)\Delta t \widetilde{\lambda}_0| \epsilon}, \right. \\
&\left. \frac{K_5 \alpha (1+\delta)}{(\mu - a)|1 + (1+i\nu)\Delta t \widetilde{\lambda}_0|} \right\}
\end{aligned} \tag{3.28}$$

and

$$K_2 := \frac{B(1+\delta)^2}{2\delta\Delta t} 2|1+(1+i\nu)\alpha|^3. \tag{3.29}$$

Before proving the conditions $C1$–$C4$ we note the following two useful equalities. For all $x \in \mathbb{R}$,

$$|1+(1+i\nu)x|^2 = 1 + 2x\widetilde{\lambda} + (1+\nu^2)x^2 \tag{3.30}$$

and for all $x, z \in \mathbb{R}$,

$$\frac{x-z}{\sqrt{x}+\sqrt{z}} \;=\; \frac{(\sqrt{x}-\sqrt{z})(\sqrt{x}+\sqrt{z})}{\sqrt{x}+\sqrt{z}} \;=\; \sqrt{x}-\sqrt{z}\,. \tag{3.31}$$

C1) By the choice of m and Δx we have that

$$\widetilde{\lambda}_{m+1} \geq \frac{\alpha(1+\delta)}{\mu} K_5\, |1+(1+i\nu)\alpha|\,.$$

using the definition of Δt, the fact that $\lambda_m < \lambda_{m+1}$ and (3.30) we find

$$\begin{aligned}
\mu &\geq \Delta t(1+\delta)K_5\,|1+(1+i\nu)\alpha| \\
&\geq \Delta t(1+\delta)K_5 \left(1 + 2\alpha\frac{\widetilde{\lambda}_m}{\widetilde{\lambda}_{m+1}} + \alpha^2\frac{\widetilde{\lambda}_m^2}{\widetilde{\lambda}_{m+1}^2}(1+\nu^2)\right)^{1/2} \\
&\geq \Delta t(1+\delta)K_5 \left(1 + 2\Delta t\widetilde{\lambda}_m + \Delta t^2\widetilde{\lambda}_m^2(1+\nu^2)\right)^{1/2}.
\end{aligned}$$

Hence

$$\mu \geq b^{-1}M(1+\delta),$$

which is exactly condition C1.

C2) By the choice of m and Δx we have

$$\widetilde{\lambda}_{m+1} \geq \frac{(1+|1+(1+i\nu)\alpha|)\,|1+(1+i\nu)\alpha|K_5}{2|1+(1+i\nu)\widetilde{\lambda}_0|\epsilon}.$$

Multiplying both sides by Δt and using the definition of Δt and B we get

$$2\epsilon\Delta t\widetilde{\lambda}_{m+1} \;\geq\; \left(1+|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|\right)|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|B. \tag{3.32}$$

If we further multiply by ϵ and add in the term $\Delta t^2\epsilon^2\widetilde{\lambda}_{q+1}$ on the left hand side (3.32) implies that

$$\begin{aligned}
&\epsilon^2\left(\Delta t\widetilde{\lambda}_{m+1} + \Delta t^2\widetilde{\lambda}_{m+1}\right) + \epsilon^2 - \epsilon^2 \\
&\geq\; \epsilon\left(1+|1+(1+i\nu)\widetilde{\lambda}_{m+1}|\right)\; |1+(1+i\nu)\widetilde{\lambda}_{m+1}|\; B,
\end{aligned}$$

from which we see

$$\frac{\epsilon^2|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|^2 - \epsilon^2}{\epsilon + \epsilon|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|} \;\geq\; |1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|B.$$

Now by the equality (3.31) this is equivalent to

$$\epsilon|1+(1+i\nu)\widetilde{\lambda}_{m+1}| - \epsilon \geq |1+(1+i\nu)\widetilde{\lambda}_{m+1}|B,$$

and so

$$\epsilon \geq B + \epsilon \frac{1}{|1+(1+i\nu)\widetilde{\lambda}_{m+1}|},$$

Hence $\epsilon \geq B + \epsilon a$ and C2 is proved.

C3) The choice of m and Δx gives us that

$$\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m \geq \frac{B(1+\delta)^2}{2\delta\Delta t} 2|1+(1+i\nu)\alpha|^3.$$

Then rearranging and using that $\widetilde{\lambda}_{m+1} > \widetilde{\lambda}_m > 0$ we find

$$\begin{aligned} 2\Delta t(\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m) \;\; &\geq \;\; \frac{B(1+\delta)^2}{\delta}|1+(1+i\nu)\widetilde{\lambda}_{m+1}||1+(1+i\nu)\widetilde{\lambda}_m| \\ &\quad \times \left(|1+(1+i\nu)\widetilde{\lambda}_{m+1}| + |1+(1+i\nu)\widetilde{\lambda}_m|\right). \end{aligned}$$

