NEW EXPLICIT SOLUTIONS OF QUASILINEAR HEAT EQUATIONS WITH GENERAL FIRST-ORDER SIGN-INVARIANTS

Victor A. Galaktionov
School of Mathematics, University of Bath
Claverton Down, Bath BA2 7AY, UK
Fax: +44 (1225) 826 492, E-mail: vag @ maths.bath.ac.uk

and

Sergey A. Posashkov
Keldysh Institute of Applied Mathematics
Russian Academy of Sciences, Miusskaya Sq. 4, 125047 Moscow, RUSSIA
Fax: (7-095) 972-0737, E-mail: posash @ applmat.msk.su

Abstract

For the one-dimensional quasilinear heat equations $u_t = \psi(u)u_{xx} + f(u)$, with smooth functions $\psi(u) \geq 0$ and $f(u)$, we derive conditions under which a general quadratic Hamilton-Jacobi operator $\mathcal{H}(u) \equiv u_t - [h_2(u)(u_x)^2 + h_1(u)u_x + h_0(u)]$ (with the functions $h_2,\ h_1,\ h_0$ to be determined) preserves both the signs, ≥ 0 and ≤ 0, on the solutions $u(x,t)$. $\mathcal{H}(u)$ is then called a sign-invariant of the given parabolic equation. Several types of different sign-invariants are established which are governed by a linear second-order ordinary differential equations. New explicit solutions via finite-dimensional dynamical systems corresponding to zero-preserving of the first order operator are constructed.

Key words. Quasilinear heat equation, Maximum Principle, explicit solutions, solitons, invariant subspaces and sets.

AMS (MOS) subject classification. 35K55, 35K65

1. Introduction

We consider the one-dimensional quasilinear heat equation

$$u_t = \psi(u)u_{xx} + f(u), \quad x \in \mathbf{R},\ t > 0,$$

where $\psi(u) \geq 0$ and $f(u)$ are given functions smooth enough. This equation is easily derived from the well-known "divergent" heat equation with source and/or absorption

$$u_t = (k(u)u_x)_x + Q(u)$$

by a nonlinear transformation of the form $u \to \int k(u)du$. It is well-known that these equations have a wide field of applications in the combustion theory, plasma physics, biophysics and mechanics of porous media. Such quasilinear evolution partial differential equations are quite popular in group-theoretical, classical and nonclassical methods for finding exact solutions which are responsible for different nonlinear phenomena. We refer to the paper by Clarkson and Mansfield [2], where a general survey, a wide list of references and a complete analysis of the *semilinear* equations (i.e., equations with $\psi(u) = k(u) \equiv 1$) are given. There are many different directions of the mathematical theory of the quasilinear heat equations concerning existence, uniqueness, regularity, asymptotics and other aspects of the qualitative properties of the solutions. See e.g. the list of references in the book by Samarskii et al [12] (we will use some of its ideas).

Let us state the quasilinear heat equation in a form more convenient for the future consideration. Namely, we perform a suitable transformation $u = \Phi(v)$ to get the quasilinear equation

$$\text{(HE)} \qquad v_t = \varphi(v)v_{xx} + m(v)(v_x)^2 + F(v),$$

with smooth nonlinear coefficients. We assume that the set $\{v(x,t)\}$ of smooth (local in time) solutions to (HE) is not empty. As the next step, we introduce a general quadratic Hamilton-Jacobi operator of the form

$$\text{(HJ)} \qquad \mathcal{H}(v) = v_t - H(v) \equiv v_t - [m(v)(v_x)^2 + s(v)v_x + M(v)],$$

where the coefficients m, s and M depend on the coefficients of (HE) and will be determined later. Without loss of generality, in (HJ) we put the coefficient of $(v_x)^2$ to be equal $m(v)$, the same function as in (HE). This representation of operator (HJ) is established for further convenience.

We begin with the main property of solutions to (HE) we are interested in.

Definition. We say that the operator (HJ) is *sign-invariant* of the parabolic equation (HE) if it preserves both the signs, ≥ 0 and ≤ 0, on the solutions $v(x,t)$ to (HE). This means that

$$(1.1) \qquad \mathcal{H}(v) \geq 0 \text{ (resp. } \leq 0) \text{ in } \mathbf{R} \text{ for } t = 0 \Rightarrow \mathcal{H}(v) \geq 0 \text{ (resp. } \leq 0) \text{ in } \mathbf{R} \text{ for } t > 0.$$

In terms of the original solutions $u(x,t)$ the sign-invariant has a similar form $\tilde{\mathcal{H}}(u) = u_t - [h_2(u)(u_x)^2 + h_1(u)u_x + h_0(u)]$.

The analysis of the sign-preservation of operator (HJ) is a consequence of a formal application of the Maximum Principle for parabolic equations. See details in the recent paper by one of the author [7]. Observe that the construction of suitable, at least one-sided, sign-invariants (the so-called *barrier techniques*) plays the fundamental role in the qualitative theory of nonlinear parabolic equations. Recall, for instance, that now classical interior regularity bounds of the Bernstein type or similar ones in the theory of parabolic equtions are derived via upper bounds on the term $\xi(x)[(\rho(v))_x]^2$ ($\xi \in C_0^\infty$, ρ depends on the coefficients) satisfying a parabolic differential inequality. See the book by Friedman [4] and references in [12] and [7].

It is clear that a suitable sign-invariant makes it possible to construct exact bounds for a wide class of general solutions to the nonlinear PDE's. This idea has a natural relation to the general theory of differential constraints, see first general statements in the book by Sidorov, Shapeev and Yanenko [14]. The lists of related references can be also found in Clarkson and Mansfield [2] and Galaktionov [7]. One of the main point of the present paper is that the acceptable structure of differential constraints comes from well-known ideas and results of the qualitative theory of nonlinear parabolic equations, and in particular from the recent blow-up research of quasilinear heat equations from the combustion theory. See [12].

For convenience, we introduce the new coefficients

$$g(v) = -\frac{s(v)}{\varphi(v)}, \quad G(v) = \frac{F(v) - M(v)}{\varphi(v)}. \tag{1.2}$$

Denote by $P(v)$ a quadratic trinomial of the form

$$P(v) = \frac{av^2}{2} + bv + c, \tag{1.3}$$

where a, b and c are arbitrary constants.

We now state the first main result.

Theorem 1.1. *The Hamilton-Jacobi operator (HJ) is a sign-invariant of the parabolic equation (HE) if the coefficients $g(v)$ and $G(v)$ satisfy the following system of ordinary differential equations*

$$2Pg'' + 3P'g' - 8P''g = 0, \tag{1.4}$$

$$2PG'' + 3P'G' - 3P''G = \frac{2}{3}(Pg^2)', \tag{1.5}$$

and the other coefficients are given by

$$m = P', \quad s' = \frac{2}{3}(2P'g + Pg'), \quad M = 2PG. \tag{1.6}$$

We call the linear ODE's (1.4) and (1.5) of the *hypergeometric* type (the second one is non-homogeneous and its right-hand side depends on the solutions to the first one) the *generating equations,* since by (1.6) the functions g and G determine all the other coefficients in (HE) and (HJ).

