SINGULAR MOMENT MAPS AND QUATERNIONIC GEOMETRY

ANDREW SWANN

School of Mathematical Sciences, University of Bath,
Claverton Down, Bath BA2 7AY,
England
Email: A.F.Swann@maths.bath.ac.uk

1. Introduction. This paper is a survey of some recent work concerning the generalisations of the moment map construction appropriate for various different quaternionic geometries. The non-singular versions of these constructions are a little under ten years old, but in many examples one considers, the interesting cases are often singular. It is therefore useful to have some sort of general theory covering the singular case.

In symplectic geometry, Sjamaar & Lerman [SL] have provided such a theory, showing how one obtains a stratification of arbitrary symplectic quotients by symplectic manifolds. In the quaternionic cases, such strong results are not yet known. Indeed, for one type of quaternionic geometry such a result is false, as will be described below. However, the ideas of Sjamaar & Lerman do enable one to divide up the quaternionic quotients into well-behaved pieces.

We will start by describing the quaternionic geometries involved and the non-singular versions of the moment map construction. The emphasis will be on the similarities with symplectic geometry. The rest of the paper will then discuss the singular cases and give illustrative examples.

It is a pleasure to thank the organisers of the workshop on symplectic singularities for their kind hospitality in Warsaw.

2. Quaternionic Geometries. There are two distinct but closely related quaternionic geometries we wish to consider.

The first is the case of a *hyperKähler* manifold. This is a Riemannian manifold (M, g) which is Kähler in three different ways. More precisely, we have three almost complex structures I, J and K such that

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i) $IJ = K = -JI$,

and

ii) if one defines $\omega_I(X,Y) = g(X, IY)$, etc., then $d\omega_I = d\omega_J = d\omega_K = 0$.

Condition (i) says that I, J and K behave like the multiplicative unit quaternions and implies that T_xM is an $\mathbb{H}$-module. In particular, the dimension of M is necessarily a multiple of four.

Hitchin [H3] showed that conditions (i) and (ii) imply that I, J and K are integrable and hence Kähler. Thus the holonomy group of a hyperKähler manifold M^{4n} is a subgroup of $Sp(n)$. Noting that $Sp(n)$ is a subgroup of $SU(2n)$, implies Berger's result [Ber] that hyperKähler manifolds are Ricci-flat.

Observe that the metric g may be regarded as a by-product of the three symplectic forms ω_I, ω_J and ω_K, since, regarding them as maps $TM \to T^*M$, we have $K = \omega_J^{-1} \circ \omega_I$ and this together with ω_K determines g. Also, note that for any point $(a,b,c) \in S^2 \subset \mathbb{R}^3$, the almost complex structure $aI + bJ + cK$ is integrable and Kähler with respect to g.

Examples of hyperKähler manifolds include flat-space $\mathbb{R}^{4n}$, with the structure obtained by identifying it with $\mathbb{H}^n$, and the torus $\mathbb{T}^{4n}$.

The main non-trivial, compact example is that of a $K3$-surface: this is a complex surface with $\pi_1 = 0$ and $c_1 = 0$; for example, $(x^4 + y^4 + z^4 + w^4 = 0) \subset \mathbb{C}P(3)$. The existence of a hyperKähler metric on these surfaces is a consequence [Bea] of Yau's proof [Yau] of the Calabi conjecture. Higher-dimensional, compact examples may be built out of these building blocks [Bea]. Although these metrics are not known explicitly, some of them may be obtained via a deformation argument [Pa] known as the Kummer construction, which Joyce [J1–3] has recently extended to obtain the first examples of compact manifolds with holonomy $Spin(7)$ (dimension 8) and G_2 (dimension 7).

Suppose a Lie group G acts on a hyperKähler manifold M, preserving the complex structures I, J and K and the metric g. Using the three symplectic structures ω_I, ω_J, ω_K we may attempt to define symplectic moment maps $\mu_I, \mu_J, \mu_K \colon M \to \mathfrak{g}^*$ by

$$d\langle \mu_I, X\rangle = i_X \omega_I, \qquad \text{etc.}$$

If these exist, they may be combined into one *hyperKähler moment map*

$$\mu \colon M \to \mathfrak{g}^* \otimes \operatorname{Im} \mathbb{H},$$
$$\mu = \mu_I i + \mu_J j + \mu_K k.$$

This definition is reasonable as we have:

THEOREM [HKLR]. *If G acts freely and properly on a hyperKähler manifold M, preserving the hyperKähler structure and with moment map μ, then $\mu^{-1}(0)/G$ is a hyperKähler manifold of dimension* $\dim M - 4\dim G$.

