Numerical Computations of Connecting Orbits in Discrete and Continuous Dynamical Systems

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Abstract

The aim of this paper is to present a numerical technique for the computation of connections between periodic orbits in non–autonomous and autonomous systems of ordinary differential equations. First the existence and computation of connecting orbits between fixed points in discrete dynamical systems is discussed; then it is shown that the problem of finding connections between equilibria and periodic solutions in continuous systems may be reduced to finding connections between fixed points in a discrete system. Implementation of the method is considered: the choice of a linear solver discussed and phase conditions are suggested for the discrete system. The paper concludes with some numerical examples: connections for equilibria and periodic orbits are computed for discrete systems and for non–autonomous and autonomous systems, including systems arising from the discretization of a partial differential equation.

KEY WORDS: dynamical system, computation and continuation, (heteroclinic, homoclinic) connecting orbits, global attractor.

CLASSIFICATION: 34B15, 34C30, 34C35, 65L10

1 Introduction

Consider the autonomous dynamical system in $\mathbb{R}^m$ given by

$$\frac{dx}{dt} = f(x), \qquad x(0) = x_0 \in \mathbb{R}^m, \tag{1.1}$$

which is possibly dependent on a parameter $\mu \in \mathbb{R}^\ell$.

Let x_- and x_+ be two invariant sets of (1.1), for example fixed points or periodic orbits. We assume that x_- and x_+ are compact. If $x(t)$ is a solution of (1.1) such that

$$\lim_{t\to-\infty} x(t) = x_-, \qquad \lim_{t\to+\infty} x(t) = x_+$$

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then $x(t)$ is a *connecting orbit*, which is *heteroclinic* if $x_- \neq x_+$ and *homoclinic* if $x_- = x_+$.

Connecting orbits play an important role in the analysis of dynamical systems. In order to understand the behaviour of a dynamical system the first step usually involves the computation of α and ω limit sets (such as x_- and x_+ above) and the orbits connecting these sets. Heteroclinic and homoclinic connections also arise naturally in the analysis of travelling wave phenomena in parabolic partial differential equations [Doedel & Kernẽvez, 1984].

In a fundamental paper, Beyn [1990] presented a new approach on the computation of homoclinic and heteroclinic connections between hyperbolic fixed points in an autonomous dynamical system. Essentially the technique involves the setting up of an approximating boundary value problem over a finite time interval with boundary conditions obtained using the asymptotic boundary condition approach of [de Hoog & Weiss, 1980] and [Lentini and Keller, 1980].

Beyn's approach was used by Bai et al. [1993] to compute heteroclinic connections in partial differential equations with a gradient structure. A different approach is used by Friedman and Doedel [1991] who use an expansion approach to approximate the boundary conditions. They further generalised their technique in [Friedman & Doedel, 1993] making use of higher order boundary conditions, which can be incorporated in AUTO, to compute a homoclinic connection at a saddle-node. Beyn's approach is extended by Bai and Champneys [1994] to compute saddle-node homoclinic orbits of both codimension one and two.

More recently Beyn [1993] has extended the work in [Beyn, 1990] to include connections between hyperbolic and non-hyperbolic compact sets, and in particular, connections between equilibria and periodic orbits, for which a rigorous analysis is given. The problem is formulated as a well posed boundary value problem that includes the computation of the periodic orbits. The boundary value problem may then be approximated numerically.

The problem we treat in this paper is related to, but different from, that in [Friedman & Doedel, 1991] and [Bai & Champneys, 1994] and our approach differs from that in [Beyn, 1993]. The aim of this paper is to present a numerical technique for the computation of heteroclinic connections between (possibly unstable) periodic orbits in both autonomous and non-autonomous dynamical systems. To achieve this we consider connections between fixed points in discrete dynamical systems.

We present in Sec. 2 an analogue of Beyn's approach in [Beyn, 1990] and [Beyn, 1993] for computing connections between hyperbolic or non–hyperbolic fixed points in discrete dynamical systems. The existence of such connections for discrete systems is briefly discussed and the problem is formulated as a discrete boundary value problem.

In Sec. 3 we show that the problem of finding connections between equilibria and periodic solutions for autonomous and non–autonomous continuous systems may be reduced to finding connections between fixed points in a discrete system.

In contrast to the method of [Beyn, 1993], in this discrete setting we do not explicitly solve for the periodic orbit, instead we asssume that the periodic orbit is either known analytically or is found computationally from some code such as AUTO [Doedel & Kernẽvez, 1984]. Since we do not solve directly for the periodic orbit this reduces the size of the system and hence larger computations are possible.

The implementation of the method is considered in Sec. 4, in particular the choice of linear solver. It is noted that the reduction of the continuous problem to the discrete setting makes the calculation of connections for periodic solutions feasible for a large system of ordinary differential equations such as those arising from the discretization of a partial differential equation. Discrete phase conditions are also suggested.

The paper concludes with some numerical examples. First we present examples of connections in discrete dynamical systems, then we compute connections between equilibria and periodic solutions for non–autonomous and autonomous systems. Our examples include computations for a system of ordinary differential equations arising from the discretization of a partial differential equations.

2 Connections for Discrete Dynamical System

In this section we discuss the existence and numerical computation of connecting orbits between hyperbolic and non–hyperbolic fixed points and periodic points for maps. These results are closely connected to the work of Beyn [1990, 1993] for the continuous case and some familiarity with those papers is assumed.

Consider the non–linear map in $\mathbb{R}^m$ given by

$$U_{n+1} = G(U_n) \tag{2.1}$$

which is possibly dependent on a parameter $\mu \in \mathbb{R}^\ell$. Then we let $S : \mathbb{R}^m \to \mathbb{R}^m$ denote the evolution semi–group for (2.1), so that given $U_0 \in \mathbb{R}^m$ we have that

$$U_n = SU_{n-1} = S^n U_0, \qquad \forall n \in \mathbb{N}. \tag{2.2}$$

Then the set $\{S^n\}_{n\in\mathbb{N}}$ enjoys the usual semi–group properties, that is $S^n S^k = S^{n+k}$ and $S^0 = I$.

