ASYMPTOTIC SELF-SIMILAR GLOBAL BLOW-UP FOR A QUASILINEAR HEAT EQUATION

by

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Abstract

We study the asymptotic behaviour near finite blow-up time $t = T$ of the solutions to the one-dimensional degenerate quasilinear parabolic equation

$$u_t = (u^\sigma u_x)_x + u^\beta \quad \text{in} \;\; \mathbf{R} \times (0,T); \; \sigma > 0, \; 1 < \beta < \sigma + 1,$$

with bounded, nonnegative, compactly supported initial data. This parameter range corresponds to global blow-up where $u(x,t) \to \infty$ as $t \to T^-$ for any $x \in \mathbf{R}$. We prove that the rescaled function

$$f(\xi,t) = (T-t)^{1/(\beta-1)} u(\xi (T-t)^m, t), \quad m = \frac{\beta - (\sigma+1)}{2(\beta-1)} < 0,$$

converges uniformly as $t \to T$ to the unique compactly supported, symmetric self-similar profile $\theta(\xi) \geq 0$ satisfying a nonlinear ODE. The proof is based on the intersections comparison (the Sturmian argument) with a two-parametric family of self-similar solutions.

Key words. Quasilinear heat equation, global blow-up, asymptotic behaviour, self-similar solution, intersection comparison.

AMS (MOS) subject classification. 35K55, 35K65

1. Introduction. Main result

1.1. Preliminary properties of global blow-up. We consider the Cauchy problem for the quasilinear parabolic equation describing a combustion process in a one-dimensional medium where both the heat conduction coefficient and the heat source depend on the temperature:

$$u_t = (u^\sigma u_x)_x + u^\beta \quad \text{for } x \in \mathbf{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in } \mathbf{R},\ u_0 \not\equiv 0,\ \sup u_0 < \infty, \tag{1.2}$$

where $\sigma > 0$ and $1 < \beta < \sigma + 1$ are fixed constants. This equation was introduced almost 20 years ago in [SZK] where it was studied in detail via formal and numerical methods. Nevertheless, the asymptotic behaviour of blow-up solutions remained an open problem.

We assume that initial data are compactly supported, the support (an open set) $\operatorname{supp} u_0 = \{x \in \mathbf{R} : u_0(x) > 0\}$ is bounded, and $u_0^\sigma(x)$ is Lipschitz continuous. Then there exists the unique local in time nonnegative weak solution $u(x,t)$ which is continuous and smooth in the positivity domain. See the survey paper [K] and the list of references in the book [SGK]. Since $u_0 \not\equiv 0$, the solution blows up in a finite time T depending on u_0 and σ, β [GKM].

We study the asymptotic behaviour of blow-up as $t \to T^-$ in the parameter range

$$1 < \beta < \sigma + 1 \quad (\sigma > 0), \tag{1.3}$$

which corresponds to *global blow-up*, see [G1], [G2] and [SGK, Chapter 4] for extensive references. Observe that the global blow-up range (1.3) does not exist for the *semilinear* heat equation with $\sigma = 0$:

$$u_t = u_{xx} + u^\beta, \quad \beta > 1.$$

Its single point blow-up properties were studied in detail last years. See a full list of references in [SGK, Chapter 4]. Global blow-up for quasilinear heat equations was less considered in the mathematical literature.

Let us state some already known properties of blowing up solutions in the range (1.3). Let

$$h_+(t) = \sup \operatorname{supp} u(\cdot,t) \ \text{ and } \ h_-(t) = \inf \operatorname{supp} u(\cdot,t) \tag{1.4}$$

be the right and left interface of the solution which are continuous functions on $[0,T)$. The fact that for $\beta \in (1, \sigma+1)$ $\operatorname{supp} u(\cdot,t)$ grows without bound as $t \to T$ was proved in [GKM]. More precise analysis shows that as $t \to T$

$$h_\pm(t) = \pm\, \xi_*(T-t)^m(1+o(1)) \to \pm\infty, \quad m = \frac{\beta - (\sigma+1)}{2(\beta-1)} < 0, \tag{1.5}$$

where $\xi_*(\sigma,\beta) > 0$ is a constant, see [G2] and [SGK, p. 266]. Moreover, for an arbitrary compactly supported u_0 (cf. [GP2])

$$u(x,t) \to \infty \quad \text{as } t \to T \text{ for all } x \in \mathbf{R}. \tag{1.6}$$

