Duality relations, correspondences, and numerical results for planar elastic composites

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Abstract

This paper addresses three topics relating to planar elastic composites with anisotropic phases. First, the duality relations of Berdichevski are generalized to a wider class of planar elastic media. These yield phase interchange relations for the effective compliance tensors of certain two phase media. Second, a simple derivation is given of the correspondence between a specific class of planar elastic problems, and the associated pairs of antiplane elastic problems. This correspondence allows one to determine the effective compliance tensor from the effective shear matrices of the associated antiplane problems. Third, a numerical algorithm is developed and implemented for computing the effective moduli of two phase composites with orthotropic phases. The numerical algorithm is based on an integral equation and can, more generally, be applied to solving for the fields in elastic bodies comprised of several phases differing in their anisotropies. A series of examples demonstrates the accuracy of the numerical method and the agreement with the theoretical findings.

1 Introduction

The objectives of this paper are threefold. To generalize the duality relations of Berdichevski (1983) to a wider class of planar elastic composite media. To obtain a simple proof showing exact correspondences between certain plane strain problems and antiplane strain problems. To present an efficient numerical method to solve for the elastic fields and effective moduli in planar composites with orthotropic phases. With unparalleled accuracy our numerical results highlight the theoretical findings.

Duality relations are well known in the context of two-dimensional electrical conductivity (or equivalently in the context of antiplane strain). Keller (1964), using the properties of harmonic functions, found a relation between the transverse effective conductivity of an array of cylinders and the conductivity when the phases are interchanged. Dykhne (1971), using

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the fact that a divergence-free field when rotated locally at each point by 90° produces a curl-free field and vice versa, generalized Keller's result to isotropic multiphase and polycrystalline media. He noted that the duality relations implied exact formulae for the conductivity of phase interchange invariant two-phase media (such as checkerboards) and for polycrystals constructed from a single crystal. These pioneering works of Keller and Dykhne stimulated a large body of research: see Benveniste (1995) for a comprehensive summary and for recent results pertaining to piezoelectricity.

Duality and phase interchange relations for planar elastic media with incompressible isotropic phases were derived by Berdichevski (1983) who used them to obtain an exact formula for the effective shear modulus of a checkerboard of two incompressible phases. Francfort (1992) subsequently observed an abstract correspondence between the equations of incompressible elasticity and those of electrical conductivity which, as a corollary, allow one to make a correspondence between the duality relations for incompressible elasticity and the duality relations for conductivity. (However this correspondence does not allow one to recover the effective shear modulus from knowledge of the effective conductivity: the problems are intrinsically different.) In section 2 of this paper we extend these duality and phase interchange relations of Berdichevski to anisotropic composite materials with compliance tensors which are not necessarily singular.

Ni and Nemat-Nasser (1995) recently obtained duality transformations for three-dimensional anisotropic bodies with stress and strain fields independent of the x_3 coordinate. They used these transformations to find interesting connections between the analytic solutions for the field around a line dislocation and around a line force, accounting for an observation of Dundurs (1968). Except in the special case where the material is incompressible in the $x_1 - x_2$ plane, their duality transformations are quite different to the ones we discuss here. Ni and Nemat-Nasser look at a broad class of mathematical equations, a subset of which derives from elasticity problems. They find duality transformations which link pairs of equations within this broad class, but which do not generally link a given elasticity problem with another elasticity problem. (In their language, one cannot usually find an elasticity tensor C corresponding to the mathematical equation obtained by interchanging the matrices A and L.) By contrast our focus is on finding duality transformations which directly link pairs of elasticity problems, and to utilize this link to relate the effective compliance tensors of different composite materials.

The phase interchange identities derived in this paper hold provided the moduli of the two phases satisfy certain relations. For example, for composites of two isotropic phases the interchange identity can be applied only when the two phases have equal bulk moduli. Gibiansky and Torquato (1995), in independent work, derived phase interchange inequalities for isotropic composites of two isotropic phases without assuming that the two phases had equal bulk moduli. Their inequalities reduce to our identity when the bulk moduli coincide.

Planar elasticity problems and antiplane elasticity problems in inhomogeneous bodies are usually regarded as being completely different. However, when the moduli of the phases satisfy certain constraints Milton and Movchan (1995) found an equivalence between the problems: the planar elasticity problem reduces to pair of uncoupled antiplane elasticity problems. For composite materials this implies that the effective compliance tensor can be determined from the effective shear matrices associated with the two antiplane problems. In section 3 of this paper we restrict our attention to two-phase media and obtain the correspondence in a direct and simple manner using the Lekhnitskii representation for the Airy stress function and displacement field in each phase. This simplified derivation clarifies the connection between the planar and antiplane elasticity problems. A different sort of correspondence between planar and antiplane elasticity problems has recently been obtained by Grabovsky (1995) for the special class of microgeometries that attain bounds on the elastic energy.

Integral equation methods are by now standard for solving boundary value problems in locally isotropic elasticity. See the Chen and Zhou (1992) for a theoretical discussions and a literature survey. A numerical example of particular interest is the paper of Greenbaum, Greengard and Mayo (1992). The authors solve the elastostatic partial differential equation rapidly and to high accuracy using iterative techniques and a biharmonic extension of the Fast Multipole Method (Rokhlin 1985; Greengard and Rokhlin 1987; Carrier Greengard and Rokhlin 1988). A recent numerical example involving inclusions is the work on Ostwald ripening by Jou, Leo and Lowengrub (1995).

There is more than one way to write the elastostatic equation in integral equation form: one can work with Green's functions that describe displacements and stresses due to point-forces or dislocations, or one can use biharmonic analysis and the complex variable representation of the Airy stress function. For orthotropic elasticity, the literature on numerical results using integral equation methods appears to be thin. Sherman (1942) derived an integral equation for an anisotropic material with the inclusions being voids. Xiwu, Liangxin and Xuiqi (1995) specialized to elliptical holes and tabulated numerical results for a finite orthotropic specimen containing two circular holes. Bhargava and Radhakrishna (1964) and Willis (1964) studied an elliptic orthotropic inclusion that underwent a stress-free transformation using complex variable techniques and obtained analytic results. Yang and Chou (1976) studied an elliptic orthotropic inclusion with a point-force method. Tan, Gao and Afagh (1992) studied an elliptic anisotropic inclusion in a finite region with an integral equation method and obtained one and two digit accurate numerical results for the elastic fields due to some external loads. Holmbom, Persson and Svanstedt (1992) obtained one and two digit accurate numerical results for square arrays of anisotropic discs using a finite element technique. We are not aware of any work, the above papers included, where integral equation methods are used to compute effective elastic moduli of two-dimensional composites with components having different anisotropies.

Section 4 of this paper presents the elastostatic equation for orthotropic two-dimensional elasticity of composites as a system of integral equations. The integral equations are derived using a point-force method. A numerical algorithm for their solution is described.

The theoretical and numerical aspects of the paper are tied together in Section 5 through a series of examples. The integral equations of Section 4 are solved numerically for some periodic composites and effective moduli are computed. In these examples we have chosen the inclusions, called amoebas, to be simply connected regions with smooth boundaries, and we have embedded them in an orthotropic elastic matrix called the filler. In all examples the effective moduli are calculated to at least ten digits of accuracy with at most 1000 discretization points on the surface of the amoeba lying within the unit cell.

2 Duality transformations for antiplane and planar elasticity

2.1 A review of the duality transformation for antiplane elasticity

The ensuing review is largely based on the analysis of Mendelson (1975), rewritten here in the language of antiplane elasticity. The constitutive law for antiplane elasticity in a periodic anisotropic medium takes the form

$$\begin{pmatrix} \tau_{13} \\ \tau_{23} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} \epsilon_{13} \\ \epsilon_{23} \end{pmatrix}, \qquad \text{where } \boldsymbol{M}(\boldsymbol{x}) = \begin{pmatrix} m_1(\boldsymbol{x}) & m_2(\boldsymbol{x}) \\ m_2(\boldsymbol{x}) & m_3(\boldsymbol{x}) \end{pmatrix} \tag{2.1}$$

is the shear matrix and τ_{13} and τ_{23} are cartesian components of the stress tensor, while ϵ_{13} and ϵ_{23} are cartesian components of the strain tensor. The relation between the volume averaged stress components and volume averaged strain components is governed by the constitutive law

$$\begin{pmatrix} \langle\tau_{13}\rangle \\ \langle\tau_{23}\rangle \end{pmatrix} = \boldsymbol{M}_* \begin{pmatrix} \langle\epsilon_{13}\rangle \\ \langle\epsilon_{23}\rangle \end{pmatrix}, \qquad \text{where } \boldsymbol{M}_* = \begin{pmatrix} m_{*1} & m_{*2} \\ m_{*2} & m_{*3} \end{pmatrix} \tag{2.2}$$

is the effective shear matrix. In the absence of body forces the stress and strain field components satisfy the differential constraints

$$\frac{\partial\tau_{13}}{\partial x_1} + \frac{\partial\tau_{23}}{\partial x_2} = 0, \qquad \frac{\partial\epsilon_{13}}{\partial x_2} - \frac{\partial\epsilon_{23}}{\partial x_1} = 0. \tag{2.3}$$

Equivalently there exists a potentials v and w ($2w$ being the displacement u_3 in the vertical direction) such that

$$\tau_{13} = -\frac{\partial v}{\partial x_2}, \quad \tau_{23} = \frac{\partial v}{\partial x_1}, \qquad \epsilon_{13} = \frac{\partial w}{\partial x_1}, \quad \epsilon_{23} = \frac{\partial w}{\partial x_2}. \tag{2.4}$$

The close similarity between the differential constraints on the stress and strain field components motivates us to consider a dual problem where the roles of the potentials v and w are interchanged. (In this dual problem we interpret $2v$ as the vertical component of a displacement field). The cartesian stress and strain field components associated with the dual problem are

$$\tau'_{13} = -\frac{\partial w}{\partial x_2}, \quad \tau'_{23} = \frac{\partial w}{\partial x_1}, \qquad \epsilon'_{13} = \frac{\partial v}{\partial x_1}, \quad \epsilon'_{23} = \frac{\partial v}{\partial x_2}. \tag{2.5}$$