Example. Let $U(1)$ act on $\mathbb{H}^n = \mathbb{R}^{4n}$ by left-multiplication by $e^{i\theta}$. Then a hyperKähler moment map for this action is given by

$$a + jb \mapsto i(\|a\|^2 - \|b\|^2 - 1) + 2k\bar{a}^t b,$$

for $a, b \in \mathbb{C}^n$. Note that although $U(1)$ does not act freely on the whole of $\mathbb{H}^n$, it does act freely on the zero set $\mu^{-1}(0)$ and so the above theorem may be applied.

The hyperKähler metric constructed this way is complete, because the metric on $\mathbb{H}^n$ is complete, $\mu^{-1}(0)$ is closed and $U(1)$ is compact. Topologically, the hyperKähler quotient is $T^* \mathbb{C}P(n-1)$, where the map to $\mathbb{C}P(n-1)$ is given by $(a, b) \mapsto [a]$, and in fact the metric obtained is one first described by Calabi [Ca1–2] via Kähler potentials.

Let us consider the four-dimensional case $T^* \mathbb{C}P(1)$. Near infinity this metric is asymptotically close to the standard metric on $\mathbb{C}^2/\{\pm 1\}$. Hitchin [H1] showed how the twistor construction could be used to obtain the metric on $T^* \mathbb{C}P(1)$ when this space is regarded as the minimal resolution of the singular space $\mathbb{C}^2/\{\pm 1\}$.

More generally one can look for hyperKähler metrics which are ALE, i.e. asymptotically locally Euclidean. Kronheimer [K2–3] showed that if the hyperKähler manifold is ALE with the topology of the minimal resolution of $\mathbb{C}^2/\Gamma$, for Γ any finite subgroup of $SU(2)$, then it may be obtained as a finite-dimensional hyperKähler quotient of flat space. Dancer [Da] extended Kronheimer's construction to obtain families of asymptotically locally flat hyperKähler metrics when Γ is dihedral.

Example. Another example may be loosely described as follows. Let $\mathcal{A}$ be the space of irreducible connections on a fixed principal bundle over $\mathbb{R}^4$, with compact structure group G, and let $\mathcal{G}$ be the gauge group. Then $\mathcal{A}$ may be identified with $\Omega^1(\mathbb{R}^4, \mathfrak{g})$, as an infinite-dimensional affine space, and this inherits a hyperKähler structure from $\mathbb{R}^4 = \mathbb{H}$. Given $A \in \mathcal{A}$, the curvature is $F_A = dA + A \wedge A$ and we may define a hyperKähler moment map for the action of $\mathcal{G}$ on $\mathcal{A}$, by

$$\mu(A) = F_A^-,$$

where $F_A^- = \frac{1}{2}(F_A - *F_A)$ is the anti-self-dual part of F_A regarded as an element of $\Omega^2(\mathbb{R}^4, \mathfrak{g})$. The resulting hyperKähler quotient $\mu^{-1}(0)/\mathcal{G}$ is the moduli space of self-dual instantons over $\mathbb{R}^4$.

Note that various technical conditions need to be added to make this example precise. Also because the construction is infinite-dimensional, rather than finite-dimensional, one has to work a little more to apply the theorem of [HKLR]. However, many particular cases of such moduli spaces may be obtained via the finite-dimensional hyperKähler quotient construction; a few of these will be seen later. The assumption of irreducibility of the connections is present merely to ensure that the quotient is non-singular.

The second type of quaternionic geometry to be considered is that of *quaternionic Kähler* manifolds: as will be explained below, these should be regarded as the non-zero scalar curvature analogue of hyperKähler manifolds.

