The Effect of Small Delays in the Feedback Loop on the Stability of Neutral Systems*

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Abstract: It is well-known that exponential stabilization of a neutral system with unstable difference operator is only possible by allowing for control laws containing derivative feedback. We show that closed-loop stability of a neutral system with unstable open-loop difference operator obtained by applying a derivative feedback scheme is extremely sensitive to arbitrarily small time-delays in the feedback loop.

Keywords: Neutral systems; Functional difference systems; Exponential stability; Small time delays; Robustness.

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1. Introduction

The phenomenon of destabilization of feedback systems by arbitrarily small delays in the loop has been well-known for many years. To the best of our knowledge the paper [1] by Barman *et al.* is the first one devoted to this topic. More recently, researchers working in control of partial differential equations, 'rediscovered' the destabilizing effect of small delays in various examples involving vibrating systems, see for example Datko [5, 6], Datko *et al.* [7] and Desch and Wheeler [8]. Whilst these paper are based on partial differential equation and related techniques, a frequency-domain point of view is taken in [1]. The approach developed by Logemann and Rebarber [18] and Logemann *et al.* [19] is similar in spirit to that in [1], but is not tied to the restrictions imposed in [1] such as the assumption that the open-loop transfer function has at most finitely many poles in the closed right-half plane. In this paper we show, using results from [19], that small delays in the feedback loop can also have a destabilizing effect on certain neutral functional differential equations.

The problems of stability, stabilizability and stabilization for neutral systems has received considerable attention in the last 15 years, see Byrnes *et al.* [2], Hale and Verduyn Lunel [10], Kamen *et al.* [12], Logemann [15, 16], Logemann and Pandolfi [17], O'Connor and Tarn [21], Salamon [22] and Spong [24], to mention just a few references. For neutral systems the problem of feedback stabilization is in general considerably more difficult than for retarded systems. The reason for this is the fact that a neutral system with unstable difference operator possesses at least one unstable infinite root chain. A simultaneous shifting of these eigenvalues to a 'stable' region of the form $\operatorname{Re} s \leq \alpha$, for some $\alpha < 0$, requires a change of the associated functional difference equation. Hence exponential stabilization of a neutral system with unstable difference operator is only possible by allowing for control schemes containing derivative feedback (this important fact has first been recognized in [21] for systems with a single point delay). Our main results shows that if a neutral system with unstable difference operator can be exponentially stabilized by a state feedback of the form

$$u(t) = F[\dot{x}(\tau), x(\tau); t - h \leq \tau \leq t]\,,$$

then there exist time delays ε_i with $\varepsilon_i > 0$ and $\lim_{i\to\infty} \varepsilon_i = 0$ such that for all $i \in \mathbb{N}$ the delayed feedback

$$u(t) = F[\dot{x}(\tau), x(\tau); t - \varepsilon_i - h \leq \tau \leq t - \varepsilon_i]$$

leads to a closed-loop system which has an unstable infinite root chain.

The paper is organized as follows. In Section 2 we present some preliminaries and state two results from [19] which are needed later in the paper. In Section 3 we prove a destabilization result for a class of feedback controlled functional difference equations and discuss how it is related to a well-known sensitivity result in the literature [10]. Finally, in Section 4 we apply the result of Section 3 to neutral systems.

2. Destabilization by small delays in the frequency domain

For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we define $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$ and

$$\mathbb{C}_{[\alpha,\beta]} := \{s \in \mathbb{C} \,|\, \alpha \leq \mathrm{Re}\, s \leq \beta\}, \qquad \mathbb{C}_{(\alpha,\beta)} := \{s \in \mathbb{C} \,|\, \alpha < \mathrm{Re}\, s < \beta\}.$$

The field of all meromorphic functions on $\mathbb{C}_\alpha$ is denoted by M_α, while H^∞_α denotes the algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$. We write H^∞ for H^∞_0. Let $\Omega \subset \mathbb{C}$. A function $\mathbf{H} : \Omega \to \mathbb{C}^{m\times m}$ is called a ($\mathbb{C}^{m\times m}$-valued) *transfer function* if there exists $\alpha \in \mathbb{R}$ such that $\mathbb{C}_\alpha \subset \Omega$ and $\mathbf{H}|_{\mathbb{C}_\alpha} \in M_\alpha^{m\times m}$.

