Green's Tensors and Lattice Sums for Elastostatics and Elastodynamics

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Abstract

We present expressions for the Green's tensors in terms of lattice sums for general two-dimensional arrays in both elastostatics and elastodynamics. We represent the lattice sums in rapidly–convergent forms by Fourier transform methods. We compare the static lattice sums with recently published, highly accurate values for square arrays. We also establish Rayleigh identities for elastostatic and elastodynamic problems, generalizing previous work in electrostatics and electromagnetism. The Rayleigh identities may be used in studies of phononic band gaps.

1 Introduction

In a classic paper, Lord Rayleigh (1892) established an identity relating electrostatic fields and sources, expressed in terms of multipole coefficients and lattice sums. Rayleigh's method has over the years found many applications in studies of scalar transport problems, involving the solution of the Laplace equation. It has recently been generalized to scalar problems in electromagnetism, involving the Helmholtz equation and scattering by singly, doubly and triply periodic sets of obstacles (McPhedran & Dawes 1992; Chin *et al.* 1994; Nicorovici & McPhedran 1994; Nicorovici *et al.* 1994, 1995a, 1995b, 1995c). In each case Green's functions have been expressed in terms of lattice sums, and computationally convenient and accurate expressions have been derived for the lattice sums.

The Rayleigh method has recently been applied in a problem of elastostatics (McPhedran & Movchan 1994), concerning the response to loading of a square array of elastic cylinders embedded in an elastic matrix. Here, we will consider both elastostatic and elastodynamic problems involving general arrays of elastic inclusions in the shape of circular cylinders, inserted in doubly periodic fashion in an elastic matrix. We will derive expressions for the doubly periodic Green's tensors in both the elastostatic and elastodynamic cases. The expressions for the Green's tensors involve lattice sums, and we use Fourier methods to derive expression for these. An interesting feature of the expressions is that they can be integrated arbitrarily often, resulting in rapid convergence. We illustrate the accuracy of our method by comparing values of lattice sums for the square array, where highly accurate results have been recently published by other authors (Berman & Greengard 1994; Helsing 1994). However, we emphasise that our static expressions are valid not only for square arrays but also for any other doubly periodic array. Also, essentially the same program can be used to establish values for dynamic lattice sums, for which no results exist in the literature. Thus, the validation of our formula for lattice sums by comparison with results of other authors in the static case provides support for its use in the dynamic case, for which no alternative results exist.

We also use the Green's tensors to establish Rayleigh identities for both elastostatic and elastodynamic fields. Our purpose here is, in the first place, to make available the Rayleigh method and its required apparatus of lattice sums, for workers in the elastostatics and elastodynamics communities. We also wish to explore the generality of the Rayleigh method by presenting it in a vector rather than a scalar case, and also for Navier's equation rather than the Helmholtz equation.

The expressions for the periodic Green's tensors we present here will be useful in applications, since they converge rapidly throughout the unit cell. They combine naturally the required periodicity with a form appropriate for the application of boundary conditions on circular interfaces. They are also expressed as single sums, rather than as the double sums one would expect in Fourier expansions for doubly-periodic problems. One of the purposes of this work is in fact to provide an alternative to the more natural Fourier expressions for the solution of doubly periodic problems. These Fourier expressions are well-adapted to satisfying periodicity conditions, but provide poor convergence when required to express boundary conditions involving (for example) a large discontinuity in a normal derivative. The expressions we present here provide a natural form for implementing such conditions.

2 Spatial and spectral domain form of the Green's tensor

The Green's tensor for two–dimensional problems in elastodynamics is the elementary solution of the equation

$$c_{ijk\ell}G_{kp,\ell j}(\mathbf{x}) + \rho^{(0)}\omega^2 G_{ip}(\mathbf{x}) = -\delta_{ip}\delta(\mathbf{x}), \tag{1}$$

where $G_{kp,\ell j}$ denotes the second order derivative of the component G_{kp} with respect to x_ℓ and x_j, and

$$c_{ijk\ell} = \lambda^{(0)}\delta_{ij}\delta_{k\ell} + \mu^{(0)}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}). \tag{2}$$

Here, $\rho^{(0)}$ is the density of the elastic medium and, $\lambda^{(0)}$ and $\mu^{(0)}$ are the Lamé constants of the elastic plane.

By applying the Fourier transform:

$$G_{ip}(\mathbf{x}) = \frac{1}{(2\pi)^2} \int \widetilde{G}_{ip}(\boldsymbol{\xi})\, e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\boldsymbol{\xi} \quad , \quad \widetilde{G}_{ip}(\boldsymbol{\xi}) = \int G_{ip}(\mathbf{x})\, e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\mathbf{x}, \tag{3}$$

we obtain:

$$\widetilde{\mathbf{G}}(\boldsymbol{\xi}) = \frac{1}{\mu^{(0)}\|\boldsymbol{\xi}\|^2 - \rho^{(0)}\omega^2}\mathbf{I} - \frac{\lambda^{(0)} + \mu^{(0)}}{(\mu^{(0)}\|\boldsymbol{\xi}\|^2 - \rho^{(0)}\omega^2)[(\lambda^{(0)} + 2\mu^{(0)})\|\boldsymbol{\xi}\|^2 - \rho^{(0)}\omega^2]}\, \boldsymbol{\Xi}, \tag{4}$$

in a convenient form (see, for example, Appendix A in Willis & Movchan, 1995). Here

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \boldsymbol{\Xi} = \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}, \tag{5}$$

or, with a and b denoting the speeds of the dilatational and shear waves, respectively,

$$a^2 = \frac{\lambda^{(0)} + 2\mu^{(0)}}{\rho^{(0)}} \quad , \quad b^2 = \frac{\mu^{(0)}}{\rho^{(0)}}, \tag{6}$$

and

$$Q = \frac{\lambda^{(0)} + \mu^{(0)}}{\lambda^{(0)} + 2\mu^{(0)}} \frac{a^2 b^2}{b^2 - a^2} = \frac{2}{1+\kappa} \frac{a^2 b^2}{b^2 - a^2} \quad , \quad \kappa = 1 + \frac{2\mu^{(0)}}{\lambda^{(0)} + \mu^{(0)}}, \tag{7}$$

we have:

$$\widetilde{\mathbf{G}}(\boldsymbol{\xi}) = \frac{1}{\mu^{(0)}(\|\boldsymbol{\xi}\|^2 - \omega^2/b^2)}\left\{\mathbf{I} + \frac{Q}{\omega^2}\boldsymbol{\Xi}\right\} - \frac{Q}{\mu^{(0)}\omega^2(\|\boldsymbol{\xi}\|^2 - \omega^2/a^2)}\,\boldsymbol{\Xi}, \tag{8}$$

the spectral domain form of the Green's tensor for two-dimensional elastodynamics. For notational convenience, we will omit for the moment the superscript indices on wave speeds and wave numbers.

In the limit $\omega \to 0$, we obtain the Fourier transform of the Green's tensor for the static case:

$$\widetilde{\mathbf{T}}(\boldsymbol{\xi}) = \frac{1}{\mu^{(0)}\|\boldsymbol{\xi}\|^2}\mathbf{I} - \frac{\lambda^{(0)} + \mu^{(0)}}{\mu^{(0)}(\lambda^{(0)} + 2\mu^{(0)})\|\boldsymbol{\xi}\|^4}\,\boldsymbol{\Xi}. \tag{9}$$

Now, we use the integrals (Abramowitz & Stegun 1972):

$$\frac{1}{(2\pi)^2}\int \frac{e^{i\boldsymbol{\xi}\cdot\mathbf{x}}}{\|\boldsymbol{\xi}\|^2 - k_b^2}\, d\boldsymbol{\xi} = \frac{i}{4} H_0^{(1)}(k_b\|\mathbf{x}\|), \tag{10}$$

$$\frac{1}{(2\pi)^2}\int \frac{\xi_\ell \xi_m e^{i\boldsymbol{\xi}\cdot\mathbf{x}}}{\|\boldsymbol{\xi}\|^2 - k_b^2}\, d\boldsymbol{\xi} = -\frac{i}{4}\frac{\partial^2}{\partial x_\ell \partial x_m} H_0^{(1)}(k_b\|\mathbf{x}\|), \tag{11}$$

where H represents the Hankel function and $k_b = \omega/b$. With these integrals, and similar relations for the terms containing $k_a = \omega/a$, we obtain the spatial domain form of the Green's tensor (8):

$$\mathbf{G}(\mathbf{x}) = \frac{i}{4\mu^{(0)}}\left\{ H_0^{(1)}(k_b\|\mathbf{x}\|)\,\mathbf{I} - \frac{Q}{\omega^2}\begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_2} \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \frac{\partial^2}{\partial x_2^2} \end{pmatrix} \left[H_0^{(1)}(k_b\|\mathbf{x}\|) - H_0^{(1)}(k_a\|\mathbf{x}\|)\right] \right\}. \quad (12)$$