Let us now give some new explicit solutions which follow from the fact that under the conditions of Theorem 1.1 the operator (HJ) is also a *zero-invariant* of equation (HE), i.e.,

$$\mathcal{H}(v) = 0 \quad \text{for all } t > 0 \quad \text{if} \quad \mathcal{H}(v) = 0 \quad \text{for } t = 0. \tag{1.7}$$

By *explicit solutions* to the nonlinear partial differential equation (HE) we mean those which can be expressed in terms of finite-dimensional dynamical systems and/or of algebraic equations.

We now state a general result. Examples of particular explicit solutions expressed by quadratures are given in the subsequent sections. For fixed constants B and C, denote by $v = V(x; B, C)$ the solution of the Cauchy problem for the following ODE

$$v_{xx} + g(v)v_x + G(v) = 0 \quad \text{for } x > 0, \tag{JE}$$

with the initial conditions

$$v(0) = B, \quad v'(0) = C. \tag{1.8}$$

We assume that this problem is locally well-posed for all $B, C \in \mathbf{R}$. The equation (JE) is easily seen to be *"joint"* to the system of equation (HE) and of the corresponding Hamilton-Jacobi equation

$$\mathcal{H}(v) = 0. \tag{HJ'}$$

These three equations are linear dependent. It is clear that the joint equation (JE) directly reduces to the first order one and also in several cases can be integrated in quadratures. See examples below. The case $g = 0$ when (JE) is always integrated in quadratures was considered in [7]. Observe that in general the structure of explicit solutions in [7] is quite different and simpler.

Theorem 1.2. *Under the hypotheses of Theorem 1.1 the set of the explicit solutions $\{v(x,t) = V(x; B(t), C(t))\}$ via zero-invariance of the operator (HJ) are governed by the following second-order dynamical system on the coefficients $B(t)$ and $C(t)$*

$$B' = P'(B)C^2 + s(B)C + 2P(B)G(B), \tag{1.9}$$

$$\begin{aligned} C' = {} & P''(B)C^3 + [s'(B) - 2P'(B)g(B)]C^2 \\ & + [2P(B)G'(B) - s(B)g(B)]C - s(B)G(B) \quad \textit{for } t > 0. \end{aligned} \tag{1.10}$$

Moreover, below we give examples where not only the dynamical system (1.9), (1.10) but also the solution's $v(x,t)$ structure admits a finite exact (quadrature) representation (this is expected cannot be done in the general case in Theorem 1.2). One can see from the easy proof of Theorem 1.2 in Section 2 that this result has a general nature, and under suitable (invariance, see below) assumptions such a dynamical (ODE's) representation exists in much more general circumstances.

In fact, in this case the stationary equation (JE) defines an *invariant set* $\mathcal{M}$ of the parabolic equation (HE) in the sense that

$$v(\cdot,0) \in \mathcal{M} \Rightarrow v(\cdot,t) \in \mathcal{M} \quad \text{for all } t > 0. \tag{1.11}$$

Therefore the dynamical system (1.9), (1.10) is the nonlinear partial differential equation (HE) on the invariant set $\mathcal{M}$. In several cases of quasilinear evolution equations the corresponding invariant sets $\mathcal{M}$ can be finite-dimensional linear subspaces invariant under nonlinear operators, see examples in the papers by the authors [8], [5]-[7]. New more complicated examples are given in Sections 3 - 6 of the present paper. Another approach was introduced in [9] where certain invariant sets on linear subspaces were considered. See also a "geometric" treatment of such solutions in Section 7.

We perform a general classification of the Hamilton-Jacobi sign and zero-invariants (HJ). Let us mention that the above solutions include as a particular case the class of explicit so called *soliton-type* solutions of the semilinear heat equation constructed by Cariello and Tabor [1], Kawahara and Tanaka [11], Kapsov [10] and in the most general case by Clarkson and Mansfield [2] by different approaches. In Section 5 we show that these soliton-type solutions to the *quasilinear* equation (HE) correspond to a particular case of the trivial choice of the quadratic trinomial in (1.3),

$$P(v) = c \quad (\text{i.e., } a = b = 0). \tag{1.12}$$

Moreover, in the case (1.12) the corresponding equation (JE) is easily linearized by the Cole-Hopf transformation $v = \kappa\rho_x/\rho$, which naturally implies a soliton's character of the solutions. In particular, this means that the invariant set introduced by Kapsov [10] is indeed a linear subspace governed by a linear third order ODE satisfied by the function ρ, and the corresponding semilinear heat equation (HE) is then Burger's equation linearized by the same Cole-Hopf transformation. Other soluton's type solutions occur in the linear case $P(v) = v$, see Section 3. We show that such linear invariant sets and similar explicit solutions exist for a wide class of quasilinear equations (HE).

In Sections 4 and 6 we present examples of essentially new explicit quadrature-type solutions to the quasilinear heat equations. We are sorry that some of them have too complicated algebraic form. In particular, in Section 6 we construct explicit solutions of the following quasilinear heat equation

$$\begin{aligned} v_t = {} & -\frac{a^2v^4 + 2acv^2 + K}{2av^2 + c}\, v_{xx} + av(v_x)^2 \\ & + \frac{1}{256}\,\frac{a^2v^4 + 3acv^2 + 2c^2 - K}{2av^2 + c}\,(4a^2v^5 + 8acv^3 + 3c^2v), \end{aligned} \tag{1.13}$$

where a, c and K are arbitrary constants.

2. Proof of the theorems: a general sign-invariant

1. We begin with the following preliminary result.

Proposition 2.1. *The operator (HJ) is a sign-invariant of the parabolic equation (HE) if the coefficients satisfy*

$$m'' = 0, \tag{2.1}$$

$$I_1(v) \equiv s'' - 4m'g - g'm = 0, \tag{2.2}$$

$$I_2(v) \equiv M'' - 2gs' - 5m'G - mG' + 2mg^2 = 0, \tag{2.3}$$

$$I_3(v) \equiv -3s'G + g'M + 4mgG = 0, \tag{2.4}$$

$$I_4(v) \equiv -\left(\frac{M}{G}\right)' + 2m = 0. \tag{2.5}$$

Proof. We set

$$J = \mathcal{H}(v). \tag{2.6}$$

Then differentiating this equality with respect to t, $J_t = v_{tt} - (H(v))_t$, and substituting the second derivative v_{tt} from (HE) differentiated in t and calculating other lower-order ones, v_t, v_{tx} and v_{txx} from equation (2.6), we arrive at a parabolic equation of the form

$$J_t = \mathcal{L}J + \mathcal{F}, \tag{2.7}$$

where $\mathcal{L}$ is a linear second order elliptic operator, and the lower-order term is

$$\mathcal{F} = \varphi\left[m''(v_x)^4 + I_1(v_x)^3 + I_2(v_x)^2 + I_3 v_x + I_4 G^2\right]. \tag{2.8}$$

Therefore, $\mathcal{F} \equiv 0$ by the system (2.1) - (2.5), and hence the function J solves the linear parabolic equation

$$J_t = \mathcal{L}J. \tag{2.9}$$

The result then follows from the Maximum Principle. Moreover, if $\mathcal{F} \not\equiv 0$ in (2.7) then at least one sign of (HJ) cannot be preserved in time on the corresponding subset of solutions. □

Remark 1. Recall that for the rigorous application of the Maximum Principle to equation (2.9), as it is well-known from the qualitative parabolic theory, we have to impose some extra conditions on the coefficients of the equation and on the boundary and initial data in the problem under consideration (see a general statement of the Maximum Principle in the book by Friedman [4]). These conditions are checked in rather general cases.