1.2. MAIN STABILITY RESULT. In the present paper we complete the study of global blow-up of compactly supported solutions. Our main result says that the asymptotic behaviour of $u(x,t)$ as $t \to T$ is described by a self-similar solution of equation (1.1) of the form

$$u_*(x,t) = (T-t)^{-1/(\beta-1)}\theta(\xi), \quad \xi = x/(T-t)^m, \tag{1.7}$$

where $\theta \geq 0$ solves the nonlinear ordinary differential equation

$$\mathbf{A}(\theta) \equiv (\theta^\sigma \theta')' - m\theta'\xi - \frac{\theta}{\beta-1} + \theta^\beta = 0 \quad \text{for } \xi \in \mathbf{R}. \tag{1.8}$$

Equation (1.8) is known to admit a unique non-constant, nonnegative solution which is a symmetric compactly supported monotone function $\theta_* = \theta_*(|\xi|) \geq 0$, see [G2] and [SGK, p. 187]. The constant ξ_* in (1.5) is exactly $\xi_* = 0.5\,\text{meas}\,(\text{supp}\,\theta_*)$. Observe that the uniqueness of such a nontrivial solution θ_* to the ODE (1.8) is a straightforward consequence of the evolution estimate (1.4), see [SGK, p. 265] and Section 2 below.

We now state the main result of the paper.

Theorem 1. *Let (1.3) hold and let T be the blow-up time of the compactly supported solution $u(x,t)$. Then the rescaled function*

$$f(\xi,t) = (T-t)^{1/(\beta-1)}u(\xi(T-t)^m, t) \tag{1.9}$$

satisfies

$$f(\xi,t) \to \theta_*(\xi) \quad \textit{as } t \to T^- \textit{ uniformly in } \mathbf{R}. \tag{1.10}$$

The self-similar rescaled function $f = f(\xi,\tau)$, where $\tau = -\log(T-t) \to \infty$ as $t \to T$ is the new time, satisfies the quasilinear parabolic equation

$$\begin{aligned} &f_\tau = \mathbf{A}(f) \quad \text{in } Q_0 = \mathbf{R} \times (\tau_0, \infty), \quad \tau_0 = -\log T, \\ &f(\xi,\tau_0) = f_0(\xi) \equiv T^{1/(\beta-1)}u_0(\xi T^m) \quad \text{in } \mathbf{R}, \end{aligned} \tag{1.11}$$

where $\mathbf{A}$ is the stationary operator in (1.8). Introducing the ω-limit set

$$\omega(f_0) = \{g \geq 0,\, g \in C(\mathbf{R}) :\ \exists\ \{\tau_j\} \to \infty\ f(\cdot,\tau_j) \to g(\cdot) \text{ uniformly in } \mathbf{R}\}, \tag{1.12}$$

que to uniqueness of θ_*, one can state (1.10) in the following form

$$\omega(f_0) = \{\theta \in C_0(\mathbf{R}) :\ \theta \text{ solves (1.8)},\ \theta \not\equiv 0\}. \tag{1.13}$$

It follows from the structure of operator (1.8) that all the nontrivial stationary solutions to equation (1.11) (including the flat one $\bar{f} \equiv (\beta-1)^{-1/(\beta-1)}$) must be *unstable* in the Lyapunov sense, that is a natural conclusion for blow-up problems. Nevertheless, due to the *finite-time* scaling (1.9) the rescaled solution is proved to lie on the *stable* (the most probably, center) manifold corresponding to the stationary profile θ_*.

In order to prove Theorem 1, we derive estimates of the rescaled solution $f(\xi,\tau)$ (Sections 2 and 3). Our *geometric stability analysis* (Section 4) is essentially based on of the intersection comparison (via the Sturmian argument) of the solution $u(x,t)$ with a two-parametric family of self-similar solutions of the form (1.7) shifted in space and time. We establish that a clear geometric structure of the family concentrated around the unique self-similar profile θ_* implies its high atractivity property on the stable manifold.

1.3. Few comments on comparison with other types of blow-up. The properties of the blowing up solutions to (1.1) in other parameters ranges are quite different.