Let (M, g) be a Riemannian manifold equipped with a subbundle $\mathcal{G}$ of $\operatorname{End} TM$ such that $\mathcal{G}$ has rank 3 and *locally* has bases $\{I, J, K\}$ such that $I^2 = J^2 = K^2 = -1$ and $IJ = K = -JI$. Then we may construct a *globally* defined 4-form Ω by the local formula

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

If the dimension of M is at least 8, we say that M is *quaternionic Kähler* if $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection of g.

Calculating $\nabla\Omega$ is not always straightforward, so the following result is useful and indicates some relationship with symplectic geometry.

PROPOSITION [Sw]. *If* $\dim M \geqslant 12$ *then* M *is quaternionic Kähler if and only if* $d\Omega = 0$.

Recall that such a result is not true in almost Hermitian geometry.

The stabiliser of Ω under the action of $GL(4n,\mathbb{R})$ is the group $Sp(n)\,Sp(1) = (Sp(n)\times Sp(1))/\{(-1,-1)\}$. Thus a quaternionic Kähler manifold is a manifold whose holonomy group is a subgroup of $Sp(n)\,Sp(1)$. This implies [Sa] that quaternionic Kähler manifolds are Einstein. The curvature of $\mathcal{G}$ is a multiple of the scalar curvature s, so quaternionic Kähler manifolds of scalar curvature zero are locally hyperKähler. For this reason we will take the term quaternionic Kähler to include the condition that s be non-zero.

If M is four-dimensional, then $\mathcal{G}$ may be identified with Λ^2_- via the metric. The four-form Ω, is just -6 times the volume form and is so always parallel. Instead one defines M^4 to be quaternionic Kähler if it is self-dual and Einstein, with non-zero scalar curvature, since this gives M^4 the same curvature properties with respect to $\mathcal{G}$ as in higher dimensions.

Examples of quaternionic Kähler manifolds include the symmetric metrics on $\mathbb{H}P(n)$, $\mathrm{Gr}_2(\mathbb{C}^n)$, $\mathrm{Gr}_4(\mathbb{R}^n)$, $G_2/\,SO(4)$ and their non-compact duals [Wo]. There is one such example for each simple, simply-connected compact Lie group G. In dimension four, the only compact examples of positive scalar curvature are $S^4 = \mathbb{H}P(1)$ and $\mathbb{C}P(2)$ [H2].

If G acts on M preserving the quaternionic Kähler structure, Galicki & Lawson [GL] defined the *quaternionic Kähler moment map* to be

$$\mu\colon M \to \mathfrak{g}^* \otimes \mathcal{G},$$
$$\langle \mu, X\rangle = \tfrac{1}{\lambda}(\nabla X)^{\mathcal{G}},$$

where $(\nabla X)^{\mathcal{G}}$ is the component of the endomorphism ∇X lying in $\mathcal{G}$ and λ is a non-zero multiple, determined by the dimension, of the (constant) scalar curvature. Note that there are no arbitrary constants in this definition, unlike the construction of the symplectic moment map, and that the quaternionic Kähler moment map always exists. Also, as μ is vector-bundle-valued, there is only one level set to consider, namely $\mu^{-1}(0)$.

THEOREM [GL]. *Suppose* G *acts freely and properly on a quaternionic Kähler manifold* M *preserving the quaternionic Kähler structure* $(g,\mathcal{G})$. *Then* $\mu^{-1}(0)/\mathcal{G}$ *is a quaternionic Kähler manifold of dimension* $\dim M - 4\dim G$.

As μ is bundle-valued, it is useful to have an alternative method to compute $\mu^{-1}(0)$. Over each quaternionic Kähler manifold M, there is a hyperKähler manifold $\mathcal{U}(M)$, which is a principal $(\mathbb{H}^*/\pm 1 = SO(3)\times\mathbb{R}_{>0})$-bundle [Sw]. The action of $\mathbb{H}^*$ on $\mathcal{U}(M)$ does not preserve the complex structures I, J and K individually, but does preserve the two sphere $\{\, aI + bJ + cK : a^2+b^2+c^2 = 1\,\}$. If a group G acts on M preserving the quaternionic Kähler structure then this action lifts to an action on $\mathcal{U}(M)$ preserving the hyperKähler structure. On $\mathcal{U}(M)$ we may choose the unique hyperKähler moment map μ_{hk} such that $\mu_{\mathrm{hk}}(x\cdot q) = \bar{q}\,\mu_{\mathrm{hk}}(x)\,q$; this map always exists. In [Sw] it was shown that $\mu_{\mathrm{hk}}^{-1}(0) = \pi^{-1}\mu^{-1}(0)$, where $\pi\colon \mathcal{U}(M) \to M$ is the projection, and that