By adding and subtracting half of the trace of (12) we may rewrite the spatial domain form of the Green's tensor as:

$$\begin{aligned} \mathbf{G}(\mathbf{x}) &= \frac{i}{4\mu^{(0)}}\left\{ \left\{ H_0^{(1)}(k_b\|\mathbf{x}\|) - \frac{Q}{2\omega^2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left[H_0^{(1)}(k_b\|\mathbf{x}\|) - H_0^{(1)}(k_a\|\mathbf{x}\|)\right] \right\} \mathbf{I} \right. \\ &\quad \left. - \frac{Q}{2\omega^2}\overline{\overline{\nabla^2}}\left[H_0^{(1)}(k_b\|\mathbf{x}\|) - H_0^{(1)}(k_a\|\mathbf{x}\|)\right] \right\} \\ &= \frac{i}{4\mu^{(0)}}\left\{ \left\{ H_0^{(1)}(k_b\|\mathbf{x}\|) + \frac{Q}{2\omega^2}\left[k_b^2 H_0^{(1)}(k_b\|\mathbf{x}\|) - k_a^2 H_0^{(1)}(k_a\|\mathbf{x}\|)\right] \right\} \mathbf{I} \right. \\ &\quad \left. - \frac{Q}{2\omega^2}\overline{\overline{\nabla^2}}\left[H_0^{(1)}(k_b\|\mathbf{x}\|) - H_0^{(1)}(k_a\|\mathbf{x}\|)\right] \right\}, \end{aligned} \quad (13)$$

where:

$$\overline{\overline{\nabla^2}} = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} & 2\frac{\partial^2}{\partial x_1 \partial x_2} \\ 2\frac{\partial^2}{\partial x_2 \partial x_1} & \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2} \end{pmatrix} \quad (14)$$

Here, we have also used the fact that the Hankel function of order zero is the elementary solution of the Helmholtz equation:

$$\left(\nabla^2 + k^2\right)\left[\frac{i}{4} H_0^{(1)}(k\|\mathbf{x}\|)\right] = -\delta(\mathbf{x}). \quad (15)$$

In the limit $\omega \to 0$, from (13) we obtain the Somigliana tensor (the Green's tensor for elastostatics, see, for example, Muskhelishvili 1963):

$$\mathbf{T}(x,y) = \frac{1}{4\pi\mu^{(0)}(1+\kappa)}\begin{pmatrix} -\kappa\ln(x^2+y^2) + \frac{x^2-y^2}{x^2+y^2} & 2\frac{xy}{x^2+y^2} \\ 2\frac{xy}{x^2+y^2} & -\kappa\ln(x^2+y^2) - \frac{x^2-y^2}{x^2+y^2} \end{pmatrix}, \quad (16)$$

where $x = x_1$ and $y = x_2$. Here, we have used the behaviour of the Hankel functions, for small arguments (Abramowitz & Stegun 1972):

$$H_0^{(1)}(z) \sim i\frac{2}{\pi}\ln z. \quad (17)$$

If we introduce the complex variable $z = x + iy$, the Somigliana tensor (16) takes the form:

$$\mathbf{T}(x,y) = \frac{1}{4\pi\mu^{(0)}(1+\kappa)}\begin{pmatrix} -\kappa\ln(z\overline{z}) + \Re e(\overline{z}/z) & -\Im m(\overline{z}/z) \\ -\Im m(\overline{z}/z) & -\kappa\ln(z\overline{z}) - \Re e(\overline{z}/z) \end{pmatrix}. \quad (18)$$

3 Periodic Green's tensors

We consider a periodic array with the fundamental translation vectors $\widehat{\mathbf{a}}_1$ and $\widehat{\mathbf{a}}_2$, and with the origin being one node of the array. These vectors are not necessarily orthogonal, nor are their lengths necessarily equal. Thus, the vectors from the origin of coordinates to the p–th node of the array are specified by a set of two integers:

$$\mathbf{R}_p = p_1\widehat{\mathbf{a}}_1 + p_2\widehat{\mathbf{a}}_2 \quad , \quad p = (p_1, p_2) \in \mathbb{Z}^2. \tag{19}$$

We denote by $\mathcal{A}$ the primitive cell of the array and by $A = \|\widehat{\mathbf{a}}_1 \times \widehat{\mathbf{a}}_2\|$ the area of $\mathcal{A}$. The primitive cell of the reciprocal array (Ashcroft & Mermin 1976) is defined by the vectors:

$$\widehat{\mathbf{b}}_1 = 2\pi\frac{\widehat{\mathbf{a}}_2 \times \widehat{\mathbf{e}}}{A} \ , \ \widehat{\mathbf{b}}_2 = 2\pi\frac{\widehat{\mathbf{e}} \times \widehat{\mathbf{a}}_1}{A} \ , \tag{20}$$

where $\widehat{\mathbf{e}} = (\widehat{\mathbf{a}}_1 \times \widehat{\mathbf{a}}_2)/A$. The vectors from the origin to the h–th node of the reciprocal array have the form:

$$\mathbf{K}_h = h_1\widehat{\mathbf{b}}_1 + h_2\widehat{\mathbf{b}}_2 \quad , \quad h = (h_1, h_2) \in \mathbb{Z}^2. \tag{21}$$

Let us consider the problem of the propagation of simple–harmonic vibrations in an elastic plane (with density $\rho^{(0)}$ and Lamé constants $\lambda^{(0)}$, $\mu^{(0)}$) containing a set $\Omega = \cup_i\Omega_i$ of circular inclusions Ω_i, with density $\rho^{(1)}$ and Lamé constants $\lambda^{(1)}$, $\mu^{(1)}$. The propagation of waves in the elastic matrix and elastic inclusions is described by the Navier equations:

$$(\lambda^{(0)} + 2\mu^{(0)})\nabla\nabla\cdot\mathbf{u}(\mathbf{x}) - \mu^{(0)}\nabla\times\nabla\times\mathbf{u}(\mathbf{x}) + \rho^{(0)}\omega^2\mathbf{u}(\mathbf{x}) = 0 \quad , \quad \mathbf{x} \in \mathbb{R}^2 \setminus \Omega, \tag{22}$$
$$(\lambda^{(1)} + 2\mu^{(1)})\nabla\nabla\cdot\mathbf{u}(\mathbf{x}) - \mu^{(1)}\nabla\times\nabla\times\mathbf{u}(\mathbf{x}) + \rho^{(1)}\omega^2\mathbf{u}(\mathbf{x}) = 0 \quad , \quad \mathbf{x} \in \Omega. \tag{23}$$

In the case of a periodic array of elastic inclusions, the center of each inclusion is defined by the vectors $\mathbf{R}_p$ and, the Green's tensor for (22) is the elementary solution of the equation:

$$c_{ijk\ell}G_{kp,\ell j}(\mathbf{x}) + \rho^{(0)}\omega^2 G_{ip}(\mathbf{x}) = -\delta_{ip}\sum_p \delta(\mathbf{x} - \mathbf{R}_p). \tag{24}$$

Now, the spatial domain form of the Green's tensor is a periodic function of the form:

$$\begin{aligned}\mathbf{G}(\mathbf{x}) = \sum_p \frac{i}{4\mu^{(0)}}\Bigg\{&\Bigg\{H_0^{(1)}(k_b\|\mathbf{x} - \mathbf{R}_p\|) \\ &+ \frac{Q}{2\omega^2}\left[k_b^2 H_0^{(1)}(k_b\|\mathbf{x} - \mathbf{R}_p\|) - k_a^2 H_0^{(1)}(k_a\|\mathbf{x} - \mathbf{R}_p\|)\right]\Bigg\}\mathbf{I} \\ &- \frac{Q}{2\omega^2}\overline{\overline{\nabla^2}}\left[H_0^{(1)}(k_b\|\mathbf{x} - \mathbf{R}_p\|) - H_0^{(1)}(k_a\|\mathbf{x} - \mathbf{R}_p\|)\right]\Bigg\}.\end{aligned} \tag{25}$$

The spatial domain form and the spectral domain form of the Green's tensor are related by the Poisson summation formula (Jones 1966):

$$\sum_p \mathbf{G}(\mathbf{x} - \mathbf{R}_p - \mathbf{x}') = \frac{1}{A}\sum_h \widetilde{\mathbf{G}}(\mathbf{K}_h)\, e^{i\mathbf{K}_h\cdot(\mathbf{x}-\mathbf{x}')}, \tag{26}$$

where $\widetilde{\mathbf{G}}$ is defined by equation (4). To obtain the relation (26) we use the formulae (3) and the equations (Jones 1966):

$$\sum_p e^{\pm i\mathbf{R}_p\cdot\mathbf{s}} = \frac{(2\pi)^2}{A}\sum_h \delta(\mathbf{s}-\mathbf{K}_h)\,, \tag{27}$$

$$\sum_h e^{\pm i\mathbf{K}_h\cdot\mathbf{r}} = A\sum_p \delta(\mathbf{r}-\mathbf{R}_p)\,. \tag{28}$$

We now recall some results due to Chin *et al.* (1994) concerning the Green's function for the scalar, doubly-periodic Helmholtz equation:

$$\left(\nabla^2+k^2\right)G_H(\mathbf{x};k) = -\sum_p \delta(\mathbf{x}-\mathbf{R}_p). \tag{29}$$

The elementary spatial form of the solution of this equation is:

$$G_H(\mathbf{x};k) = \sum_p \frac{i}{4}H_0^{(1)}(k_b\|\mathbf{x}-\mathbf{R}_p\|), \tag{30}$$

which should compared with (25). This Green's function may be expressed as a Neumann series with coefficients the dynamic lattice sums $S_\ell^Y(k)$:

$$G_H(\mathbf{x};k) = -\frac{1}{4}Y_0(kx) - \frac{1}{2}\Re e\left[\sum_{\ell=0}^{\infty}\epsilon_\ell S_{2\ell}^Y(k)J_{2\ell}(kx)e^{-i2\ell\theta}\right], \tag{31}$$

where the polar representation of the vector $\mathbf{x}$ is (x,θ), and ϵ_ℓ denotes the Neumann symbol ($\frac{1}{2}$ if $\ell=0$, 1 otherwise). Only even order lattice sums $S_{2\ell}^Y$ occur in equation (31), since the odd order sums are zero as a result of the fact that, for every vector $\mathbf{R}_p$ in a two-dimensional array, $-\mathbf{R}_p$ also lies in the array. Here, the $S_{2\ell}^Y$ may be computed in terms of sums over the reciprocal array:

$$\begin{aligned} S_{2\ell}^Y(k)J_{2\ell+q}(ku) = &-\left[Y_q(ku)+\frac{1}{\pi}\sum_{m=1}^{q}\frac{(q-m)!}{(m-1)!}\left(\frac{2}{ku}\right)^{q-2m+2}\right]\delta_{\ell,0} \\ &-\frac{4}{A}(-1)^\ell\sum_h\left(\frac{k}{K_h}\right)^q\frac{J_{2\ell+q}(K_h u)}{K_h^2-k^2}e^{-i2\ell\theta_h}, \end{aligned} \tag{32}$$

where $\mathbf{K}_h=(K_h,\theta_h)$, q is an arbitrary integer, and u does not exceed the distance between the nearest neighbours in the lattice. Also, J and Y represent the Bessel functions of the first and second kind, respectively.