Using the relations

$$\begin{pmatrix} \epsilon'_{13} \\ \epsilon'_{23} \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} \tau_{13} \\ \tau_{23} \end{pmatrix}, \quad \begin{pmatrix} \tau'_{13} \\ \tau'_{23} \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} \epsilon_{13} \\ \epsilon_{23} \end{pmatrix} \quad \text{where } \boldsymbol{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.6}$$

implied by (2.5) we see that the stress and strain field components for the dual problem are related via the constitutive law

$$\begin{pmatrix} \tau'_{13} \\ \tau'_{23} \end{pmatrix} = \boldsymbol{M}' \begin{pmatrix} \epsilon'_{13} \\ \epsilon'_{23} \end{pmatrix}, \qquad \text{where } \boldsymbol{M}'(\boldsymbol{x}) = \boldsymbol{R}_\perp[\boldsymbol{M}(\boldsymbol{x})]^{-1}\boldsymbol{R}_\perp^T. \tag{2.7}$$

Explicit calculation shows that

$$\boldsymbol{M}'(\boldsymbol{x}) = \frac{1}{m_1 m_3(\boldsymbol{x}) - (m_2)^2} \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} = \boldsymbol{M}(\boldsymbol{x})/\text{det}\boldsymbol{M}(\boldsymbol{x}). \tag{2.8}$$

In other words, the vertical displacement $2v(\boldsymbol{x})$ solves the antiplane elasticity problem in a material with shear matrix $\boldsymbol{M}'(\boldsymbol{x})$. It is easy to check that the constitutive equation governing the relation between the average stress and strain fields in the dual problem takes a similar form:

$$\begin{pmatrix} \langle \tau'_{13} \rangle \\ \langle \tau'_{23} \rangle \end{pmatrix} = \boldsymbol{M}'_* \begin{pmatrix} \langle \epsilon'_{13} \rangle \\ \langle \epsilon'_{23} \rangle \end{pmatrix}, \quad \text{where } \boldsymbol{M}'_* = \boldsymbol{M}_*/\mathrm{det}\boldsymbol{M}_* \tag{2.9}$$

is the effective shear matrix for the dual antiplane problem. Thus the duality transformation provides a connection between the effective moduli of different antiplane elasticity problems. It may happen that the quantity

$$\Delta = -\mathrm{det}\boldsymbol{M}(\boldsymbol{x}) \tag{2.10}$$

takes a value independent of $\boldsymbol{x}$. (We have defined Δ to be the negative of the determinant to be consistent with previous work (Milton & Movchan, 1995).) Then $\boldsymbol{M}'(\boldsymbol{x})$ differs from $\boldsymbol{M}(\boldsymbol{x})$ only by a constant scale factor of $-1/\Delta$. Consequently the effective shear matrix $\boldsymbol{M}'_*$ must differ from $\boldsymbol{M}_*$ by the same scale factor $-1/\Delta$. In view of the relation (2.9) we are forced to conclude that $\boldsymbol{M}_*$ shares the same value of the determinant,

$$\mathrm{det}\boldsymbol{M}_* = -\Delta \tag{2.11}$$

Another interesting application of duality transformations is to two phase media. We let the phases be denoted by two subscripts a and f. (In the subsequent numerical work these will represent the inclusion or *amoeba* phase and the matrix or *filler* phase, but at present we do not place any topological constraint on the positioning of the two phases.) The geometrical configuration of the a and f phases are represented by their associated characteristic functions χ_a and χ_f,

$$\begin{aligned} \chi_a = 1 - \chi_f &= 1 \quad \text{in phase a} \\ &= 0 \quad \text{in phase f.} \end{aligned} \tag{2.12}$$

Let us suppose the shear matrix in phase a is proportional to the shear matrix in phase f. Then $\boldsymbol{M}(\boldsymbol{x})$ takes the special form

$$\boldsymbol{M}(\boldsymbol{x}) = (\alpha_a \chi_a(\boldsymbol{x}) + \alpha_f \chi_f(\boldsymbol{x}))\boldsymbol{A}, \tag{2.13}$$

in which $\boldsymbol{A}$ is a constant matrix and α_a and α_f are constants of proportionality. The shear matrix of the dual material is

$$\boldsymbol{M}'(\boldsymbol{x}) = (\alpha_f \chi_a(\boldsymbol{x}) + \alpha_a \chi_f(\boldsymbol{x}))\boldsymbol{A}/(\alpha_f \alpha_a \mathrm{det}\boldsymbol{A}). \tag{2.14}$$

Aside from the constant proportionality constant of $1/(\alpha_f \alpha_a \mathrm{det}\boldsymbol{A})$ the dual material is the same as the original material but with the phases interchanged. Thus if we consider the effective stress matrix $\boldsymbol{M}_*(\alpha_a, \alpha_f)$ as a function of the constants α_a and α_f then the formula (2.9) for $\boldsymbol{M}'_*$ implies the phase interchange identity

$$\boldsymbol{M}_*(\alpha_f, \alpha_a) = \frac{\alpha_a \alpha_f \mathrm{det}\boldsymbol{A}}{\mathrm{det}\boldsymbol{M}_*(\alpha_a, \phi_f)} \boldsymbol{M}_*(\alpha_a, \alpha_f) \tag{2.15}$$

which gives the effective shear matrix when we interchange the phases.

2.2 A duality transformation for planar elasticity

To begin with let us consider a two-dimensional periodic medium which is locally orthotropic, having the crystal axes aligned with the coordinate axes, and which is locally rigid with respect to shear forces directed perpendicular to the axes (i.e. with respect to shears with $\tau_{11} = \tau_{22} = 0$ and $\tau_{12} \neq 0$). The constitutive relation in such a medium takes the special form

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \sqrt{2}\epsilon_{12} \end{pmatrix} = \begin{pmatrix} s_1 & s_2 & 0 \\ s_2 & s_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \sqrt{2}\tau_{12} \end{pmatrix} \tag{2.16}$$

where the moduli s_1, s_2 and s_4 of the compliance tensor depend upon $\boldsymbol{x}$. This constitutive relation imposes the differential constraint $\epsilon_{12} = 0$ on the stress component ϵ_{12} and it imposes no constraint on the strain component τ_{12}. The remaining stress and strain components are linked through the reduced constitutive relation

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \end{pmatrix} = \boldsymbol{S}_r \begin{pmatrix} \tau_{11} \\ \tau_{22} \end{pmatrix} \quad \text{where } \boldsymbol{S}_r(\boldsymbol{x}) = \begin{pmatrix} s_1(\boldsymbol{x}) & s_2(\boldsymbol{x}) \\ s_2(\boldsymbol{x}) & s_4(\boldsymbol{x}) \end{pmatrix}. \tag{2.17}$$

The associated effective tensor $\boldsymbol{S}_{r*}$ which governs the relation between the average fields,

$$\begin{pmatrix} \langle \epsilon_{11} \rangle \\ \langle \epsilon_{22} \rangle \end{pmatrix} = \boldsymbol{S}_{r*} \begin{pmatrix} \langle \tau_{11} \rangle \\ \langle \tau_{22} \rangle \end{pmatrix} \tag{2.18}$$

determines the elements of effective compliance tensor $\boldsymbol{S}_*$:

$$\boldsymbol{S}_* = \begin{pmatrix} \boldsymbol{S}_{r*} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{S}_{r*} = \begin{pmatrix} s_{*1} & s_{*2} \\ s_{*2} & s_{*4} \end{pmatrix} \tag{2.19}$$

Now we ask, what are the differential constraints on the stress and strain components which enter the reduced constitutive relation? Since $\epsilon_{12} = 0$ the infinitesimal strain compatibility relation reduces to

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 0. \tag{2.20}$$

Equivalently, there must exist a potential ψ such that

$$\epsilon_{11} = \frac{\partial^2 \psi}{\partial x_1^2}, \qquad \epsilon_{22} = -\frac{\partial^2 \psi}{\partial x_2^2}. \tag{2.21}$$

To eliminate the stress component τ_{12} from the differential restrictions $\partial\tau_{11}/\partial x_1 + \partial\tau_{12}/\partial x_2 = 0$ and $\partial\tau_{12}/\partial x_1 + \partial\tau_{22}/\partial x_2 = 0$ on the stress we take the derivative of the first equation with respect to x_1 and the derivative of the second equation with respect to x_2 and subtract them. This yields the differential constraint

$$\frac{\partial^2 \tau_{11}}{\partial x_1^2} - \frac{\partial^2 \tau_{22}}{\partial x_2^2} = 0. \tag{2.22}$$

Equivalently, there must exist a potential ϕ (known as the Airy stress function) such that

$$\tau_{11} = \frac{\partial^2 \phi}{\partial x_2^2}, \qquad \tau_{22} = \frac{\partial^2 \phi}{\partial x_1^2}. \tag{2.23}$$

The key to obtaining a duality transformation is to notice the close similarity between the differential constraints (2.20) and (2.21) on the strain components and the differential constraints (2.22) and (2.23) on the stress components. This observation motivates us to define new fields

$$\begin{pmatrix} \epsilon'_{11} \\ \epsilon'_{22} \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} \tau_{11} \\ \tau_{22} \end{pmatrix} = \begin{pmatrix} \tau_{22} \\ -\tau_{11} \end{pmatrix}, \quad \begin{pmatrix} \tau'_{11} \\ \tau'_{22} \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \end{pmatrix} = \begin{pmatrix} \epsilon_{22} \\ -\epsilon_{11} \end{pmatrix}. \tag{2.24}$$