$$\mathcal{U}(\mu^{-1}(0)/G) = \mu_{\mathrm{hk}}^{-1}(0)/G,$$

i.e. the hyperKähler quotient of $\mathcal{U}(M)$ is the hyperKähler manifold associated to the quaternionic Kähler quotient of M.

Example. Let M be quaternionic projective space $\mathbb{H}P(n)$. Then $\mathcal{U}(M)$ is (the $\mathbb{Z}/2$-quotient of) $\mathbb{H}^{n+1} \setminus \{0\}$ with $\mathbb{H}^*$ acting by right-multiplication.

Consider the action of S^1 on $\mathbb{H}P(n)$ given by

$$e^{i\theta} \cdot [q_0, \dots, q_n] = [e^{i\theta} q_0, \dots, e^{i\theta} q_n].$$

Then the moment map is

$$\mu(q) = \bar{q}^t i q = i(\|a\|^2 - \|b\|^2) + 2k\bar{a}^t b,$$

where $q = a + jb$ with $a, b \in \mathbb{C}^{n+1}$. Note that the above formula for μ is valid both for the quaternionic Kähler moment map on $\mathbb{H}P(n)$ and for the hyperKähler moment map μ_{hk} on $\mathbb{H}^{n+1} \setminus \{0\}$, in the former case it just needs to be interpreted as being $\mathcal{G}$-valued. If $q = a + jb$ lies in $\mu^{-1}(0)$, then a and b are orthogonal and of equal length. It is now straightforward to show [Ga] that the quaternionic Kähler quotient is

$$\mu^{-1}(0)/S^1 = \mathrm{Gr}_2(\mathbb{C}^{n+1}).$$

Example. Similarly the diagonal action of $Sp(1)$ on $\mathbb{H}P(n)$, gives $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{n+1})$ as the quaternionic Kähler quotient.

The above two examples are very special as the resulting quaternionic Kähler quotients are smooth and compact. For S^1-actions there is the following recent result:

THEOREM [Ba]. *Suppose N is a compact quaternionic Kähler manifold of positive scalar curvature such that N is the quaternionic Kähler quotient of some quaternionic Kähler manifold M by an S^1-action. Then $N = \mathrm{Gr}_2(\mathbb{C}^n)$.*

The proof of this may be briefly sketched as follows. If X is the vector field generated by the S^1-action and μ is the moment map, then

$$\frac{1}{\|X\|^2}((\nabla X) - \mu)$$

may be regarded as an element of $\Lambda^2 T^* M$ and defines a non-zero harmonic 2-form on N. This implies that the second Betti number $b_2(N)$ is non-zero. However, LeBrun & Salamon [LS] have shown that the only compact quaternionic Kähler manifold of positive scalar curvature with $b_2 > 0$ is the complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$. Their result in turn relies on Wiśniewski's classification of Fano manifolds of dimension $2n+1$, index $n+1$ and Picard number at least 2 [Wi] which is proved using techniques from Mori theory.

The above result partly reflects the lack of known examples of quaternionic Kähler manifolds of positive scalar curvature. In fact the following question is still open.

QUESTION. Is every compact quaternionic Kähler manifold of positive scalar curvature necessarily symmetric?

In dimensions 4 and 8 this question has been answered affirmatively by Hitchin [H2] and Poon & Salamon [PS] (see also [LS]), respectively.

Note that the situation for complete quaternionic Kähler metrics of negative scalar curvature is very different: LeBrun has shown [Le] that $\mathbb{C}^{2n}$ has infinitely many such metrics.

Thus, we are naturally lead to considering the quotient construction in the singular case.

3. Singular Quotients in HyperKähler Geometry. Let us start by recalling the results of Sjamaar & Lerman [SL] concerning the symplectic case.