By applying the Graf's addition theorem to the Hankel functions in (25), we may express the components of the Green's tensor in terms of lattice sums:

$$\mathbf{G}(\mathbf{x}) = -\frac{1}{4\mu^{(0)}}\Bigg\{\Big\{Y_0(k_b\|\mathbf{x}\|)+\frac{Q}{2\omega^2}\left[k_b^2Y_0(k_b\|\mathbf{x}\|)-k_a^2Y_0^(k_a\|\mathbf{x}\|)\right]\Big\}\,\mathbf{I}$$

$$- \frac{Q}{2\omega^2}\overline{\overline{\nabla^2}}\left[Y_0(k_b\|\mathbf{x}\|) - Y_0(k_a\|\mathbf{x}\|)\right]\Bigg\}$$

$$- \sum_{\ell=-\infty}^{\infty} \frac{1}{4\mu^{(0)}}\Bigg\{\Bigg\{S_{2\ell}^Y(k_b)J_{2\ell}(k_b\|\mathbf{x}\|)$$

$$+ \frac{Q}{2\omega^2}\left[k_b^2 S_{2\ell}^Y(k_b)J_{2\ell}(k_b\|\mathbf{x}\|) - k_a^2 S_{2\ell}^Y(k_a)J_{2\ell}(k_a\|\mathbf{x}\|)\right]\Bigg\} e^{-i2\ell\theta}\,\mathbf{I}$$

$$- \frac{Q}{2\omega^2}\overline{\overline{\nabla^2}}\left[S_{2\ell}^Y(k_b)J_{2\ell}(k_b\|\mathbf{x}\|) - S_{2\ell}^Y(k_a)J_{2\ell}(k_a\|\mathbf{x}-\mathbf{R}_p\|)\right]\, e^{-i2\ell\theta}\Bigg\}. \tag{33}$$

We may let k tend to zero in ((29), (31), (32)), to obtain the static Green's function:

$$\nabla^2 G_S(\mathbf{x}) = -\sum_p \delta(\mathbf{x}-\mathbf{R}_p), \tag{34}$$

where

$$G_S(\mathbf{x}) = -\frac{1}{2\pi}\ln\left(\frac{x}{2}\right) + \frac{1}{2\pi}\mathfrak{Re}\left[\sum_{\ell=1}^{\infty}\frac{x^{2\ell}}{2\ell}S_{2\ell}e^{-i2\ell\theta}\right], \tag{35}$$

and

$$S_{2\ell} = \sum_{p\neq 0}\frac{1}{z_p^{2\ell}}, \tag{36}$$

with z_p the representation in the complex plane of the vector $\mathbf{R}_p$. Thus, for $\mathbf{R}_p = (R_p, \varphi_p)$ and $R_p = \|\mathbf{R}_p\|$, we have $z_p = R_p \exp(i\varphi_p)$. The relation (35) for the Green's function makes obvious the central role of the lattice sums in periodic electrostatic problems.

In electrostatics, only the lattice sums (36) are required, whereas in elastostatics the lattice sums $S_{2\ell}$ of (36) are complemented by a second set of lattice sums (McPhedran & Movchan 1994; Berman & Greengard 1994; Helsing 1994):

$$S_{2\ell}^{(1)} = \sum_{p\neq 0}\frac{|z_p|^2}{z_p^{2\ell}}. \tag{37}$$

We will next consider the derivation of convenient expressions for $S_{2\ell}$ and $S_{2\ell}^{(1)}$, analogous to the result (32) for the dynamic lattice sums.

4 Static lattice sums

In this section we present some formulae for the lattice sums $S_{2\ell}$ and $S_{2\ell}^{(1)}$ which occur in problems of electrostatics and elastostatics. Many authors have discussed these sums, and have presented accurate and fast algorithms for their evaluation (see, for example, Helsing 1994, and Berman & Greengard 1994). The expressions quoted below are of interest since they correspond directly to the lattice sums occurring in problems of dynamics for which the relation (32) is required. This correspondence is advantageous when we consider the homogenization procedure based on the static limit of the dynamic Green's tensor (McPhedran *et al.* 1995).

We present only the principal formulae here, while the detailed calculations are given in Appendix A. The quantities $S_{2\ell}$ and $S_{2\ell}^{(1)}$ are both expressed as sums over the reciprocal lattice:

$$u^{2\ell} S_{2\ell} = (-1)^\ell \pi \frac{u^2}{A} \frac{(2\ell+q)!}{(2\ell-1)!} \left[\sum_{h\neq 0} \frac{J_{2\ell+q}(K_h u)}{(K_h u/2)^{q+2}} e^{-i2\ell\theta_h} + \frac{1}{(q+2)!} e^{-i2\theta_0}\, \delta_{\ell,1} \right], \tag{38}$$

for $\ell \geq 1$, and

$$\begin{aligned} u^{2\ell-2} S_{2\ell}^{(1)} &= (-1)^\ell \pi \frac{u^2}{A} \frac{(2\ell+q-2)!}{(2\ell-2)!} \\ &\times \left[\sum_{h\neq 0} \frac{(2\ell+q) J_{2\ell+q}(K_h u) + J_{2\ell+q-2}(K_h u)}{(K_h u/2)^{q+2}} e^{-i2\ell\theta_h} + \frac{1}{(q+2)!} e^{-i4\theta_0}\, \delta_{\ell,2} \right], \end{aligned} \tag{39}$$

for $\ell \geq 2$. Note, that these expressions contain two arbitrary parameters, u and q. One of the numerical tests we have carried out on them is to verify that the lattice sum values are independent of the choices of u (u has to be a length of a vector within the unit cell) and $q > 0$.

For a square array, when $\hat{\mathbf{a}}_i \cdot \hat{\mathbf{a}}_j = d^2 \delta_{ij}$, and only the lattice sums of order multiples of four are nonzero, and real, we find:

$$u^{4\ell} S_{4\ell} = \pi \frac{u^2}{d^2} \frac{(4\ell+q)!}{(4\ell-1)!} \left[\sum_{h\neq 0} \frac{J_{4\ell+q}(K_h u)}{(K_h u/2)^{q+2}} \cos(4\ell\theta_h) \right], \tag{40}$$

and

$$\begin{aligned} u^{4\ell-2} S_{4\ell}^{(1)} &= \pi \frac{u^2}{d^2} \frac{(4\ell+q-2)!}{(4\ell-2)!} \\ &\times \left[\sum_{h\neq 0} \frac{(4\ell+q) J_{4\ell+q}(K_h u) + J_{4\ell+q-2}(K_h u)}{(K_h u/2)^{q+2}} \cos(4\ell\theta_h) + \frac{1}{(q+2)!} e^{-i4\theta_0}\, \delta_{\ell,1} \right], \end{aligned} \tag{41}$$

for $\ell \geq 1$.

We mention that, even in the case of a square array, the lattice sum S_2 does not vanish. From (38), we have:

$$u^2 S_2 = -\pi \frac{u^2}{d^2} (q+2)! \left[\sum_{h\neq 0} \frac{J_{2+q}(K_h u)}{(K_h u/2)^{q+2}} \cos(2\theta_h) + \frac{1}{(q+2)!} e^{-i2\theta_0} \right]. \tag{42}$$

The series in (42) vanishes. Consequently, for an infinite square array:

$$S_2 = -\frac{\pi}{d^2}\, e^{-i2\theta_0}. \tag{43}$$

For a rectangular array, with $\theta_0 = \pi/2$, we obtain $S_2 = \pi/A$. This result has been derived by Lord Rayleigh (1892) and Perrins *et al.* (1979), by summing directly the series (36) over a needle–shaped region of the direct array, elongated along the x–axis. They considered an incident field parallel to the x–axis, which corresponds to $\theta_0 = \pm\pi/2$. Also, the exponential in (43) represents Helsing's J operator justified in a different way (Helsing 1994).

The lattice sum $S_4^{(1)}$ from equation (41) also has a directionally dependent term (the second term in square brackets). This is combined with a constant term, nonzero even for a square array. Its value for $\theta_0 = 0$ may be obtained from Table 2.

The correspondence between the dynamic lattice sums $S_\ell^Y(k)$, defined in (32), and the static lattice sums S_ℓ for $k \to 0$ is discussed in Appendix B. We note that, for small k,

$$D_\ell(k) = S_\ell^Y(k)/Y_\ell(k) = S_\ell + O(k^2) \tag{44}$$

(see also Table 3).