The pair $(\epsilon'_{11}, \epsilon'_{22})$ clearly satisfies the same differential constraints as the pair $(\epsilon_{11}, \epsilon_{22})$ while the pair (τ'_{11}, τ'_{22}) satisfies the same differential constraints as the pair (τ_{11}, τ_{22}). Under this duality transformation the roles of the potentials ψ and ϕ are interchanged. The new fields $\boldsymbol{\epsilon}'$ and $\boldsymbol{\tau}'$ are linked by the constitutive relation

$$\begin{pmatrix} \epsilon'_{11} \\ \epsilon'_{22} \end{pmatrix} = \boldsymbol{S}'_r \begin{pmatrix} \tau'_{11} \\ \tau'_{22} \end{pmatrix} \quad \text{where } \boldsymbol{S}'_r(\boldsymbol{x}) = \boldsymbol{R}_\perp [\boldsymbol{S}_r(\boldsymbol{x})] \boldsymbol{R}_\perp^T = \boldsymbol{S}_r(\boldsymbol{x}) / \det \boldsymbol{S}_r(\boldsymbol{x}). \tag{2.25}$$

Hence all the duality results for the antiplane elasticity problem immediately extend to the planar elasticity problem. In particular a medium with compliance tensor

$$\boldsymbol{S}'(\boldsymbol{x}) = \boldsymbol{S}(\boldsymbol{x}) / \det \boldsymbol{S}_r(\boldsymbol{x}) = \frac{1}{s_1 s_4 - s_2^2} \begin{pmatrix} s_1 & s_2 & 0 \\ s_2 & s_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.26}$$

will have effective tensor

$$\boldsymbol{S}'_* = \boldsymbol{S}_* / \det \boldsymbol{S}_{r*} = \frac{1}{s_{*1} s_{*4} - s_{*2}^2} \begin{pmatrix} s_{*1} & s_{*2} & 0 \\ s_{*2} & s_{*4} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.27}$$

It follows that if $\det \boldsymbol{S}_r(\boldsymbol{x})$ takes a value $-\Delta$ independent of $\boldsymbol{x}$, then necessarily $\det \boldsymbol{S}_{r*}$ takes the same value:

$$\det \boldsymbol{S}_{r*} = -\Delta \quad \text{when } \det \boldsymbol{S}_r(\boldsymbol{x}) = -\Delta \quad \text{for all } \boldsymbol{x}. \tag{2.28}$$

These duality relations, derived here for media where $\boldsymbol{S}(\boldsymbol{x})$ varies smoothly with $\boldsymbol{x}$ naturally extend to multiphase media containing interfaces across which $\boldsymbol{S}(\boldsymbol{x})$ and the stress and strain fields are discontinuous. The continuity of the displacement field and normal component of the traction across the interface implies the continuity of $\nabla\psi$ and $\nabla\phi$ across the interface. These continuity conditions are maintained when we interchange ψ and ϕ.

In particular, if the medium is two-phase with $\boldsymbol{S}_r(\boldsymbol{x})$ taking the form

$$\boldsymbol{S}_r(\boldsymbol{x}) = (\alpha_a \chi_a(\boldsymbol{x}) + \alpha_f \chi_f(\boldsymbol{x})) \boldsymbol{A} \tag{2.29}$$

for some choice of matrix $\boldsymbol{A}$ and constants α_a and α_f, and we consider $\boldsymbol{S}_{r*}$ and $\boldsymbol{S}_*$ as functions $\boldsymbol{S}_{r*}(\alpha_a, \alpha_f)$ and $\boldsymbol{S}_*(\alpha_a, \alpha_f)$ of α_a and α_f, then we have the relation

$$\boldsymbol{S}_*(\alpha_f, \alpha_a) = \frac{\alpha_a \alpha_f \det \boldsymbol{A}}{\det \boldsymbol{S}_{r*}(\alpha_a, \alpha_f)} \boldsymbol{S}_*(\alpha_a, \alpha_f) \tag{2.30}$$

which gives the effective elasticity tensor when we interchange the phases. If the geometry is invariant under phase interchange (like a checkerboard), then we have

$$\det \boldsymbol{S}_{r*}(\alpha_a, \alpha_f) = \alpha_a \alpha_f \det \boldsymbol{A}. \tag{2.31}$$

If the medium is a simply connected elastic body rather than a periodic composite material then the duality transformations can still be applied. Under the duality transformation displacement boundary conditions map to traction boundary conditions and vice-versa. To see this it is sufficient to recognize that specifying displacement around the boundary is equivalent to specifying the value of $\nabla\psi$ around the boundary while specifying traction around the boundary is (after integration along the boundary) equivalent to specifying the value of $\nabla\phi$ around the boundary. Clearly these two types of boundary conditions are interchanged under the interchange of ψ and ϕ which occurs in a duality transformation. A similar sort of mapping between displacement boundary conditions and traction boundary conditions occurs in the duality transformations of Ni and Nemat-Nasser (1995).

2.3 Other duality transformations for planar elasticity

Duality relationships apply to other elastic media besides orthotropic media locally rigid with respect to shear. They can be applied whenever the compliance tensor $\boldsymbol{S}(\boldsymbol{x})$ is locally rigid with respect to stresses proportional to a constant matrix $\boldsymbol{v}_1$, that is whenever the relation $\boldsymbol{S}(\boldsymbol{x})\boldsymbol{v}_1 = 0$ holds for all $\boldsymbol{x}$ for some matrix $\boldsymbol{v}_1$ independent of $\boldsymbol{x}$.

Without loss of generality (by rotating the co-ordinate system and multiplying $\boldsymbol{v}^{(1)}$ by a constant, if necessary) let us assume that $\boldsymbol{v}_1$ is diagonal, of the form

$$\boldsymbol{v}_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{with } a_1^2 + a_2^2 = 1. \tag{2.32}$$

Then the constraint that $\boldsymbol{S}(\boldsymbol{x})\boldsymbol{v}_1 = 0$ becomes

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_4 & s_5 \\ s_3 & s_5 & s_6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} = 0 \quad \text{for all } \boldsymbol{x} \tag{2.33}$$

Let us introduce the matrices

$$\boldsymbol{v}_2 = \begin{pmatrix} a_2 & 0 \\ 0 & -a_1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.34}$$

which together with $\boldsymbol{v}_1$ form an orthonormal basis on the space of symmetric matrices. The coefficients

$$\begin{aligned} \epsilon_1 &= a_1\epsilon_{11} + a_2\epsilon_{22}, \quad \epsilon_2 = a_2\epsilon_{11} - a_1\epsilon_{22}, \quad \epsilon_3 = \sqrt{2}\epsilon_{12}, \\ \tau_1 &= a_1\tau_{11} + a_2\tau_{22}, \quad \tau_2 = a_2\tau_{11} - a_1\tau_{22}, \quad \tau_3 = \sqrt{2}\tau_{12}, \end{aligned} \tag{2.35}$$

which enter the expansions

$$\boldsymbol{\epsilon}(\boldsymbol{x}) = \epsilon_1(\boldsymbol{x})\boldsymbol{v}_1 + \epsilon_2(\boldsymbol{x})\boldsymbol{v}_2 + \epsilon_3(\boldsymbol{x})\boldsymbol{v}_3, \qquad \boldsymbol{\tau}(\boldsymbol{x}) = \tau_1(\boldsymbol{x})\boldsymbol{v}_1 + \tau_2(\boldsymbol{x})\boldsymbol{v}_2 + \tau_3(\boldsymbol{x})\boldsymbol{v}_3 \tag{2.36}$$

of the stress and strain fields in this basis are linked through the constitutive equation

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{S}_r \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \quad \text{where } \boldsymbol{S}_r(\boldsymbol{x}) = \begin{pmatrix} s_1 + s_4 & s_3/a_2 \\ s_3/a_2 & s_6 \end{pmatrix}. \tag{2.37}$$

The reduced constitutive relation takes the form

$$\begin{pmatrix} \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \boldsymbol{S}_r \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} \tag{2.38}$$

and the associated effective tensor $\boldsymbol{S}_{r*}$ has matrix elements

$$\boldsymbol{S}_{r*} = \begin{pmatrix} s_{*1} + s_{*4} & s_{*3}/a_2 \\ s_{*3}/a_2 & s_{*6} \end{pmatrix}. \tag{2.39}$$

The components of the strain and stress fields entering the reduced constitutive relation satisfy the differential constraints

$$\begin{aligned} &[a_1\frac{\partial^2}{\partial x_1^2} - a_2\frac{\partial^2}{\partial x_2^2}]\epsilon_2 + [\sqrt{2}\frac{\partial^2}{\partial x_1 \partial x_2}]\epsilon_3 = 0 \\ &[\sqrt{2}\frac{\partial^2}{\partial x_1 \partial x_2}]\tau_2 - [a_1\frac{\partial^2}{\partial x_1^2} - a_2\frac{\partial^2}{\partial x_2^2}]\tau_3 = 0. \end{aligned} \tag{2.40}$$

Equivalently, there exist potentials ψ and ϕ ($\phi(\boldsymbol{x})$ being the Airy stress function) such that

$$\begin{aligned} \epsilon_2 &= -[\sqrt{2}\frac{\partial^2}{\partial x_1 \partial x_2}]\psi, \qquad \epsilon_3 = [a_1\frac{\partial^2}{\partial x_1^2} - a_2\frac{\partial^2}{\partial x_2^2}]\psi \\ \tau_2 &= -[a_1\frac{\partial^2}{\partial x_1^2} - a_2\frac{\partial^2}{\partial x_2^2}]\phi, \qquad \tau_3 = -[\sqrt{2}\frac{\partial^2}{\partial x_1 \partial x_2}]\phi. \end{aligned} \tag{2.41}$$

The similarity between the differential constraints on the strain fields and the differential constraints on the stress fields motivates us to introduce new fields

$$\begin{pmatrix} \epsilon_2' \\ \epsilon_3' \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} \tau_3 \\ -\tau_2 \end{pmatrix}, \qquad \begin{pmatrix} \tau_2' \\ \tau_3' \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} \epsilon_3 \\ -\epsilon_2 \end{pmatrix}. \tag{2.42}$$

The pair $(\epsilon_2', \epsilon_3')$ satisfies the same differential constraints as the pair (ϵ_2, ϵ_3) while the pair (τ_2', τ_3') satisfies the same differential constraints as the pair (τ_2, τ_3). Consequently all the duality results extend to this problem: a medium with compliance tensor

$$\boldsymbol{S}'(\boldsymbol{x}) = \boldsymbol{S}(\boldsymbol{x})/\det \boldsymbol{S}_r(\boldsymbol{x}) \tag{2.43}$$

will have effective tensor

$$\boldsymbol{S}_*' = \boldsymbol{S}_*/\det \boldsymbol{S}_{r*}; \tag{2.44}$$

the identity (2.28) will hold when $\det \boldsymbol{S}_r(\boldsymbol{x})$ is constant; and when the material is two-phase with $\boldsymbol{S}_r(\boldsymbol{x})$ of the form (2.29) then (2.30) gives the effective elasticity tensor of the phase interchanged geometry and (2.31) will hold if the material is interchange invariant. In practice it is not necessary to find $\boldsymbol{S}_r(\boldsymbol{x})$ and $\boldsymbol{S}_{r*}$ in order to determine $\det \boldsymbol{S}_r(\boldsymbol{x})$ and $\det \boldsymbol{S}_{r*}$. Instead one can calculate the product of the two non-zero eigenvalues of $\boldsymbol{S}(\boldsymbol{x})$ and $\boldsymbol{S}_*$ to obtain $\det \boldsymbol{S}_r(\boldsymbol{x})$ and $\det \boldsymbol{S}_{r*}$.