Remark 2. If we consider the identity $\mathcal{F} \equiv 0$ as an algebraic fourth-order equation for v_x, we obtain something like $v_x = R(v)$ (i.e., the set $1, v_x, ..., v_x^4$ is linear dependent in (2.8)). This kind of zero-invariance reduces to $\int dv/R(v) = x + C(t)$. Substituting into (HE) we arrive at the equation C' =const, whence well-known travelling wave solutions

$v(x,t) = \theta(x+\lambda t)$. Therefore, in order to get a nontrivial result, in the proof we assume that all the five terms in (2.8) are linear independent, that gives five ODE's (2.1) - (2.5).

2. *Proof of Theorem 1.1.* (2.1) immediately implies that $m(v) = av + b \equiv P'(v)$, see the first equation (1.6). It follows from (2.5) that

$$\frac{M}{G} = 2\int m(v)dv \equiv 2P(v), \tag{2.10}$$

cf. the third equation (1.6). Substituting M from (2.10) into (2.4) we obtain

$$3s' - 4P'g - 2Pg' = 0, \tag{2.11}$$

and the second equation (1.6) follows. The system of equations (2.2) and (2.11) gives the first generating equation (1.4). Finally, the second one, (1.5), is the result of substituting the function M from (2.10) and s' from (2.11) into (2.3). $\square$

3. *Proof of Theorem 1.2.* Setting $x = 0$ in the Hamilton-Jacobi equation (HJ') and in the equation (differentiated in x) $v_{xt} = (H(v))_x$ and using (1.8), (JE) and (1.6) we arrive at the dynamical system (1.9), (1.10). $\square$

3. The linear case

1. General structure of sign-invariants. In Sections 3 and 4 we present a full description of sign-invariants and the corresponding explicit solutions in the *linear* case $a = 0$ where the quadratic trinomial is the linear function $P(v) = bv + c$ with the constant $b \neq 0$. Without loss of generality, we put $c = 0$ which corresponds to translation. Moreover, one can see from the generating equations (1.4), (1.5) that the constant b plays no important role in the analysis, and we put $b = 1$.

Finally, we consider the system of generating equations (1.4), (1.5) with the linear function

$$P(v) = v. \tag{3.1}$$

The first equation (1.4) is then easily integrated:

$$g(v) = \mu v^{-1/2} + \nu, \tag{3.2}$$

with arbitrary constants μ and ν.

We begin with the simpler case $\nu \neq 0$. The *critical* case $\nu = 0$ is considered in the next section. Substituting (3.2) in the right-hand side of the second equation (1.5) we derive the general solution

$$G(v) = \alpha v^{-1/2} + \beta + K_1 v^{1/2} + K_2 v, \tag{3.3}$$

where

$$K_1 = \frac{2}{3}\,\mu\nu, \quad K_2 = \frac{2}{9}\,\nu^2. \tag{3.4}$$

Thus, having the general solution (3.2), (3.3) of the generating equations we now study the structure of the invariant set given by the stationary equation (JE) and the dynamical system (1.9), (1.10).

2. Linearization of the joint equation. The linearization consists of several steps.
(i) We put

$$v = 1/w^2 \quad (w > 0) \tag{3.5}$$

in the joint equation

$$v_{xx} + (\mu v^{-1/2} + \nu)v_x + \alpha v^{-1/2} + \beta + K_1 v^{1/2} + K_2 v = 0,$$

that yields the equation

$$w w_{xx} - 3(w_x)^2 + (\mu w + \nu) w w_x - \frac{K_2}{2} w^2 - \frac{K_1}{2} w^3 - \frac{\beta}{2} w^4 - \frac{\alpha}{2} w^5 = 0 \tag{3.6}$$

with a quadratic differential operator.
(ii) We set

$$w = z_x, \tag{3.7}$$

and then (3.6) reads

$$\begin{aligned} &z_x z_{xxx} - 3(z_{xx})^2 + (\mu z_x + \nu) z_x z_{xx} \\ &- \frac{K_2}{2}(z_x)^2 - \frac{K_1}{2}(z_x)^3 - \frac{\beta}{2}(z_x)^4 - \frac{\alpha}{2}(z_x)^5 = 0. \end{aligned} \tag{3.8}$$

(iii) We now introduce the inverse function with respect to the space variable

$$x = \Psi(y,t) \equiv z^{-1}(y,t), \tag{3.9}$$

so that $z(\Psi(y,t),t) \equiv y$, and

$$z_x = \frac{1}{\Psi_y}, \quad z_{xx} = -\frac{\Psi_{yy}}{(\Psi_y)^3}, \quad z_{xxx} = -\frac{\Psi_{yyy}}{(\Psi_y)^4} + 3\,\frac{(\Psi_{yy})^2}{(\Psi_y)^5}\,. \tag{3.10}$$

Substituting the derivatives from (3.10) into (3.8) we get

$$\Psi_{yyy} + (\mu + \nu\Psi_y)\Psi_{yy} + \frac{K_2}{2}(\Psi_y)^3 + \frac{K_1}{2}(\Psi_y)^2 + \frac{\beta}{2}\Psi_y + \frac{\alpha}{2} = 0. \tag{3.11}$$

(iv) Let

$$V = \Psi_y. \tag{3.12}$$

Then V solves

$$V_{yy} + (\mu + \nu V)V_y + \frac{K_2}{2}V^3 + \frac{K_1}{2}V^2 + \frac{\beta}{2}V + \frac{\alpha}{2} = 0. \tag{3.13}$$

(v) Finally, we apply the standard Cole-Hopf transformation

$$V = \kappa \frac{\rho_y}{\rho} \tag{3.14}$$

to obtain the equation with cubic nonlinearities

$$\begin{aligned} &\rho^2 \left(\rho_{yyy} + \mu \rho_{yy} + \frac{\beta}{2}\rho_y + \frac{\alpha}{2\kappa}\rho \right) \\ &+ (\nu\kappa - 3) \left(\rho \rho_y \rho_{yy} + \frac{1}{9}(\nu\kappa - 6)(\rho_y)^3 + \frac{\mu}{3}\rho(\rho_y)^2 \right) = 0. \end{aligned} \tag{3.15}$$

Therefore, choosing in (3.14)

$$\kappa = 3/\nu \tag{3.16}$$

(recall that $\nu \neq 0$) we arrive at the linear ODE

$$\rho_{yyy} + \mu \rho_{yy} + \frac{\beta}{2}\rho_y + \frac{\alpha\nu}{6}\rho = 0. \tag{3.17}$$

The characteristic equation derived by substituting into (3.17)

$$\rho = e^{\lambda y} \tag{3.18}$$

has the form

$$P(\lambda) \equiv \lambda^3 + \mu\lambda^2 + \beta\lambda/2 + \alpha\nu/6 = 0. \tag{3.19}$$