Let $U_- := U_-(\mu)$ and $U_+ := U_+(\mu)$ be two invariant sets for (2.2), so that $\forall n \in \mathbb{Z}, S^n U_\pm = U_\pm$. Then the orbit $\gamma = \{U_n\}_{n\in\mathbb{Z}}$ is said to be a *connecting orbit* from U_- to U_+ if

$$\mathrm{dist}(U_n, U_\pm) \to 0 \text{ as } n \to \pm\infty. \tag{2.3}$$

When $U_+ = U_-$ the orbit γ is called a *homolinic connection*, whereas if $U_+ \neq U_-$, γ is termed a *heteroclinic connection*.

Remark 2.1 *Note that to find connections between a set of periodic points* Π_- *of period* k_1 *and a set of periodic points* Π_+ *of period* k_2 *for the mapping* S *we simply form the map* $\widetilde{S} = S^{k_1 k_2}$ *and seek connections between the fixed points* $U_- \in \Pi_-$ *and* $U_+ \in \Pi_+$ *of* $\widetilde{S}$. *Thus there is no restriction in considering* $U_\pm$ *as fixed points in the discussion below.*

Let the points $U_\pm$ have centre manifold $\mathcal{M}_{\pm c}$ and stable and unstable manifolds $\mathcal{M}_{\pm s}$ and $\mathcal{M}_{\pm u}$ of dimension $m_{\pm c} + m_{\pm s}$ and $m_{\pm c} + m_{\pm u}$ respectively (independent of any parameter μ). Then

$$m = m_{-c} + m_{-u} + m_{-s} = m_{+c} + m_{+u} + m_{+s}, \tag{2.4}$$

and a connecting orbit, γ, lies in the intersection of the unstable manifold for $\mathcal{M}_{-u}$ and the stable manifold of $\mathcal{M}_{+s}$, i.e.

$$\gamma \subset \mathcal{M}_{-u} \bigcap \mathcal{M}_{+u}.$$

We expect the connecting orbit γ to lie in a $q+1$ dimensional manifold if

$$m_{-u} + m_{-c} + m_{+s} + m_{+c} = m + 1 + q. \tag{2.5}$$

For $q = 0$, γ is contained in a 1 dimensional manifold, whereas for $q > 0$ we require q further *determining conditions* in order to compute a single connection. Any determining conditions are denoted by

$$\Psi_d(\gamma) = 0. \tag{2.6}$$

For $q < 0$ the connecting orbit is not structurally stable but may be stabilized by introducing $-q$ further parameters into the system (see Beyn [1990] for details). For the examples taken in this paper it is generally found that $q \geq 0$.

A connecting orbit γ satisfying (2.2), (2.3) and (2.6) is constrained to lie in a one dimensional submanifold $\mathcal{M}_\gamma \subset \mathcal{M}_{-u} \bigcap \mathcal{M}_{+s}$. Suppose we were given a connection $\gamma \in \mathcal{M}_\gamma$ and any perturbation $\varepsilon \in \mathcal{M}_\gamma$, then $\gamma + \varepsilon \in \mathcal{M}_\gamma$ is also a connection from U_- to U_+. Therefore γ is not uniquely determined by (2.2), (2.3) and (2.6). This leads to the introduction of a discrete version of the continuous phase condition which fixes one connection from the family of connections given by the one dimensional submanifold $\mathcal{M}_\gamma$. We call this a *discrete phase condition* and denote it by

$$\Psi_p(\gamma) = 0. \tag{2.7}$$

The numerical implementation of a discrete phase condition is discussed in Sec. 4.1.

We conclude that a connection $\gamma \subset \mathcal{M}_{-u} \bigcap \mathcal{M}_{+s}$ from U_- to U_+ satisfies the following discrete boundary value problem:

$$\begin{array}{rcl} U_n = S^n U_0 & \text{for} & n \in \mathbb{Z}\ ; \\ \lim\limits_{n \to -\infty} U_n = U_- & \text{and} & \lim\limits_{n \to \infty} U_n = U_+\ ; \\ & \Psi_p(\gamma) = 0; & \\ \text{and, if required,} & & \Psi_d(\gamma) = 0. \end{array} \tag{2.8}$$

In order to solve this problem numerically we need to truncate the system (2.9) to a finite dimensional problem. Hence we consider

$$\begin{array}{rcl} U_n = S^n U_0 & \text{for} & n \in [N_-, N_+] \; ; \\ b_-(U_{N_-}) = 0 & \text{and} & b_+(U_{N_+}) = 0 \; ; \\ & \Psi_p(\gamma) = 0; & \\ \text{and, if required,} & & \Psi_d(\gamma) = 0 \; ; \end{array} \tag{2.9}$$

where $b(U_{N_-})$ and $b_+(U_{N_+})$ denote the boundary conditions at U_{N_-} and U_{N_+} respectively. We choose to consider projection boundary conditions such as are taken in [Beyn, 1990] for the continuous case. To this end we let $L_{-s} \in I\!R^{m_{-s} \times m}$ denote the projection operator from $I\!R^m$ onto $\mathcal{M}_{-u}$ and $L_{+u} \in I\!R^{m_{+u} \times m}$ denote the projection operator from $I\!R^m$ onto $\mathcal{M}_{+s}$. Then,

$$b_-(U_{N_-}) = L_{-s}\left[U_- - U_{N_-}\right] \tag{2.10}$$

$$b_+(U_{N_+}) = L_{+u}\left[U_+ - U_{N_+}\right] \tag{2.11}$$

and the number of boundary conditions is given by $m_{-s} + m_{+u}$ which by (2.4) and (2.5) becomes

$$\begin{aligned} m_{-s} + m_{+u} &= m - (m_{-u} + m_{-c}) + m - (m_{+s} + m_{+c}) \\ &= 2m - (m + 1 + q) \\ &= m - 1 - q. \end{aligned}$$

Hence the truncated problem is a well determined problem.

3 Connections between Periodic Orbits in Continuous Dynamical Systems

The aim now is to show how the problem of computing heteroclinic connections between equilibria and periodic orbits, and heteroclinic or homoclinic connections between periodic orbits may be reduced to one of finding connections between fixed points of a map. The case of connections between periodic solutions in a non-autonomous system is simplest and we discuss this case first.