Following Berdichevski (1983), let us now consider a two-phase (transversely) isotropic composite of two isotropic incompressible phases. In an isotropic incompressible material the compliance matrix takes the form

$$\boldsymbol{S} = \begin{pmatrix} 1/(4\mu) & -1/(4\mu) & 0 \\ -1/(4\mu) & 1/(4\mu) & 0 \\ 0 & 0 & 1/(2\mu) \end{pmatrix}, \tag{2.45}$$

where μ is the shear modulus. When the compliance matrices S_a and S_f of the two phases take this form then the condition (2.33) is satisfied with $a_1 = a_2 = 1$. Consequently, if we consider the effective shear modulus μ_* as a function $\mu_*(\mu_a, \mu_f)$ of the shear moduli μ_a and μ_f of the two phases the duality result (2.30) implies the phase interchange identity

$$\mu_*(\mu_f, \mu_a)\mu_*(\mu_a, \mu_f) = \mu_a\mu_f. \tag{2.46}$$

If the geometry is phase interchange invariant, like a checkerboard, then this identity yields Berdichevski's exact formula $\mu_* = \sqrt{\mu_a\mu_f}$ for the effective shear modulus.

A related example is that of an isotropic two-dimensional polycrystal of incompressible crystals. The individual crystals, being incompressible, necessarily have square symmetry and are characterized by two shear moduli $\mu^{(1)}$ and $\mu^{(2)}$. The duality result (2.28) implies that the effective shear modulus μ_* of the polycrystal is given by the formula

$$\mu_* = \sqrt{\mu^{(1)}\mu^{(2)}} \tag{2.47}$$

of Lurie and Cherkaev(1984).

2.4 Using translations to extend the use of the duality transformations

In the duality transformations discussed so far the compliance matrices have a zero eigenvalue. This restriction can be relaxed. It is sufficient that there exists a constant c such that one can apply the duality transformations of the previous section to the *translated medium* with tensor

$$\boldsymbol{S}''(\boldsymbol{x}) = \boldsymbol{S}(\boldsymbol{x}) - c\boldsymbol{\mathcal{R}}_\perp, \tag{2.48}$$

where

$$\boldsymbol{\mathcal{R}}_\perp = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.49}$$

The required constraint that $\boldsymbol{S}''(\boldsymbol{x})\boldsymbol{v} = 0$ can be rewritten as

$$\boldsymbol{\mathcal{R}}_\perp\boldsymbol{S}(\boldsymbol{x})\boldsymbol{v} = c\boldsymbol{v}. \tag{2.50}$$

In other words, $\boldsymbol{\mathcal{R}}_\perp\boldsymbol{S}(\boldsymbol{x})$ must have an eigenvector $\boldsymbol{v}$ and an associated eigenvalue c which are both independent of $\boldsymbol{x}$. The elasticity problem in the original medium and the translated medium are equivalent. This observation was made by Lurie and Cherkaev (1984) in the context of the plate equation, and was further developed by Cherkaev, Lurie and Milton (1992), Thorpe and Jasiuk (1992), Christensen (1993), and Dundurs and Markenscoff (1993). If $\boldsymbol{u}$ and ϕ are the displacement and Airy stress function which solve the elasticity equations in the medium $\boldsymbol{S}(\boldsymbol{x})$ then it is easy to verify that the displacement $\boldsymbol{u}'' = \boldsymbol{u} - c\nabla\phi$ and the Airy stress function $\phi'' = \phi$ solve the elasticity equations in the medium $\boldsymbol{S}''(\boldsymbol{x})$. Due to this correspondence the two media are said to be equivalent. It follows that the medium $\boldsymbol{S}''(\boldsymbol{x})$ has effective tensor

$$\boldsymbol{S}''_* = \boldsymbol{S}_* - c\boldsymbol{\mathcal{R}}_\perp. \tag{2.51}$$

After applying the duality transformation one can make a further translation to obtain an equivalent dual material with a non-singular compliance matrix.

Using such translations it is easy to check that for a two-phase (transversely) isotropic composite of two isotropic phases with equal in plane bulk moduli $\kappa_a = \kappa_f = \kappa$ the phase interchange relation (2.46) generalizes to

$$E_*(E_f, E_a)E_*(E_a, E_f) = E_a E_f \tag{2.52}$$

in which $E_*(E_a, E_f)$ is the effective in plane Young's modulus ($E_* = 4/\kappa + 4/\mu_*$) expressed as a function of the in plane Young's moduli E_a and E_f of the two phases. When the two phases do not have the same bulk modulus then the phase interchange identity (2.52) gets replaced by the phase interchange inequalities of Gibiansky and Torquato (1995). As remarked in the introduction, their inequalities reduce to the phase interchange identity (2.52) in the limit as κ_a approaches κ_f.

Similar considerations enabled Lurie and Cherkaev (1984) to generalize (2.47). Using translations, they found that the effective shear modulus μ_* of a two-dimensional polycrystal, comprised of grains with square symmetry is given by the formula

$$\mu_* = \frac{\kappa}{-1 + \sqrt{(\kappa + \mu^{(2)})(\kappa + \mu^{(1)})/(\mu^{(1)}\mu^{(2)})}}, \tag{2.53}$$

where κ, $\mu^{(1)}$ and $\mu^{(2)}$ are the planar bulk and two shear moduli of the square symmetric crystal.

As a another example, suppose we start with a two-dimensional medium which has a compliance tensor of the form

$$\boldsymbol{S}(\boldsymbol{x}) = \begin{pmatrix} s_1(\boldsymbol{x}) & s_2(\boldsymbol{x}) & 0 \\ s_2(\boldsymbol{x}) & s_4(\boldsymbol{x}) & 0 \\ 0 & 0 & s_6 \end{pmatrix} \tag{2.54}$$

where s_6 does not depend upon $\boldsymbol{x}$. By applying the duality result to the translated medium $\boldsymbol{S}''(\boldsymbol{x})$ with $c = -s_6$ one sees that when

$$\Delta = (s_2(\boldsymbol{x}) + s_6)^2 - s_1(\boldsymbol{x})s_4(\boldsymbol{x}) \tag{2.55}$$

takes a value independent of $\boldsymbol{x}$ then the moduli of the effective compliance tensor $\boldsymbol{S}_*$ associated with $\boldsymbol{S}(\boldsymbol{x})$ satisfy

$$(s_{*2} + s_{*6})^2 - s_{*1}s_{*4} = \Delta, \quad s_{*6} = s_6. \tag{2.56}$$

For a homogeneous medium $\boldsymbol{S}$ one can always find both a positive and negative value of c such that $\boldsymbol{S}''$ is singular and positive semidefinite. Consequently if the medium is a simply connected homogeneous body rather than a periodic composite then one can use a translation followed by a duality transformation to convert any traction boundary value problem to a displacement boundary value problem with a compliance matrix which is singular and positive semi-definite. The converse, however, is not true: the strain (unlike the stress under traction boundary conditions) does not remain invariant under translation.

3 A correspondence between planar and antiplane elasticity

In this section we present the correspondence between the plane strain problem and a pair of anti–plane shear problems for a certain type of anisotropic two-phase medium.

Previously Milton and Movchan (1995), through a lengthy analysis, obtained such a correspondence assuming the the compliance tensor had the form

$$\boldsymbol{S}(\boldsymbol{x}) = \begin{pmatrix} s_1(\boldsymbol{x}) & s_2(\boldsymbol{x}) & 0 \\ s_2(\boldsymbol{x}) & s_4(\boldsymbol{x}) & 0 \\ 0 & 0 & s_6 \end{pmatrix}, \tag{3.1}$$

where s_6 and the quantity $\Delta = (s_2(\boldsymbol{x}) + s_6)^2 - s_1(\boldsymbol{x})s_4(\boldsymbol{x})$ are independent of $\boldsymbol{x}$. It was established that if Δ is positive and $\Delta \neq s_6^2$ then there is a mapping from the planar elasticity problem to a pair of dielectric problems (or equivalently to a pair of anti–plane shear problems). Without loss of generality, by multiplying $\boldsymbol{S}(\boldsymbol{x})$ if necessary, one can assume that $s_6 = 1/2$. Then the associated dielectric tensor fields are

$$\boldsymbol{\epsilon}^{(1)}(\boldsymbol{x}) = \begin{pmatrix} \gamma(\boldsymbol{x}) - \varphi(\boldsymbol{x}) & 0 \\ 0 & \beta - 1 \end{pmatrix}, \qquad \boldsymbol{\epsilon}^{(2)}(\boldsymbol{x}) = \begin{pmatrix} \gamma(\boldsymbol{x}) + \varphi(\boldsymbol{x}) & 0 \\ 0 & \beta + 1 \end{pmatrix}, \tag{3.2}$$

where

$$\beta = (1 + 4\Delta)/4\sqrt{\Delta}, \quad \gamma = \gamma'\beta/\beta', \quad \varphi = \frac{s_2}{s_1} + \frac{1}{2s_1(\beta' + 1)}, \tag{3.3}$$

and the quantities β' and $\gamma'(\boldsymbol{x})$ are given by

$$\beta' = (1 + 4\Delta)/(1 - 4\Delta), \qquad \gamma' = \beta'\frac{s_2}{s_1} + \frac{1}{2s_1}. \tag{3.4}$$

The two effective dielectric tensors were shown to be related to the effective compliance tensor through a similar set of equations.