Finally, the general solution takes the form $\rho(y) = H_1\rho_1(y) + H_2\rho_2(y) + H_3\rho_3(y)$, where ρ_1, ρ_2 and ρ_3 are linear independent solutions of the ODE (3.17). In view of the representation (3.14) of the solution V, we may put $H_1 = 1$ (the trivial choice $H_1 = 0$ leads to the significant simplification of the solution). We write the general solution of (3.17) in the form

$$\rho(y) = \rho_1(y) + H_2\rho_2(y) + H_3\rho_3(y). \tag{3.20}$$

3. LINEARIZATION OF THE TRANSFORMED (HE). We now consider the (HE) with the coefficients constructed from (1.6) with the functions (3.1) - (3.3). This yields

$$m(v) = 1, \quad s(v) = 2\mu v^{1/2} + \frac{4}{3}\nu v + K_3, \quad M(v) = 2vG(v), \tag{3.21}$$

where K_3 is a free constant. We then obtain from (1.2)

$$\varphi(v) = -\frac{s(v)}{g(v)} = -\frac{2\mu v^{1/2} + \frac{4}{3}\nu v + K_3}{\mu v^{-1/2} + \nu}, \quad F(v) = (\varphi(v) + 2v)G(v). \tag{3.22}$$

Substituting $G(v)$ from (JE) into (HE) we derive the following parabolic equation

$$v_t = -2vv_{xx} + v_x^2 + (K_3 - \frac{2}{3}\nu v)v_x. \tag{3.23}$$

We now apply to (3.23) the transformations (i) - (v). The first one, (3.5), yields the equation

$$w_t = -2(w^{-2}w_x)_x + \left(K_3 w + \frac{2\nu}{3w}\right)_x. \tag{3.24}$$

The corresponding equation without the convection term,

$$w_t = -2(w^{-2}w_x)_x \tag{3.25}$$

was first proved by Storm [13] to be transformed to the linear heat equation by the transformation (3.7). Moreover, the same holds for (3.24) as was shown by Fokas and Yortsos in [3] (the linear convection term Kw_x is easily eliminated). Indeed, we integrate (3.24) over $(0, x)$ to obtain the equation

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\int_0^x w(s,t)ds + \int_0^t \left(-2\frac{w_x}{w^2} + K_3 w + \frac{2\nu}{3w}\right)(0,\tau)d\tau\right] \\ &= -2\frac{w_x}{w^2} + K_3 w + \frac{2\nu}{3w}. \end{aligned} \tag{3.26}$$

Denote by z the function in the square brackets in the left-hand side, so that (3.7) holds. Then we arrive at the equation

$$z_t = -2\frac{z_{xx}}{(z_x)^2} + K_3 z_x + \frac{2\nu}{3z_x}. \tag{3.27}$$

Next, (3.9) with $\Psi = \Psi(y,t)$, so that $z_t = -\Psi_t/\Psi_y$, reduces (3.27) to the equation

$$\Psi_t = -2\Psi_{yy} - K_3 - \frac{2}{3}\nu(\Psi_y)^2. \tag{3.28}$$

It follows from (3.12) that $V(y,t) = \Psi_y(y,t)$ solves Burger's equation

$$V_t = -2V_{yy} - \frac{4}{3}\nu V V_y. \tag{3.29}$$

Finally, one can easily check that by transformation (3.14), (3.16) equation (3.29) reduces to the heat equation with the extra linear term

$$\rho_t = -2\rho_{yy} + N_0(t)\rho, \tag{3.30}$$

where $N_0(t)$ is arbitrary. We thus have proved

Theorem 3.1. *In the linear case (3.1) with $\nu \neq 0$ the quasilinear heat equation (HE) with coefficients (3.21), (3.22) admits explicit solutions given by the system of the linear ODE (3.17) and the heat equation (3.30).*

Substituting the general solution (3.20) into (3.30) we obtain a linear dynamical system on the coefficient $H_2(t)$ and $H_3(t)$. Let us consider two examples. A general description of such explicit solutions was performed by Clarkson and Mansfield [2].

Example 3.2. Let $\alpha = -6/\nu$, $\beta = 2$ and $\mu = -1$ in (3.3). Then the characteristic equation (3.19) has the form $(\lambda - 1)(\lambda^2 + 1) = 0$ whence the general solution

$$\rho(y,t) = e^y + H_2(t)\cos y + H_3(t)\sin y. \tag{3.31}$$

Substituting (3.31) into equation (3.30) we get that $N_0(t) \equiv 2$ and the coefficients solve the linear dynamical system

$$H_2' = 4H_2, \quad H_3' = 4H_3, \quad t > 0,$$

which can be treated as the nonlinear heat equation (HE) on the corresponding invariant set. Therefore, from (3.14) we obtain the explicit solution

$$V(y,t) = \frac{3}{\nu}\,\frac{e^y + e^{4t}(-A\sin y + B\cos y)}{e^y + e^{4t}(A\cos y + B\sin y)}, \tag{3.32}$$

where A and B are arbitrary constants. $\square$

Example 3.3. Set now $\alpha = 0$, $\beta = 8$ and $\mu = -5$. Then from (3.20)

$$\rho(y,t) = 1 + H_2 e^y + H_3 e^{4y}, \tag{3.33}$$

and substituting into (3.30) yields $N_0 = 0$ and the solution

$$V(y,t) = \frac{3}{\nu}\,\frac{Ae^{y-2t} + 4Be^{4y-32t}}{1 + Ae^{y-2t} + Be^{4y-32t}}. \qquad \square \tag{3.34}$$

4. The linear critical case

Let $\nu = 0$. Then the final transformation (3.14) with the coefficient (3.16) does not make any sense, and as a result the explicit solutions *are not of a soliton-type structure.*

If $\nu = 0$ the joint equation

$$v_{xx} + \mu v^{-1/2} v_x + \alpha v^{-1/2} + \beta = 0 \tag{4.1}$$

is easily integrated by setting

$$v_x = (\alpha + \beta v^{1/2})Y(v) \tag{4.2}$$

whence the first order ODE in separate variables

$$v^{1/2}(\alpha+\beta v^{1/2})\frac{dY}{dv} = -\frac{1}{2Y}(\beta Y^2+2\mu Y+2). \tag{4.3}$$

We need to consider two cases.