(i) If $\beta > \sigma+1$, then under certain hypotheses on u_0 there exists *single point blow-up* of the solution to (1.1), (1.2), see different approaches in [G3], [GP1] and [SGK, p. 370]. The asymptotic self-similar behaviour was proved in [G5] to be self-similar via the construction of a standard Lyapunov function. (Let us mention that a similar construction also works for the range (1.3) [G6], but an extra intersection hypothesis on initial data appears which does not seem to be natural; the present geometric proof is general and simpler than that in [G6]).

(ii) The case $\beta = \sigma+1$ ($m=0$ in (1.7) and (1.9)) corresponds to *regional blow-up* in the localization domain of the length $L_s = 2\pi(\sigma+1)^{1/2}/\sigma$ [G1], see also [G4]. The asymptotic behaviour as $t \to T$ of symmetric nonincreasing compactly supported solutions is shown in [G2] to be of self-similar separable form. Then (1.10) is valid with $\xi \equiv x$ and

$$\theta_*(x) = \left[\frac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\left(\frac{\pi x}{L_s}\right)\right]^{1/\sigma} \quad \text{for } |x| \le L_s/2, \tag{1.14}$$

and $\theta_*(x) = 0$ for $x > L_s/2$. See also [SGK, p. 260]. A complete classification of compactlly supported blow-up patterns for (1.1), $\beta = \sigma+1$, is given in [BG].

(iii) For a class of the semilinear heat equations $u_t = u_{xx} + f(u)$ in $\mathbf{R}$, the global blow-up behaviour is known to exist in the case $u_t = u_{xx} + u\log^\beta(1+u)$ in the parameter range $1 < \beta < 2$. The asymptotic behaviour as $t \to T$ is described by a unique self-similar solution to a nonlinear evolution equation of the Hamilton-Jacobi type. See [SGK, p. 292] and [GV].

(iv) Finally, we expect that a similar to Theorem 1 global stability of blow-up self-similarity is true for the quasilinear equation with gradient-type diffusivity $u_t = (|u_x|^\sigma u_x)_x + u^\beta$, where $\sigma > 0$ and $1 < \beta < 1+\sigma$.

2. Eventual connectedness of the support

We begin with the property of eventual (in time) connectedness of the support. This analysis illustrates the basic ideas of intersections comparison of $u(x,t)$ with a family of self-similar solutions (1.7).

Lemma 2.1. *Let* $\operatorname{supp} u_0$ *be bounded. Then there exists* $t_c \in [0,T)$ *such that*

$$\operatorname{supp} u(\cdot,t) \quad \textit{is connected for all } t \in [t_c, T). \tag{2.1}$$

Proof. First of all, it is well-known that if $u(x_0,t_0) > 0$ for some $(x_0,t_0) \in Q = \mathbf{R} \times (0,T)$, then $u(x_0,t) > 0$ for all $t \in (t_0,T)$ [K].

We argue by contradiction. Assuming that (2.1) does not hold, without loss of generality, we may choose, say, a component $\tilde{\kappa}$ of the support $\operatorname{supp} u_0$ such that, for instance, the right interface $\tilde{h}_+(0)$ of the function

$$\tilde{u}_0(x) = \{u_0(x) \ \text{ on } \tilde{\kappa}, \ \ 0 \ \text{ on } \mathbf{R} \setminus \tilde{\kappa}\} \tag{2.2}$$

satisfies $\tilde{h}_+(0) < h_+(0)$, and moreover $\tilde{h}_+(t) < h_+(t)$ for all $t \in (0,T)$. Here $\tilde{h}_+(t)$ is the right interface of $\tilde{u}(x,t)$ being the solutions to (1.1) with the initial function (2.2). We suppose also that the solutions $\tilde{u}$ and u have the same blow-up time T. In fact, we are assuming that for all $t \in (0,T)$ $\operatorname{supp} u(\cdot,t)$ consists of at least two connected components, and $x = \tilde{h}_+(t) < h_+(t)$ is the right interface of the solution $\tilde{u}$. Then the solutions to the left and to the right of the curve $x = \tilde{h}_+(t)$ are independent (for all times there is a $\{u \equiv 0\}$-gap in between). We assume that the left one has the same blow-up time $T = T(u_0)$ (this plays no role in the analysis below).

For convenience, we omit tildes and denote $\tilde{u}$ by $u(x,t)$ with initial data u_0. Without loss of generality, we prove estimates of the right interface $h_+(t)$ of $u(x,t)$. We compare $u(x,t)$ with the self-similar solutions (1.7) shifted in space. Namely, given T and a we denote by $v(x,t) \equiv v(x,t;T,a)$ the function $u_*(x-a,t)$ which blows up at $t = T$, and $x = a$ is the maxima of the spatial profile for all $t \in (0,T)$.