5 Rayleigh's identities

In wave propagation problems, the displacement field $\mathbf{u}$ may be represented in terms of Lamé potentials (Atkin & Fox 1980):

$$\mathbf{u}(\mathbf{x}) = \nabla\varphi(\mathbf{x}) + \nabla \times \boldsymbol{\psi}(\mathbf{x}) \quad , \quad \nabla \cdot \boldsymbol{\psi}(\mathbf{x}) = 0 \ , \tag{45}$$

where $\varphi(\mathbf{x})$ and $\boldsymbol{\psi}(\mathbf{x})$ are scalar and vector fields, respectively. In the case of wave propagation through an elastic matrix, containing an array of inclusions, the displacement field in the matrix satisfies the equation (22). Therefore, the Lamé potentials are solutions of the Helmholtz equations:

$$\left[\nabla^2 + (k_a^{(0)})^2\right] \varphi(\mathbf{x}) = 0, \tag{46}$$

$$\left[\nabla^2 + (k_b^{(0)})^2\right] \boldsymbol{\psi}(\mathbf{x}) = 0, \tag{47}$$

where $k_a^{(0)} = \omega/a$ and $k_b^{(0)} = \omega/b$ are the wave numbers for dilatational and shear waves in the matrix. The velocities of these waves, a and b, are defined in (6).

For an isotropic and homogeneous matrix the system of equations for the components of $\boldsymbol{\psi}$ consists of uncoupled equations. Consequently, the components of the vector field $\boldsymbol{\psi}$ are solutions of the scalar Helmholtz equation:

$$\left[\nabla^2 + (k_b^{(0)})^2\right] f(\mathbf{x}) = 0, \tag{48}$$

and for the problem at hand the vector potential $\boldsymbol{\psi}$ has only its third component nonzero, with this component ψ depending only on x_1, x_2. We use in the matrix material the series expansions in terms of Bessel functions:

$$\varphi^{(0)}(\mathbf{x}) = \sum_{n=-\infty}^{\infty} \left[A_n^{(a)} J_n(k_a^{(0)}\|\mathbf{x}\|) + B_n^{(a)} Y_n(k_a^{(0)}\|\mathbf{x}\|)\right] e^{in\theta}, \tag{49}$$

$$\psi^{(0)}(\mathbf{x}) = \sum_{n=-\infty}^{\infty} \left[A_n^{(b)} J_n(k_b^{(0)}\|\mathbf{x}\|) + B_n^{(b)} Y_n(k_b^{(0)}\|\mathbf{x}\|)\right] e^{in\theta}, \tag{50}$$

and in the circular cylindrical inclusion we write

$$\varphi^{(1)}(\mathbf{x}) = \sum_{n=-\infty}^{\infty} C_n^{(a)} J_n(k_a^{(1)}\|\mathbf{x}\|)e^{in\theta}, \tag{51}$$

$$\psi^{(1)}(\mathbf{x}) = \sum_{n=-\infty}^{\infty} C_n^{(b)} J_n(k_b^{(1)}\|\mathbf{x}\|)e^{in\theta}, \tag{52}$$

where $k_a^{(1)}$, $k_b^{(1)}$ are the wave numbers for dilatational and shear waves in the inclusion material. Taking the cylinders to have a radius r_c, the following displacement boundary conditions apply at the interface $r := \|\mathbf{x}\| = r_c$:

$$\frac{\partial\varphi^{(0)}}{\partial r} + \frac{1}{r_c}\frac{\partial\psi^{(0)}}{\partial\theta} = \frac{\partial\varphi^{(1)}}{\partial r} + \frac{1}{r_c}\frac{\partial\psi^{(1)}}{\partial\theta}, \tag{53}$$

$$\frac{1}{r_c}\frac{\partial\varphi^{(0)}}{\partial\theta} - \frac{\partial\psi^{(0)}}{\partial r} = \frac{1}{r_c}\frac{\partial\varphi^{(1)}}{\partial\theta} - \frac{\partial\psi^{(1)}}{\partial r}, \tag{54}$$

expressing the continuity of the normal and tangential displacements, respectively. The traction interface boundary conditions are

$$\begin{aligned}
&\mu^{(0)} \left(-\frac{2}{r_c^2}\frac{\partial\varphi^{(0)}}{\partial\theta} + \frac{2}{r_c}\frac{\partial^2\varphi^{(0)}}{\partial\theta\partial r} + (k_b^{(0)})^2\psi^{(0)} + \frac{2}{r_c}\frac{\partial\psi^{(0)}}{\partial r} + \frac{2}{r_c^2}\frac{\partial^2\psi^{(0)}}{\partial\theta^2}\right) \\
= &\mu^{(1)} \left(-\frac{2}{r_c^2}\frac{\partial\varphi^{(1)}}{\partial\theta} + \frac{2}{r_c}\frac{\partial^2\varphi^{(1)}}{\partial\theta\partial r} + (k_b^{(1)})^2\psi^{(1)} + \frac{2}{r_c}\frac{\partial\psi^{(1)}}{\partial r} + \frac{2}{r_c^2}\frac{\partial^2\psi^{(1)}}{\partial\theta^2}\right),
\end{aligned} \tag{55}$$

$$\begin{aligned}
&2\mu^{(0)} \left(\frac{1}{r_c}\frac{\partial^2\psi^{(0)}}{\partial\theta\partial r} - \frac{1}{r_c^2}\frac{\partial\psi^{(0)}}{\partial\theta} - \frac{1}{r_c}\frac{\partial\varphi^{(0)}}{\partial r} + \frac{1}{r_c^2}\frac{\partial^2\varphi^{(0)}}{\partial\theta^2}\right) - (k_a^{(0)})^2(2\mu^{(0)} + \lambda^{(0)})\varphi^{(0)} \\
= &2\mu^{(1)} \left(\frac{1}{r_c}\frac{\partial^2\psi^{(1)}}{\partial\theta\partial r} - \frac{1}{r_c^2}\frac{\partial\psi^{(1)}}{\partial\theta} - \frac{1}{r_c}\frac{\partial\varphi^{(1)}}{\partial r} + \frac{1}{r_c^2}\frac{\partial^2\varphi^{(1)}}{\partial\theta^2}\right) - (k_a^{(1)})^2(2\mu^{(1)} + \lambda^{(1)})\varphi^{(1)},
\end{aligned} \tag{56}$$

expressing the continuity of the stress tensor components $\sigma_{r\theta}$ and σ_{rr}, respectively.

These interface conditions can be used to eliminate the coefficients $C_n^{(a)}$, $C_n^{(b)}$ and to express the coefficients $A_n^{(a)}$, $A_n^{(b)}$ in terms of $B_n^{(a)}$, $B_n^{(b)}$ as follows

$$\begin{pmatrix} A_n^{(a)} \\ A_n^{(b)} \end{pmatrix} = \begin{pmatrix} M_n^{(aa)} & M_n^{(ab)} \\ M_n^{(ba)} & M_n^{(bb)} \end{pmatrix} \begin{pmatrix} B_n^{(a)} \\ B_n^{(b)} \end{pmatrix}, \tag{57}$$

where the coefficients $M_n^{(\alpha\beta)}$ are defined in Appendix C. Note that $M_n^{(aa)}, M_n^{(bb)}$ are real, while $M_n^{(ab)} = -M_n^{(ba)}$ are pure imaginary. Thus, the matrix $(M_n^{(\alpha\beta)})$ is Hermitian as one would expect for a scattering matrix from a non-dissipative object.

Using this solution of the interface conditions and Rayleigh's method extended for dynamic problems (McPhedran & Dawes 1992; Chin *et al.* 1994), we obtain the dynamic Rayleigh identities:

$$\sum_\beta M_n^{(\alpha\beta)} B_n^{(\beta)} + \sum_{\ell=-\infty}^{\infty} (-1)^{n+\ell} S_{\ell-n}^{Y}(k_\alpha) B_\ell^{(\alpha)} = 0, \tag{58}$$

where, $\alpha, \beta = a, b$, and only the dynamic lattice sums of even order are non-zero, Note that only one type of lattice sum is required in dynamics (with two different argument values, corresponding to the two different wave numbers), whereas two types are required in statics.

The dynamic Rayleigh identities constitute an infinite set of homogeneous equations. We may isolate frequencies for which elastic waves propagate in the elastic composite material by truncating the equations, and then searching numerically for frequencies for which the determinant of the truncated matrix vanishes. This will provide us the dispersion curves for the propagation of elastic waves through periodic structures (their phononic band structure). The null vectors of the matrix at points for which its determinant is zero define the structure of the elastic waves which can propagate in the composite. We will leave the numerical investigation of these points for future work.

6 Numerical results

To illustrate the accuracy of the formulae (38) and (39), we consider a square array with constant $d = 1$. We also consider that $\theta_0 = \pm\pi/2$. In this case, from (43) we have $S_2 = \pi$ and the analytic expression of the lattice sum $S_4^{(1)}$ is given by (41) with $\ell = 1$:

$$u^2 S_4^{(1)} = \pi \frac{u^2}{d^2} \frac{(2+q)!}{2} \left[\sum_{h \neq 0} \frac{(4+q)J_{4+q}(K_h u) + J_{2+q}(K_h u)}{(K_h u/2)^{q+2}} \cos(4\theta_h) + \frac{1}{(q+2)!} \right]. \quad (59)$$

Also, in what follows we choose $u = 1$.

From (36) and (37) we may see that both these lattice sums are represented, in the direct array space, by conditionally convergent series. Following the method suggested by Lord Rayleigh (1892), we evaluate the series in (36) and (37) over a needle shaped region, elongated along the x–axis. This means that we consider partial sums of these series, for a number of nodes along the x-axis (n_x) bigger than the number of nodes along the y-axis (n_y).