(i) $\beta \neq 0$. Let us consider the case when

$$q^2 = \frac{2\beta-\mu^2}{\beta^2} > 0, \quad \text{and set } \ p = \frac{\mu}{\beta}. \tag{4.4}$$

Then integrating the ODE (4.3) we obtain

$$\begin{aligned} R(Y) &\equiv \frac{1}{2\beta}\log[(Y+p)^2+q^2] - \frac{\mu}{q\beta^2}\arctan\left(\frac{Y+p}{q}\right) \\ &= -\frac{1}{\beta}\log(\alpha+\beta v^{1/2}) + E \equiv \eta(v), \end{aligned} \tag{4.5}$$

where $E=E(t)$ is the constant of integration. It then follows from (4.2) that

$$v_x = (\alpha+\beta v^{1/2})R^{-1}(\eta(v)), \tag{4.6}$$

and hence the solution $v=v(x,t)$ is given by the quadrature

$$\int_0^{v(x,t)} \frac{dz}{(\alpha+\beta z^{1/2})R^{-1}(\eta(z))} = x + H(t), \tag{4.7}$$

with $H=H(t)$ being a free function. Differentiating (4.7) with respect to t yields the time derivative

$$v_t = (\alpha+\beta v^{1/2})R^{-1}(\eta)\left[H'(t) - E'(t)\int_0^v \left(\frac{1}{R^{-1}(\eta(z))}\right)'_\eta \frac{dz}{(\alpha+\beta z^{1/2})}\right]. \tag{4.8}$$

Substituting now the derivatives (4.6) and (4.8) into the corresponding Hamilton-Jacobi equation

$$v_t = (v_x)^2 + (2\mu v^{1/2}+K_3)v_x + 2\alpha v^{1/2} + 2\beta v \tag{4.9}$$

we obtain the identity

$$H'(t) - E'(t)\int_0^v (\cdot) = (\alpha+\beta v^{1/2})R^{-1}(\eta(v)) + 2\mu v^{1/2} + K_3 + \frac{2v^{1/2}}{R^{-1}(\eta(v))} \tag{4.10}$$

(the integral in the left-hand side is the same as in (4.8)), which should be valid for a.e. $t \in \mathbf{R}$ and $v>0$. Setting in (4.10) $v=0$ we obtain the first dynamical equation

$$H'(t) = \alpha R^{-1}(E(t) - \frac{1}{\beta}\log\alpha) + K_3. \tag{4.11}$$

(Here we are assuming that $\alpha \neq 0$; if $\alpha = 0$ we integrate in (4.7) from the limit, say, 1). Differentiating (4.10) with respect to v, one can check that this gives an identity if

$$E'(t) = 1, \tag{4.12}$$

which is the second dynamical equation.

(ii) $\beta = 0$. This case is easier. Integrating (4.3) we get

$$v_x = \alpha R^{-1}(\eta(v)), \quad \eta(v) = -\frac{1}{\alpha} v^{1/2} + E, \tag{4.13}$$

where

$$R(Y) = \frac{1}{2\mu}\left[Y - \frac{1}{\mu}\log(\mu Y + 1)\right], \quad Y = \frac{v_x}{\alpha}. \tag{4.14}$$

Therefore, instead of (4.7) we obtain the solution

$$\int_0^{v(x,t)} \frac{dz}{R^{-1}(\eta(z))} = \alpha x + H(t). \tag{4.15}$$

By the same calculations we then derive the following dynamical equation (cf. (4.11))

$$H'(t) = \alpha^2 R^{-1}(E(t)) + \alpha K_3, \tag{4.16}$$

with $E(t)$ satisfying (4.12).

We thus have proved the following

Theorem 4.1. *If $\nu = 0$ then under the given hypotheses, the explicit solutions (4.7) or (4.15) are governed by the ODE (4.11) or (4.16) with the coefficient $E(t) = t$.*

5. The degenerate case

1. In this section we consider the simplest case of the completely degenerate quadratic trinomial (1.3), where $a = b = 0$, and

$$P(v) = c \neq 0. \tag{5.1}$$

Then $P'' = P' \equiv 0$ and the first generating equation (1.4) yields

$$g'' = 0, \tag{5.2}$$

whence $g = dv + h$ ($d \neq 0$), and then putting $h = 0$ by translation we obtain

$$g(v) = dv. \tag{5.3}$$

From (1.5) we then get $G''(v) = \frac{2}{3} d^2 v$, and hence

$$G(v) = \frac{1}{9} d^2 v^3 + K_1 v + K_2, \tag{5.4}$$

with arbitrary constants K_1, K_2. It then follows from (1.6) that

$$m(v) = 0, \quad s(v) = \frac{2}{3} cdv + K, \quad M(v) = 2cG(v), \tag{5.5}$$

where K is arbitrary. Finally, from (1.2) we derive the following class of *quasilinear* heat equations

$$v_t = -\frac{\frac{2}{3}cdv + K}{dv} v_{xx} + \frac{\frac{4}{3}cdv - K}{dv} G(v), \tag{5.6}$$

which admits the sign-invariant

$$\mathcal{H}(v) = v_t - \left(\frac{2}{3}cdv + K\right) v_x - 2cG(v). \tag{5.7}$$

Remark. In the particular case $K = 0$ and $c = -3/2$ where equation (5.6) becomes the *semilinear* heat equation of the form

$$v_t = v_{xx} - 2G(v), \tag{5.6'}$$

the invariant set governed by the corresponding joint equation (see below) was introduced by Kapsov [10]. Notice that explicit solutions from [10] have a strong soliton's nature and have been constructed by different approaches (by the Hirota transformation) e.g. by Kawahara and Tanaka [11] and Cariello and Tabor [1]. A complete analysis of the types of such explicit solutions of equation (5.6') with the cubic function (5.4) is given by Clarkson and Mansfield [2].

The construction of the explicit solutions of the quasilinear equation (5.6) is much easier than in the previous section.

2. The corresponding joint equation

$$v_{xx} + dvv_x + \frac{1}{9} d^2 v^3 + K_1 v + K_2 = 0 \tag{5.8}$$

is linearized by the transformation

$$v = \frac{3}{d} \frac{\rho_x}{\rho}. \tag{5.9}$$

This yields the linear third order equation

$$\rho_{xxx} + K_1 \rho_x + \frac{1}{3} dK_2 \rho = 0, \tag{5.10}$$

whence the linear representation (3.20) of the general solution.

3. Substituting the function $G(v)$ from the joint equation (5.8) into (5.6) we arrive at Burger's equation

$$v_t = -2cv_{xx} - \left(\frac{4}{3}cdv - K\right) v_x, \tag{5.11}$$

which by (5.9) reduces to the linear heat equation with lower order terms

$$\rho_t = -2c\rho_{xx} + K\rho_x + N_0(t)\rho. \tag{5.12}$$

The rest of the analysis of the compatible system (5.10) and (5.12) is quite similar.

Example 5.1. Setting $K_1 = -1$ and $K_2 = 0$ in (5.10) we get the solution of the joint equation $\rho = 1 + H_2 e^x + H_3 e^{-x}$. Substituting into (5.12) we obtain $N_0 = 0$ and the explicit solution

$$v(x,t) = \frac{3}{d}\,\frac{Ae^{x+(K-2c)t} - Be^{-x-(K+2c)t}}{1 + Ae^{x+(K-2c)t} + Be^{-x-(K+2c)t}}. \qquad \square$$

Example 5.2. Let now $K_1 = 0$ and $dK_2 = -3$. Then

$$\rho = e^x + e^{-x/2}\left[H_2 \sin(\sqrt{3}x/2) + H_3 \cos(\sqrt{3}x/2)\right].$$

Substituting into the heat equation (5.12) we deduce that $N_0 = 2c - K$ and the coefficients $H_2(t)$ and $H_3(t)$ solve the linear dynamical system

$$H_2' = 3(c - K/2)H_2 - \sqrt{3}(c + K/2)H_3, \quad H_3' = \sqrt{3}(c + K/2)H_2 + 3(c - K/2)H_3,$$

which is easily integrated. In particular, if $K = -2c$ we have the solution

$$v = \frac{3}{2d}\,\frac{2e^x + e^{6ct - x/2}[(A\sqrt{3} - B)\cos(\sqrt{3}x/2) - (B\sqrt{3} + A)\sin(\sqrt{3}x/2)]}{e^x + e^{6ct-x/2}[A\sin(\sqrt{3}x/2) + B\cos(\sqrt{3}x/2)]}$$

satisfying the quasilinear equation (5.6). $\square$

6. The quadratic case

In this section we consider the most interesting *quadratic* case $a \neq 0$ where (1.3) represents a quadratic trinomial.