THEOREM [SL]. *Suppose G acts properly and smoothly preserving a symplectic structure ω on M. If there is a moment map μ for this action, then the symplectic quotient $M//G = \mu^{-1}(0)/G$ is the union*

$$\bigcup_{(H)<G} S_{(H)},$$

where (H) runs over conjugacy classes of subgroups of G and

$$S_{(H)} = \left(M_{(H)} \cap \mu^{-1}(0)\right)/G,$$

with $M_{(H)}$ denoting the subset of M consisting of points whose stabilisers are conjugate in G to H. Moreover, $S_{(H)}$ is a smooth symplectic manifold (possibly not of pure dimension).

In fact, Sjamaar & Lerman go on to prove that this is also a stratification of the symplectic quotient.

The essence of the proof of the above theorem may be described as follows. Suppose $\mathcal{O} = G \cdot p \cong G/H$ is the orbit of a point p lying in the zero set of μ. Let

$$V = \frac{(T_p\mathcal{O})^\perp}{T_p\mathcal{O} \cap (T_p\mathcal{O})^\perp}$$

be the fibre of the symplectic normal bundle at p, where $\cdot^\perp$ denotes the complement with respect to the symplectic form. Then H acts freely on the product $T^*G \times V$ and $\mathcal{O}$ has a neighbourhood in M which is equivariantly symplectomorphic to the symplectic quotient $(T^*G \times V)//H$. On this manifold one now has a canonical form for the moment map and study of this leads to the above result (see also [GS]).

If we consider hyperKähler geometry, the above approach has a promising start.

THEOREM [DS1]. *Consider $\mathbb{H}^m$ as a flat hyperKähler manifold. Suppose G is a compact Lie group and H is a closed subgroup of G acting on $\mathbb{H}^m$ preserving all three complex structures. Then $T^*G^{\mathbb{C}}$ has a G-invariant hyperKähler structure such that the hyperKähler quotient of $T^*G^{\mathbb{C}} \times \mathbb{H}^m$ by H is a hyperKähler manifold.*

The crucial part of this result is the construction of the hyperKähler metric on $T^*G^{\mathbb{C}}$ which was originally done by Kronheimer [K1]. The idea is that $T^*G^{\mathbb{C}}$ may be identified G-equivariantly with a moduli space of solutions of Nahm's equations. These equations are for four functions $T_0, \dots, T_3\colon [0,1] \to \mathfrak{g}$ and state

$$\frac{dT_i}{dt} = [T_j, T_k] - [T_0, T_i],$$

where (i,j,k) runs over cyclic permutations of $(1,2,3)$. These equations may be interpreted as a hyperKähler moment map for the action of the group $\mathcal{G}_0^0$ of smooth maps from $f\colon [0,1] \to G$ which are the identity at $t = 0, 1$, on the affine space of all $\frac{d}{dt} + T_0 + T_1 i + T_2 j + T_3 k$, by

$$T_0 \mapsto f\,T_0 f^{-1} - \frac{df}{dt} f^{-1} \qquad \text{and} \qquad T_i \mapsto f\,T_i f^{-1}, \quad \text{for } i = 1,2,3.$$

One can now show that the quotient is indeed a hyperKähler manifold.

Nahm's equations are precisely the self-duality equations for the curvature of the $\mathbb{R}^3$-invariant connection $T_0\,dt + T_1\,dx_1 + T_2\,dx_2 + T_3\,dx_3$ on the trivial principal G-bundle over $[0,1] \times \mathbb{R}^3$ (for example, see [AH]). Thus the above moduli space is a particular example of a hyperKähler moduli space of instantons as mentioned above.

Unfortunately, unlike the symplectic case, the above manifold no longer provides a normal form for the hyperKähler moment map: note that any hyperKähler equivalence is necessarily an isometry and thus is much stronger than a symplectomorphism.