This implies that

$$\begin{aligned} &2\Delta t(\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m) + \Delta t^2(1+\nu^2)(\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m) \\ &\quad \geq \;\; \frac{B(1+\delta)^2}{\delta}|1+(1+i\nu)\widetilde{\lambda}_{m+1}||1+(1+i\nu)\widetilde{\lambda}_m| \\ &\qquad \times \left(|1+(1+i\nu)\widetilde{\lambda}_{m+1}| + |1+(1+i\nu)\widetilde{\lambda}_m|\right) \end{aligned}$$

where we have added the positive term $\Delta t^2(1+\nu^2)(\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m)$. Taking terms over to the left hand side we see that

$$\begin{aligned} &\frac{\left(1+2\Delta t\widetilde{\lambda}_{m+1} + \Delta t(1+\nu^2)\right) - \left(1+\Delta t\widetilde{\lambda}_m + \Delta t^2(1+\nu^2)\widetilde{\lambda}_m\right)}{|1+(1+i\nu)\widetilde{\lambda}_{m+1}||1+(1+i\nu)\widetilde{\lambda}_m|\left(|1+(1+i\nu)\widetilde{\lambda}_{m+1}| + |1+(1+i\nu)\widetilde{\lambda}_m|\right)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \geq \frac{B(1+\delta)^2}{\delta}. \end{aligned}$$

Now use the equality (3.31) to get

$$\begin{aligned} \frac{|1+(1+i\nu)\widetilde{\lambda}_{m+1}| - |1+(1+i\nu)\widetilde{\lambda}_m|}{|1+(1+i\nu)\widetilde{\lambda}_{m+1}||1+(1+i\nu)\widetilde{\lambda}_m|} \;\; &\geq \;\; \frac{B(1+\delta)^2}{\delta} \\ \frac{1}{|1+(1+i\nu)\widetilde{\lambda}_m|} - \frac{1}{|1+(1+i\nu)\widetilde{\lambda}_{m+1}|} \;\; &\geq \;\; \frac{B(1+\delta)^2}{\delta} \\ \delta(a-b) \;\; &\geq \;\; B(1+\delta)^2. \end{aligned}$$

Noting that $(1+\delta)^2 = (1+\delta) + \delta(1+\delta)$ we find

$$\delta a + (1+\delta)B \leq \delta b - \delta(1+\delta)B,$$

and hence C3 is proved.

C4) The choice of m and Δx gives us from (3.28) that

$$\widetilde{\lambda}_{m+1} \geq \frac{K_5 \alpha(1+\delta)}{(\mu - a)|1 + (1+i\nu)\widetilde{\lambda}_0|}.$$

Then by the definition of Δt and dividing through by $\widetilde{\lambda}_{m+1}$ we have

$$1 \geq \frac{B(1+\delta)}{\mu - a},$$

and so

$$\mu \geq B(1+\delta) + a,$$

which is exactly C4. $\square$

We now state the main result for this section, the existence of an inertial manifold for the scheme **DI**.

Theorem 3.5 *There exists $m \in I\!N$, $\Delta x_0 \in I\!R^+$ such that $\forall \Delta x < \Delta x_0$ the scheme given by equation* (3.17) *has a global inertial manifold which may be represented as a graph $\Phi : Y_{\Delta x} \to Z_{\Delta x}$ and hence the scheme* (1.21) *has an inertial manifold which may be represented as a graph $\Phi : Y_{\Delta x} \to Z_{\Delta x}$ within $\mathcal{B}_1(\rho_1)$.*

Proof Lemmas 3.4 and 3.5 ensure that all the conditions are satisfied for Theorem 1.1. $\square$

Note that although not required for the proof of an inertial manifold, it was shown in [26] that the fully–discrete Ginzburg–Landau equation given by (1.21) satisfies a cone condition.

3.3 The Explicit–Implicit Scheme DEI

In this section we consider the scheme **DEI** (1.23) and prove existence of an absorbing ball in $L^2_{\Delta x}$. We commence with some lemmas which establish bounds that will be used to control the non–linear term in the proofs of **DEI2**.