Let $\mathbf{H}$ be a transfer function and consider the feedback system shown in Figure 2.1, where u is the input function, y is the output function and the block with transfer function $e^{-\varepsilon s}$ represents a delay of length $\varepsilon \geq 0$

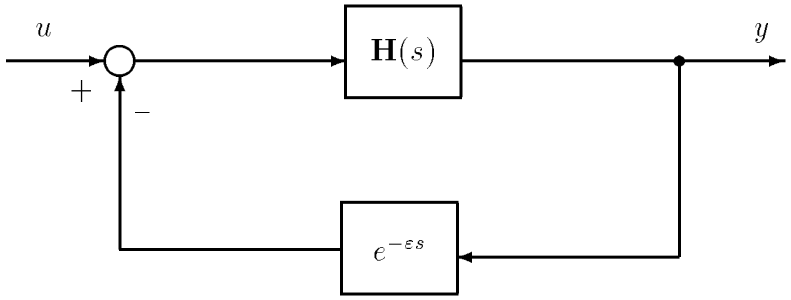

Figure 2.1: Feedback system with delay

If $\det(I + e^{-\varepsilon s}\mathbf{H}(s)) \not\equiv 0$, then the function $\mathbf{G}^\varepsilon$ defined by

$$\mathbf{G}^\varepsilon(s) := \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1} \tag{2.1}$$

is a transfer function, the so-called closed-loop transfer function of the feedback system shown in Figure 2.1.

As usual we say that $\mathbf{G}^\varepsilon$ is L^2-*stable* if $\mathbf{G}^\varepsilon \in (H^\infty)^{m\times m}$. It is well known that L^2-stability of $\mathbf{G}^\varepsilon$ is equivalent to the existence of a constant $K > 0$ such that

$$\|y\|_{L^2(0,\infty)} \leq K\|u\|_{L^2(0,\infty)} \qquad \text{for all} \ \ u \in L^2(0,\infty;\mathbb{C}^m),$$

whenever u and y are related as in Figure 2.1. It is easy to see that if $\mathbf{G}^{\varepsilon_0}$ is L^2-stable for some $\varepsilon_0 \in [0,\infty)$, then $\mathbf{H} \in M_0^{m\times m}$.

Definition 2.1 Let $\mathbf{H}$ be a transfer function. $\mathbf{H}$ is called *well-posed* if $\mathbf{H} \in (H^\infty_\alpha)^{m\times m}$ for some $\alpha \in \mathbb{R}$. Moreover, $\mathbf{H}$ is called *regular* if it is well-posed and if the limit $\lim_{\xi\to+\infty} \mathbf{H}(\xi)$ exists (where $\xi \in \mathbb{R}$). This limit is called the *feedthrough matrix* of $\mathbf{H}$. $\diamond$

Well-posed transfer functions correspond to shift-invariant operators on $L^2_{loc}(0,\infty;\mathbb{C}^m)$ with finite growth bound, see Weiss [26], section 2. In particular, the transfer function of any abstract linear system (in the sense of Weiss [25]) is well-posed, as follows from [26], proposition 4.1. Conversely, for any well-posed transfer function $\mathbf{H}$ we can find an abstract linear system whose transfer function is $\mathbf{H}$, as follows from results in Salamon [23].

Practically all well-posed transfer functions of interest are regular. (In fact, it is a non-trivial exercise in complex analysis to construct an example of a well-posed and non-regular transfer function.) If the transfer function of an abstract linear system is regular (such systems are called regular), then the system has a simple and convenient state-space representation, analogous to a finite-dimensional state space system, see [26], section 2.

For a transfer function $\mathbf{H}$ let $\mathfrak{P}_{\mathbf{H}}$ denote the set of poles of $\mathbf{H}$. We define

$$\gamma(\mathbf{H}) := \limsup_{|s|\to\infty,\, s\in\mathbb{C}_0\setminus\mathfrak{P}_{\mathbf{H}}} r(\mathbf{H}(s))\,, \tag{2.2}$$

where $r(\mathbf{H}(s))$ denotes the spectral radius of $\mathbf{H}(s)$.

The following destabilization result for regular transfer functions was proved by Logemann *et al.* [19].

Theorem 2.2 *Let $\mathbf{H}$ be a regular transfer function. If $\mathbf{G}^0$ is L^2-stable, and if $\gamma(\mathbf{H}) > 1$, then there exists sequences (ε_i) and (s_i) with*

$$\varepsilon_i > 0\,, \quad \lim_{i\to\infty} \varepsilon_i = 0\,, \quad s_i \in \mathbb{C}_0\,, \quad \lim_{i\to\infty} |\mathrm{Im}\, s_i| = \infty\,,$$

so that for all $i\in\mathbb{N}$, s_i is a pole of $\mathbf{G}^{\varepsilon_i}$.

Moreover, the instability of the transfer functions $\mathbf{G}^{\varepsilon_i}$ is robust in the sense that there exist numbers $\delta_i \in (0,\varepsilon_i)$ such that $\mathbf{G}^{\varepsilon}$ has at least one pole in $\mathbb{C}_0$ for all $\varepsilon \in \bigcup_{i\in\mathbb{N}} (\varepsilon_i - \delta_i, \varepsilon_i + \delta_i)$.