Consider the non-autonomous system,

$$\frac{dx}{dt} = g(x, t, \mu), \qquad x \in I\!R^m, \; t \in I\!R, \; \mu \in I\!R^\ell. \tag{3.1}$$

with g being T-periodic, i.e.

$$g(x, t, \mu) = g(x, t + T, \mu), \qquad \forall x \in I\!R^m, \mu \in I\!R^\ell.$$

Assume (3.1) has two T-periodic orbits π_-, π_+ given by

$$\pi_\pm := \{x \in I\!\!R^m \mid x = p_\pm(t), p_\pm(t) = p_\pm(t+T), \forall t \in I\!\!R\}.$$

The case of two orbits with periods T_- and T_+ can be considered by an obvious extension (as discussed in Remark 2.1), so the assumption that both orbits have period T is not restrictive. Let $S(t) : I\!\!R^m \to I\!\!R^m$ denote the evolution semigroup defined by (3.1), so that $x(t) = S(t)x(0)$. Then one may define the map

$$U_{n+1} := S(T)U_n, \qquad U_n \in I\!\!R^m. \tag{3.2}$$

Then any point of π_- and π_+ is a fixed point of the (3.2). It is clear that heteroclinic connections between two periodic orbits of (3.1) can be found by computing heteroclinic connections between fixed points of the map (3.2). A numerical example illustrating this technique is given in Sec. 5.

For the autonomous system

$$\frac{dx}{dt} = g(x, \mu), \qquad x \in I\!\!R^m,\ \mu \in I\!\!R^\ell, \tag{3.3}$$

the technique is essentially the same as that for non–autonomous system (3.1). To illustrate this suppose we wish to find a connection γ_c between two equilibria x_- and x_+ of (3.3). Then both x_- and x_+ correspond to fixed points of the map

$$U_{n+1} = S(T_n)U_n, \tag{3.4}$$

where T_n could either vary with n or remain fixed. Furthermore given the connection γ_c exists for (3.3) there clearly exists a connection between the fixed points for the map (3.4). We see from the numerical discussion in Sec. 4 below, that the numerical approach using the map is equivalent to a multiple shooting method over interval of length T_n for solving the differential system in [Beyn, 1990].

Connections between a fixed point and a periodic solution or between periodic solutions for the continuous system (3.3) are determined in a similar way. For simplicity of presentation consider the connection from a fixed point x_- to a periodic orbit π_+ of period T for the system (3.3). We make use of the following observation :

Remark 3.1 *Given a connection γ_c between a fixed point x_- and periodic orbit π_+ of period T for* (3.3) *there exists a connection γ for the map* (3.4) *between any point x_- and any point $x_+ \in \pi_+$.*

Fix $T_n = T$ in (3.4) and suppose we are given any point $x_+ \in \pi_+$. Then x_- and x_+ are fixed points of the map defined by $U_{n+1} = S(T)U_n$, and any connection γ from x_- to x_+ satisfies

$$U_{n+1} = S(T)U_n, \qquad n \in Z\!\!\!Z \tag{3.5}$$

$$\lim_{n\to-\infty} U_n = x_- \quad \text{and} \quad \lim_{n\to+\infty} U_n = x_+. \tag{3.6}$$

We now show that a connection between fixed points for the map (3.4) uniquely determines, up to a suitable phase shift, a connection for (3.3) between a fixed point and a periodic solution. Given any $\widetilde{x}_+ \in \pi_+$ there exists a unique phase shift $\tau \in [0, T)$ such that

$$\widetilde{x}_+ = S(\tau)x_+. \tag{3.7}$$

Consider the map given by

$$\widetilde{U}_n \;=\; S(T+\tau)U_{n-1} \;=\; S(T)S(\tau)U_{n-1} \;=\; S(\tau)U_n.$$

Then we see that

$$\lim_{n\to\pm\infty} \widetilde{U}_n \;=\; \lim_{n\to\pm\infty} S(\tau)U_{n-1} \;=\; S(\tau)x_\pm \;=\widetilde{x}_\pm.$$

Thus, given the connection to one point on the periodic orbit, x_+, the connection to any other point $\widetilde{x}_+$ is determined by a suitable phase shift.

Note that it is not necessary to restrict $T_n = T$ in map (3.4). In fact all that is required is that $\lim_{n\to\infty} T_n = T$.

Clearly a similar argument holds for connections between periodic orbits π_- of period T_- and periodic orbits π_+ of period T_+. We do not present any details here but note that in this case T_n may be taken to vary, so that $lim_{n\to\pm\infty} T_n = T_\pm$, or be taken as fixed so that $U_{n+1} = S(T)U_n$ with $T = T_+T_-$.

4 Numerical Implementation

In this section we discuss the computational techniques used in the determination of the connecting orbits for maps. Recall that we seek a solution of the following system of nonlinear equations:

$$\begin{array}{rcl} U_n = S^n U_0 & \text{for} & n \in [N_-, N_+]; \\ b_-(U_{N_-}) = 0 & \text{and} & b_+(U_{N_+}) = 0\ ; \\ & \Psi_p(\gamma) = 0 & \\ \text{and, if required,} & & \Psi_d(\gamma) = 0. \end{array} \tag{4.1}$$

We solve the nonlinear system (4.1) by Newton or chord Newton method. At each iteration we have to solve sequence of linear system of the form:

$$Ax = b \tag{4.2}$$

where A has the block form

$$A = \begin{pmatrix} J_{N_-} & -I & & & & & \\ & J_{(N_-+1)} & -I & & & & \\ & & \cdots & \cdots & & & \\ & & & & \cdots & \cdots & \\ & & & & & J_{N_+-1} & -I \\ L_- & & & & & & L_+ \\ B_{N_-} & B_{(N_-+1)} & \cdots & B_0 & \cdots & B_{(N_+-1)} & B_{N_+} \end{pmatrix} \tag{4.3}$$

with $J_j = DS[U_j]$, the linearization of the map $S : I\!R^m \to I\!R^m$ with respect to U_j, and $B_j = D\Psi_{d/p}(\mathcal{U})$ the linearization of the determining and phase conditions. The blocks L_- and L_+ arise from the linearization of the boundary conditions.

If m is small then it is likely that (4.2) is best solved using gauss elimination, ignoring the structure in A. However for large m an efficient, stable direct method is needed. The obvious block elimination (i.e. do not interchange rows from different blocks) would retain structure, but is unstable [Ascher *et al.*, 1988]. The matrix A has a similar form to that arising from a multiple shooting or finite difference approach to the solution of two-point boundary value problems, except for the last two (block) rows. A combination of the ideas of stable block elimination, [Govaerts & Pryce, 1993], [Chan, 1984a], [Chan, 1984b] and [Moore, 1987]. and stable algorithms for boundary value problem, [Wright, 1992], [Wright, 1993], [Ascher *et al.*, 1988], can be used to solve (4.2).