Here we focus on two-phase media. This restriction allows us to easily prove the correspondence between the plane strain problem and the associated antiplane problems using the representation of Lekhnitskii (1968) for the Airy stress function and displacement fields in each phase in terms of analytic functions. We will see that the equations decouple when the compliance tensors of the two phases have the same value of Δ and s_6.

Consider a two phase medium where the phases are orthotropic with their axes aligned with the co-ordinate axes. The compliance matrices $\boldsymbol{S}_a$ and $\boldsymbol{S}_f$ have constant moduli s_{a1}, s_{a2}, s_{a4}, s_{a6} and s_{f1}, s_{f2}, s_{f4}, s_{f6}, satisfying the constraints

$$s_{a6} = s_{f6} = s_6, \qquad (s_{a2} + s_{a6})^2 - s_{a1}s_{a4} = (s_{f2} + s_{f6})^2 - s_{f1}s_{f4} = \Delta > 0. \tag{3.5}$$

The remaining moduli are zero. The Airy stress function ϕ and displacement (u_1, u_2) in either phase can be expressed in the form

$$\begin{aligned} \phi &= \Re e\{\phi_\alpha(z_\alpha) + \phi_\beta(z_\beta)\}, \\ u_1 &= \Re e\{p_\alpha {\phi'}_\alpha(z_\alpha) + p_\beta {\phi'}_\beta(z_\beta)\}, \qquad u_2 = -\Im m\{p_\beta\mu_\alpha {\phi'}_\alpha(z_\alpha) + p_\alpha\mu_\beta {\phi'}_\beta(z_\beta)\}, \end{aligned} \tag{3.6}$$

where ϕ_α and ϕ_β are analytic functions with derivatives ${\phi'}_\alpha$ and ${\phi'}_\beta$, and

$$z_\alpha = x_1 + i\mu_\alpha x_2, \quad z_\beta = x_1 + i\mu_\beta x_2, \qquad p_\alpha = s_2 - \mu_\alpha^2 s_1, \quad p_\beta = s_2 - \mu_\beta^2 s_1. \tag{3.7}$$

Here μ_α and μ_β are the two real positive roots of the polynomial

$$s_1\mu^4 - 2(s_2 + s_6)\mu^2 + s_4 = 0, \tag{3.8}$$

ordered with $\mu_\alpha < \mu_\beta$. Notice that p_α and p_β only depend on s_6 and Δ,

$$p_\alpha = -s_6 + \sqrt{\Delta}/2, \qquad p_\beta = -s_6 - \sqrt{\Delta}/2. \tag{3.9}$$

Therefore the constraints (3.5) ensure that p_α and p_β are the same in both phases. Along any interface between the phases the conditions of continuity of displacement take the form

$$\begin{aligned} p_\alpha \Re e\{{\phi'}^a_\alpha(z^a_\alpha)\} + p_\beta \Re e\{{\phi'}^a_\beta(z^a_\beta)\} &= p_\alpha \Re e\{{\phi'}^f_\alpha(z^f_\alpha)\} + p_\beta \Re e\{{\phi'}^f_\beta(z^f_\beta)\}, \\ \mu^a_\alpha p_\alpha \Im m\{{\phi'}^a_\alpha(z^a_\alpha)\} + \mu^a_\beta p_\beta \Im m\{{\phi'}^a_\beta(z^a_\beta)\} &= \mu^f_\alpha p_\alpha \Im m\{{\phi'}^f_\alpha(z^f_\alpha)\} + \mu^f_\beta p_\beta \Im m\{{\phi'}^f_\beta(z^f_\beta)\} \end{aligned} \tag{3.10}$$

while the conditions of continuity of $\nabla\phi$ (which ensures continuity of traction) are given by

$$\begin{aligned} \Re e\{{\phi'}^a_\alpha(z^a_\alpha)\} + \Re e\{{\phi'}^a_\beta(z^a_\beta)\} &= \Re e\{{\phi'}^f_\alpha(z^f_\alpha)\} + \Re e\{{\phi'}^f_\beta(z^f_\beta)\}, \\ \mu^a_\alpha \Im m\{{\phi'}^a_\alpha(z^a_\alpha)\} + \mu^a_\beta \Im m\{{\phi'}^a_\beta(z^a_\beta)\} &= \mu^f_\alpha \Im m\{{\phi'}^f_\alpha(z^f_\alpha)\} + \mu^f_\beta \Im m\{{\phi'}^f_\beta(z^f_\beta)\}. \end{aligned} \tag{3.11}$$

Notice that by adding and subtracting appropriate multiples of these continuity conditions we obtain a system of equations,

$$\begin{aligned} \Re e\{{\phi'}^a_\alpha(z^a_\alpha)\} &= \Re e\{{\phi'}^f_\alpha(z^f_\alpha)\}, \quad \mu^a_\alpha \Im m\{{\phi'}^a_\alpha(z^a_\alpha)\} = \mu^f_\alpha \Im m\{{\phi'}^f_\alpha(z^f_\alpha)\}, \\ \Re e\{{\phi'}^a_\beta(z^a_\beta)\} &= \Re e\{{\phi'}^f_\beta(z^f_\beta)\}, \quad \mu^a_\beta \Im m\{{\phi'}^a_\beta(z^a_\beta)\} = \mu^f_\beta \Im m\{{\phi'}^f_\beta(z^f_\beta)\}, \end{aligned} \tag{3.12}$$

where the equations involving ${\phi'}^a_\alpha$ and ${\phi'}^f_\alpha$ are decoupled from the equations involving ${\phi'}^a_\beta$ and ${\phi'}^f_\beta$. This decoupling is what makes a correspondence with antiplane problems possible.

Now we consider the antiplane shear formulation, where the shear matrices have the diagonal form

$$\boldsymbol{M}^a = \begin{pmatrix} k(\mu^a)^2 & 0 \\ 0 & k \end{pmatrix}, \; \boldsymbol{M}^f = \begin{pmatrix} k(\mu^f)^2 & 0 \\ 0 & k \end{pmatrix}, \tag{3.13}$$

with k being a positive constant. The potentials v and w in either phase can be expressed in the form

$$w = \Re e\{\Phi(z)\}, \; v = -\Im m\{k\mu\Phi(z)\}, \tag{3.14}$$

for a suitable choice of analytic function $\Phi(z)$, where $z = x_1 + i\mu x_2$, where μ takes the values μ^a or μ^f according to the phase being considered.

Along the interface the condition of continuity of $w(\boldsymbol{x})$ (which ensures continuity of vertical displacement $u_3(\boldsymbol{x}) = 2w(\boldsymbol{x})$) and the condition of continuity of $v(\boldsymbol{x})$ (which ensures continuity of shear tractions) are

$$\Re e\{\Phi^a(z^a)\} = \Re e\{\Phi^f(z^f)\} \tag{3.15}$$

and

$$\mu^a \Im m\{\Phi^a(z^a)\} = \mu^f \Im m\{\Phi^f(z^f)\}. \tag{3.16}$$

Choosing two such pairs of antiplane problems where the constants $(\mu^a,\ \mu^f)$ are replaced by $(\mu^a_\alpha,\ \mu^f_\alpha)$ and $(\mu^a_\beta,\ \mu^f_\beta)$, while the potential functions $(\Phi^a,\ \Phi^f)$ are replaced by $({\phi'}^a_\alpha,\ {\phi'}^f_\alpha)$ and $({\phi'}^a_\beta,\ {\phi'}^f_\beta)$, we arrive at the correspondence between the plane strain formulation for the two-phase medium and a pair of uncoupled antiplane strain problems.

Using this correspondence one can deduce the effective compliance matrix for the planar elasticity problem from the effective shear matrices for the antiplane problems. The effective shear matrices take the form

$$\boldsymbol{M}_{*\alpha} = \begin{pmatrix} k(\mu_{*\alpha})^2 & 0 \\ 0 & k \end{pmatrix}, \quad \boldsymbol{M}_{*\beta} = \begin{pmatrix} k(\mu_{*\beta})^2 & 0 \\ 0 & k \end{pmatrix}, \tag{3.17}$$

for suitable values of $\mu_{*\alpha}$ and $\mu_{*\beta}$ with $\mu_{*\alpha} > \mu_{*\alpha} > 0$. The effective compliance matrix has moduli such that $\mu_{*\alpha}$ and $\mu_{*\beta}$ are roots of the polynomial $s_{*1}\mu^4 - 2(s_{*2} + s_{*6})\mu^2 + s_{*4} = 0$. Also they must be such that the identities (2.56) are satisfied. Solving these equations gives the formulae

$$\begin{aligned} s_{*1} &= \frac{2\sqrt{\Delta}}{(\mu_{*\beta}^2 - \mu_{*\alpha}^2)}, \quad s_{*4} = \frac{2\mu_{*\alpha}^2\mu_{*\beta}^2\sqrt{\Delta}}{(\mu_{*\beta}^2 - \mu_{*\alpha}^2)}, \\ s_{*6} &= s_6, \qquad s_{*2} = -s_6 + \frac{(\mu_{*\beta}^2 + \mu_{*\alpha}^2)\sqrt{\Delta}}{\mu_{*\beta}^2 - \mu_{*\alpha}^2} \end{aligned} \tag{3.18}$$

for the effective compliance moduli in terms of the parameters $\mu_{*\alpha}$ and $\mu_{*\beta}$ that enter the expressions (3.17) for the effective shear matrices.