As an example, in Table 1, we display the numerical values of different partial sums, for the series (36) and (37), when $n_x = n_y^2$. (If $\mathbf{R}_p = (m, n)$, then $-n_x \le m \le n_x$ and $-n_y \le n \le n_y$). The partial sums of the series over the direct array converge slowly to the values given by the corresponding series over the reciprocal array. The exact value of the lattice sum S_2 is π. While the total number of terms in the series over the direct array is $(2n_x + 1)(2n_x^2 + 1)$, the series over the reciprocal array gives a very accurate result for $S_4^{(1)}$ with only $n_x = n_y = 100$ and $q = 20$ in (59). (Now, for $\mathbf{K}_h = 2\pi(m, n)$, we have $-n_x \le m \le n_x$ and $-n_y \le n \le n_y$). Compared with the numerical value of $S_4^{(1)}$ given by Helsing (1994), we obtained a 14 figure agreement (see first line, columns 4 and 5, in Table 2).

In Table 2 (columns 2 and 4) we present the numerical values of the lattice sums S_{4m} and $S_{4m}^{(1)}$, for a square array ($d = 1$), as given by (40) and (41), respectively. In these formulae, we have also used $u = 1$, $n_x = n_y = 600$ and $q = 12$. Compared with the numerical results given by Berman & Greengard (1994) and Helsing (1994), displayed in columns 3 and 5 (Table 2), the agreement is to 14 figures for low order lattice sums and to 12 figures for high orders. The computation time required by our method is large compared with that required by the method of Berman & Greengard (1994) or Helsing (1994). However, we stress once more that the interest of our method is that essentially the same procedure works for both static and dynamic lattice sums, and in the latter case alternative techniques are not available.

Actually, the series over the direct array converge as R_p^{-4m}, while the series over the reciprocal array converge as $K_h^{-q-2.5}$. In our case, $q = 12$ and the results given by the series over the reciprocal array are very accurate for $m \leq 12$. For higher orders, $m \geq 14$, even the series over the direct array converge rapidly. We may increase the value of q but too large values of q means evaluating Bessel functions of high orders and large arguments. Therefore, the accuracy of the numerical values of the series over the reciprocal lattice depends on the algorithms used to evaluate the Bessel functions. At the same time, we have to deal with large values introduced by the factorials and powers of K_h at the edge of the array, which may lead to catastrophic cancellations between different terms in the series.

We may conclude that the formulae involving series over the reciprocal array are preferable in the cases when the series over the direct array are conditionally convergent or converge slowly.

Finally, we present in Table 3 some values of the coefficients $D_l(k)$, illustrating their convergence to the static lattice sums S_ℓ, as given by equation (44).

7 Conclusions

On the basis of analysis of spatial and spectral forms of the Green's tensor for two–dimensional problems of elastodynamics and series representations for Green's tensor in periodic medium we have obtained effective means for calculations of dynamic elasticity lattice sums which are essential in the twin expansions technique in linear elasticity. Taking the limit case we derived rapidly convergent series representations for the static lattice sums. The discussions of Rayleigh's identities is presented for static and dynamic cases; the dynamic Rayleigh's identities are derived on the basis of series expansions of the Lamé potentials in terms of Bessel functions. In analogy with the method developed for the study of photonic band gaps (Nicorovici *et al.* 1995b, 1995c), this technique is capable of studying, numerically and analytically, the dispersion curves for the propagation of elastic waves.

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A Convenient Forms for Static Lattice Sums

We start by taking the limit of (26) as $\omega \to 0$:

$$\sum_p \mathbf{T}(\mathbf{x}-\mathbf{R}_p-\mathbf{x}') = \frac{1}{A}\sum_h \widetilde{\mathbf{T}}(\mathbf{K}_h)\, e^{i\mathbf{K}_h\cdot(\mathbf{x}-\mathbf{x}')}. \tag{60}$$

The tensor in the right hand side of (60), defined in (9), may be written in the form:

$$\begin{aligned}\widetilde{\mathbf{T}}(\mathbf{K}_h) &= \frac{1}{\mu^{(0)}K_h^2}\mathbf{I} - \frac{2}{\mu^{(0)}(1+\kappa)K_h^4}\begin{pmatrix} K_{h,x}^2 & K_{h,x}K_{h,y} \\ K_{h,x}K_{h,y} & K_{h,y}^2\end{pmatrix} \\ &= \frac{\kappa}{\mu^{(0)}(1+\kappa)K_h^2}\mathbf{I} - \frac{1}{\mu^{(0)}(1+\kappa)K_h^4}\begin{pmatrix} K_{h,x}^2 - K_{h,y}^2 & 2K_{h,x}K_{h,y} \\ 2K_{h,x}K_{h,y} & K_{h,y}^2 - K_{h,x}^2\end{pmatrix} \\ &= \frac{1}{\mu^{(0)}(1+\kappa)K_h^2}\begin{pmatrix} \kappa - \cos(2\theta_h) & -\sin(2\theta_h) \\ -\sin(2\theta_h) & \kappa+\cos(2\theta_h)\end{pmatrix},\end{aligned} \tag{61}$$

where $\mathbf{K}_h = (K_h, \theta_h)$ and $K_h = \|\mathbf{K}_h\|$.

Now, we denote by $w = z - z'$ the representation in the complex plane, of the vector $\mathbf{u} = \mathbf{x} - \mathbf{x}'$. Consequently, $|w| = \|\mathbf{u}\| \equiv u$. We also denote by z_p the representation in complex of the lattice vector R_p. These notations together with (18) enable one to write (60) in the form:

$$\begin{aligned}&\sum_p \begin{pmatrix} -\kappa \ln|w-z_p|^2 + \Re e[(\overline{w-z_p})/(w-z_p)] & -\Im m[(\overline{w-z_p})/(w-z_p)] \\ -\Im m[(\overline{w-z_p})/(w-z_p)] & -\kappa\ln|w-z_p|^2 - \Re e[(\overline{w-z_p})/(w-z_p)]\end{pmatrix} \\ &= \frac{4\pi}{A}\sum_h \begin{pmatrix} \kappa-\cos(2\theta_h) & -\sin(2\theta_h) \\ -\sin(2\theta_h) & \kappa+\cos(2\theta_h)\end{pmatrix}\frac{e^{i\mathbf{K}_h\cdot\mathbf{u}}}{K_h^2}.\end{aligned} \tag{62}$$

From (10) we have the Fourier transform of the Hankel function and, for $\mathbf{u} = \mathbf{x} - \mathbf{x}'$, by means of formula (27), we obtain:

$$\begin{aligned}\frac{i\pi}{2}\sum_p H_0^{(1)}(k\|\mathbf{u}-\mathbf{R}_p\|) &= \frac{1}{2\pi}\sum_p \int \frac{e^{-i\boldsymbol{\xi}\cdot\mathbf{R}_p + i\boldsymbol{\xi}\cdot\mathbf{u}}}{\|\boldsymbol{\xi}\|^2 - k^2}d\boldsymbol{\xi} \\ &= \frac{1}{2\pi}\int \frac{e^{i\boldsymbol{\xi}\cdot\mathbf{u}}}{\|\boldsymbol{\xi}\|^2-k^2}\left[\sum_p e^{-i\boldsymbol{\xi}\cdot\mathbf{R}_p}\right] d\boldsymbol{\xi}\end{aligned}$$

$$= \frac{1}{2\pi}\int \frac{e^{i\boldsymbol{\xi}\cdot\mathbf{u}}}{\|\boldsymbol{\xi}\|^2-k^2}\left[\frac{(2\pi)^2}{A}\sum_h \delta(\boldsymbol{\xi}-\mathbf{K}_h)\right]d\boldsymbol{\xi}$$

$$= \frac{2\pi}{A}\sum_h \frac{e^{i\mathbf{K}_h\cdot\mathbf{u}}}{K_h^2-k^2}. \tag{63}$$

In the limit $k\to 0$, (63) becomes:

$$\sum_p \ln|w-z_p|^2 = -\frac{4\pi}{A}\sum_h \frac{e^{i\mathbf{K}_h\cdot\mathbf{u}}}{K_h^2}, \tag{64}$$

where we have used $\|\mathbf{u}-\mathbf{R}_p\| = |w-z_p|$. Therefore, the set of three equations from (62) reduces to only two equations:

$$\Re e\sum_p \frac{\overline{w-z_p}}{w-z_p} = -\ \frac{4\pi}{A}\sum_h \frac{e^{i\mathbf{K}_h\cdot\mathbf{u}}}{K_h^2}\cos(2\theta_h), \tag{65}$$

$$\Im m\sum_p \frac{\overline{w-z_p}}{w-z_p} = \quad \frac{4\pi}{A}\sum_h \frac{e^{i\mathbf{K}_h\cdot\mathbf{u}}}{K_h^2}\sin(2\theta_h), \tag{66}$$

or:

$$\sum_p \frac{\overline{w-z_p}}{w-z_p} = -\frac{4\pi}{A}\sum_h \frac{e^{i\mathbf{K}_h\cdot\mathbf{u}-2i\theta_h}}{K_h^2}. \tag{67}$$