Apart from the solution of (4.2), a major cost in the algorithm is the setting up of J_j blocks in the matrix A. This is particularly expensive for the continuous systems (3.1) or (3.3) since each evolution of the map $S(T)U_n$ involves the solution of a p dimensional initial value problem over T time units. Also it is very likely that we will not be able to evaluate the elements in J_j efficiently. To decrease the cost of setting up A we approximate the elements in J_j by the simplest finite difference approximation, which involves an extra evolution of the map per component in J_j. This makes our algorithm feasible for large problems, such as those obtained by semi-discretization of PDE's. Also, as can be seen from the results in Example 5.3, Chord Newton method is preferred instead of full Newton to eliminate the repeated re-evaluation of A. This again cuts down the work dramatically.

Remark 4.1: We note that as $j \to N_-$, $U_j \to U_{N_-}$ and $J_j \to J_{N_-}$. Hence if $\|U_j - U_-\| < TOL$, a given tolerance, we set $J_j = J_{N_-}$. A similar strategy is used at $n = N_+$. The effect of this approximation is also discussed in Example 5.3.

4.1 Implementation of the Phase Condition

In order to solve the truncated boundary value problem (4.1) a form of phase condition (2.7) is required.

Suppose that some approximation $\tilde{\gamma} = \left\{\tilde{U}_n\right\}_{n\in Z\!\!\!Z}$ to the connecting orbit γ is known and that this approximation is constrained to lie in a one dimensional manifold $\mathcal{M}_{\tilde{\gamma}}$. Furthermore suppose that the tangent vector V_n at $\tilde{U}_n$ to the manifold $\mathcal{M}_{\tilde{\gamma}}$ is known. Then the simplest form of phase condition is the discrete analogue of the classical phase condition which fixes the orbit at one point :

$$\Psi_p^c(\gamma) := V_0^T \left[U_0 - \tilde{U}_0\right]. \tag{4.4}$$

An alternative to this condition is the discrete analogue of the integral phase condition

$$\Psi_p^{\infty}(\gamma) := \sum_{n=-\infty}^{\infty} V_n^T \left[U_n - \tilde{U}_n \right]. \tag{4.5}$$

Note that if the mapping arises from the flow of a differential equation such as (3.1) or (3.3) then the tangent vectors are given by the time derivative as in [Beyn, 1990], [Beyn, 1993] Therefore in the case where the discrete system (2.2) arises from a continuous system as described in Sec. 3 the phase conditions (4.4) and (4.5) are implemented using tangent vectors from the underlying continuous system.

However, for a general discrete dynamical system (2.2), the tangents to the manifold containing the connecting orbit γ are not known analytically or readily available.

The method we employed for approximating the tangent for a general mapping is based on the observation that the connecting orbit γ is constrained to lie in a one dimensional C^1 submanifold $\mathcal{M}_\gamma \subset \mathcal{M}_{-u} \bigcap \mathcal{M}_{+s}$ and that the sequences $\{U_n\}_{n\in\mathbb{N}}$ and $\{U_{-n}\}_{n\in\mathbb{N}}$ are Cauchy sequences. Thus in an ε–neighbourhood of the fixed points $U_\pm$ we may approximate the tangent $V_{n_\pm}$ at $U_{n_\pm} \in N(U_\pm, \varepsilon)$ by $\tilde{V}_{n_\pm}$ for $n_+ < N_+$ and $n_- > N_-$ where

$$\tilde{V}_{n_-} := \frac{U_{n_-+1} - U_{n_-}}{\|U_{n_-+1} - U_{n_-}\|} \quad \text{and} \quad \tilde{V}_{n_+} := \frac{U_{n_++1} - U_{n_+}}{\|U_{n_++1} - U_{n_+}\|}. \tag{4.6}$$

From which we form the following approximations to the classical phase condition

$$\Psi_p^{c+} = \tilde{V}_{n_+} \left[U_{n_+} - \tilde{U}_{n_+} \right] = 0, \tag{4.7}$$

$$\Psi_p^{c-} = \tilde{V}_{n_-} \left[U_{n_-} - \tilde{U}_{n_-} \right] = 0, \tag{4.8}$$

which may be combined to give a composite condition

$$\Psi_p^{\pm} = \Psi_p^{c+} + \Psi_p^{c-} = 0. \tag{4.9}$$

We note that the phase conditions (4.7–4.9) require few additional computations.

5 Numerical Examples

Now we illustrate the numerical method with six examples, with our special interest being in the computations for the connections to periodic solutions. We emphasize that in all cases the connections are to unstable fixed points for the map (and hence unstable equilibria or periodic solutions in the continuous dynamical system). When computations were done involving continuation, the code PITCON [Rheinboldt, 1986] was used. In all cases of maps derived from continuous problem, the computation of U_{n+1} that is the evolution of $S(T)U_n$ was carried out using VODE [Brown *et al.*, 1989].

Example 5.1 : As a first illustration we consider connections between two unstable fixed points in the simple map:

$$U_{n+1} = aU_n(1-U_n) + \mu V_n^2, \quad (5.1)$$
$$V_{n+1} = bV_n + \mu U_n^2(U_n - 1 + 1/a)^2, \quad (5.2)$$

For $1 < a < 3$ and $b > 1, \mu = 0$, it is clear that

$$(U,V) = W_\pm = (U_\pm, 0), \quad \text{where } \bar{U}_- = 0, \quad \bar{U}_+ = 1 - 1/a$$

are fixed points of the mapping (5.1)—(5.2). Restricting to the one-dimensional subspace $V = 0$, U_- is unstable and U_+ is stable. The connecting orbit $U_- \to U_+$ can be found readily by merely iterating the map forward with a very small positive starting value. However in $\mathbb{R}^2$, both W_- and W_+ are unstable. They are fixed points of mapping (5.1)—(5.2) for all $\mu \geq 0$ and their stability does not change as μ changes. The connections for various values of μ were computed using continuation. Figure 1 shows the connections, (a) for $\mu = 0$, (b) for $\mu = 74.5808$ and (c) for $\mu = -15.7227$. In all cases $N_- = 0$ and $N_+ = 100$.