1. Single-term trinomial. Let us first restrict our attention to the particular case of the single-term quadratic trinomial

$$P(v) = \frac{a}{2}v^2, \quad \text{with } a \neq 0 \ \ (\text{i.e., } b = c = 0). \tag{6.1}$$

Then the generating equations (1.4) and (1.5) are solved explicitly

$$g(v) = \mu v^2 + \frac{\nu}{v^4}, \tag{6.2}$$

$$G(v) = \frac{\alpha}{v^3} + \beta v + \frac{1}{16}\left(\mu^2 v^5 - \frac{\nu^2}{v^7}\right), \tag{6.3}$$

where the function $G_0(v) = \alpha/v^3 + \beta v$ in (6.3) denotes the general solution of the homogeneous equation (1.5) while the last term is a particular solution of the non-homogeneous equation.

We now explain in more detail the structure of the explicit solutions in the case $\alpha = \beta = 0$ in (6.3) when the joint equation (JE),

$$v_{xx} + \frac{1}{v^4}(\mu v^6 + \nu)v_x + \frac{1}{16 v^7}(\mu v^6 - \nu)(\mu v^6 + \nu) = 0, \tag{6.4}$$

can be integrated in quadratures. Indeed, setting

$$v_x = \frac{1}{16}\frac{\mu v^6 - \nu}{v^3} Y \tag{6.5}$$

we arrive at the first order equation

$$vYY'_v = -\frac{\mu v^6 + \nu}{\mu v^6 - \nu}(Y+4)(3Y+4). \tag{6.6}$$

Integrating yields

$$Y = R^{-1}(\eta(v)), \quad \eta(v) = E\frac{v^2}{(\mu v^6 - \nu)^{2/3}}, \tag{6.7}$$

where E denotes a free constant and R^{-1} is the inverse function to

$$R(Y) = \frac{Y+4}{(3Y+4)^{1/3}}. \tag{6.8}$$

Integrating equation (6.5) with Y given by (6.7) we arrive at the following structure of the explicit solution $v(x,t)$

$$16\int_1^{v(x,t)} \frac{z^3}{\mu z^6 - \nu}\frac{dz}{R^{-1}(\eta(z))} = x + H, \tag{6.9}$$

where $E = E(t)$ and $H = H(t)$ are some smooth functions to be determined by substituting (6.9) into the corresponding Hamilton-Jacobi equation

$$\mathcal{H}(v) \equiv v_t - \left[av(v_x)^2 + \left(\frac{1}{2}a\mu v^4 + K\right)v_x + \frac{a}{16}\left(\mu^2 v^7 - \frac{\nu^2}{v^5}\right)\right] = 0, \tag{6.10}$$

with the coefficients easily calculated by (1.6). Using (1.2) we obtain the following quasilinear heat equation

$$\begin{aligned} v_t = &- v^4\frac{\frac{1}{2}a\mu v^4 + K}{\mu v^6 + \nu}v_{xx} + av(v_x)^2 \\ &+ \frac{1}{16v^5}(\mu v^6 - \nu)\left(\frac{1}{2}a\mu v^6 + a\nu - Kv^2\right). \end{aligned} \tag{6.11}$$

Theorem 6.1. *The quasilinear parabolic equation (6.11) admits the sign-invariant (6.10) and moreover the explicit solution (6.9) where the functions $E(t)$ in (6.7) and $H(t)$ satisfy the dynamical system*

$$H' = \frac{1}{16} a(\mu - \nu) g_E(t) + \frac{a\mu}{2} + K + \frac{a(\mu+\nu)}{g_E(t)}, \tag{6.12}$$

with $g_E(t) = R^{-1}(E(t)(\mu-\nu)^{-2/3})$, *and*

$$E' = \frac{1}{2} a\mu\nu E + \frac{a}{128} E^4, \quad t > 0. \tag{6.13}$$

This dynamical system is the PDE (6.11) on the invariant set given by the ODE (6.4).

Proof. We follow the same lines as in the linear critical case, Section 4. Calculating the derivative v_t from (6.9),

$$v_t = \frac{\mu v^6 - \nu}{v^3} R^{-1}(\eta(v)) \left[\frac{H'}{16} - E' \int_1^v \frac{z^5}{(\mu z^6 - \nu)^{5/3}} \left(\frac{1}{R^{-1}(\eta(z))} \right)'_\eta dz \right], \tag{6.14}$$

and substituting into the Hamilton-Jacobi equation (6.10) together with v_x from (6.5) we get the identity

$$\frac{H'}{16} - E' \int_1^v (\cdot) = \frac{1}{16} \left[\frac{a}{16} \frac{\mu v^6 - \nu}{v^2} R^{-1} + \frac{a\mu}{2} v^4 + K + a \frac{\mu v^6 + \nu}{v^2} \frac{1}{R^{-1}} \right]. \tag{6.15}$$

Setting here $v = 1$ yields the first dynamical equation (6.12). Differentiating (6.15) with respect to v and using (6.7) we arrive at the following ODE

$$vYY'_v = - \frac{2(2\mu v^6 + \nu) Y^2 \left[Y + \frac{4(2\mu v^6 - \nu)}{2\mu v^6 + \nu} \right] (Y + 4)}{(\mu v^6 - \nu) Y^2 - \left[\frac{256\gamma v^6}{\mu v^6 + \nu} + 16(\mu v^6 + \nu) \right]}, \tag{6.16}$$

where $\gamma = -E'/2aE$.

Let us compare the ODE's (6.16) and (6.6) which must admit a common solution given by (6.7), (6.8). One can check that the right-hand sides of both the equations (6.16), (6.6) coincide if the function $Y = Y(v)$ solves the cubic equation

$$Y^3 + 12Y^2 + \left[16 + \frac{64 v^6 (4\gamma + \mu\nu)}{(\mu v^6 - \nu)^2} \right] (3Y + 4) = 0. \tag{6.17}$$

On the other hand, expressions (6.7), (6.8) can be written in the form of the following cubic equation

$$Y^3 + 12Y^2 + \left[16 - \frac{E^3 v^6}{(\mu v^6 - \nu)^2} \right] (3Y + 4) = 0. \tag{6.18}$$

Therefore, the last coefficients of the term $(3Y+4)$ of equations (6.17) and (6.18) coincide provided that

$$256\gamma = -64\mu\nu - E^3,$$

whence the second dynamical equation (6.13). This completes the proof. $\square$

2. A SOLUTION FOR GENERAL TRINOMIAL. We now consider another solution of the generating equations. Namely, we again assume that $a \neq 0$, and by translation we consider the quadratic trinomial in the most general form

$$P(v) = \frac{1}{2}(av^2 + 2c), \tag{6.19}$$

where $c \neq 0$. Then the system (1.4), (1.5) has the general solution

$$g(v) = \frac{1}{4}(2av^2 + c), \tag{6.20}$$

$$G(v) = \frac{1}{64}(a^2v^5 + 2acv^3) + \alpha G_0(v) + \beta v, \tag{6.21}$$

where α, β are arbitrary constants and $G_0(v)$ denotes the second solution linear independent of v of the homogeneous equation (1.5).