4 Evaluating the effective moduli using an integral equation method

4.1 Displacement and traction due to a point force

In this section we compute the displacement and the traction in a homogeneous orthotropic material due to a point force at the origin. The treatment partly follows Mura (1987).

The elastic properties of a homogeneous orthotropic material is described by a compliance matrix S containing coefficients s_1, s_2, s_4, and s_6, and, equivalently, by an elasticity matrix C containing coefficients c_1, c_2, c_4, and c_6. The elasticity matrix is the inverse of the compliance matrix. The displacement $\boldsymbol{u} = (u_1, u_2)$ satisfies the following second-order elliptic PDE, known as the elastostatic equation,

$$\begin{aligned} \left\{ \frac{\partial}{\partial x_1} c_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \frac{c_6}{2} \frac{\partial}{\partial x_2} \right\} u_1 + \left\{ \frac{\partial}{\partial x_2} \frac{c_6}{2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} c_2 \frac{\partial}{\partial x_2} \right\} u_2 = f_1, \\ \left\{ \frac{\partial}{\partial x_2} c_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} \frac{c_6}{2} \frac{\partial}{\partial x_2} \right\} u_1 + \left\{ \frac{\partial}{\partial x_1} \frac{c_6}{2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} c_4 \frac{\partial}{\partial x_2} \right\} u_2 = f_2, \end{aligned} \tag{4.1}$$

where $\boldsymbol{f} = (f_1, f_2)$ is applied force. The traction $\boldsymbol{t} = (t_1, t_2)$ on the plane with normal $\boldsymbol{n} = (n_1, n_2)$ is given by

$$\begin{aligned} t_1 &= \left\{ n_1 c_1 \frac{\partial}{\partial x_1} + n_2 \frac{c_6}{2} \frac{\partial}{\partial x_2} \right\} u_1 + \left\{ n_2 \frac{c_6}{2} \frac{\partial}{\partial x_1} + n_1 c_2 \frac{\partial}{\partial x_2} \right\} u_2 \\ t_2 &= \left\{ n_2 c_2 \frac{\partial}{\partial x_1} + n_1 \frac{c_6}{2} \frac{\partial}{\partial x_2} \right\} u_1 + \left\{ n_1 \frac{c_6}{2} \frac{\partial}{\partial x_1} + n_2 c_4 \frac{\partial}{\partial x_2} \right\} u_2 \end{aligned} \tag{4.2}$$

We shall use the Green's tensor whose columns represent solutions to (4.1) for $\boldsymbol{f}$ being a point force oriented along the coordinate axes. For convenience we assume $(s_2 + s_6)^2 > s_1 s_4$.

The basis for constructing this Green's tensor is the observation that for an appropriate ratio of constants a and b, the complex valued vector function $(u_1, u_2) = (a \log z, b \log z)$, where $z = x + i\mu y$, satisfies the homogeneous equation (4.1), except at the origin and along the branch cut of the logarithm; here μ is a root of the polynomial (3.8).

Now let μ_α and μ_β be the two positive distinct roots of (3.8). The solution $\boldsymbol{u}$ to (4.1) for $\boldsymbol{f}$ being a point force at $\boldsymbol{r}'$ can be written as

$$u_i(\boldsymbol{r}) = G_{ij}(S, \boldsymbol{r} - \boldsymbol{r}') f_j. \tag{4.3}$$

The four entries of the symmetric matrix G_{ij} representing the Green's tensor are

$$G_{11}(S, \boldsymbol{r}) = -\frac{p_\alpha^2}{q\mu_\alpha} \Re\mathrm{e}\left\{\log z_\alpha\right\} + \frac{p_\beta^2}{q\mu_\beta} \Re\mathrm{e}\left\{\log z_\beta\right\}, \tag{4.4}$$

$$G_{12}(S, \boldsymbol{r}) = G_{21}(S, \boldsymbol{r}) = \frac{p_\alpha p_\beta}{q} \Im\mathrm{m}\left\{\log z_\alpha - \log z_\beta\right\}, \tag{4.5}$$

$$G_{22}(S, \boldsymbol{r}) = \frac{p_\beta^2 \mu_\alpha}{q} \Re\mathrm{e}\left\{\log z_\alpha\right\} - \frac{p_\alpha^2 \mu_\beta}{q} \Re\mathrm{e}\left\{\log z_\beta\right\}, \tag{4.6}$$

where p_α, p_β, z_α, and z_β are defined by (3.7) and $q = 2\pi(\mu_\beta^2 - \mu_\alpha^2)s_1$. Notice that while the imaginary parts of $\log z_\alpha$ and $\log z_\beta$ are multiple valued, their difference, and hence the Green's tensor-function, is single valued.

The traction $\boldsymbol{t}$ in direction $\boldsymbol{n}$ at $\boldsymbol{r}$ due to a point force at $\boldsymbol{r}'$ can be expressed in a way analogous to that of the displacement, namely

$$t_i(\boldsymbol{r}) = {}^*G_{ij}(S, \boldsymbol{r} - \boldsymbol{r}') f_j. \tag{4.7}$$

The four entries of ${}^*G_{ij}$ are

$${}^*G_{11}(S, \boldsymbol{r}) = \frac{p_\alpha}{q} \Re\mathrm{e}\left\{\frac{n_\alpha}{z_\alpha}\right\} - \frac{p_\beta}{q} \Re\mathrm{e}\left\{\frac{n_\beta}{z_\beta}\right\}, \tag{4.8}$$

$${}^*G_{12}(S, \boldsymbol{r}) = -\frac{p_\beta \mu_\alpha}{q} \Im\mathrm{m}\left\{\frac{n_\alpha}{z_\alpha}\right\} + \frac{p_\alpha \mu_\beta}{q} \Im\mathrm{m}\left\{\frac{n_\beta}{z_\beta}\right\}, \tag{4.9}$$

$${}^*G_{21}(S, \boldsymbol{r}) = -\frac{p_\alpha}{q\mu_\alpha} \Im\mathrm{m}\left\{\frac{n_\alpha}{z_\alpha}\right\} + \frac{p_\beta}{q\mu_\beta} \Im\mathrm{m}\left\{\frac{n_\beta}{z_\beta}\right\}, \tag{4.10}$$

$${}^*G_{22}(S, \boldsymbol{r}) = -\frac{p_\beta}{q} \Re\mathrm{e}\left\{\frac{n_\alpha}{z_\alpha}\right\} + \frac{p_\alpha}{q} \Re\mathrm{e}\left\{\frac{n_\beta}{z_\beta}\right\}, \tag{4.11}$$

where $n_\alpha = \mu_\alpha n_1 + i n_2$ and $n_\beta = \mu_\beta n_1 + i n_2$.

4.2 Jump in traction over a line force

Assume that a line force $\boldsymbol{\rho} = (\rho_1, \rho_2)$ is present at a boundary B of a bounded domain D in the material. The traction $\boldsymbol{t}$ in direction $\boldsymbol{n}$ at a point $\boldsymbol{r}$ is then

$$t_i(\boldsymbol{r}) = \int_B {}^*G_{ij}(\boldsymbol{S}, \boldsymbol{r} - \boldsymbol{r}') \rho_j(\boldsymbol{r}') \mathrm{d}S'. \tag{4.12}$$

If $\boldsymbol{r}_0$ is a point on B and if $\boldsymbol{n}$ is taken to be the outward unit normal on B, then the following jump relations hold

$$\lim_{\substack{\boldsymbol{r}\to\boldsymbol{r}_0\\ \boldsymbol{r}\in cD}} t_i(\boldsymbol{r}) = t_i(\boldsymbol{r}_0) + \frac{\rho_i}{2}, \tag{4.13}$$

$$\lim_{\substack{\boldsymbol{r}\to\boldsymbol{r}_0\\ \boldsymbol{r}\in D}} t_i(\boldsymbol{r}) = t_i(\boldsymbol{r}_0) - \frac{\rho_i}{2}, \tag{4.14}$$

where cD is the complement of the closure of D, and $t_i(\mathbf{r}_0)$ is the "traction on the interface" defined by (4.12) taking the Cauchy principal value of the integral.

4.3 An integral equation for composite materials

In this section we transform the elastostatic equation (4.1) for the displacement $\boldsymbol{u}$ into an integral equation for an unknown force density $\boldsymbol{\rho}$. This is similar to a previous treatment of isotropic elasticity (Helsing 1995*a*).

We study a doubly periodic composite with a square unit cell. A unit cell has a unit area. The unit cell consists of one or more amoebas embedded in a filler. The filler is described by a compliance matrix $\boldsymbol{S}_{\rm f}$ and a corresponding elasticity matrix $\boldsymbol{C}_{\rm f}$. $\boldsymbol{S}_{\rm f}$ has elements $s_{\rm f1}$, $s_{\rm f2}$, $s_{\rm f4}$ and $s_{\rm f6}$. $\boldsymbol{C}_{\rm f}$ has elements $c_{\rm f1}$, $c_{\rm f2}$, $c_{\rm f4}$, and $c_{\rm f6}$. The amoeba material is described by a compliance matrix $\boldsymbol{S}_{\rm a}$ and an elasticity matrix $\boldsymbol{C}_{\rm a}$. $\boldsymbol{S}_{\rm a}$ has elements $s_{\rm a1}$, $s_{\rm a2}$, $s_{\rm a4}$ and $s_{\rm a6}$. $\boldsymbol{C}_{\rm a}$ has elements $c_{\rm a1}$, $c_{\rm a2}$, $c_{\rm a4}$, and $c_{\rm a6}$. The interfaces of the amoebas in the unit cell are called B. Their periodic images outside the unit cell are called B^∞.