However, there is an alternative approach. Consider the symplectic case. Given $H \leqslant G$, set

$$M_H = \{\, x \in M : \operatorname{stab}_G x = H \,\}.$$

Then M_H is symplectic. Let M'_H denote the union of the connected components of M_H which meet the zero set of μ. Sjamaar & Lerman prove that $\mu|_{M'_H}$ is a moment map for the free action of $L := N_G(H)/H$ on M'_H and hence

$$S_{(H)} = M'_H /\!/ L.$$

This argument carries directly over to the hyperKähler case and we get

Theorem [DS2]. *If G acts properly and smoothly on a hyperKähler manifold M, then the hyperKähler quotient of M is a union*

$$\bigcup_{(H)<G} S_{(H)}$$

of smooth hyperKähler manifolds $S_{(H)}$, given as the hyperKähler quotients of M'_H by $L = N_G(H)/H$.

We do not yet know whether the above union is a stratification, or even a decomposition, of the hyperKähler quotient; though it would not be surprising if one does get a stratification this way.

Example. Suppose G is a classical Lie group ($U(n)$, $SU(n)$, $O(n)$ or $Sp(n)$). Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of the Lie algebra of G. Inside $\mathfrak{g}^{\mathbb{C}}$ we have the nilpotent variety $\mathcal{N}$ which consists of all nilpotent matrices in $\mathfrak{g}^{\mathbb{C}}$. This variety is preserved by the adjoint action of $G^{\mathbb{C}}$ and decomposes as a union of nilpotent orbits. Each orbit is a smooth complex manifold carrying the Kostant-Kirillov-Souriau complex symplectic structure, so is of even complex dimension. Kronheimer [K4] showed that each nilpotent orbit carries a hyperKähler structure. As above this is proved by identifying the orbit with a moduli space of solutions to Nahm's equations.

However, the nilpotent variety as a whole may be obtained as a finite-dimensional hyperKähler quotient of flat space [KS]. In the case of $G = SU(n)$ the construction is as follows (see also [Na]). The flat space we consider is

$$M = \bigoplus_{a=0}^{k-1} \left(\mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^{a+1}) \oplus \mathrm{Hom}(\mathbb{C}^{a+1}, \mathbb{C}^a)\right).$$

A point of this space may be conveniently represented as a diagram

$$\{0\} \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} \mathbb{C}^n.$$

This is given the structure of a finite-dimensional right $\mathbb{H}$-module by defining

$$(\alpha_a, \beta_a)j := (-\beta_a^*, \alpha_a^*),$$

where $\cdot^*$ denotes the Hermitian adjoint with respect to the standard inner products on $\mathbb{C}^a$ and $\mathbb{C}^{a+1}$. The group $K = U(1) \times U(2) \times \cdots \times U(n-1)$ acts in the usual way, factor by factor, on $\mathbb{C} \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^{n-1}$ and this induces an action of K on M preserving the hyperKähler structure. A hyperKähler moment map for this action is given at a point (α_a, β_a) of M by

$$\mu = i(\alpha_a\alpha_a^* - \beta_a^*\beta_a + \beta_{a+1}\beta_{a+1}^* - \alpha_{a+1}^*\alpha_{a+1}) + 2k(\alpha_a\beta_a - \beta_{a+1}\alpha_{a+1}).$$

Here the Lie algebra of $U(a)$ is identified with the skew-adjoint endomorphisms of $\mathbb{C}^a$. The hyperKähler quotient $\mu^{-1}(0)/K$ is then precisely the nilpotent variety in $\mathfrak{sl}(n, \mathbb{C})$. The identification with the nilpotent variety is via the map

$$(\alpha_a, \beta_a) \mapsto X := \alpha_{n-1}\beta_{n-1}.$$

As defined, X is an endomorphism of $\mathbb{C}^n$. However, when (α_a, β_a) lies in $\mu^{-1}(0)$ it follows from the moment map equations that $X^n = 0$ and hence X is nilpotent. In particular, X is trace-free and so lies in $\mathfrak{sl}(n, \mathbb{C})$.

Note that by considering a non-zero value of the above moment map, Nakajima [Na] has obtained the cotangent bundle

$$T^*\left(\frac{U(n)}{U(1)^n}\right)$$

as a (non-singular) hyperKähler quotient. In fact, Nakajima obtains a complete hyperKähler metric on the cotangent bundle of any flag manifold for $U(n)$ this way, but the method does not extend to flag manifolds for other groups.