To evaluate the lattice sums $S_{2\ell}$, defined in (36), we expand the logarithm and the exponential from (64). We denote by $\mathbf{u}=(u,\theta)$, $w=u\exp(i\theta)$, $\mathbf{R}_p=(R_p,\varphi_p)$, $z_p = R_p\exp(i\varphi_p)$ and $\mathbf{K}_h=(K_h,\theta_h)$. We assume that $|w|=u<|z_p|=R_p$, $\forall p\neq 0$. In this case we obtain:

$$\sum_p \ln|w-z_p|^2 \;= 2\ln u + 2\sum_{p\neq 0}\ln R_p$$

$$-2\ \sum_{n=1}^{\infty}\frac{u^n}{n}\left[\sum_{p\neq 0}\frac{\cos(n\varphi_p)}{R_p^n}\right]\cos(n\theta) - 2\sum_{n=1}^{\infty}\frac{u^n}{n}\left[\sum_{p\neq 0}\frac{\sin(n\varphi_p)}{R_p^n}\right]\sin(n\theta). \tag{68}$$

We also have the series expansion in terms of Bessel functions (Abramowitz & Stegun 1972):

$$e^{i\mathbf{K}_h\cdot\mathbf{u}} = J_0(K_hu) + 2\sum_{\ell=1}^{\infty}(-1)^\ell J_{2\ell}(K_hu)\cos[2\ell(\theta-\theta_h)]. \tag{69}$$

In (68) we may identify the real and imaginary parts of the lattice sums S_n and, by substituting (68) and (69) in (64) we find that all the lattice sums of odd order ($S_{2\ell+1}$) vanish, and:

$$u^{2\ell}S_{2\ell} = (-1)^\ell\frac{8\pi\ell}{A}\sum_h \frac{J_{2\ell}(K_hu)}{K_h^2}\,e^{-i2\ell\theta_h} \quad , \quad \text{for } \ell\geq 1. \tag{70}$$

Also, for $\ell=0$, we obtain:

$$2\ln u + 2\sum_{p\neq 0}\ln R_p = -\frac{4\pi}{A}\sum_h \frac{J_0(K_hu)}{K_h^2}. \tag{71}$$

Here, u may be considered as a free parameter and (70) and (71) are true for arbitrary u in the unit cell. The equation (71) is nugatory, since both left and right-hand sides diverge (the latter because of the term from $K_h = 0$). However, this equation is irrelevant in the sequel.

The condition $u < R_p$, $\forall p \neq 0$, is satisfied for u restricted to the central unit cell, and we may choose $u \leq 1$, adjusting the vectors $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$, accordingly. The series in (70) converge as $K_h^{-2.5}$. We may accelerate the convergence using the method developed by Chin *et al.* (1994). Thus, we multiply both sides in (70) by $u^{2\ell+1}$ and integrate over u from 0 to η, where $\eta \leq 1$. The integrals involving Bessel functions have the closed form (Abramowitz & Stegun 1972):

$$\int_0^\eta u^{2\ell+1} J_{2\ell}(K_h u) du = \eta^{2\ell+1} \frac{J_{2\ell+1}(K_h \eta)}{K_h}, \tag{72}$$

for $\ell \geq 0$. Then, we change η into u and repeat the same procedure, this time multiplying the equation by $u^{2\ell+2}$. After q such steps, we obtain:

$$u^{2\ell} S_{2\ell} = (-1)^\ell \pi \frac{u^2}{A} \frac{(2\ell+q)!}{(2\ell-1)!} \left[\sum_{h \neq 0} \frac{J_{2\ell+q}(K_h u)}{(K_h u/2)^{q+2}} e^{-i2\ell\theta_h} + \frac{1}{(q+2)!} e^{-i2\theta_0} \delta_{\ell,1} \right], \tag{73}$$

for $\ell \geq 1$. Note that (73) is also the direct limit of (32) as $k \to 0$ (see Appendix B). This equation may be used to evaluate the lattice sums $S_{2\ell}$.

The expression of the lattice sums $S_{2\ell}^{(1)}$ results from (67). Thus, for $|w| < |z_p|$, $\forall p \neq 0$, we expand the left hand side in (67):

$$\begin{aligned}
\sum_p \frac{\overline{w - z_p}}{w - z_p} &= \frac{\overline{w}}{w} + \sum_{p \neq 0} \frac{\overline{w - z_p}}{w - z_p} = \frac{\overline{w}}{w} - \sum_{p \neq 0} \frac{\overline{w - z_p}}{z_p} \frac{1}{1 - w/z_p} \\
&= e^{-2i\theta} - \overline{w} \sum_{\ell=0}^{\infty} w^\ell \sum_{p \neq 0} \frac{1}{z_p^{\ell+1}} + \sum_{\ell=0}^{\infty} w^\ell \sum_{p \neq 0} \frac{\overline{z_p}}{z_p^{\ell+1}} \\
&= e^{-2i\theta} - \sum_{\ell=0}^{\infty} u^{\ell+1} e^{i(\ell-1)\theta} S_{\ell+1} + \sum_{\ell=0}^{\infty} u^\ell e^{i\ell\theta} S_{\ell+2}^{(1)} \\
&= e^{-2i\theta} - u e^{-i\theta} S_1 + \sum_{m=0}^{\infty} u^m e^{im\theta} \left[S_{m+2}^{(1)} - u^2 S_{m+2} \right] \\
&= e^{-2i\theta} - u e^{-i\theta} S_1 + \sum_{n=2}^{\infty} u^{n-2} e^{i(n-2)\theta} \left[S_n^{(1)} - u^2 S_n \right].
\end{aligned} \tag{74}$$

In the right hand side of (67) we use the expansion (69), so that (67) becomes:

$$\begin{aligned}
e^{-2i\theta} - u e^{-i\theta} S_1 &+ \sum_{n=2}^{\infty} u^{n-2} e^{i(n-2)\theta} \left[S_n^{(1)} - u^2 S_n \right] \\
&= -\frac{4\pi}{A} \sum_{\ell=-\infty}^{\infty} i^\ell e^{i\ell\theta} \sum_h \frac{J_\ell(K_h u)}{K_h^2} e^{-i(\ell+2)\theta_h}.
\end{aligned} \tag{75}$$

For $\ell = 0$ we have:

$$S_2^{(1)} - u^2 S_2 = -\frac{4\pi}{A} \sum_h \frac{J_0(K_h u)}{K_h^2} e^{-i2\theta_h}. \tag{76}$$

Again, this equation is nugatory, as both sides diverge, but the lattice sum $S_2^{(1)}$ is not involved in the sequel.

The central symmetry of the general array implies that all the lattice sums $S_{2\ell+1}$ and $S_{2\ell+1}^{(1)}$, of odd order, vanish. Consequently, from (75) we obtain:

$$u^{2\ell-2} \left[S_{2\ell}^{(1)} - u^2 S_{2\ell} \right] = (-1)^\ell \frac{4\pi}{A} \sum_h \frac{J_{2\ell-2}(K_h u)}{K_h^2} e^{-i2\ell\theta_h}, \tag{77}$$

for $\ell \geq 2$. For $\ell \leq -2$ we obtain a similar relation for the complex conjugate lattice sums. By means of (70), the relation (77) may be written in the form:

$$u^{2\ell-2} S_{2\ell}^{(1)} = (-1)^\ell \frac{4\pi}{A} \sum_h \frac{2\ell J_{2\ell}(K_h u) + J_{2\ell-2}(K_h u)}{K_h^2} e^{-i2\ell\theta_h}, \tag{78}$$

for $\ell \geq 2$. Then, we apply the same method as in the case of (70) to accelerate the convergence of the series in (78). This gives us the formula:

$$\begin{aligned} u^{2\ell-2} S_{2\ell}^{(1)} &= (-1)^\ell \pi \frac{u^2}{A} \frac{(2\ell+q-2)!}{(2\ell-2)!} \\ &\times \left[\sum_{h\neq 0} \frac{(2\ell+q) J_{2\ell+q}(K_h u) + J_{2\ell+q-2}(K_h u)}{(K_h u/2)^{q+2}} e^{-i2\ell\theta_h} + \frac{1}{(q+2)!} e^{-i4\theta_0}\, \delta_{\ell,2} \right], \end{aligned} \tag{79}$$

for $\ell \geq 2$. This expression allows one to evaluate the lattice sums $S_{2\ell}^{(1)}$.

B Static Limit of Lattice Sums

For $\ell \geq 2$, the dynamic lattice sums are given by the formula (Chin *et al.* 1994):

$$S_\ell^Y(k)\, J_\ell(ku) \;=\; -i^\ell\, \frac{4}{A} \sum_h \frac{J_\ell(K_h u)}{{K_h}^2 - k^2}\, e^{i\ell\theta_h}\,, \tag{80}$$

and, in the limit $k \to 0$, we equate lowest order terms on the left and right–hand sides, to obtain:

$$S_\ell^Y(k) \left(\frac{ku}{2}\right)^\ell \frac{1}{\ell!} \;=\; -i^\ell\, \frac{4}{A} \sum_h \frac{J_\ell(K_h u)}{{K_h}^2}\, e^{i\ell\theta_h}\,. \tag{81}$$

To evaluate the right hand side in (81), we use the Poisson summation formula (Jones 1966):

$$\sum_h f(\mathbf{K}_h) = \frac{A}{(2\pi)^2} \sum_p F(\mathbf{R}_p)\,. \tag{82}$$

Now,

$$
\begin{aligned}
F(\mathbf{R}_p) &= \int f(\mathbf{K})\, e^{-i\mathbf{K}\cdot\mathbf{R}_p}\, d\mathbf{K} \\
&= \int \frac{J_\ell(Ku)}{K^2}\, e^{i\ell\theta_h}\, e^{-iKR_p\cos(\theta-\varphi_p)}\, K dK d\theta \\
&= 2\pi(-i)^\ell\, e^{i\ell\varphi_p} \int_0^\infty dK\, \frac{J_\ell(Ku)\, J_\ell(KR_p)}{K}\,. \qquad (83)
\end{aligned}
$$