Now apply an average displacement field $\boldsymbol{A}$ to the composite. To describe the elastic response in the filler we introduce a homogeneous comparison material with compliance matrix $\boldsymbol{S}_{\rm f}$. In the homogeneous material we impose a force density $\boldsymbol{\rho}^{\rm f}$ along B and along its periodic images B^∞ in such a way as to reproduce the displacement field in the original composite outside the amoebas. To describe the elastic response within the amoeba we introduce a homogeneous comparison material with compliance matrix $\boldsymbol{S}_{\rm a}$. In the homogeneous material we impose a force density $\boldsymbol{\rho}^{\rm a}$ only along B in such a way as to reproduce the displacement field inside the amoeba. To describe this in mathematical terms we need to introduce two displacement integral operators

$$(M_1)_{ij}\rho_j = \int_{B+B^\infty} G_{ij}(\boldsymbol{S}_{\rm f}, \boldsymbol{r}-\boldsymbol{r}')\rho_j(\boldsymbol{r}')\mathrm{d}S', \tag{4.15}$$

$$(M_2)_{ij}\rho_j = \int_{B} G_{ij}(\boldsymbol{S}_{\rm a}, \boldsymbol{r}-\boldsymbol{r}')\rho_j(\mathbf{r}')\mathrm{d}S'. \tag{4.16}$$

In terms of these operators, the applied average displacement $\boldsymbol{A}$ and the two force densities $\boldsymbol{\rho}^{\rm f}$ and $\boldsymbol{\rho}^{\rm a}$ the displacement $\boldsymbol{u}$ in the filler is represented as

$$u_i = A_i + (M_1)_{ij}\rho_j^{\rm f}. \tag{4.17}$$

The displacement in the amoeba is represented as

$$u_i = A_i + (M_2)_{ij}\rho_j^{\rm a}. \tag{4.18}$$

Since we are interested in ensuring continuity of the displacement field across the boundary B let us introduce operators M_3 and M_4 which represent the restriction of M_1 and M_2 to B:

$$(M_3)_{ij}\rho_j = \int_{B+B^\infty} G_{ij}(\boldsymbol{S}_{\rm f}, \boldsymbol{r}-\boldsymbol{r}')\rho_j(\boldsymbol{r}')\mathrm{d}S' \qquad \boldsymbol{r}\in B, \tag{4.19}$$

$$(M_4)_{ij}\rho_j = \int_B G_{ij}(\boldsymbol{S}_\mathrm{a}, \boldsymbol{r} - \boldsymbol{r}')\rho_j(\boldsymbol{r}')\mathrm{d}S' \qquad \boldsymbol{r} \in B. \tag{4.20}$$

Also since we are interested in ensuring continuity of tractions across B let us introduce the traction integral operators

$$(M_5)_{ij}\rho_j = \int_{B+B^\infty} {}^*G_{ij}(\boldsymbol{S}_\mathrm{f}, \boldsymbol{r} - \boldsymbol{r}')\rho_j(\boldsymbol{r}')\mathrm{d}S' \qquad \boldsymbol{r} \in B, \tag{4.21}$$

$$(M_6)_{ij}\rho_j = \int_B {}^*G_{ij}(\boldsymbol{S}_\mathrm{a}, \boldsymbol{r} - \boldsymbol{r}')\rho_j(\boldsymbol{r}')\mathrm{d}S' \qquad \boldsymbol{r} \in B. \tag{4.22}$$

Together with the applied field these operators give "the traction on the interface". To determine the traction on each side of the interface we use (4.13) and (4.14).

The requirements of continuity of displacements and tractions (which are implied by the elastostatic equation) lead to the relations

$$(M_3)_{ij}\rho_j^\mathrm{f} = (M_4)_{ij}\rho_j^\mathrm{a}, \tag{4.23}$$

$$D_{ij}n_j = (M_5)_{ij}\rho_j^\mathrm{f} - (M_6)_{ij}\rho_j^\mathrm{a} + \frac{\rho_i^\mathrm{f} + \rho_i^\mathrm{a}}{2}. \tag{4.24}$$

Here D_{ij} is a symmetric matrix with entries

$$D_{11} = (c_{\mathrm{a}1} - c_{\mathrm{f}1})\partial A_1/\partial x_1 + (c_{\mathrm{a}2} - c_{\mathrm{f}2})\partial A_2/\partial x_2, \tag{4.25}$$

$$D_{12} = D_{21} = \frac{(c_{\mathrm{a}6} - c_{\mathrm{f}6})}{2}(\partial A_1/\partial x_2 + \partial A_2/\partial x_1), \tag{4.26}$$

$$D_{22} = (c_{\mathrm{a}2} - c_{\mathrm{f}2})\partial A_1/\partial x_1 + (c_{\mathrm{a}4} - c_{\mathrm{f}4})\partial A_2/\partial x_2. \tag{4.27}$$

The equations (4.23) and (4.24) are coupled first and second kind Fredholm integral equations. These could be solved in many ways. We use the first equation to express $\boldsymbol{\rho}^\mathrm{a}$ in terms of $\boldsymbol{\rho}^\mathrm{f}$ and substitute the result into the second equation, yielding

$$D_{ij}n_j = \left\{\frac{\delta_{ij}}{2} + \frac{((M_4)^{-1}M_3)_{ij}}{2} + (M_5)_{ij} - (M_6(M_4)^{-1}M_3)_{ij}\right\}\rho_j^\mathrm{f}. \tag{4.28}$$

4.4 Effective properties

Once (4.28) is solved the effective properties can easily be computed from $\boldsymbol{\rho}^\mathrm{f}$. From Gauss' theorem we have

$$\int \sigma_{ij}\mathrm{d}V = \int r_j\sigma_{ik}n_k\mathrm{d}S. \tag{4.29}$$

Using this identity and the continuity of traction over interfaces a calculation (which we do not reproduce here) shows that

$$\int \sigma_{11}\mathrm{d}V = \int_{B_\mathrm{cell}} (c_{\mathrm{f}1}A_1\mathrm{d}S_1 + c_{\mathrm{f}2}A_2\mathrm{d}S_2) + \int_B x_1\rho_1^\mathrm{f}\mathrm{d}S, \tag{4.30}$$

$$\int \sigma_{22}\mathrm{d}V = \int_{B_\mathrm{cell}} (c_{\mathrm{f}2}A_1\mathrm{d}S_1 + c_{\mathrm{f}4}A_2\mathrm{d}S_2) + \int_B x_2\rho_2^\mathrm{f}\mathrm{d}S \tag{4.31}$$

$$\int \sigma_{12} \mathrm{d}V = \int_{B_{\mathrm{cell}}} \frac{c_{\mathrm{f}6}}{2}(A_1 \mathrm{d}S_1 + A_2 \mathrm{d}S_2) + \int_B x_1 \rho_2^{\mathrm{f}} \mathrm{d}S, \tag{4.32}$$

where B_{cell} denotes the boundary of the unit cell. In particular, with $\boldsymbol{A} = (x_1, 0)$ we can compute the effective elastic moduli

$$c_{*1} = c_{\mathrm{f}1} + \int_B x_1 \rho_1^{\mathrm{f}} \mathrm{d}S \qquad \text{and} \qquad c_{*2} = c_{\mathrm{f}2} + \int_B x_2 \rho_2^{\mathrm{f}} \mathrm{d}S. \tag{4.33}$$

With $\boldsymbol{A} = (0, x_2)$ we can compute the effective elastic moduli

$$c_{*2} = c_{\mathrm{f}2} + \int_B x_1 \rho_1^{\mathrm{f}} \mathrm{d}S \qquad \text{and} \qquad c_{*4} = c_{\mathrm{f}4} + \int_B x_2 \rho_2^{\mathrm{f}} \mathrm{d}S. \tag{4.34}$$

With $\boldsymbol{A} = (x_2, x_1)$ we can compute the effective elastic modulus

$$c_{*6} = c_{\mathrm{f}6} + \int_B x_1 \rho_2^{\mathrm{f}} \mathrm{d}S. \tag{4.35}$$

4.5 Evaluating the integral operators numerically

Let us represent the unknown force densities $\boldsymbol{\rho}^{\mathrm{f}}$ and $\boldsymbol{\rho}^{\mathrm{a}}$ of (4.23) and (4.24) by their values at $2N$ equi-spaced points on the interface B. We can think of $\boldsymbol{\rho}^{\mathrm{f}}$ and $\boldsymbol{\rho}^{\mathrm{a}}$ as being represented by vectors. In this representation the integral operators of the preceding sections become matrices with $16N^2$ elements. The action of an integral operator on a force density corresponds to matrix vector multiplication. An individual matrix element gives the contribution of a force density at some point to the displacement or the traction at some other point. Equi-spaced discretization of a boundary integral equation in two-dimensions corresponds to evaluating the integral with the trapezoidal quadrature rule. The trapezoidal rule achieves superalgebraic convergence for smooth periodic functions once the spacing between discretization points is sufficiently small. In our applications we can take advantage of this because the interface is closed.

For numerical purposes it is useful to view the unit cell centers of the doubly periodic composite as lattice points in a complex plane. Furthermore, for a given value of μ, it useful to consider a stretched complex lattice where the lattice points form a set given by $k_1 + i\mu k_2$ with k_1 and k_2 being integers.

The computation of matrix elements corresponding to the integral operators of section 4.3 involves discretization of kernels of the type $\log[z - \omega - z']$ and $n/(z - \omega - z')$, where ω is a lattice vector on the stretched lattice. The discretized kernels are evaluated using different techniques depending on the magnitude of ω (Helsing and Samuelsson 1995). When $|\omega|$ is small the kernels are evaluated directly. For larger $|\omega|$ series expansion is used.

Now we comment on a few technical details which apply to the evaluation of the kernels when ω is the lattice point at the origin. These details differ from the treatment of the conductivity problem described in Helsing and Samuelsson (1995).

Kernels of the type $n/(z - z')$ have Cauchy type singularities. These kernels are evaluated using only every other discretization point: this is one reason why the number of discretization points was chosen to be even. Discretization points on B are assigned consecutive numbers. Matrix elements giving contributions from odd points to odd points and from even points to even points are omitted. Contributions from odd points to even points, and from even points to odd points are counted twice.