Let us restrict our attention to the *quadrature* case

$$\alpha = 0, \quad \beta = \frac{3}{256}c^2. \tag{6.22}$$

Then setting in the joint equation (cf. (6.5))

$$v_x = \frac{1}{192}(2av^3 + 3cv)Y \tag{6.23}$$

we arrive at the ODE

$$YY'_v = -\frac{6av^2 + 3c}{2av^3 + 3cv}(Y+4)(Y+12), \tag{6.24}$$

whence

$$Y = R^{-1}(\eta(v)), \quad \eta(v) = \frac{E}{144}(2av^3 + 3cv)^2 \tag{6.25}$$

with

$$R(Y) = \frac{Y+4}{(Y+12)^3}. \tag{6.26}$$

Integrating (6.23) we obtain the following structure of the explicit solution

$$192\int_1^{v(x,t)} \frac{1}{2az^3 + 3cz}\frac{dz}{R^{-1}(\eta(z))} = x + H. \tag{6.27}$$

Observe that the Hamilton-Jacobi equation is now

$$\begin{aligned}\mathcal{H}(v) \equiv v_t - \Big[& av(v_x)^2 + \frac{1}{4}(a^2v^4 + 2acv^2 + K)v_x \\ & + \frac{1}{256}(av^2 + 2c)(4a^2v^5 + 8acv^3 + 3c^2v)\Big] = 0.\end{aligned} \tag{6.28}$$

The quasilinear heat equation takes the form (1.13).

Theorem 6.2. *Equation (1.13) admits sign-invariant (6.28) and the explicit solution (6.27) where the coefficients $E(t)$ in (6.25) and $H(t)$ satisfy the dynamical system*

$$H' = \frac{a}{192}(2a+3c)g_E(t) + \frac{1}{4}(a^2+2ac+K) + \frac{3(a+2c)(2a+c)}{4g_E(t)}, \tag{6.29}$$

with $g_E(t) = R^{-1}[E(t)(2a+3c)^2/144]$, *and*

$$E' = -\frac{a}{128} - \frac{3c^3}{64}E, \quad t > 0. \tag{6.30}$$

Proof. It is quite similar to the proof of the previous theorem. Calculating v_t from (6.27) and substituting both the derivatives, v_x and v_t, into (6.28) (then setting in the identity $v = 1$ gives (6.29)), after differentiating in v we obtain the ODE

$$YY'_v = -\frac{2av(4av^2+3c)Y^2\left(Y + 12\frac{4av^2+5c}{4av^2+3c}\right)(Y+12)}{av^2(2av^2+3c)Y^2 + \frac{12288\gamma}{2av^2+c} - 144(av^2+2c)(2av^2+c)}, \tag{6.31}$$

where $\gamma = -E'/2E$. One can see that the ODE (6.31) admits the solution (6.25), (6.26) satisfied by (6.24) if

$$\frac{1}{E} - \frac{256}{a}\gamma + 6\frac{c^3}{a} = 0, \tag{6.32}$$

whence the second dynamical equation (6.30). $\square$

Finally, we note that due to Theorem 1.2 the dynamical system (1.9), (1.10) governing the solution on the invariant set is given explicitly in all the cases (as well as in other cases of invariant sets described by stationary (JE)), but in the most general case (6.19) the corresponding solution $v(x,t)$ in general does not have an explicit quadrature representation (excluding the cases considered).

7. Invariant subspaces and sets

In this section we give another "geometric" interpretation of some of new explicit solutions. Let $\mathcal{W}_s = \mathcal{L}\{f_1(x), f_2(x), ..., f_s(x)\}$ be a linear subspace in the space of smooth functions $X = \{f(x)\}$, where $\mathcal{L}\{\cdot\}$ denotes the linear span of given functions, which are assumed to be linear independent. We use a standard concept of invariant subspaces.

A *linear* subspace $\mathcal{W}_s$ is called *invariant* under the given *nonlinear* differential (in x) operator $\mathbf{A}$ if

$$\mathbf{A}(\mathcal{W}_s) \subseteq \mathcal{W}_s. \tag{7.1}$$

It follows from (7.1) that the evolution equation

$$v_t = \mathbf{A}(v) \quad \text{for } t > 0 \tag{7.2}$$

on the invariant subspace $\mathcal{W}_s$ directly reduces to a s-dimensional nonlinear *dynamical system*. Different such examples for quasilinear evolution equations can be found in the author's papers [8], [5] - [7]. An interesting new example was constructed by Titov [15] for the porous medium equation.

Let us now introduce another definition. A set $\mathcal{M} \subseteq \mathcal{W}_s$ is said to be *invariant on the linear subspace* $\mathcal{W}_s$ under the operator $\mathbf{A}$ if

$$\mathbf{A}(\mathcal{M}) \subseteq \mathcal{W}_s. \tag{7.3}$$

(Obviously, according to this definition, an arbitrary invariant subspace $\mathcal{W}_s$ can be treated as the set being invariant on itself).

It was shown in a paper of the authors and Svirshchevskii [9] that in general, if $\mathcal{M} \neq \mathcal{W}_s$, then equation (7.2) on $\mathcal{M}$ reduces to an *overdetermined dynamical system* unlike the case of the invariant subspace (7.1). Several examples of solvability or non-solvability of such systems are given in this paper.

We now treat some examples of explicit solutions (see Galaktionov [7]) from the point of view of the above definition. Unfortunately, for more conplicated explicit solutions constructed in Sections 4 and 5 we still do not have a clear geometric treatment.