Thus, for $\ell > 0$, we have (Abramowitz & Stegun 1972):

$$
F(\mathbf{R}_p) = 2\pi(-i)^\ell\, e^{i\ell\varphi_p} \begin{cases} 0 & ;\ R_p < u, \\ \\ \frac{1}{2\ell}\left(\frac{u}{R_p}\right)^\ell & ;\ R_p \geq u, \end{cases} \qquad (84)
$$

and

$$
\sum_h \frac{J_\ell(K_h u)}{{K_h}^2}\, e^{i\ell\theta_h} \;=\; \frac{A}{(2\pi)^2}\, 2\pi(-i)^\ell\, \frac{1}{2\ell} \sum_{p\neq 0} \left(\frac{u}{R_p}\right)^\ell e^{i\ell\varphi_p}\,. \qquad (85)
$$

Consequently, in the limit $k \to 0$, we obtain:

$$
S_\ell^Y(k) \left(\frac{ku}{2}\right)^\ell \frac{1}{\ell!} \;=\; -i^\ell\, \frac{4}{A}\, \frac{A}{4\pi\ell}\, (-i)^\ell \sum_{p\neq 0} \left(\frac{u}{R_p}\right)^\ell e^{i\ell\varphi_p}\,, \qquad (86)
$$

and

$$
S_\ell^Y(k) \;=\; -\frac{2^\ell\,(\ell-1)!}{\pi\, k^\ell} \sum_{p\neq 0} \left(\frac{1}{R_p}\right)^\ell e^{i\ell\varphi_p}\,. \qquad (87)
$$

Thus, we establish to leading order for small k (McPhedran & Dawes 1992):

$$
S_\ell^Y(k) = Y_\ell(k) \sum_{p\neq 0} \left(\frac{1}{R_p}\right)^\ell e^{i\ell\varphi_p} = Y_\ell(k) S_\ell. \qquad (88)
$$

Here, we have used the behaviour of the Bessel functions for small arguments (see Abramowitz & Stegun 1972):

$$
J_\ell(z) = \left(\frac{z}{2}\right)^\ell \frac{1}{\ell\,!}\,, \text{ for } \ \ell \geq 0\,, \qquad (89)
$$

$$
Y_\ell(z) = -\frac{1}{\pi} \left(\frac{2}{z}\right)^\ell (\ell-1)!\,, \text{ for } \ \ell > 0\,. \qquad (90)
$$

Now, for $\ell \geq 1$, Eq.(32) takes the form:

$$
S_{2\ell}^Y(k) J_{2\ell+q}(ku) = -\frac{4}{A}(-1)^\ell \sum_h \left(\frac{k}{K_h}\right)^q \frac{J_{2\ell+q}(K_h u)}{K_h^2 - k^2} e^{-i2\ell\theta_h}, \qquad (91)
$$

and, in the limit $k \to 0$, by substituting (88) and (89) in (91) we have:

$$S_{2\ell}\left[-\frac{1}{\pi}\left(\frac{2}{k}\right)^{2\ell}(2\ell-1)!\right]\ \left[\left(\frac{ku}{2}\right)^{2\ell+q}\frac{1}{(2\ell+q)!}\right]$$
$$= -\frac{4}{A}(-1)^{\ell}\sum_h\left(\frac{k}{K_h}\right)^q\frac{J_{2\ell+q}(K_hu)}{K_h^2}e^{-i2\ell\theta_h}. \qquad (92)$$

This leads us to the formula:

$$u^{2\ell}S_{2\ell} = (-1)^{\ell}\pi\frac{u^2}{A}\frac{(2\ell+q)!}{(2\ell-1)!}\sum_h\frac{J_{2\ell+q}(K_hu)}{(K_hu/2)^{q+2}}e^{-i2\ell\theta_h}. \qquad (93)$$

From (93), separating out the term for $h=(0,0)$, we obtain exactly (73).

C Evaluation of the Scattering Matrix

In this appendix we present, in a form suitable for programming, the explicit values of the components of the scattering matrix ($M_n^{(\alpha\beta)}$). This form was also verified by a numerical test.

$$M_n^{(aa)} = (F_6F_3+F_2F_7)/(F_1F_6+F_2F_5), \qquad (94)$$
$$M_n^{(ab)} = -\mathrm{i}(F_6F_4-F_2F_8)/(F_1F_6+F_2F_5), \qquad (95)$$
$$M_n^{(ba)} = \mathrm{i}(F_5F_3-F_1F_7)/(F_1F_6+F_2F_5), \qquad (96)$$
$$M_n^{(bb)} = (F_5F_4+F_1F_8)/(F_1F_6+F_2F_5). \qquad (97)$$

The $n-$ dependence of the coefficients F_i has been suppressed in the above equations and further in the text of this appendix. The expressions for the coefficients F_i are given by

$$F_1 = E_1-(C_1D_1+C_5D_2)/\mathcal{D}, \qquad (98)$$
$$F_2 = E_2-(-C_2D_1+C_6D_2)/\mathcal{D}, \qquad (99)$$
$$F_3 = -E_3+(C_3D_1+C_7D_2)/\mathcal{D}, \qquad (100)$$
$$F_4 = -E_4+(-C_4D_1+C_8D_2)/\mathcal{D}, \qquad (101)$$
$$F_5 = E_5-(C_1D_3-C_5D_4)/\mathcal{D}, \qquad (102)$$
$$F_6 = E_6-(C_2D_3+C_6D_4)/\mathcal{D}, \qquad (103)$$
$$F_7 = -E_7+(C_3D_3-C_7D_4)/\mathcal{D}, \qquad (104)$$
$$F_8 = -E_8+(C_4D_3+C_8D_4)/\mathcal{D}, \qquad (105)$$

where

$$\mathcal{D} = \frac{n^2}{r_c^2}J_n(k_b^{(1)}r_c)J_n(k_a^{(1)}r_c)-k_a^{(1)}k_b^{(1)}J_n'(k_b^{(1)}r_c)J_n'(k_a^{(1)}r_c), \qquad (106)$$

and

$$E_1 = \frac{2n\mu^{(0)}}{r_c}\left[k_a^{(0)}J_n'(k_a^{(0)}r_c) - \frac{1}{r_c}J_n(k_a^{(0)}r_c)\right], \tag{107}$$

$$E_2 = \mu^{(0)}\left\{\left[(k_b^{(0)})^2 - 2\frac{n^2}{r_c^2}\right]J_n(k_b^{(0)}r_c) + 2\frac{k_b^{(0)}}{r_c}J_n'(k_b^{(0)}r_c)\right\}, \tag{108}$$

$$E_3 = \frac{2n\mu^{(0)}}{r_c}\left[k_a^{(0)}Y_n'(k_a^{(0)}r_c) - \frac{1}{r_c}Y_n(k_a^{(0)}r_c)\right], \tag{109}$$

$$E_4 = \mu^{(0)}\left\{\left[(k_b^{(0)})^2 - 2\frac{n^2}{r_c^2}\right]Y_n(k_b^{(0)}r_c) + 2\frac{k_b^{(0)}}{r_c}Y_n'(k_b^{(0)}r_c)\right\}, \tag{110}$$

$$E_5 = \frac{2\mu^{(0)}}{r_c}\left[-k_a^{(0)}J_n'(k_a^{(0)}r_c) + \frac{n^2}{r_c}J_n(k_a^{(0)}r_c)\right] - (k_a^{(0)})^2(2\mu^{(0)} + \lambda^{(0)})J_n(k_a^{(0)}r_c), \tag{111}$$

$$E_6 = 2\mu^{(0)}\frac{n}{r_c}\left[k_b^{(0)}J_n'(k_b^{(0)}r_c) - \frac{1}{r_c}J_n(k_b^{(0)}r_c)\right], \tag{112}$$

$$E_7 = \frac{2\mu^{(0)}}{r_c}\left[-k_a^{(0)}Y_n'(k_a^{(0)}r_c) + \frac{n^2}{r_c}Y_n(k_a^{(0)}r_c)\right] - (k_a^{(0)})^2(2\mu^{(0)} + \lambda^{(0)})Y_n(k_a^{(0)}r_c), \tag{113}$$

$$E_8 = 2\mu^{(0)}\frac{n}{r_c}\left[k_b^{(0)}Y_n'(k_b^{(0)}r_c) - \frac{1}{r_c}Y_n(k_b^{(0)}r_c)\right]. \tag{114}$$

Also,

$$D_1 = \frac{2n\mu^{(1)}}{r_c}\left[k_a^{(1)}J_n'(k_a^{(1)}r_c) - \frac{1}{r_c}J_n(k_a^{(1)}r_c)\right], \tag{115}$$

$$D_2 = \mu^{(1)}\left\{\left[(k_b^{(1)})^2 + 2\frac{n^2}{r_c^2}\right]J_n(k_b^{(1)}r_c) + 2\frac{k_b^{(1)}}{r_c}J_n'(k_b^{(1)}r_c)\right\}, \tag{116}$$

$$D_3 = \frac{2\mu^{(1)}}{r_c}\left[-k_a^{(1)}J_n'(k_a^{(1)}r_c) + \frac{n^2}{r_c}J_n(k_a^{(1)}r_c)\right] - (k_a^{(1)})^2(2\mu^{(1)} + \lambda^{(1)})J_n(k_a^{(1)}r_c), \tag{117}$$

$$D_4 = 2\mu^{(1)}\frac{n}{r_c}\left[k_b^{(1)}J_n'(k_b^{(1)}r_c) - \frac{1}{r_c}J_n(k_b^{(1)}r_c)\right]. \tag{118}$$