Lemma 3.6 *Suppose that given $u^0 \in L^2_{\Delta x}$ there exists a solution u^n of* **DEI** (1.23) *for all $n > 0$. Then the following two inequalities hold,*

$$|u_j^{n+1}|^2 - |u_j^n|^2 \leq 2R\Delta t |u_j^{n+1}|^2 + 2(1+\nu^2)^{1/2}\Delta t \left|(\delta^2 u_j^{n+1})\overline{u}_j^{n+1}\right| \tag{3.33}$$

and,

$$\sum_{j=0}^{J-1} \Delta x \left|\left(\delta^2 u_j^{n+1}\right) \overline{u_j^{n+1}}\right| |u_j^{n+1}|^2 \leq \frac{4}{\Delta x^2} |u^{n+1}|^4_{L^4_{\Delta x}}. \tag{3.34}$$

Proof Inequality (3.33) is proved by writing the scheme (1.23) in component form multiplying by $\overline{u}_j^{n+1}$ and taking the real part to get

$$\begin{aligned}|u_j^{n+1}|^2 &= \frac{1}{2}(u_j^n\overline{u}_j^{n+1} + u_j^{n+1}\overline{u}_j^n) + R\Delta t|u_j^{n+1}|^2 + \Delta t\mathrm{Re}\left\{(1+i\nu)(\delta^2 u_j^{n+1})\overline{u}_j^{n+1}\right\}\\ &\quad -\Delta t|u_j^n|^2|u_j^{n+1}|^2.\end{aligned}$$

Completing the square and neglecting the term from the non–linearity yields the result.

Inequality (3.34) requires a little more work.

$$\begin{aligned}&\sum_{j=0}^{J-1}\Delta x\left|\left(\delta^2 u_j^{n+1}\right)\overline{u}_j^{n+1}\right||u_j^{n+1}|^2\\ &\quad= \sum_{j=0}^{J-1}\Delta x\left|\frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})}{\Delta x^2}\overline{u}_j^{n+1}\right||u_j^{n+1}|^2\\ &\quad\leq \sum_{j=0}^{J-1}\Delta x\frac{(|u_{j+1}^{n+1}||u_j^{n+1}| + 2|u_j^{n+1}|^2 + |u_{j-1}^{n+1}||u_j^{n+1}|)}{\Delta x^2}|u_j^{n+1}|^2.\end{aligned}$$

Complete the square twice to get:

$$\begin{aligned}&\sum_{j=0}^{J-1}\Delta x\left|\delta^2 u_j^{n+1}\overline{u_j^{n+1}}\right||u_j^{n+1}|^2\\ &\leq \frac{1}{\Delta x^2}\sum_{j=0}^{J-1}\Delta x\frac{1}{2}(|u_{j+1}^{n+1}|^2 + |u_j^{n+1}|^2 + 4|u_{j+1}^{n+1}|^2 + |u_{j-1}^{n+1}|^2 + |u_j^{n+1}|^2)|u_j^{n+1}|^2\\ &\leq \frac{1}{\Delta x^2}\sum_{j=0}^{J-1}\Delta x\frac{1}{4}(|u_{j+1}^{n+1}|^4 + |u_j^{n+1}|^4 + 2|u_j^{n+1}|^4 + 2|u_j^{n+1}|^2\\ &\qquad + |u_{j-1}^{n+1}|^4 + |u_j^{n+1}|^4 + |u_{j-1}^{n+1}|^4 + |u_j^{n+1}|^4);\end{aligned}$$

and use the periodic boundary conditions to get (3.34). □

We are now in a position to establish **DEI2** : the existence of an absorbing ball $\mathcal{B}_0(\rho_0)$ where ρ_0 is independent of Δx.

Theorem 3.6 (DEI2) *Suppose that given $u^0 \in L^2_{\Delta x}$ we have a solution u^n of* (1.23) $\forall n > 0$. *Then there exists a constant $\rho_0 = \rho_0(R) > 0$, independent of Δx, such that $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant provided Δt satisfies*

$$\Delta t < \Delta t_1 := \frac{\Delta x^2}{4R\Delta x^2 + 16(1+\nu^2)^{1/2}}. \tag{3.35}$$

Proof For $R \leq 0$ the proof is exactly like that of Theorem 3.2, and we may take ρ_0 to be any positive constant.

For $R > 0$ we take the $L^2_{\Delta x}$ inner product of (1.23) with u^{n+1}, take the real part, and complete the square to find

$$\frac{|u^{n+1}|^2_{L^2_{\Delta x}} - |u^n|^2_{L^2_{\Delta x}}}{2\Delta t} \le R|u^{n+1}|^2_{L^2_{\Delta x}} - \|u^{n+1}\|_1^2 + \sum_{j=0}^{J-1} \Delta x |u_j^n|^2 |u_j^{n+1}|^2. \tag{3.36}$$

Now add in and subtract out $\Delta t|u^{n+1}|^4_{L^4_{\Delta x}}$ on the right–hand side. This converts (3.36) to the fully implicit case (3.2) with an error term. We successfully dealt with (3.2) so our main concern will be the error term.