Kernels of the type $\Re\mathrm{e}\{\log[z - z']\}$ are evaluated with a Fourier method. An even number of Fourier terms, $2N$, is used (matching the number of discretization points). In the previous work an odd number of discretization points and Fourier terms were used.

The kernel $\Im\mathrm{m}\{\log[(z_\alpha - z'_\alpha)/(z_\beta - z'_\beta)]\}$ is evaluated with straight-forward pointwise discretization. When $z_\alpha = z'_\alpha$ and $z_\beta = z'_\beta$ this involves taking the limit

$$\lim_{\boldsymbol{r}' \to \boldsymbol{r}} \log \left[\frac{z_\alpha - z'_\alpha}{z_\beta - z'_\beta} \right] = \log \left[\frac{\dot{\xi}(s) + i\mu_\alpha \dot{\eta}(s)}{\dot{\xi}(s) + i\mu_\beta \dot{\eta}(s)} \right], \tag{4.36}$$

where $(x_1, x_2) = (\xi(s), \eta(s))$ is a parameterization of the interface B and the dot denotes differentiation.

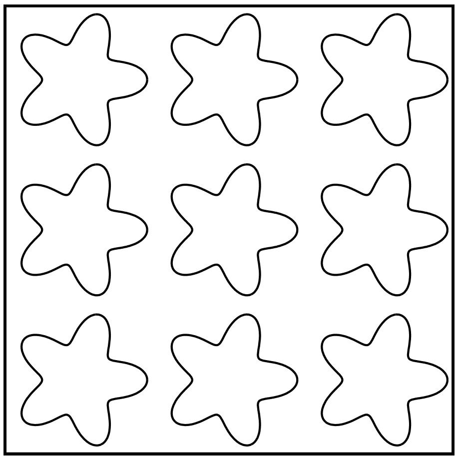

Fig.1 A unit cell and its nearest neighbours
in a periodic array of amoebas

5 Numerical Examples

Following Holmbom, Persson and Svanstedt (1992) we evaluate $\boldsymbol{S}_*$ for a square array of cylinders with the cylinders occupying a volume fraction of 0.45. The amoeba and filler have elasticity tensors as in Table 1. Our results, accurate to the number of digits specified, are compared with their results from finite element method computations.

To illustrate the main results of sections 2 and 3 we considered two examples for the geometry in Fig. 1. The amoeba has five arms and its interface has the parameterization

$$(x_1, x_2) = R(1 + \epsilon \cos n\phi)(\cos n\phi, \sin n\phi) \tag{5.1}$$

with $R = 0.35$, $\epsilon = 0.3$, and $n = 5$. The first example illustrates the use of the duality relation (2.24). The compliance moduli $\boldsymbol{S}_a$ and $\boldsymbol{S}_f$ have been chosen so they (and hence $\boldsymbol{S}_*$) share the same value of s_6. The original material is translated until $s_6 = 0$ in both the amoeba and the filler phases. The duality transformation is applied, and the sign of all moduli reversed. The resulting material is then translated back so that the value of s_6 coincides with the original value. (Any other choice for the final value of s_6 would have been equally appropriate: this choice was merely an example.) The compliance moduli of the resulting composite are given by the formulae

$$s_{*1}^{\text{dual}} = \frac{s_{*1}}{\Delta_*}, \quad s_{*2}^{\text{dual}} = -s_{*6} + \frac{s_{*2} + s_{*6}}{\Delta_*}, \quad s_{*4}^{\text{dual}} = \frac{s_{*4}}{\Delta_*}, \quad s_{*6}^{\text{dual}} = s_{*6}, \tag{5.2}$$

where $\Delta_* = (s_{*2} + s_{*6})^2 - s_{*1}s_{*4}$. The numerical results of Table 2 are consistent with these equations, as the reader may check.

The second example illustrates the use of the phase interchange relation (2.30). The compliance moduli of the two phases have been chosen so that the relations

$$s_{a1} = \alpha s_{f1}, \quad s_{a4} = \alpha s_{f4}, \quad s_{a2} - \sqrt{s_{a1}s_{a4}} = s_{f2} - \sqrt{s_{f1}s_{f4}} = c, \quad s_{a6} + c = \alpha(s_{f6} + c) \tag{5.3}$$

are satisfied for some choice of constants α and c, specifically for $\alpha = 2$ and $c = -1$. The original material is translated until the determinant of the leading two by two block of the compliance matrix of both amoeba and filler simultaneously vanish. The constraints (5.3) ensure that the resulting compliance matrices of moduli and filler are proportional sharing the same null-vector. The associated effective tensor must also share this null-vector. It follows that there exists a constant α_* such that

$$s_{*1} = \alpha_* s_{f1}, \quad s_{*4} = \alpha_* s_{f4}, \quad s_{*2} = c + \alpha_* \sqrt{s_{f1}s_{f4}}, \tag{5.4}$$

For the example of Table 3 we have $\alpha_* = 1.258380336451$. Next the phase interchange relation (2.30) is applied to the translated medium.

The resulting compliance matrices are then translated back so that they match the original compliance tensors, but with the phases interchanged. The compliance moduli of the resulting composite are given by the formulae

$$\begin{aligned} s_{*1}^{\text{int}} &= \alpha_*^{\text{int}} s_{f1}, \quad s_{*2}^{\text{int}} = c + \alpha_*^{\text{int}} \sqrt{s_{f1}s_{f4}}, \\ s_{*4}^{\text{int}} &= \alpha_*^{\text{int}} s_{f4}, \quad s_{*6}^{\text{int}} = (s_{f6} + c)\alpha/\alpha_* - c, \end{aligned} \tag{5.5}$$

where

$$\alpha_*^{\text{int}} = \alpha(s_{f6} + c)/(s_{*6} + c). \tag{5.6}$$

The numerical results of Table 3 are consistent with these equations, as the reader may check.

To illustrate the correspondence between planar elasticity and antiplane elasticity we choose a set of moduli so that the relations (3.5) are satisfied. The numerical results for the effective compliance moduli for the five armed amoeba of Fig 1 and for a nearly touching square array of disks of radius 0.48 are given in Table 3. For comparison the effective shear matrices of the two associated antiplane problems given by (3.13) are also tabulated. These are calculated using the algorithm of Helsing and Samuelsson (1995). The reader may check that one can recover the effective compliance moduli from the effective shear moduli using the formulae (3.18).

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Table 1: Comparison with previous results: Elements of the effective elastic matrix C_* for a square array of disks in a filler. Both materials have elastic tensors with square symmetry ($c_2 = c_4$). The area fraction of disks is 0.45. The elastic constants for filler and disks are given in the first two columns. The effective values of the third column is computed via discretization and solution of (4.28) and evaluation of (4.33)–(4.35). 96 discretization points are needed for full convergence. The fourth column is from Holmbom, Persson and Svanstedt (1992).

	filler	disks	effective[a]	effective[b]
c_1	3800	45000	9010.311864769	8900
c_2	1200	14000	1952.438074613	2000
c_6	1300	16000	2610.516059647	2500

[a] Refers to the integral equation method
[b] Refers to Holmbom, Persson and Svanstedt (1992)

Table 2: Duality in plane elasticity: Elements of the effective compliance matrix S_* for square arrays of amoebas in a filler. The geometry of the amoebas is shown in Fig. 1. Two compliance matrices are considered, corresponding to the original and to the dual problem discussed in section 2. 1000 discretization points are used for convergence of the effective constants to the number of digits displayed.

	filler	amoebas	effective
s_1	0.5	0.5	0.4943932823394
s_2	0.5	0.25	0.3867428369260
s_4	1.5	1	1.258468650918
s_6	0.5	0.5	0.5000000000000
s_1^{dual}	2	8	3.0121244957
s_2^{dual}	3.5	11.5	4.9025406814
s_4^{dual}	6	16	7.6673053334
s_6^{dual}	0.5	0.5	0.5000000000000

Table 3: Phase interchange relations in plane elasticity: Elements of the effective compliance matrix S_* for square arrays of amoebas in a filler. The geometry of the amoebas is shown in Fig. 1. Two compliance matrices are considered, corresponding to the original and to the interchanged problem discussed in section 2. 1000 discretization points are used for convergence of the effective constants to the number of digits displayed.

	filler	amoebas	effective
s_1	2	4	2.51676067290
s_2	0	1	0.258380336451
s_4	0.5	1	0.629190168225
s_6	1.5	2	1.69640856095
$s_1^{\rm int}$	4	2	2.87187738940
$s_2^{\rm int}$	1	0	0.435938694701
$s_4^{\rm int}$	1	0.5	0.717969347350
$s_6^{\rm int}$	2	1.5	1.79467230298

Table 4: Linking planar and antiplane elasticity: Elements of the effective compliance matrix S_* and effective shear matrix M_* for square arrays of disks and for square arrays of amoebas. The geometry of the amoebas is shown in Fig. 1. In the square array of disks the disk radius is $R = 0.48$. One compliance matrix for the planar problem and two shear matrices for the antiplane problems are considered. 600 discretization points are used for the disk and 900 for the amoeba to obtain convergence of the effective constants to the number of digits displayed.

	filler	inclusion	effective disks[e]	effective amoebas[f]
s_1	1.25	0.5	0.6468727571121	0.8656478246863
s_2	0.5	0.25	0.3110206003286	0.3877402150152
s_4	0.75	1	0.9202032511230	0.8381961678449
s_6	0.5	0.5	0.5000000000000	0.5000000000000
m_{11}	0.25	0.5	0.4100576924376	0.3285805678041
m_{22}	0.25	0.25	0.2500000000000	0.2500000000000
m_{11}	1.35	2.25	1.951382767107	1.657620388874
m_{22}	2.25	2.25	2.250000000000	2.250000000000

[e] Effective constants for inclusions being disks
[f] Effective constants for inclusions being amoebas