It was also shown in [19] that if $\mathbf{H}$ is regular, then L^2-stability of $\mathbf{G}^0$ is robust with respect to small delays in the loop (*i.e.* there exists $\varepsilon^* > 0$ such that $\mathbf{G}^{\varepsilon} \in (H^\infty)^{m\times m}$ for all $\varepsilon \in (0,\varepsilon^*)$), provided that $\gamma(\mathbf{H}) < 1$.

There are many partial differential equation models of physical and technical systems which have transfer functions which are not well-posed, and hence are not regular. It has been shown by Logemann and Rebarber [18] that if $\mathbf{H}$ is not well-posed, then the robustness properties of $\mathbf{G}^0$ with respect to small delays are extremely poor, in fact worse than in the case when $\mathbf{H}$ is regular. However, neutral systems have regular transfer functions, and thus, for the pupose of this paper, we do not need to consider the results of [18] in more detail.

We close this section with the statement of a simple lemma which has been proved in [19].

Lemma 2.3 *Let $U \subset \mathbb{C}$ and $\mathbf{H}$ be a transfer function and suppose that $\mathbf{G}^0$ is bounded and holomorphic on U. If $\sup_{s\in U} r(\mathbf{H}(s)) < \infty$, then $\sup_{s\in U} \|\mathbf{H}(s)\| < \infty$.*

3. A destabilization result for functional difference systems

Consider a functional difference equation of the form

$$x(t) = \sum_{i=1}^{\infty} D_i x(t-h_i)\,, \qquad t \geq 0\,, \tag{3.1}$$

where $0 < h_i \leq h$, for some $h > 0$, and the matrices $D_i \in \mathbb{R}^{n\times n}$ satisfy $\sum_{i=1}^{\infty} \|D_i\| < \infty$. Defining the bounded linear operator

$$\mathcal{D}_d: \quad C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n\,, \qquad \psi \mapsto \psi(0) - \sum_{i=1}^{\infty} D_i \psi(-h_i)\,, \tag{3.2}$$

and setting $x_t(\tau) = x(t+\tau)$ for $\tau \in [-h,0]$, equation (3.1) can be written as

$$\mathcal{D}_d x_t = 0\,, \qquad t \geq 0\,. \tag{3.3}$$

An operator of the form (3.2) is called a *difference operator*. The concept of a solution for (3.1) or, equivalently, for (3.3) must include initial functions defined on $[-h,0]$. As the set of all admissible initial functions we choose

$$C_{\mathcal{D}_d} := \{\psi \in C([-h,0],\mathbb{R}^n) \mid \mathcal{D}_d \psi = 0\}\,.$$

It is well-known (see Hale and Verduyn Lunel [10], pp. 274) that for any initial function $\psi \in C_{\mathcal{D}_d}$, (3.3) admits a unique solution $x(\cdot,\psi) \in C([-h,\infty),\mathbb{R}^n)$ satisfing $x_0(\cdot,\psi) = \psi$. Moreover, the family of operators defined by $S_d(t)\psi = x_t(\cdot,\psi)$, $t \geq 0$, defines a strongly continuous semigroup on $C_{\mathcal{D}_d}$. Setting

$$\mathbf{\Delta}_d(s) := I - \sum_{i=1}^{\infty} D_i e^{-h_i s}\,, \tag{3.4}$$

we have that the exponential growth of $S_d(t)$ is given by

$$\omega_d := \sup\{\operatorname{Re} s \mid \det \mathbf{\Delta}_d(s) = 0\}\,.$$

The difference operator $\mathcal{D}_d$ is called *stable* if $\omega_d < 0$.

Consider the controlled functional difference equation

$$\mathcal{D}_d x_t = Bu(t)\,, \qquad \text{where} \quad B \in \mathbb{R}^{n\times m}\,, \tag{3.5}$$

and apply the feedback $u(t) = \mathcal{F}_d x_{t-\varepsilon}$, where $\varepsilon \geq 0$ and

$$\mathcal{F}_d: \quad C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n\,, \qquad \psi \mapsto \sum_{i=1}^{\infty} F_i \psi(-k_i)\,. \tag{3.6}$$

Here $0 < k_i \le h$ and the feedback matrices $F_i \in \mathbb{R}^{m\times n}$ satisfy $\sum_{i=1}^{\infty} \|F_i\| < \infty$. The resulting closed-loop difference operator $\mathcal{D}_d^\varepsilon$ is given by

$$\mathcal{D}_d^\varepsilon : \quad C([-h-\varepsilon,0],\mathbb{R}^n) \to \mathbb{R}^n, \qquad \psi \mapsto \psi(0) - \sum_{i=1}^{\infty} D_i\psi(-h_i) - B\sum_{i=1}^{\infty} F_i\psi(-k_i-\varepsilon)\,. \tag{3.7}$$

Clearly, $\mathcal{D}_d^\varepsilon$ is stable if

$$\omega_d^\varepsilon := \sup\{\operatorname{Re} s \mid \det \mathbf{\Delta}_d^\varepsilon(s) = 0\} < 0\,,$$

where

$$\mathbf{\Delta}_d^\varepsilon(s) = I - \sum_{i=1}^{\infty} D_i e^{-h_i s} - e^{-\varepsilon s} B \sum_{i=1}^{\infty} F_i e^{-k_i s}\,. \tag{3.8}$$

We can now formulate the main result of this section.