3. Singular Quotients in Quaternionic Kähler Geometry. In the quaternionic Kähler case, despite the close links with the hyperKähler quotient construction, we do not get the quaternionic Kähler quotient as a union of quaternionic Kähler manifolds.

Suppose G acts properly on a quaternionic Kähler manifold M preserving the quaternionic Kähler structure. At point x of M, the stabiliser $H = \mathrm{stab}_G(x)$ acts on the bundle $\mathcal{G}$ of local almost complex structures. As H acts isometrically and preserves the orientation we have a map $\phi\colon H \to SO(3)$. We will consider two such maps to be equivalent if they are conjugate in $SO(3)$. Let $M_{(H),\phi}$ denote the set of points with stabiliser conjugate to H and representation $H \to SO(3)$ conjugate to ϕ. A similar notation without brackets will be used when conjugation is not to be considered. Write $S_{(H),\phi}$ for the corresponding part $(M_{(H),\phi} \cap \mu^{-1}(0))/G$ of the quaternionic Kähler quotient.

Two cases are quite straightforward [DS2]:

a) If $\dim \phi(H)$ is non-zero, then the set $M_{(H),\phi}$ does not meet the zero set of the moment. So these submanifolds do not contribute to the quaternionic Kähler quotient.

b) If $\phi(H)$ is trivial, then $M_{H,1}$ is a quaternionic Kähler submanifold of M and the restriction of μ to the components of $M_{H,1}$ meeting $\mu^{-1}(0)$ is the quaternionic Kähler moment map for the free action of $L = N_G(H)/H$ on $M_{H,1}$. Thus $S_{(H),1}$ is a quaternionic Kähler manifold.

As a consequence of the above two results, if all the stabilisers of the action of G on M are connected, then the quaternionic Kähler quotient of M by G is a union of quaternionic Kähler manifolds.

So far we have had no surprises, but:

c) If $\phi(H)$ is a cyclic subgroup of $SO(3)$, then $S_{(H),\phi}$ is locally Kähler.

Example. Let S^1 act on $\mathbb{H}P(n)$ by

$$e^{i\theta} \cdot [u, v] = [e^{pi\theta}u, e^{qi\theta}v],$$

where $(u, v) \in \mathbb{H}^{n+1} = \mathbb{H}^n \oplus \mathbb{H}$ and q and p are coprime integers with $q < p$. The moment map is

$$[u, v] \mapsto p\bar{u}^t iu + q\bar{v}iv.$$

Then the quaternionic Kähler quotient splits into three pieces

$$U \cup \mathrm{Gr}_2(\mathbb{C}^n) \cup \mathbb{C}P(n-1),$$

where U is open, dense and quaternionic Kähler. The part $\mathbb{C}P(n-1)$ comes from the points $[ja, 1]$, where $a \in \mathbb{C}^n$ has $|a|^2 = 1$. Such points have the cyclic group $\mathbb{Z}/p+q$ as stabiliser. Thus, if n is even we obtain a submanifold of the quotient which does not even have the correct dimension to be quaternionic Kähler.

The final case to consider is:

d) If $\phi(H)$ is finite but not cyclic, then $M_{H,\phi}$ is a totally real submanifold of M and hence $S_{(H),\phi}$ does not inherit any local almost complex structures from M.

This last case actually occurs, as the following example illustrates.

Example. Let $Sp(1) \cong SU(2)$ act on $\mathbb{H}P(4)$ via its irreducible representation on $\mathbb{H}^5 = S^9\mathbb{C}^2$. Suppose $\{x, y\}$ is an orthonormal basis for $\mathbb{C}^2 = \mathbb{H}$ such that $y = jx$. Then the element $[x^9 + 6\sqrt{7}x^3y^6]$ lies in the zero set of the moment for the action of $Sp(1)$ and for this point, $\phi(H)$ is the dihedral group D_6.

Thus we can write an arbitrary quaternionic Kähler quotient as a union of manifolds, but in general these will not be quaternionic Kähler. It is intriguing to ask what special properties the geometric structure on the non-quaternionic parts has.

References

[AH] M. F. Atiyah and N. J. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press, Princeton, 1988.

[Ba] F. Battaglia, *S^1-quotients of quaternion-Kähler manifolds*, Proc. Amer. Math. Soc. (to appear).