Next consider $a = 4$ and $b < 1$. With $\mu = 0$, and restricting to the subspace $V = 0$, we are in the chaotic regime for the famous quartic map. There is a homoclinic orbit for this map, see [Henry, 1981]. Using this as the initial condition for the two dimensional map at $\mu = 0$ we compute a path of homoclinic orbit for $\mu \neq 0$. Figure 2 shows the connections for different values of μ, namely (a) for $\mu = 0$, (b) for $\mu = 2.3953$, and (c) for $\mu = -9.0277$. Again $N_- = 0$, $N_+ = 100$.

Example 5.2 : This following example is also based on the quadratic map and is used to illustrate the connection between a fixed point and periodic orbit for maps. Consider the simple system

$$U_{n+1} = aU_n(1-U_n) \quad (5.3)$$
$$V_{n+1} = bV_n/(c + dV_n) \quad (5.4)$$
$$W_{n+1} = \alpha W_n/(\beta + \gamma W_n). \quad (5.5)$$

For $a = 3.2$ the map (5.3) has a stable period 2 solution π and the origin is unstable. The map given by (5.4) has two equilibria, one at the origin, the other at $V := (b-c)/d$, and for $b = 2$, $c = 1$, $d = 4$ the origin is unstable and V is stable. Similarly the map given by (5.5) has two equilibria, one at the origin, the other at $W := (\alpha - \beta)/\gamma$, and for $\alpha = 5$, $\beta = 2$, $\gamma = 0.25$ the origin is unstable and W is stable. We consider connections from the origin to the unstable solution $(\pi, 0, W)$ which lie in a 2 dimensional manifold. We introduce a parameter μ given by

$$\mu = \sum_{j=N_-}^{N_+} W_j^2$$

Figure 1: The heteroclinic connection at $a = 3$

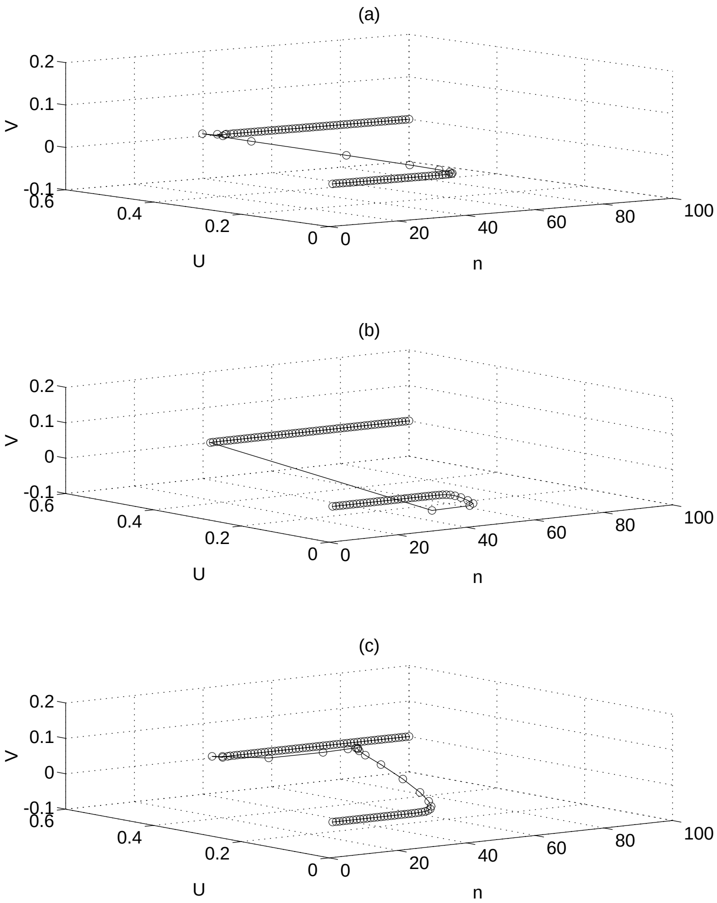
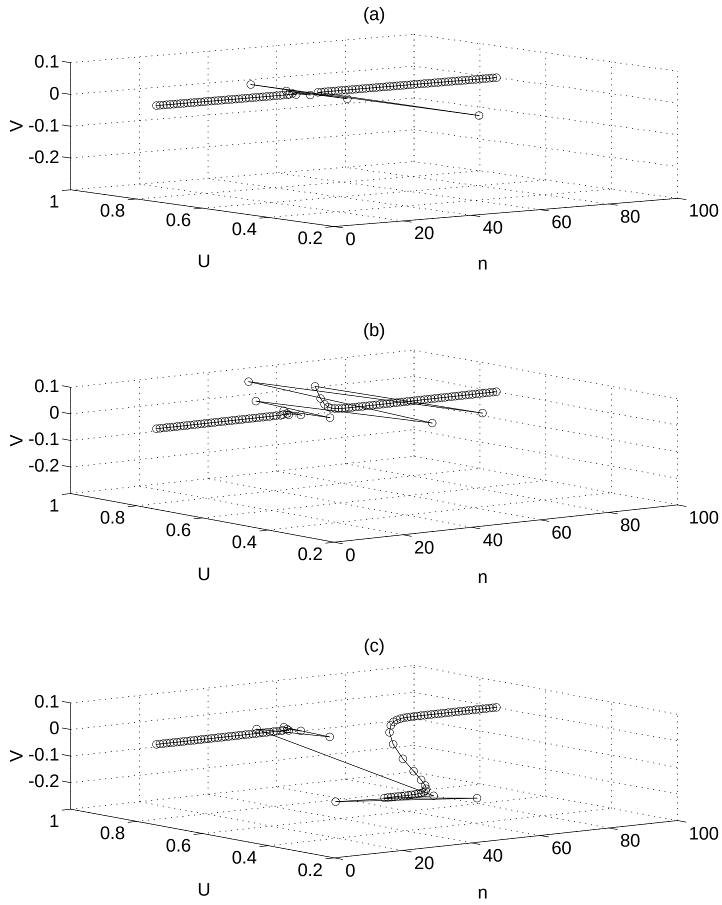


Figure 2: The homoclinic connection at $a = 4$



and use numerical continuation in μ to force the connection to move in the 2 dimensional manifold. In Fig. 3 we see the results of the such a computation. The circles mark the start of the continuation and the crosses the end of the continuation. In Fig. 4 we have plotted the connections at the start 'o' and end 'x' of the continuation process.