Theorem 3.1 *Suppose that $\omega_d \ge 0$ and $\omega_d^0 < 0$. Then there exist sequences (ε_i) and (s_i) with $\varepsilon_i > 0$, $\lim_{i\to\infty}\varepsilon_i = 0$, $s_i \in \mathbb{C}_0$ and $\lim_{i\to\infty} |\operatorname{Im} s_i| = \infty$, so that for all $i \in \mathbb{N}$*

$$\det \mathbf{\Delta}_d^{\varepsilon_i}(s_i) = 0\,,$$

i.e. for all $i \in \mathbb{N}$ we have that $\omega_d^{\varepsilon_i} > 0$. Moreover, there exist numbers $\delta_i \in (0,\varepsilon_i)$ such that $\det \mathbf{\Delta}_d^\varepsilon$ has at least one zero in $\mathbb{C}_0$ for all $\varepsilon \in \bigcup_{i\in\mathbb{N}}(\varepsilon_i - \delta_i, \varepsilon_i + \delta_i)$.

Before we prove Theorem 3.1 we make some comments concerning its interpretation and give a short discussion of related results in the literature.

Remark 3.2 (i) The assumption $\omega_d \ge 0$ says that the open-loop system is unstable, whilst $\omega_d^0 < 0$ means that the feedback $u(t) = \mathcal{F}_d x_t$ is stabilizing. Neccessary and sufficient conditions for the existence of a stabilizing feedback operator $\mathcal{F}_d$ in terms of $\operatorname{rank}(\mathbf{\Delta}_d(s), B)$ have been given in special cases by Salamon [22] and by Spong [24]. In [22] a difference operator $\mathcal{D}_0$ with finitely many commensurable delays h_i is considered. The case of finitely many noncommensurable delays is studied in [24] using a stronger concept of stability. The conclusion of Theorem 3.1 is that closed-loop stability is not robust with respect to small delays in the feedback loop, *i.e.* closed-loop stability can be destroyed by arbitrarily small delays in the feedback loop.

(ii) Let $h_i \in [0,\infty)$ and $D_i \in \mathbb{R}^{n\times n}$ for $i = 1,\dots,\ell$, and set $h := \max_{1\le i\le \ell} h_i$, $\vec{h} := (h_1,\dots,h_\ell)$ and $\vec{D} := (D_1,\dots,D_\ell)$. It is well-known that the stability of a 'finite' difference operator of the form

$$\mathcal{D}(\vec{h},\vec{D}) : \quad C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n, \qquad \psi \mapsto \psi(0) - \sum_{i=1}^{\ell} D_i\psi(-h_i)$$

can be very sensitive to small changes in the delay parameters h_i, see Carvalho [3], Hale [9], Hale and Verduyn Lunel [10] and Melvin [20]. The main result is that the stability of $\mathcal{D}(\vec{h},\vec{D})$ is insensitive to small perturbations in the parameters h_i if and only if

$\mathcal{D}(\vec{h},\vec{D})$ is *strongly stable, i.e.* $\mathcal{D}(\vec{h},\vec{D})$ is stable for each $\vec{h} \in [0,\infty)^\ell$, see [9] and [10]. However, Theorem 3.1 does not follow from this result, since absence of strong stability does not guarantee that $\mathcal{D}_d^0$ can be destabilized by the highly structured one-parametric perturbation $k_i \rightsquigarrow k_i + \varepsilon$.

(iii) For 'finite' difference operators it is shown in [22] that in the case of independent delays in the given equation and in the feedback loop, any control law of the form $u(t) = \mathcal{F}_d x_t$ leads to a degradation in the strong stability behaviour of the closed-loop system. However, this does not imply that the closed-loop system is actually destabilized by arbitrarily small delays in the feedback loop. $\diamond$

Proof of Theorem 3.1: We proceed in several steps.