Finally

$$C_1 = -k_b^{(1)}J_n'(k_b^{(1)}r_c)k_a^{(0)}J_n'(k_a^{(0)}r_c) + \frac{n^2}{r_c^2}J_n(k_b^{(1)}r_c)J_n(k_a^{(0)}r_c), \tag{119}$$

$$C_2 = -\frac{n}{r_c}\left[k_b^{(1)}J_n'(k_b^{(1)}r_c)J_n(k_b^{(0)}r_c) - k_b^{(0)}J_n'(k_b^{(0)}r_c)J_n(k_b^{(1)}r_c)\right], \tag{120}$$

$$C_3 = -k_b^{(1)}J_n'(k_b^{(1)}r_c)k_a^{(0)}Y_n'(k_a^{(0)}r_c) + \frac{n^2}{r_c^2}J_n(k_b^{(1)}r_c)Y_n(k_a^{(0)}r_c), \tag{121}$$

$$C_4 = -\frac{n}{r_c}\left[k_b^{(1)}J_n'(k_b^{(1)}r_c)Y_n(k_b^{(0)}r_c) - k_b^{(0)}Y_n'(k_b^{(0)}r_c)J_n(k_b^{(1)}r_c)\right], \tag{122}$$

$$C_5 = \frac{n}{r_c}\left[k_a^{(1)}J_n'(k_a^{(1)}r_c)J_n(k_a^{(0)}r_c) - k_a^{(0)}J_n'(k_a^{(0)}r_c)J_n(k_a^{(1)}r_c)\right], \tag{123}$$

$$C_6 = -k_a^{(1)}J_n'(k_a^{(1)}r_c)k_b^{(0)}J_n'(k_b^{(0)}r_c) + \frac{n^2}{r_c^2}J_n(k_a^{(1)}r_c)J_n(k_b^{(0)}r_c), \tag{124}$$

$$C_7 = \frac{n}{r_c}\left[k_a^{(1)}J_n'(k_a^{(1)}r_c)Y_n(k_a^{(0)}r_c) - k_a^{(0)}Y_n'(k_a^{(0)}r_c)J_n(k_a^{(1)}r_c)\right], \tag{125}$$

$$C_8 = -k_a^{(1)}J_n'(k_a^{(1)}r_c)k_b^{(0)}Y_n'(k_b^{(0)}r_c) + \frac{n^2}{r_c^2}J_n(k_a^{(1)}r_c)Y_n(k_b^{(0)}r_c). \tag{126}$$

References

Abramowitz, M. & Stegun, I. A. 1972 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* New York: Dover.

Ashcroft, N.W. & Mermin, N.D. 1976 *Solid State Physics.* Philadelphia: Saunders College.

Atkin, R. J. & Fox, N. 1980 *An Introduction to the Theory of Elasticity.* New York: Longman.

Berman, C. L. & Greengard, L. 1994 A renormalization method for the evaluation of lattice sums. *J. Math. Phys.* **35**, 6036-6048.

Chin, S. K., Nicorovici, N. A. & McPhedran, R. C. 1994 Green's function and lattice sums for electromagnetic scattering by a square array of perfectly conducting cylinders. *Phys. Rev. E* **49**, 4590-4602.

Helsing, J. 1994 Bounds on the shear modulus of composites by interface integral methods. *J. Mech. Phys. Solids* **42**, 1123-1138.

Honein, T. & Herrmann, G 1990 On bonded inclusions with circular or straight boundaries in plane elastostatics. *Trans. ASME* **57**, 850-856.

Jones, D. S. 1966 *Generalized Functions.* London: McGraw-Hill.

McPhedran, R. C. & Dawes, D. H. 1992 Lattice sums for an electromagnetic problem. *J. Electromagn. Waves Appl.* **6**, 1327-1340.

McPhedran, R. C. & Movchan, A. B. 1994 The Rayleigh multipole method for linear elasticity. *J. Mech. Phys. Solids* **42**, 711-727.

McPhedran, R. C., Poulton, C. G., Nicorovici, N. A. & Movchan, A. B. 1995 Low Frequency Corrections to the Static Effective Dielectric Constant of a Two Dimensional Composite Material. Submitted to *Proc. R. Soc. Lond. A.*

Muskhelishvili, N. I. 1963 *Some Basic Problems of Mathematical Theory of Elasticity.*

Noordhoff: Groningen.

Nicorovici, N. A., McPhedran, R. C. & Petit, R. 1994 Efficient calculation of the Green's function for electromagnetic scattering by gratings. *Phys. Rev. E* **49**, 4563-4577.

Nicorovici, N. A. & McPhedran, R. C. 1994 Lattice sums for off–axis electromagnetic scattering by gratings. *Phys. Rev. E* **50**, 3143-3160.

Nicorovici, N. A., McPhedran, R. C. & Bao Ke-Da 1995a Propagation of electromagnetic waves in periodic lattices of spheres: Green's function and lattice sums. *Phys. Rev. E* **51**, 690-702.

Nicorovici, N. A., McPhedran, R. C. & Botten, L. C. 1995b Photonic bad gaps for arrays of perfectly conducting cylinders, *Phys. Rev. E* **52**, 1135-1145.

Nicorovici, N. A., McPhedran, R. C. & Botten, L. C. 1995c Photonic band gaps: Non-commuting limits and the "acoustic band", *Phys. Rev. Lett.* **75**, 1507-1510 (1995).

Perrins, W. T., McKenzie, D. R. & McPhedran, R. C. 1979 Transport properties of regular arrays of cylinders. *Proc. R. Soc. Lond.* A **369**, 207-225.

Rayleigh, Lord 1892 On the influence of obstacles arranged in rectangular order upon the properties of medium. *Phil. Mag.* **34**, 481-502.

Willis, J. R. & Movchan, A. B. 1995 Dynamic Weight Functions for a Moving Crack. I. Mode I Loading. *J. Mech. Phys. Solids* **43**, 319-341.

Table 1: The static lattice sums S_2 and $S_4^{(1)}$ for a square array ($d = 1$, $\theta_0 = \pm\pi/2$) obtained from (36) and (37), respectively, with $n_x = n_y^2$. For the summation over the reciprocal array, we have used $u = 1$, $n_x = n_y = 100$ and $q = 20$ in (59).

n_y	(36)	(43)	(37)	(59)
100	3.1013960166380	3.1415926535898	4.0382572302759	4.0784511611614
300	3.1282372218614		4.0650958286863	
600	3.1349204468239		4.0717789667710	
1200	3.1382579332707		4.0751164422996	

Table 2: The static lattice sums S_{4m} and $S_{4m}^{(1)}$ for a square array, as given by (38) and (39), and the corresponding values presented by Berman & Greengard (1994) and Helsing (1994). The slight differences between values are due to small inaccuracies in Bessel function evaluations required in (38) and (39).

m	S_{4m}	B&G, H	$S_{4m}^{(1)}$	B&G, H
4	3.1512120021539	3.1512120021539	4.0784511611614	4.0784511611614
8	4.2557730353652	4.2557730353652	4.5155154350371	4.5155154350371
12	3.9388490128280	3.9388490128280	3.8807308458546	3.8807308458546
16	4.0156950330248	4.0156950330250	4.0315403138194	4.0315403138195
20	3.9960967531758	3.9960967531763	3.9921986988966	3.9921986988967
24	4.0009768053041	4.0009768053038	4.0019541008378	4.0019541008376
28	3.9997558754741	3.9997558754744	3.9995117843433	3.9995117843435
32	4.0000610360529	4.0000610360538	4.0001220738684	4.0001220738691
36	3.9999847412676	3.9999847412680	3.9999694826486	3.9999694826489
40	4.0000038147011	4.0000038147010	4.0000076294096	4.0000076294095
44	3.9999990463246	3.9999990463259	3.9999980926513	3.9999980926523
48	4.0000002384159	4.0000002384186	4.0000004768351	4.0000004768372
52	3.9999999403955	3.9999999403954	3.9999998807911	3.9999998807907
56	4.0000000149016	4.0000000149012	4.0000000298027	4.0000000298023
60	3.9999999962742	3.9999999962747	3.9999999925489	3.9999999925494
64	4.0000000009333	4.0000000009313	4.0000000018644	4.0000000018626
68	3.9999999997608	3.9999999997672	3.9999999995284	3.9999999995343
72	4.0000000000562	4.0000000000582	4.0000000001154	4.0000000001164
76	3.9999999999857	3.9999999999854	3.9999999999720	3.9999999999709
80	4.0000000000067	4.0000000000036	4.0000000000095	4.0000000000073
84	4.0000000000003	3.9999999999991	3.9999999999997	3.9999999999982
88	3.9999999999930	4.0000000000002	3.9999999999945	4.0000000000005
92	3.9999999999943	3.9999999999999	3.9999999999963	3.9999999999999
96	4.0000000000015	4.0000000000000	4.0000000000017	4.0000000000000
100	3.9999999999946	4.0000000000000	3.9999999999950	4.0000000000000

Table 3: The dynamic lattice sum quantities $D_l(k) = S_l^Y(k)/Y_l(k)$, defined in (44), are given for various values of l and k, illustrating their convergence towards the static lattice sums S_l.

l	$D_l(0.5)$	$D_l(0.05)$	$D_l(0.005)$	$D_l(0.0005)$	S_l
4	3.137568796	3.151077905	3.151210662	3.151211989	3.151212002
8	4.258107588	4.255796229	4.255773269	4.255773039	4.255773035
12	3.938517762	3.938845722	3.938848991	3.938849024	3.938849013
16	4.015761285	4.015695752	4.015695098	4.015695092	4.015695033
20	3.996084125	3.996096844	3.996096971	3.996096973	3.996096753