Figure 3: Continuation of a heteroclinic connection.

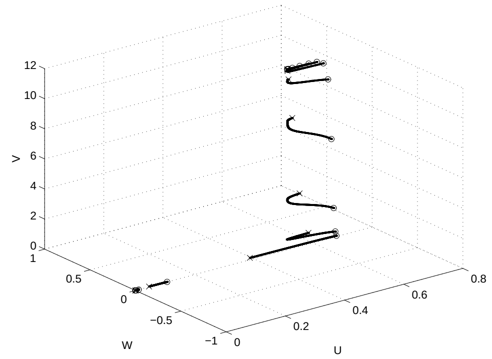
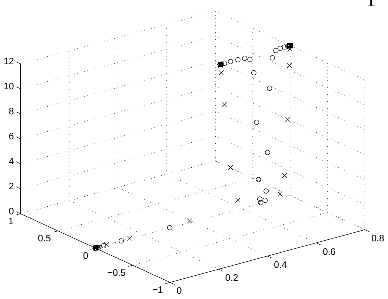


Figure 4: Heteroclinic connections to a period 2 orbit.



Example 5.3 : As a final example of connections for discrete systems we compute the connection between a hyperbolic and a nonhyperbolic fixed point of a map. Consider the map

$$x_{n+1} = \sqrt{\frac{3}{4}} x_n (2 - R_n^2)^{1/2} + \mu z_n^2, \tag{5.6}$$

$$y_{n+1} = \sqrt{\frac{3}{4}} y_n (2 - R_n^2)^{1/2} - \mu z_n^2, \tag{5.7}$$

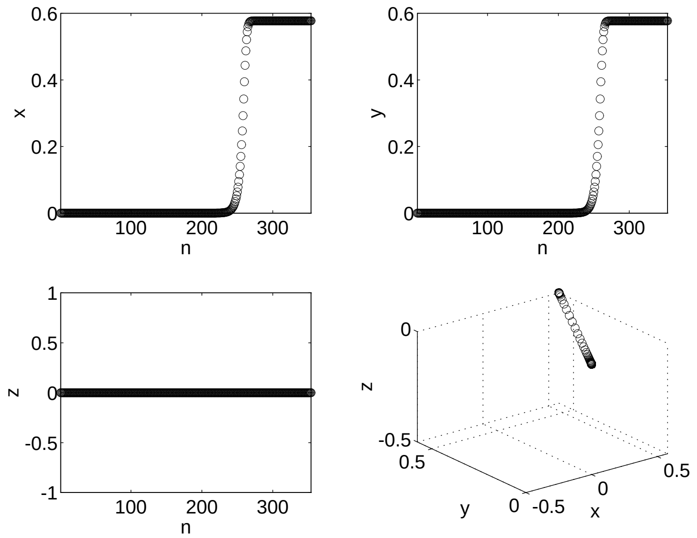
$$z_{n+1} = 2 z_n + \mu (\frac{2}{3} - R_n^2) R_n^2, \tag{5.8}$$

where $R_n^2 = x_n^2 + y_n^2$. We observe that this map has fixed point $(x, y, z) = (0, 0, 0)$ which is hyperbolic and nonhyperbolic fixed points

$$(x, y, z) = \{(x, y, 0) : x^2 + y^2 = \frac{2}{3}\}. \tag{5.9}$$

When $\mu = 0$, the subspace $z_n = 0$ is invariant and the fixed points in (5.9) are stable. Hence we can find a connection from $(0, 0, 0)$ to one of those in (5.9), say U_+, by starting at the unstable manifold of $(0, 0, 0)$ and iterating forward. This connection is used as starting value in a continuation process as μ varies. The projection boundary condition at U_+ asks that the connection be orthogonal to the tangent space at U_+ of the circle of fixed points given by (5.9). Figure 5 shows the connection at $\mu = 0$; Fig. 6 shows that for $\mu = 1.7782$.

Figure 5: The heteroclinic connection at $\mu = 0$



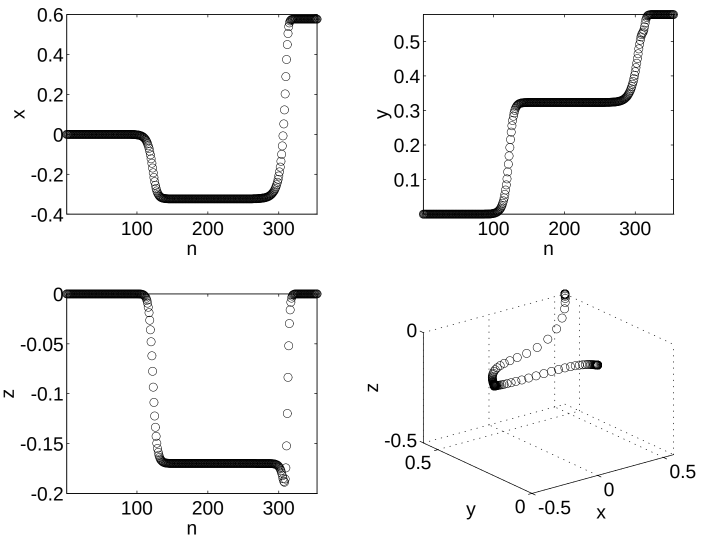
Example 5.4 : Next we discuss the computation of connections between hyperbolic equilibria in an autonomous dynamical system. First consider the well-known Chafee—Infante [Chafee & Infante, 1974]. problem,

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} - f(u), \qquad x \in (0, 1), \quad t > 0, \tag{5.10}$$

$$u(0, t) = u(1, t) = 0, \qquad t > 0, \tag{5.11}$$

$$u(x, 0) = u_0(x), \qquad x \in [0, 1], \tag{5.12}$$

Figure 6: The heteroclinic connection at $\mu = 1.7782$



where it is assumed that

$$f(s) = \sum_{j=0}^{p} b_j s^{2j+1}, \qquad b_p > 0, \qquad b_0 = -1. \tag{5.13}$$