STEP 1: For the controlled system (3.5) we introduce the following 'artificial' output

$$y(t) = -\mathcal{F}_d x_t\,. \tag{3.9}$$

Setting

$$\mathbf{F}_d(s) := \sum_{i=1}^{\infty} F_i e^{-k_i s}$$

we obtain the following expression for the transfer function $\mathbf{H}$ of the controlled and observed system given by (3.5) and (3.9)

$$\mathbf{H}(s) = -\mathbf{F}_d(s)\boldsymbol{\Delta}_d^{-1}(s)B\,. \tag{3.10}$$

It is easy to see that $\mathbf{H}$ is a regular transfer function with feedthrough matrix $0_{m\times m}$.

Consider the delayed negative output feedback

$$u(t) = v(t) - y(t-\varepsilon)\,,$$

where v is the external input for the closed-loop system. In the frequency-domain the relation between v and the closed-loop output y is given by the closed-loop transfer function

$$\mathbf{G}^\varepsilon(s) = \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1} = -\mathbf{F}_d(s)\left(\boldsymbol{\Delta}_d(s) - e^{-\varepsilon s}B\mathbf{F}_d(s)\right)^{-1} B\,. \tag{3.11}$$

Using (3.8), it follows that

$$\mathbf{G}^\varepsilon(s) = -\mathbf{F}_d(s)\left(\boldsymbol{\Delta}_d^\varepsilon(s)\right)^{-1} B\,. \tag{3.12}$$

STEP 2: We show that $\mathbf{G}^0 \in (H^\infty)^{m\times m}$. By combining (3.12) with Cramer's rule we obtain

$$\mathbf{G}^0 = -\mathbf{F}_d\left(\frac{1}{\det \boldsymbol{\Delta}_d^0}\,\mathrm{adj}\,\boldsymbol{\Delta}_d^0\right)B\,, \tag{3.13}$$

where adj M denotes the adjugate of the matrix M. It follows from (3.8) and the summability of the sequences (D_i) and (F_i) that there exists a number $\beta > 0$ such that

$$\inf_{s\in\mathbb{C}_\beta} |\det \mathbf{\Delta}_d^0(s)| > 0\,. \tag{3.14}$$

Since by assumption $\omega_d^0 < 0$, there exists a constant $\alpha < 0$ such that $\det \mathbf{\Delta}_d^0(s) \neq 0$ for all $s \in \mathbb{C}_{[\alpha,\beta]}$. Now $\det \mathbf{\Delta}_d^0(s)$ is an almost periodic function in every vertical strip in the complex plane (cf. Corduneanu [4], pp. 73), and therefore, by a result in Levin's book [14], pp. 268, we obtain that

$$\inf_{s\in\mathbb{C}_{[\alpha,\beta]}} |\det \mathbf{\Delta}_d^0(s)| > 0\,. \tag{3.15}$$

Obviously $\mathbf{\Delta}_d^0$ is in $(H^\infty)^{n\times n}$, and combining (3.14) and (3.15) shows that $\mathbf{\Delta}_d^0$ is a unimodular matrix in $(H^\infty)^{n\times n}$, *i.e.*

$$\inf_{s\in\mathbb{C}_0} |\det \mathbf{\Delta}_d^0(s)| > 0\,. \tag{3.16}$$

Since $\mathbf{F}_d \in (H^\infty)^{m\times n}$ and $\operatorname{adj}\mathbf{\Delta}_d^0 \in (H^\infty)^{n\times n}$ it follows from (3.13) and (3.16) that $\mathbf{G}^0 \in (H^\infty)^{m\times m}$.

STEP 3: We claim that there exists a sequence (s_i) in $\mathbb{C}_0^{cl}$ with $\lim_{i\to\infty} |\operatorname{Im} s_i| = \infty$, such that

$$\lim_{i\to\infty} \det \mathbf{\Delta}_d(s_i) = 0\,. \tag{3.17}$$

To this end we distinguish between two cases.
CASE (A): If there exists $z \in \mathbb{C}_0^{cl}$ such that $\det \mathbf{\Delta}_d(z) = 0$, then the claim follows from the almost periodicity of $\mathbf{\Delta}_d$.
CASE (B): If $\det \mathbf{\Delta}_d(s) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$, then, by the assumption that $\omega_d \geq 0$, there exists a sequence $z_i \in \mathbb{C}$ with $\operatorname{Re} z_i < 0$, $\lim_{i\to\infty} \operatorname{Re} z_i = 0$, $\lim_{i\to\infty} \operatorname{Im} z_i = \infty$ and such that for all $i \in \mathbb{N}$

$$\det \mathbf{\Delta}_d(z_i) = 0\,. \tag{3.18}$$

Now for any $\alpha < \beta$, $\det \mathbf{\Delta}_d(s)$ is holomorphic and bounded on the open vertical strip $\mathbb{C}_{(\alpha,\beta)}$, and thus, by a result in [4], p. 72, $\det \mathbf{\Delta}_d(s)$ is uniformly continuous on any closed vertical strip $\mathbb{C}_{[\alpha,\beta]}$. Therefore it follows from (3.18) and the fact that $\lim_{i\to\infty} \operatorname{Re} z_i = 0$ that (3.17) holds true with $s_i = \imath \operatorname{Im} z_i$, where $\imath$ denotes the imaginary unit.