Consider the solution $v(x,t;T,a_-)$ with $a_- = h_-(0) - \xi_* T^m$. Then the corresponding initial functions satisfy

$$\operatorname{supp} u_0 \cap \operatorname{supp} v(x,0) = \emptyset, \tag{2.3}$$

and moreover the supports touch each other at the end point $x = h_-(0)$. Therefore, the number of intersections $J(0)$ of the profiles $u_0(x)$ and $v(x,0)$ in $\mathbf{R}$ or, which is the same, the number of sign changes of the difference $w(x,0) = u_0(x) - v(x,0)$, satisfies

$$J(0) = 1. \tag{2.4}$$

Then the Sturmian (intersection comparison) argument says that the number of intersections does not increase in time, so that

$$J(t) \le 1 \quad \text{for all } t \in (0,T). \tag{2.5}$$

See applications for the quasilinear degenerate parabolic equations in [SGK, p. 240] and a full list of references therein.

On the other hand, since both the solutions $u(x,t)$ and $v(x,t)$ have the *same* blow-up time, (2.5) provides us with the lower bound on the interface

$$h_+(t) \ge a_- + \xi_*(T-t)^m \quad \text{for all } t \in (0,T), \tag{2.6}$$

where the function on the right-hand side is the right interface of the solution $v(x,t)$. Indeed, if (2.6) is false at some $t_1 \in (0,T)$ then the unique intersection must disappear at the interface $x = h_+(t)$ at a time $t_2 < t_1$, so that $J(t_1) = 0$ by (2.5), and hence

$$u(x,t_1) \le v(x,t_1) \quad \text{in } \mathbf{R} \text{ and } \overline{\operatorname{supp}}\, u(\cdot,t_1) \subset \operatorname{supp} v(\cdot,t_1) \tag{2.7}$$

(the second condition is true after, if necessary, a small shifting to the left of the profile $v(x, t_1)$). See [SGK, p. 242]. A similar estimate from above is derived for the left interface

$$h_-(t) \le a_+ - \xi_*(T-t)^m, \quad \text{with} \;\; a_+ = h_+(0) + \xi_* T^m. \tag{2.8}$$

Moreover, comparing $u(x,t)$ with $v(x,t;T,a_+)$ one can derive a lower bound on $h_+(t)$ of the form (cf. (2.6))

$$h_+(t) \le a_+ + \xi_*(T-t)^m, \tag{2.9}$$

and a similar one for h_-:

$$h_-(t) \ge a_- - \xi_*(T-t)^m. \tag{2.10}$$

Finally, since

$$\operatorname{supp} u_0 \subset [h_-(0), h_+(0)], \tag{2.11}$$

one can see from (2.3) that the $\operatorname{supp} u(\cdot, t_c)$ becomes connected if

$$|h_\pm(t_c) - h_\pm(0)| \ge S_0 = h_+(0) - h_-(0). \tag{2.12}$$

It then follows from (2.6) and (2.9) that (2.12) is true if

$$\xi_*[(T-t_c)^m - T^m] \ge S_0, \tag{2.13}$$

whence an upper bound on $t_c < T$. $\square$

Remark. Comparing in a similar way the solution $u(x,t)$ to the problem (1.1), (1.2) with the self-similar ones $v(x,t;T,a_\pm)$, from (2.6), (2.8) - (2.10) we easily derive estimate (1.5) with $o(1) = O((T-t)^{-m}) \to 0$ as $t \to T$.

3. Bounds on the rescaled solution

Bounds on the rescaled orbit and our stability analysis are based on the properties of the self-similar ODE (1.8).