Applying the Galerkin spectral technique gives the following system of ODE's: (see [Bai *et al., 1993] for details)*

$$\frac{dA(t)}{dt} = G(A(t), \gamma), \qquad\qquad A(0) = A_0, \tag{5.14}$$

where

$$A_0 = (a_1^0, a_2^0, \cdots, a_{\mathrm{m}}^0)^T.$$

are coefficients in the spectral expansion of $u_0(x)$. We note here that in this case the projection boundary conditions for the map are precisely the same as those obtained by the method of [Bai et al., 1993] and [Beyn, 1990]. The difference in the methods is only in how the truncated systems are solved. For this example we can use the map in the form $U_{n+1} = S(T_n)U_n$ for varying T_n. Prescribing the values of T_n is precisely the same as prescribing the length of the intervals in a multiple shooting approach to solving the corresponding boundary value problem in [Beyn, 1990]. The heteroclinic connections for this problem are very well known. Theoretical results can be found in for example, [Hale, 1988], [Henry, 1981], or [Temam, 1988] and numerical computation may be found in [Bai et *al., 1993]. The starting value for a connecting orbit is found using the same approach as [Bai et al., 1993]. The results from the approach of this paper agree exactly with what we have presented in [Bai et al., 1993] and are not reproduced here. In fact, if we wished to merely compute connections between equilibria in autonomous systems, we recommend the use of the technique in [Beyn, 1990]. However we use this example to illustrate the effect of the various options for the setting up of the matrix A in* (4.2). *Equation* (5.14) *with* $m = 20$ *was used in a continuation procedure to compute heteroclinic connections between unstable equilibria for* 20 *different values of* μ *with* $\gamma = 130$. *To construct the map we use* $N_- = 0, N_+ = 20$. *In Table 5.1 we list the average time taken to compute one connection using the following five different methods:*

(1) Full Newton solution of (4.1).

(2) Chord-Newton solution of (4.1) *with the Jacobian formed only once.*

(3) As (2) but use the saving strategy given by Remark 4.1 with $TOL = 5 \times 10^{-5}$.

(4) As (3) but with $TOL = 10^{-5}$.

(5) As (3) but with $TOL = 5 \times 10^{-6}$.

Method	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*
Average CPU	*6962.9*	*3861.1*	*2699.5*	*2854.2*	*2925.1*

Table 5.1: *Comparison for different methods*

Computations were done on SGI workstation and CPU time are in seconds. In all cases over 98% *of the cost was taken up by ODE solver, both to evaluate* J_j *blocks and the right hand sides.*

Example 5.5 : *Next we consider the computation of heteroclinic connections in a non-autonomous system. Consider the following problem*

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} - f(u) + \beta u \sin 2p\pi t, \qquad x \in (0,1), \quad t > 0, \tag{5.15}$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{5.16}$$

$$u(x,0) = u_0(x), \qquad x \in [0,1], \tag{5.17}$$

which is a forced Chafee–Infante equation. Here $f(u)$ *is defined as* (5.13), p *is a positive integer,* β *is another parameter. Applying the Galerkin spectral technique to* (5.15)—(5.17), *gives the following system of ODE's:*

$$\frac{dA(t)}{dt} = G(A(t), \gamma) + g(A(t), t, \beta), \qquad A(0) = A_0, \tag{5.18}$$

where G *is as in* (5.14) *and*

$$g(A(t), t, \beta) = (g_1, g_2, \cdots, g_{\mathrm{m}})^T, \qquad g_j = \beta a_j(t) \sin 2p\pi t.$$

At $\beta = 0$, both the steady states and connecting orbits are well known, see Bai et. al. [1993]. The first question to ask is what happen to the steady states as β becomes non-zero. Analytically, $T-$periodic solutions are born from each of the steady states for β small enough. Numerically, we find periodic solutions for values of up to $\beta \approx 0$. Next we can look for connections between those periodic solutions. We note that the invariant subspace technique discussed in [Bai et al., 1993] Sec. 4.2 remains valid for $\beta \neq 0$ in this example to find good starting values for continuation on connections. Again we use $m = 20$, $\gamma = 130$ here and $N_- = 0, N_+ = 20$ to construct the corresponding map. After we find connections for the discrete dynamical system, we could recover the connecting orbits for the system of ODE (5.18) *using the semigroup $S(t)$. Finally solutions for* (5.15)–(5.17) *are recovered by spectral transformation. We denote $U_j^{(k)}$ as the kth element of the vector U_j. The parameter μ in this problem is introduced by*

$$\mu = \sum_{j=\mathrm{N}_-}^{\mathrm{N}_+} U_j^{(3)} * U_j^{(3)}.$$

Figures 7–8 show connections for $\beta = 0.1$. Specifically, in Fig. 7 $\mu = 0.0116$ and in Fig. 8 $\mu = 5.00$. In the latter case we have forced the connection from a equilibrium to periodic orbit to approach closely a second periodic orbit in the same way that heteroclinic connections between equilibria were manipulated in [Bai et al., 1993].

Example 5.6 : *Next we discuss the computation of connections between a steady state and a periodic solution in an autonomous differential equation. Consider the following system,*

$$\begin{aligned}
\frac{dx}{dt} &= \nu x + y - (x+y)(x^2+y^2) + \mu z^2, &(5.19)\\
\frac{dy}{dt} &= -x + \nu y - (y-x)(x^2+y^2) - \mu z^2, &(5.20)\\
\frac{dz}{dt} &= z + \mu(x^2+y^2)[(x^2+y^2) - \mu], &(5.21)
\end{aligned}$$

When $\mu = 0$, the equations decouple and the solutions are easily found with $z = 0$ being invariant subspace. If $\nu > 0$, the set

$$(x, y, z) = \{(x, y, 0) : x^2 + y^2 = \nu\}. \tag{5.22}$$

is a periodic solution of (5.19)—(5.21) *with period $T = 2\pi/|\nu - 1|$. For $0 < \nu < 1$, the periodic solution rotates clockwise on the xy plane; for $\nu > 1$ this periodic solution rotates anti-clockwise, see [Stuart, 1990]. Also $x_- = (x, y, z) = (0, 0, 0)$ is a steady state solution of* (5.19)—(5.21)*. We also note that, restricted to the (x, y) plane, the periodic solution given by* (5.22) *is stable. Hence we could find a connection from x_- to the periodic solution* (5.22) *easily. We remark that x_-*