$$\begin{aligned}\frac{|u^{n+1}|^2_{L^2_{\Delta x}} - |u^n|^2_{L^2_{\Delta x}}}{2\Delta t} \;\le\; & R|u^{n+1}|^2_{L^2_{\Delta x}} - \|u^{n+1}\|_1^2 - |u^{n+1}|^4_{L^4_{\Delta x}} \\ & + \sum_{j=0}^{J-1} \Delta x (|u_j^{n+1}|^2 - |u_j^n|^2)|u_j^{n+1}|^2.\end{aligned}$$

use the point–wise bound that we found in Lemma 3.6 on the error term

$$\begin{aligned}&\frac{|u^{n+1}|^2_{L^2_{\Delta x}} - |u^n|^2_{L^2_{\Delta x}}}{2\Delta t} \\ &\le\; R|u^{n+1}|^2_{L^2_{\Delta x}} - |u^{n+1}|^4_{L^4_{\Delta x}} - \|u^{n+1}\|_1^2 \\ &\quad + \sum_{j=0}^{J-1} \Delta x \left(2R\Delta t |u_j^{n+1}|^2 + 2(1+\nu^2)^{1/2}\Delta t \left|\left(\delta^2 u_j^{n+1}\right) \overline{u}_j^{n+1}\right|\right) |u_j^{n+1}|^2 \\ &\le\; R|u^{n+1}|^2_{L^2_{\Delta x}} - |u^{n+1}|^4_{L^4_{\Delta x}} - \|u^{n+1}\|_1^2 + 2R\Delta t \sum_{j=0}^{J-1} \Delta x |u_j^{n+1}|^4 \\ &\quad +2(1+\nu^2)^{1/2}\Delta t \sum_{j=0}^{J-1} \Delta x \left|\delta^2 u_j^{n+1} \overline{u}_j^{n+1}\right| |u_j^{n+1}|^2.\end{aligned}$$

By inequality (3.34) we find

$$\begin{aligned}\frac{|u^{n+1}|^2_{L^2_{\Delta x}} - |u^n|^2_{L^2_{\Delta x}}}{2\Delta t} \;\le\; & R|u^{n+1}|^2_{L^2_{\Delta x}} - \|u^{n+1}\|_1^2 - \frac{1}{2}|u^{n+1}|^4_{L^4_{\Delta x}} - \frac{1}{2}|u^{n+1}|^4_{L^4_{\Delta x}} \\ & + \left(2R\Delta t + 8(1+\nu^2)^{1/2}\frac{\Delta t}{\Delta x^2}\right)|u^{n+1}|^4_{L^4_{\Delta x}}.\end{aligned}$$

Hence

$$\begin{aligned}|u^{n+1}|^2_{L^2_{\Delta x}} \;\le\; & |u^n|^2_{L^2_{\Delta x}} + 2R\Delta t|u^{n+1}|^2_{L^2_{\Delta x}} - \Delta t|u^{n+1}|^4_{L^4_{\Delta x}} - 2\Delta t\|u^{n+1}\|_1^2 \\ & + \Delta t\left\{4R\Delta t + 16(1+\nu^2)^{1/2}\frac{\Delta t}{\Delta x^2} - 1\right\}|u^{n+1}|^4_{L^4_{\Delta x}},\end{aligned}$$

and we can control the error term provided the restriction (3.35) holds. Thus

$$|u^{n+1}|^2_{L^2_{\Delta x}} \le |u^n|^2_{L^2_{\Delta x}} - 2\Delta t\|u^{n+1}\|_1^2 + 2R\Delta t|u^{n+1}|^2_{L^2_{\Delta x}} - \Delta t|u^{n+1}|_{L^4_{\Delta x}}. \tag{3.37}$$

We are now in exactly the same position as in Theorem 3.1: inequality (3.5), and the existence of an absorbing set in $L^2_{\Delta x}$ follows by the same analysis. □

Calling upon the finite dimensionality of the problem we conclude the following theorem.

Theorem 3.7 (DEI4) *Suppose that $\Delta t < \Delta t_1$ where Δt_1 is given by* (3.35) *and assume that the scheme* **DEI** *forms a dynamical system. Then the dynamical system given by the explicit implicit discrete Ginzburg–Landau equation* (1.23) *possesses a global attractor $\mathcal{A}_{\Delta x}$,*

$$\mathcal{A}_{\Delta x} = \omega\left(\mathcal{B}_1(\rho_0)\right).$$

In Section 4.5.3 of [26] numerical evidence is presented that Δt may, in fact, be taken to be independent of the spatial mesh Δx and that the results found here are an artifact of the analysis rather than a true reflection of the scheme.

4 Conclusion

In summary we have shown the existence of an inertial manifold for the semi–discrete approximation and proved convergence over a finite time to the true inertial manifold inside a Gevrey ball of regularity. We have also proved existence of a discrete global attractor and inertial manifold for the fully implicit discrete scheme and existence of a global attractor in L^2 for the semi–implicit scheme. These results lend confidence to long–time finite difference simulations of the Ginzburg–Landau equation and in particular show that by taking an appropriate discretization the correct qualitative behaviour may be observed.

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