3.1. Properties of self-similar profiles. Fix an arbitrary $\mu > 0$ and denote by $\theta(\xi;\mu)$ the solutions to (1.8) for $\xi > 0$ with the conditions

$$\theta(0;\mu) = \mu, \quad \theta'_\xi(0;\mu) = 0. \tag{3.1}$$

In the nonnegativity domain we set $\theta(-\xi;\mu) = \theta(\xi;\mu)$. Let us recall some peoperties of the family $\{\theta(\xi;\mu)\}$, which follow from the construction of the *unique* solutions θ_*. See [G1] and [SGK, p. 187]. Denote

$$k = (\beta-1)^{-1/(\beta-1)}, \quad \bar{\mu} = \left[\frac{\beta+\sigma+1}{(\beta-1)(\sigma+2)}\right].$$

Lemma 3.1. *There exists* $\mu_* \in (k, \bar{\mu})$ *such that*

(i) $\theta(\xi;\mu) > 0$ *for all* $\xi \in \mathbf{R}$ *if* $0 < \mu < \mu_*$.

(ii) $\theta(\xi;\mu) > 0$ *on the finite interval* $(-\xi_\mu, \xi_\mu)$ *and* $\theta(\pm\xi_\mu;\mu) = 0$ *if* $\mu \geq \mu_*$. *Moreover, the flux* $(\theta^{\sigma+1})'_\xi(\xi_\mu^-;\mu) < 0$ *if* $\mu > \mu_*$ *and* $= 0$ *if* $\mu = \mu_*$. *There holds*

$$\xi_\mu \to \infty \ \textit{and} \ \theta(\xi;\mu) \to \infty \quad \textit{as} \ \mu \to \infty \tag{3.2}$$

uniformly on compact subsets.

The value μ_* corresponds to the unique global nontrivial solution $\theta_*(\xi) \equiv \theta(\xi;\mu_*)$, $\theta_* \equiv 0$ for $|\xi| \geq \xi_*$, which is given by

$$\begin{aligned} \mu_* &= \sup\{\nu > 0 : \ \theta(\xi;\mu) > 0 \quad \text{in } \mathbf{R} \text{ for all } \mu \in (0,\nu)\} \\ &= \inf\{\nu > k : \ \theta(\xi;\mu) \quad \text{vanishes for all } \mu \geq \nu\}. \end{aligned} \tag{3.3}$$

3.2. Bounds. Denote by $\xi_+(\tau)$ and $\xi_-(\tau)$ the right and the left interface of the rescaled solution $f(\xi,\tau)$. It follows from (1.5) that

$$\xi_\pm(\tau) = \pm\,\xi_*(1 + o(1)) \quad \text{as } \tau \to \infty. \tag{3.4}$$

Therefore, the Cauchy problem is equivalent to the initial boundary value problem in a bounded domain with zero Dirichlet boundary condition. Moreover, due to Lemma 2.1 we have that

$$\text{meas}\,[\text{supp}\, f(\cdot,\tau) \setminus (-\xi_*, \xi_*)] \to 0 \quad \text{as } \tau \to \infty. \tag{3.5}$$

Proposition 3.2. *There exists a constant* $C_* > \mu_*$ *such that*

$$k < \|f(\cdot,\tau)\|_\infty \leq C_* \quad \textit{for all } \tau > \tau_0. \tag{3.6}$$

Proof. The lower bound follows by comparison of $u(x,t)$ with the flat solution to (1.1)

$$v_T(t) = k(T-t)^{-1/(\beta-1)}, \tag{3.7}$$

having the same blow-up time $T = T(u_0)$. Indeed, given arbitrary $t \in (0,T)$ the profile $u(x,t)$ must intersect $v_T(t)$, whence the estimate.

In order to prove the upper bound in (3.6), we choose $\mu \gg 1$ such that due to (3.2) $f_0(\xi) \leq \theta(\xi;\mu)$ in $(-\xi_\mu, \xi_\mu)$, and also $|\xi_\pm(\tau)| \leq \xi_\mu$ for all $\tau > \tau_0$ by (3.4). Then $f(\xi,\tau) \leq \theta(\xi;\mu)$ in $(-\xi_\mu,\xi_\mu) \times (\tau_0,\infty)$ by comparison, whence the estimate with $C_* = \mu$. $\square$

From (3.4) and the uniform upper bound (3.7) using the Bernstein estimate [K] we derive the gradient bound

$$|(f^\sigma)_\xi| \leq C \quad \text{in } Q, \tag{3.8}$$

and hence the ω-limit set (1.12) is well defined.