STEP 4: Our aim is to show that $\gamma(\mathbf{H}) = \infty$, where $\gamma(\mathbf{H})$ is defined by (2.2). By STEP 3 there exists a sequence (s_i) in $\mathbb{C}_0^{cl}$ with $\lim_{i\to\infty} |\operatorname{Im} s_i| = \infty$ and such that

$$\lim_{i\to\infty} \|\mathbf{\Delta}_d^{-1}(s_i)\| = \infty\,. \tag{3.19}$$

Setting $\mathbf{X} := (\mathbf{\Delta}_d^0)^{-1}$ it follows that $\mathbf{\Delta}_d(s)\mathbf{X}(s) - B\mathbf{F}_d(s)\mathbf{X}(s) = I$ and hence

$$\mathbf{X}(s) - \mathbf{\Delta}_d^{-1}(s)B\mathbf{F}_d(s)\mathbf{X}(s) = \mathbf{\Delta}_d^{-1}(s)\,.$$

Now, by (3.16), the entries of $\mathbf{X}(s)$ are in H^∞, and so are the entries of $\mathbf{F}_d(s)$. Combining the above equation with (3.19) we obtain that

$$\lim_{i\to\infty} \|\mathbf{\Delta}_d^{-1}(s_i)B\| = \infty\,. \tag{3.20}$$

Moreover, multiplying the identity $\mathbf{X}(s)\mathbf{\Delta}_d(s) - \mathbf{X}(s)B\mathbf{F}_d(s) = I$ from the right by $\mathbf{\Delta}_d^{-1}(s)B$ and using (3.10) leads to

$$\mathbf{X}(s)B + \mathbf{X}(s)B\mathbf{H}(s) = \mathbf{\Delta}_d^{-1}(s)B\,.$$

It follows from the boundedness of $\mathbf{X}(s)$ and from (3.20) that

$$\lim_{i\to\infty} \|\mathbf{H}(s_i)\| = \infty\,,$$

and thus, by Lemma 2.3

$$\lim_{i\to\infty} r(\mathbf{H}(s_i)) = \infty\,.$$

STEP 5: We have shown that $\mathbf{G}^0$ is L^2-stable (STEP 2) and that $\gamma(\mathbf{H}) > 1$ (STEP 4), and hence we may apply Theorem 2.2 to obtain that there exists sequences (ε_i) and (s_i) with $\varepsilon_i > 0$, $\lim_{i\to\infty} \varepsilon_i = 0$, $s_i \in \mathbb{C}_0$ and $\lim_{i\to\infty} |\mathrm{Im}\, s_i| = \infty$ and such that for any $i \in \mathbb{N}$, s_i is a pole of the transfer function $\mathbf{G}^{\varepsilon_i}$ given by (3.11). Consequently, using (3.12), s_i is a pole of $(\mathbf{\Delta}_d^{\varepsilon_i})^{-1}$ and hence a zero of $\det \mathbf{\Delta}_d^{\varepsilon_i}$. Moreover, it follows from Theorem 2.2 that there exist numbers $\delta_i \in (0, \varepsilon_i)$ such that $\det \mathbf{\Delta}_d^{\varepsilon}$ has at least one zero in $\mathbb{C}_0$ for all $\varepsilon \in \bigcup_{i\in\mathbb{N}} (\varepsilon_i - \delta_i, \varepsilon_i + \delta_i)$. $\square$

4. An application to neutral systems

Consider the neutral system

$$\frac{d}{dt}\mathfrak{D}x_t = \mathfrak{L}x_t\,, \qquad x_0 = \psi \in C([-h,0],\mathbb{R}^n)\,, \tag{4.1}$$

where $\mathfrak{D}$ and $\mathfrak{L}$ are linear bounded operators from $C([-h,0],\mathbb{R}^n)$ to $\mathbb{R}^n$. We asssume that $\mathfrak{D}$ is atomic at 0 (cf. [10], p. 52). Using the Riesz representation theorem, we write

$$\mathfrak{D}\psi = \psi(0) - \int_{-h}^0 d\Delta(\tau)\psi(\tau)\,, \quad \mathfrak{L}\psi = \int_{-h}^0 d\Lambda(\tau)\psi(\tau)\,,$$

where Δ, $\Lambda \in BV([-h,0],\mathbb{R}^{n\times n})$ and $\mathrm{Var}_{[\tau,0]}\Delta \to 0$ as $\tau \to 0$. We define

$$\mathbf{D}(s) := I - \int_{-h}^0 d\Delta(\tau)e^{s\tau}\,, \quad \mathbf{L}(s) := \int_{-h}^0 d\Lambda(\tau)e^{s\tau}\,, \quad \text{and} \quad \mathbf{\Delta}(s) := s\mathbf{D}(s) - \mathbf{L}(s)\,.$$

The function $\det \mathbf{\Delta}(s)$ is called the *characteristic function* of (4.1).