[Bea] A. Beauville, *Variétés Kähleriennes dont la 1ère classe de Chern est nulle*, J. Differential Geom. 18 (1983), 755–782.

[Ber] M. Berger, *Sur les groupes d'holonomie des variétés à connexion affines et des variétés riemanniennes*, Bull. Soc. Math. France 83 (1955), 279–330.

[Ca1] E. Calabi, *Métriques Kählériennes et Fibrés Holomorphes*, Ann. Scient. Éc. Norm. Sup. 12 (1979), 269–294.

[Ca2] E. Calabi, *Isometric Families of Kähler Structures*, in: The Chern Symposium, 1979, W.-Y. Hsiang, et al. (eds.), Springer-Verlag, 1980, 23–39.

[Da] A. S. Dancer, *Dihedral singularities and gravitational instantons*, J. Geom. Phys. 12 (1993) 77–91.

[DS1] A. S. Dancer and A. F. Swann, *Hyperkähler metrics associated to compact Lie groups*, Math. Proc. Camb. Phil. Soc. (to appear).

[DS2] A. S. Dancer and A. F. Swann, *The structure of quaternionic Kähler quotients*, preprint, 1995.

[Ga] K. Galicki, *A generalization of the momentum mapping construction for quaternionic Kähler manifolds*, Comm. Math. Phys. 108 (1987), 117–138.

[GL] K. Galicki and H. B. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. 282 (1988), 1–21.

[GS] V. Guillemin and S. Sternberg, *A normal form for the moment map*, In: Differential geometric methods in mathematical physics, S. Sternberg (ed.), Reidel Publishing Company, Dordrecht, 1984, 161–175.

[H1] N. J. Hitchin, *Polygons and Gravitons*, Math. Proc. Camb. Phil. Soc. 85 (1979), 465–476.

[H2] N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. 43 (1981), 133–150.

[H3] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. 55 (1987), 59–126.

[HKLR] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *HyperKähler metrics and supersymmetry*, Comm. Math. Phys. 108 (1987), 535–589.

[J1] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2. I*, J. Differential Geom. (to appear).

[J2] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2. II*, J. Differential Geom. (to appear).

[J3] D. D. Joyce, *Compact Riemannian 8-manifolds with holonomy* $\mathrm{Spin}(7)$, preprint, 1995.

[KS] P. Z. Kobak and A. F. Swann, *Classical nilpotent orbits as hyperKähler quotients*, International J. Math. (to appear).

[K1] P. B. Kronheimer, *A hyperkähler structure on the cotangent bundle of a complex Lie group*, MSRI preprint (1988).

[K2] P. B. Kronheimer, *The construction of* ALE *spaces as hyperKähler quotients*, J. Differential Geom. 29 (1989), 665–683.

[K3] P. B. Kronheimer, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. 29 (1989), 685–697.

[K4] P. B. Kronheimer, *Instantons and the geometry of the nilpotent variety*, J. Differential Geom. 32 (1990), 473–490.

[Le] C. R. LeBrun, *On complete quaternionic-Kähler manifolds*, Duke Math. J. 63 (1991), 723–743.

[LS] C. R. LeBrun and S. M. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. 118 (1994), 109–132.

[Na] H. Nakajima, *Instantons on* ALE *spaces, quiver varieties and Kac-Moody algebras*, Duke Math. J. 76 (1994), 365–416.

[Pa] D. N. Page, *A physical picture of the $K3$ gravitational instanton*, Phys. Lett. 80B (1978), 55–57.

[PS] Y. S. Poon and S. M. Salamon, *Eight-dimensional quaternionic Kähler manifolds with positive scalar curvature*, J. Differential Geom. 33 (1991), 363–378.

[Sa] S. M. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. 67 (1982), 143–171.

[SL] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. Math. 134 (1991), 375–422.

[Sw] A. F. Swann, *HyperKähler and quaternionic Kähler geometry*, Math. Ann. 289 (1991), 421–450.

[Wi] J. A. Wiśniewski, *On Fano manifolds of large index*, Manuscripta Math. 70 (1991), 145–152.

[Wo] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. 14 (1965), 1033–1047.

[Yau] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure and Appl. Math. 31 (1978), 339–411.