Figure 7: The heteroclinic connection for forced CI

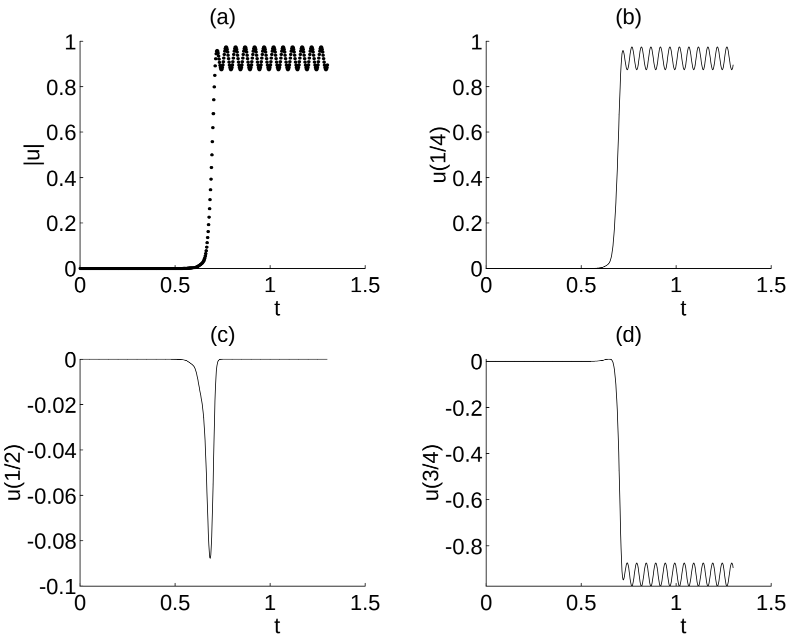
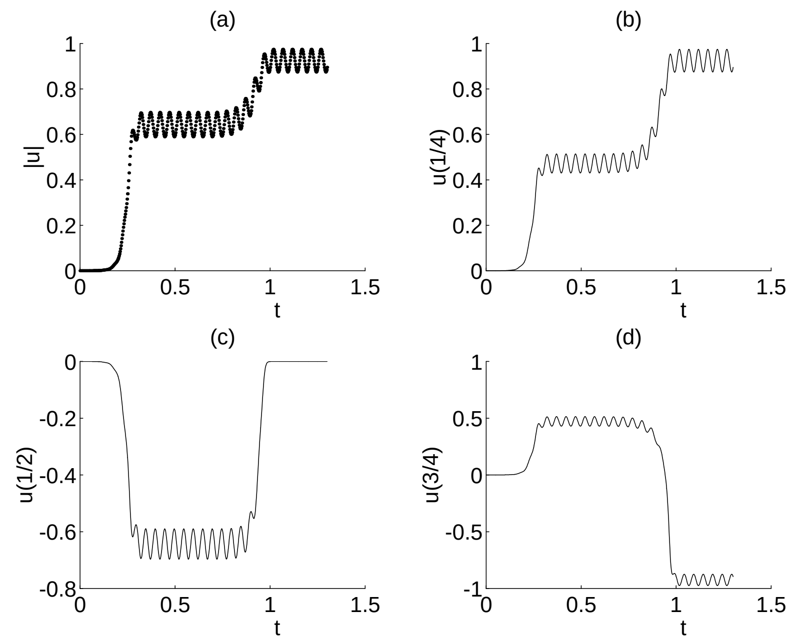


Figure 8: The heteroclinic connection for forced CI



remains a steady state solution and (5.22) *is still a periodic solution of* (5.19)—(5.21) *for any* $\mu \neq 0$. *Also their stability properties do not change as* μ *changes. Starting at the connection for* $\mu = 0$, *we carry out continuation on* μ *to get connecting orbits for larger* μ *values. Figure 9 shows the connecting orbit at* $\mu = 0$; *and Fig. 10 shows the connections for* $\mu = 0.91$ *which is no longer inside the invariant subspace. Thus we have computed a connection in* (5.19)–(5.21) *from an unstable steady state to an unstable periodic orbit.*

Figure 9: The heteroclinic connection at $\mu = 0$

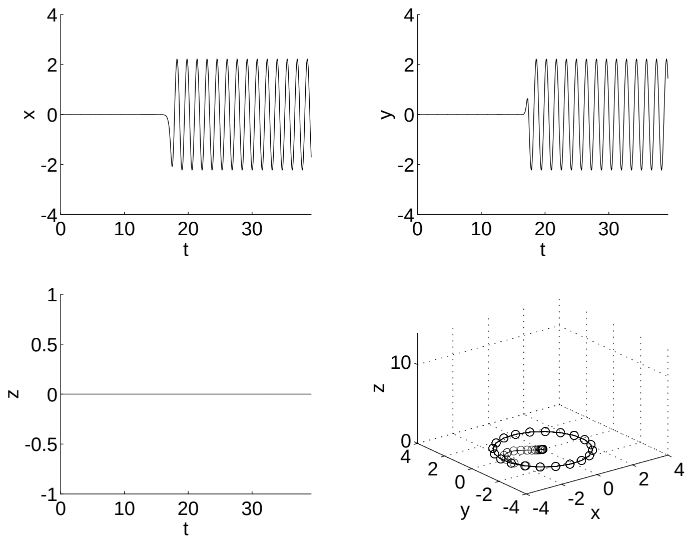


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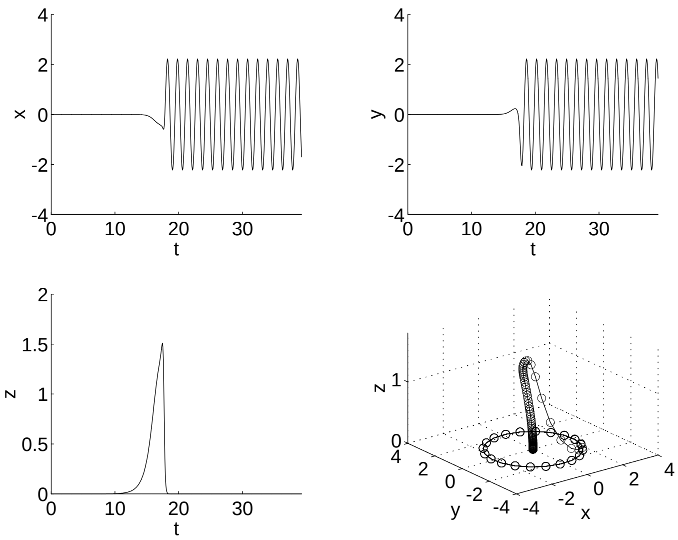
Figure 10: The heteroclinic connection at $\mu = 0.91$



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