Example 7.1. Consider the simplest quasilinear case. Namely, fix $a = 1$ in (6.1) and set $\mu = \nu = 0$ in (6.2) (i.e., $g \equiv 0$) and $\alpha = 1$, $\beta = 0$ in (6.3) (i.e., $G(v) = 1/v^3$). Then we arrive in particular at the following semilinear heat equation (with $\varphi(v) \equiv 1$ in (HE))

$$v_t = v_{xx} + v(v_x)^2 + \frac{1+v^2}{v^3}, \tag{7.4}$$

which admits the explicit solution of the typical self-similar form

$$v_*(x,t) = \left(2t - \frac{x^2}{2t}\right)^{1/2} \tag{7.5}$$

(obviously, this solution is not self-similar in the sense that it cannot be constructed by a standard group-invariant analysis). Using (7.5) it is natural to introduce the new function

$$V = v^2, \tag{7.6}$$

which satisfies the equation

$$V_t = \mathbf{A}(V) \equiv \left[V_{xx} - \frac{1}{2}\frac{(V_x)^2}{V} + \frac{2}{V}\right] + \left[\frac{1}{2}(V_x)^2 + 2\right] \equiv \mathbf{B}_1(V) + \mathbf{B}_2(V). \tag{7.7}$$

It is clear that

$$V_*(x,t) = 2t - \frac{x^2}{2t} \in \mathcal{W}_2 = \mathcal{L}\{1, x^2\}, \tag{7.8}$$

for all $t > 0$, where the two-dimensional subspace $\mathcal{W}_2$ is invariant under the Hamilton-Jacobi operator $\mathbf{B}_2$ (but not under $\mathbf{B}_1$). Indeed, for all $V = C_0 + C_1 x^2 \in \mathcal{W}_2$ we have that

$$\mathbf{B}_2(V) = 2 + 2C_1^2 x^2 \in \mathcal{W}_2. \tag{7.9}$$

On the other hand, let us introduce the following set

$$\mathcal{M}=\{V=C_0+C_1x^2:\ \mathbf{B}_1(V)=0,\ \text{i.e., } C_0C_1+1=0\}. \tag{7.10}$$

One can see that $\mathcal{M}$ is invariant on $\mathcal{W}_2$ under the operator $\mathbf{A}$. Hence, substituting $V=C_0(t)+C_1(t)x^2$ we deduce that equation (7.7) on the set $\mathcal{M}$ is equivalent to the overdetermined dynamical system

$$\begin{gathered} C_0'=2, \quad C_1'=2C_1^2, \\ C_0C_1+1=0 \ (\text{i.e., } V\in\mathcal{M}), \end{gathered} \tag{7.11}$$

which is easily solved to obtain solution (7.8). $\square$

Remark. The explicit solution (7.8) satisfies also the following quasilinear parabolic equation

$$V_t=\psi(V)\mathbf{B}_1(V)+\mathbf{B}_2(V), \tag{7.12}$$

with an arbitrary function $\psi(V)$.

Example 7.2. Setting again $a=1$, $\mu=\nu=0$, we now take $\alpha=\beta=1$, i.e., according to (6.3) $G(v)=v+1/v^3$. Then setting $V=v^2$ in the semilinear equation

$$v_t=v_{xx}+v(v_x)^2+(1+v^2)\left(v+\frac{1}{v^3}\right), \tag{7.13}$$

we get the equation

$$V_t=\left[V_{xx}-\frac{1}{2}\frac{(V_x)^2}{V}+\frac{2}{V}+2V\right]+\left[\frac{1}{2}(V_x)^2+2V^2+2\right]\equiv\mathbf{B}_1(V)+\mathbf{B}_2(V), \tag{7.14}$$

with the explicit solution

$$V_*(x,t)=\frac{\sin 4t}{\cos 4t}+\frac{1}{\cos 4t}\sin 2x\in\tilde{\mathcal{W}}_2=\mathcal{L}\{1,\sin 2x\}, \tag{7.15}$$

where $\tilde{\mathcal{W}}_2$ is invariant under $\mathbf{B}_2$. The invariant set $\mathcal{M}\subset\tilde{\mathcal{W}}_2$ such that $\mathbf{B}_1(V)=0$ on $\mathcal{M}$ takes the form

$$\mathcal{M}=\{V=C_0+C_1\sin 2x:\ C_0^2-C_1^2+1=0\}. \tag{7.16}$$

Then substituting $V=C_0(t)+C_1(t)\sin 2x\in\tilde{\mathcal{W}}_2$ into equation (7.14) we arrive at the overdetermined dynamical system

$$\begin{gathered} C_0'=2(C_0^2+C_1^2)+2, \quad C_1'=4C_0C_1, \\ C_0^2-C_1^2+1=0, \end{gathered} \tag{7.17}$$

which admits a nontrivial solution given in (7.15). Recall that (7.15) solves the more general equation (7.12). $\square$

Acknowledgments

This paper was written in May, 1994, while the first author was a visiting professor at the Department of Mathematics, University Autónoma of Madrid, and the second author visited this institution, to which they are deeply grateful for its hospitality. The authors have been partially supported by EEC Grant SC1-0019-C(TT).

References

[1] F. Cariello and M. Tabor, *Painleve expansions for nonintegrable evolution equations*, Physica D **39** (1989), 77-94.

[2] P.A. Clarkson and E.L. Mansfield, *Symmetry reductions and exact solutions of a class of nonlinear heat equations*, Physica D **70** (1993), 250-288.

[3] A.S. Fokas and Y.C. Yortsos, *On the exactly solvable equations* $S_t = ((\beta S+\gamma)^{-2}S_x)_x + \alpha(\beta S+\gamma)^2 S_x$ *occurring in the two-phase flow in porous media*, SIAM J. Appl. Math. **42** (1982), 318-332.

[4] A. Friedman, Partial Differential Equations of Parabolic Type, Englewood Cliffs, NJ, Prentice Hall, 1964.

[5] V.A. Galaktionov, *On new exact blow-up solutions for nonlinear heat conduction equations with source and applications*, Differ. and Integr. Equat. **3** (1990), 863-874.

[6] V.A. Galaktionov, *Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities*, (1991), Report School of Math., Univ. of Bristol, No. AM-91-11; Proc. Roy Soc. Edinburgh **125A** (1995), 225-246.

[7] V.A. Galaktionov, *Quasilinear heat equations with first-order sign-invariants and new explicit solutions*, Nonlinear Anal., Theory, Meth. and Appl. **23** (1994), 1595-1621.

[8] V.A. Galaktionov and S.A. Posashkov, *New exact solutions of parabolic equations with quadratic nonlinearities*, Zh. Vychisl. Mat. i Mat. Fiz. **29** (1989), 497-506; English translation: USSR Comput. Math. and Math. Phys., **29** (1989), 112-119.

[9] V.A. Galaktionov, S.A. Posashkov, and S.R. Svirshchevskii, *On invariant sets and explicit solutions of nonlinear evolution equations with quadratic nonlinearities*, (1994), preprint; Differ. and Integr. Equat., to appear.

[10] O.V. Kapsov, *Invariant sets of evolution equations*, Nonlinear Anal., Theory, Meth. and Appl. **19** (1992), 753-761.

[11] T. Kawahara and M. Tanaka, *Interactions of travelling fronts: an exact solution of a nonlinear diffusion equation*, Physics Letters **97A** (1983), 311-314.

[12] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Nauka, Moscow, 1987 (in Russian); English translation: Walter de Gruyter, Berlin, 1995.

[13] M.L. Storm, *Heat conduction in simple metals*, J. Appl. Phys. **22** (1951), 940-951.

[14] A.F. Sidorov, V.P. Shapeev, and N.N. Yanenko, Method of Differential Constraints and its Applications in Gas Dynamics, Nauka, Novosibirsk, 1984 (in Russian).

[15] S.S. Titov, *A method of finite-dimensional rings to solve nonlinear equations of mathematical physics*, In: "Aerodynamics of Plane and Axis-Symmetric Flows of Liquids", Saratov University, Saratov, 1988, 104-109 (in Russian).