It is well known that any function of bounded variation can be written as the sum of an absolutely continuous function, a singular function and a jump function (cf. for example Kolmogorov and Fomin [13], p. 341). We will need the following assumption

(**NS**) The function Δ does not have a singular part.

If the assumption (**NS**) holds, then $\mathcal{D} = \mathcal{D}_d - \mathcal{D}_a$, where $\mathcal{D}_d$ is given by (3.2) and

$$\mathcal{D}_a : \quad C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n, \qquad \psi \mapsto \int_{-h}^{0} D_a(\tau)\psi(\tau)\,d\tau\,,$$

with $D_a \in L^1([-h,0],\mathbb{R}^{n\times n})$. Correspondingly, using (3.4), we obtain

$$\mathbf{D}(s) = \mathbf{\Delta}_d(s) - \int_{-h}^{0} D_a(\tau)e^{s\tau}\,d\tau\,.$$

The neutral system (4.1) defines a strongly continuous semigroup $S(t)$ on $C([-h,0],\mathbb{R}^n)$, see [10], p. 263. Under the assumption (**NS**), $S(t)$ satisfies the spectrum determined growth assumption and the exponential growth constant ω of $S(t)$ is given by

$$\omega = \sup\{\operatorname{Re} s \mid \det \mathbf{\Delta}(s) = 0\}\,,$$

see Henry [11] (cf. also [10], p. 269).

The following well-known lemma (cf. [22], p. 160) shows that the stability behaviour of (4.1) and its associated functional difference equation $\mathcal{D}_d x_t = 0$ are closely related.

Lemma 4.1 *Suppose that* (**NS**) *holds and let $\alpha < \beta$ be given. Then the following statements are equivalent:*

(i) There exists $z \in \mathbb{C}_{(\alpha,\beta)}$ such that $\det \mathbf{\Delta}_d(z) = 0$.

(ii) There exists some $\delta > 0$ and a sequence (s_i) such that $s_i \in \mathbb{C}_{[\alpha+\delta,\beta-\delta]}$, $\lim_{i\to\infty} |\operatorname{Im} s_i| = \infty$ *and* $\det \mathbf{\Delta}(s_i) = 0$ *for all $i \in \mathbb{N}$.*

The above lemma says, in particular, that if z is a zero of $\det \mathbf{\Delta}_d$, then there exists a sequence (s_i) of zeros of $\det \mathbf{\Delta}$ such that $\lim_{i\to\infty} \operatorname{Re} s_i = \operatorname{Re} z$ and $\lim_{i\to\infty} |\operatorname{Im} s_i| = \infty$. Such a sequence is called an *infinite root chain* of $\det \mathbf{\Delta}$ (or of (4.1)). The infinite root chain (s_i) is called *exponentially unstable* if $\operatorname{Re} z > 0$.

Let us consider the controlled neutral system

$$\frac{d}{dt}\mathcal{D}x_t = \mathcal{L}x_t + Bu(t)\,, \qquad \text{where} \quad B \in \mathbb{R}^{n\times m}\,.$$

Since Lemma 4.1 implies that $\omega \geq \omega_d$, it follows that in the case $\omega_d \geq 0$ exponential stabilization by state feedback requires a stabilization of the difference operator $\mathcal{D}_d$. This is only possible by allowing for derivative feedback, and therefore we consider control laws of the form

$$u(t) = \mathcal{F}_d \dot{x}_{t-\varepsilon} + \mathcal{F}_a \dot{x}_{t-\varepsilon} + \mathcal{F} x_{t-\varepsilon}\,, \tag{4.2}$$

where ε represents a time-delay in the feedback loop. Here $\mathcal{F}_d$ is given by (3.6) and $\mathcal{F}_a$ and $\mathcal{F}$ are bounded linear operators from $C([-h,0],\mathbb{R}^n)$ to $\mathbb{R}^m$ given by

$$\mathcal{F}_a\psi = \int_{-h}^{0} F_a(\tau)\psi(\tau)\,d\tau\,, \qquad \mathcal{F}\psi = \int_{-h}^{0} d\Phi(\tau)\psi(\tau)\,,$$

respectively, where $F_a \in L^1([-h,0],\mathbb{R}^{m\times n})$ and $\Phi \in BV([-h,0],\mathbb{R}^{m\times n})$. We define