4. Proof of Theorem 1

Fix an arbitrary monotone sequence $\{\tau_j\} \to \infty$ such that $f(\cdot, \tau_j) \to g(\cdot) \in \omega(f_0)$ uniformly. It follows from (3.4), (3.5) that supp g is connected and

$$\text{supp } g \subseteq (-\xi_*, \xi_*). \tag{4.1}$$

4.1. Two-parametric family of rescaled solutions. For a fixed $a \in \mathbf{R}$ and a blow-up time $\tilde{T} > 0$ (in general, $\tilde{T} \neq T$), consider the self-similar solution $u_*(x-a,t)$ of the form (1.7) with the blow-up time $\tilde{T}$. The corresponding rescaled solution defined as in (1.9) with fixed $T = T(u_0)$ has the form

$$f_*(\xi, \tau; a, \lambda) = (1 + \lambda e^\tau)^{-1/(\beta-1)} \theta_*((\xi - a e^{m\tau})(1 + \lambda e^\tau)^{-m}), \tag{4.2}$$

where $\lambda = \tilde{T} - T$. Observe that all the functions from the two-parametric family $W = \{f_*(\cdot; a, \lambda)\}$ are weak solutions to the rescaled equation (1.11). One can see that if $\lambda = 0$ (i.e. $\tilde{T} = T$) then solution (4.2) is globally defined and as $\tau \to \infty$

$$f_*(\xi, \tau; a, 0) = \theta_*(\xi - a e^{m\tau}) \to \theta_*(\xi) \ \text{ uniformly on } \mathbf{R}. \tag{4.3}$$

On the other hand, if $\lambda < 0$ then $f_*(\xi, \tau)$ blows up in the finite time $\tau = -\log|\lambda|$. If $\lambda > 0$ then as $\tau \to \infty$

$$f_*(\xi, \tau) \to 0 \quad \text{uniformly on } \mathbf{R}. \tag{4.4}$$

By $J(\tau; a, \lambda)$, with $\lambda e^\tau > -1$, we denote the number of intersections (in ξ) of the profiles $f_*(\xi, \tau; a, \lambda)$ and $f(\xi, \tau)$.

Let us now introduce the family of perturbations of the self-similar profile $\theta_*(\xi)$:

$$\theta_*(\xi; b, \rho) = (1 + \rho)^{-1/(\beta-1)} \theta_*((\xi - b)(1 + \rho)^{-m}), \tag{4.5}$$

where $b \in \mathbf{R}$ and $\rho > -1$ are parameters. We have that $\theta_*(\xi; 0, 0) \equiv \theta_*(\xi)$ and

$$\theta_*(\xi; b, \rho) \to \theta_*(\xi) \quad \text{as } b, \rho \to 0 \text{ uniformly on } \mathbf{R}. \tag{4.6}$$

By the definitions of the families (4.2) and (4.5) we have

$$\theta_*(\xi; a e^{m\tau}, \lambda e^\tau) \equiv f_*(\xi, \tau; a, \lambda). \tag{4.7}$$

4.2. Proof of Theorem 1. We begin with the main intersection property of the family (4.2).

Proposition 4.1. *For any $|a| \gg 1$ there holds*

$$J(\tau; a, \lambda) \leq 1 \quad \textit{provided that } \ 1 + \lambda e^\tau > 0. \tag{4.8}$$

Proof. One can see that

$$\text{supp } u_0(x) \cap \text{supp } u_*(x - a, 0; T + \lambda) = \emptyset \tag{4.9}$$

provided that

$$a \geq h_+(0) + \xi_*(T + \lambda)^m \quad \text{or} \quad a \leq h_-(0) - \xi_*(T + \lambda)^m. \tag{4.10}$$

This means that $J(\tau_0; a, \lambda) = 1$ whence (4.8) by the Sturmian argument. $\square$

As a straigntforward consequence, we obtain the first intersection property of the ω-limit set.

Corollary 4.2. *Any $g \in \omega(f_0)$ intersects $\theta_*(\xi)$ at most once.*

Proof. Let $f(\tau_j) \to g$ as $\tau_j \to \infty$ uniformly. Then the result follows from (4.8) (which means that $J(\tau_j; a, 0) \le 1$), by passing to the limit $\tau_j \to \infty$ and using (4.3). $\square$

Let $g \in \omega(f_0)$. Denote by $I(b, \rho)$ the number of intersections of the profile $g(\xi)$ and the function $\theta_*(\xi; b, \rho)$ from the family (4.5). The main intersection stability property of the ω-limit set can be stated as follows.