$$\mathbf{F}(s) := \int_{-h}^{0} d\Phi(\tau)e^{s\tau}\,.$$

The difference operator $\mathcal{D}_d^\varepsilon$ of the closed-loop system is then given by (3.7) and for the exponential growth rate ω^ε of the closed-loop neutral system we have

$$\omega^\varepsilon = \sup\{\operatorname{Re} s \mid \det \boldsymbol{\Delta}^\varepsilon(s) = 0\}\,,$$

where

$$\boldsymbol{\Delta}^\varepsilon(s) := s\left(\boldsymbol{\Delta}_d^\varepsilon(s) - \int_{-h}^{0} D_a(\tau)e^{s\tau}d\tau - e^{-\varepsilon s}\int_{-h}^{0} F_a(\tau)e^{s\tau}d\tau\right) - \mathbf{L}(s) - e^{-\varepsilon s}\mathbf{F}(s)\,,$$

with $\boldsymbol{\Delta}_d^\varepsilon$ given by (3.8).

A combination of Lemma 4.1 and Theorem 3.1 yields the following result.

Corollary 4.2 *Suppose that* (**NS**) *holds. If $\omega_d \geq 0$ and $\omega^0 < 0$, then there exists a sequence (ε_i) with $\varepsilon_i > 0$ and $\lim_{i\to\infty} \varepsilon_i = 0$ and such that for any $i \in \mathbb{N}$, the closed-loop characteristic function* $\det \boldsymbol{\Delta}^{\varepsilon_i}$ *has an exponentially unstable infinite root chain. In particular, $\omega^{\varepsilon_i} > 0$ for all $i \in \mathbb{N}$. Moreover, there exist numbers $\delta_i \in (0, \varepsilon_i)$ such that* $\det \boldsymbol{\Delta}^\varepsilon$ *has at least one exponentially unstable infinite root chain for all $\varepsilon \in \bigcup_{i\in\mathbb{N}}(\varepsilon_i - \delta_i, \varepsilon_i + \delta_i)$.*

We close this section with a simple example.

Example 4.3 Consider the neutral system

$$\dot{x}(t) - D\dot{x}(t-h) = Lx(t) + Mx(t-k) + u(t)\,, \tag{4.3}$$

where $D, L, M \in \mathbb{R}$ and $h, k > 0$. Clearly, $\boldsymbol{\Delta}_d(s) = 1 - De^{-hs}$, and hence, if $D \geq 1$, the difference equation $x(t) - Dx(t-h) = 0$ is unstable, and so is the neutral system (4.3). An application of the control law

$$u(t) = -D'\dot{x}(t-h) - L'x(t)$$

leads to the following closed-loop system

$$\dot{x}(t) - (D - D')\dot{x}(t-h) = (L - L')x(t) + Mx(t-k)\,. \tag{4.4}$$

It is straightforward to show that (4.4) is exponentially stable for all choices of the parameters D' and L' satisfying

$$-1 < D - D' < 0\,, \qquad L - L' < -|M|\,, \qquad \frac{|D-D'|\,|L-L'|}{(1-|D-D'|)^2} < \frac{\pi}{2h}\,.$$

However, Corollary 4.2 guarantees that there exists a sequence of delays (ε_i) with $\varepsilon_i > 0$ and $\lim_{i\to\infty} \varepsilon_i = 0$ and such for any $i \in \mathbb{N}$ the delayed feedback control

$$u(t) = -D'\dot{x}(t-h-\varepsilon_i) - L'x(t-\varepsilon_i)$$

leads to a closed-loop system which has an exponentially unstable infinite root chain.

Finally, consider (4.3) for $D = 1$ and $k = h$. Then $\mathbf{\Delta}_d$ and $\mathbf{\Delta}$ are given by

$$\mathbf{\Delta}_d(s) = 1 - e^{-hs}\,, \qquad \mathbf{\Delta}(s) = s(1-e^{-hs}) - Me^{-hs} - L\,.$$

It is easy to show that if $L < -|M|$, then $\mathbf{\Delta}(s) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$, and moreover, the feedback

$$u(t) = -\frac{1}{2}\dot{x}(t-h)$$

is exponentially stabilizing. However, by Corollary 4.2 there exist delays $\varepsilon_i > 0$ with $\lim_{i\to\infty} \varepsilon_i = 0$ and such that for any $i \in \mathbb{N}$ the delayed feedback control

$$u(t) = -\frac{1}{2}\dot{x}(t-h-\varepsilon_i)$$

leads to a closed-loop characteristic function $\mathbf{\Delta}^{\varepsilon_i}$ which has an exponentially unstable infinite root chain. Hence the stability behaviour of the delayed closed-loop system is worse than the stability behaviour of the original open-loop system. $\diamond$

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