Lemma 4.3. *Given $g \in \omega(f_0)$, for any b and $\rho > -1$,*

$$I(b, \rho) \le 1. \tag{4.11}$$

Proof. Let $f(\tau_j) \to g$ uniformly. Assume for contradiction that there exist b, ρ such that

$$I(b, \rho) \ge 2. \tag{4.12}$$

We take the sequence of functions $f_*(\xi, \tau_j; a_j, \lambda_j)$ with $a_j e^{m\tau_j} = b$ and $\lambda_j e^{\tau_j} = \rho$ so that

$$f_*(\xi, \tau_j; a_j, \lambda_j) \equiv \theta_*(\xi; b, \rho). \tag{4.13}$$

Then

$$|a_j| = |b| e^{-m\tau_j} \to \infty \quad \text{as } j \to \infty \quad (\ m < 0). \tag{4.14}$$

In view of (4.13) and (4.12) we conclude that

$$J(\tau_j; a_j, \lambda_j) \ge 2 \quad \text{for all } \tau_j \gg 1. \tag{4.15}$$

Since $|a_j| \gg 1$ by (4.14), this contradicts Proposition 4.1. $\square$

4.3. End of proof of Theorem 1. According to (4.1) we need to consider the following two cases.

Case 1: $\operatorname{meas}\operatorname{supp} g < 2\xi_*$. Then there exists a small $|b|$ such that

$$\overline{\operatorname{supp}}\ g \subset \operatorname{supp}\ \theta_*(\cdot; b, 0) = (-\xi_* + b, \xi_* + b). \tag{4.16}$$

Then $g(\xi)$ must intersect $\theta_*(\xi; b, 0)$, otherwise the corresponding solutions to equation (1.1) have different blow-up times. See Section 2. Then (4.16) implies that $I(b, 0) \ge 2$ contradicting (4.11).

Case 2: $\operatorname{meas}\operatorname{supp} g = 2\xi_*$. Then by continuity $g(\xi) > 0$ on $(-\xi_*, \xi_*)$ and $g = 0$ for $\xi = \pm\xi_*$. If $I(0, 0) = 1$ then $g(\xi) \ge \theta_*(\xi)$ near one of the end point $\xi = \pm\xi_*$, say, at the right one $\xi = \xi_*$. By the Strong Maximum Principle applied in the positivity domain, we have that $g(\xi) > \theta_*(\xi)$ for all $\xi_* - \xi > 0$ small, and $g(\xi) < \theta_*(\xi)$ for all $\xi_* + \xi > 0$ small. One can see that for all fixed $b > 0$ small, the function $\theta_*(\xi; b, 0)$ intersects $g(\xi)$ at least twice which contradicts (4.11).

Finally, in view of Corollary 4.2, we need to consider the last case when $I(0,0) = 0$. Then either $g(\xi) > \theta_*(\xi)$ or $g(\xi) < \theta_*(\xi)$ on the interval $(-\xi_*, \xi_*)$. (This is expected to be impossible for solutions having the same blow-up times, but we again use the main intersection property). Let e.g.

$$g > \theta_* \quad \text{on } (-\xi_*, \xi_*). \tag{4.17}$$

Then for any fixed small $b > 0$ and

$$\rho = \left(1 + \frac{2b}{\xi_*}\right)^{1/m} - 1 = \frac{2b}{m\xi_*}(1 + o(1)) \to 0 \quad \text{as } b \to 0, \tag{4.18}$$

we have that

$$\theta_*(\xi; b, \rho) > 0 \quad \text{on } [-\xi_*, \xi_*]. \tag{4.19}$$

On the other hand, from (4.17), (4.19) and (4.6) one can conclude that $I(b,\rho) \geq 2$ contradicting (4.11). This completes the proof of Theorem 1. $\square$

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November 15, 1995

Dear Professor Aftabizadeh,

Enclosed, please find my paper: "Asymptotic self-similar global blow-up for a quasilinear heat equation" by V.A. Galaktionov,
which I would like to submit to Differential and Integral Equations.

It is devoted to the study of the asymptotic behaviour of solutions to a quasilinear heat equation introduced 20 years ago, a problem which remained open during these years. The result proved in the present paper completes the research in this aerea.

I am looking forward to hearing from you soon.

With best regards.

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V.A. Galaktionov

Asymptotic self-similar global blow-up for a quasilinear...

Abbreviated version of the title

SELF-SIMILAR GLOBAL BLOW-UP

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