Dressing orbits of super-horizontal holomorphic maps.*

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Abstract

By studying dressing orbits a complete account is given of the (full) super-horizontal holomorphic maps of finite type from the 2-plane into a full flag manifold. It is shown that precisely one dressing orbit contains all 2-spheres of finite type. From this follows a simple description of all harmonic 2-spheres in $\mathbf{CP}^n$ of finite type.

1 Introduction.

In [3] Burstall & Pedit described the dressing action of a loop group on the space of primitive harmonic maps $\psi : \mathbf{R}^2 \to G/K$ from the 2-plane into a k-symmetric space. Their results were particularly aimed at understanding the dressing orbits of maps arising from the 'Symes map' construction, which contains all primitive harmonic maps of finite type [2]. Their principal motivation was that maps of *semisimple* finite type play a leading role in the study of harmonic 2-tori in, for example, spheres or complex projective spaces [1]. One of the characteristics of such maps is that the $(1,0)$-component $\partial\psi$ of the differential is semisimple-valued (indeed, thought of as a 1-form with values in the Lie algebra $\mathfrak{g}^{\mathbf{C}}$, it takes values in the $G^{\mathbf{C}}$-orbit of a single semisimple element).

We might now ask if there are any harmonic 2-spheres of finite type (if so, they must arise from the Symes map). This leads to an investigation of the dressing orbits of primitive *holomorphic* (also called super-horizontal holomorphic) maps which arise from the Symes map: in this case $\partial\psi$ must be nilpotent-valued. Some results about these maps were obtained in [3] but they were (to quote the authors) "less complete and somewhat confused". My aim here is to remove some of the confusion by giving a complete account of the dressing orbits for full primitive holomorphic maps into a full flag manifold G/T (i.e. for T a maximal torus in the simple compact Lie group G) which arise from the Symes map. By definition, the map ψ is full when $\partial\psi$ is principal nilpotent-valued.

The results presented here show that the Symes map can only give us a finite dimensional space of primitive holomorphic maps (indeed this is the case for any flag manifold) and that the set of dressing orbits of the full maps is in bijective

*Research supported by SERC Grant GR/J33043.

correspondence with the set of $G^{\mathbf{C}}$-orbits of maximal dimension in $\mathfrak{g}^{\mathbf{C}}$. From [6] one knows the latter is parameterised by $\mathbf{C}^n$, where n =rank(G). All these maps have finite uniton number (a result from [3]) and we obtain a canonical form for these maps using Kostant's description [6] of the quasi-regular elements of $\mathfrak{g}^{\mathbf{C}}$ (i.e. elements whose $G^{\mathbf{C}}$-orbit has maximal dimension). From this we see immediately that the dressing group acts as the group of (base-point preserving) holomorphic automorphisms of G/T.

Finally, we see that *all* these maps are of finite type and this accounts for all full primitive holomorphic maps of finite type in G/T. Moreover, primitive holomorphic 2-spheres of finite type do exist and occupy precisely one of the dressing orbits: when $G = SU_{n+1}$ each 2-sphere of finite type is the primitive lift of a rational normal (i.e. smooth) curve of degree n in $\mathbf{CP}^n$. From this we deduce the satisfying fact that the only (full) harmonic 2-spheres of finite type in $\mathbf{CP}^n$ are those in the Frenet frame of a rational normal curve of degree n.

Acknowledgment. Much of the motivation for this arose out of discussions with Fran Burstall, who also helped clarify the classification of maps of finite type.

2 Preliminaries.

Here we will gather together all the preliminary results we require from [3] concerning the Symes map and from [5] concerning principal nilpotents in $\mathfrak{g}^{\mathbf{C}}$. Wherever possible the notation will be consistent with these two references.

Recall that G/T is a k-symmetric space for the Coxeter automorphism τ, whose eigenspaces present us with a $\mathbf{Z}_k$-grading on $\mathfrak{g}^{\mathbf{C}}$ which we write

$$\mathfrak{g}^{\mathbf{C}} = \sum_{j\in\mathbf{Z}_k} \mathfrak{g}_j,$$

where $\mathfrak{g}_0 = \mathfrak{t}^{\mathbf{C}}$, the Lie algebra for $T^{\mathbf{C}}$. By fixing a Borel subgroup $B \supset T^{\mathbf{C}}$ in $G^{\mathbf{C}}$ we fix an isomorphism $G/T \simeq G^{\mathbf{C}}/B$ which exhibits the complex structure on the full flag manifold. The choice of B (with Lie algebra $\mathfrak{b}$) also fixes a compatible $\mathbf{Z}$-grading on $\mathfrak{g}^{\mathbf{C}}$

$$\mathfrak{g}^{\mathbf{C}} = \sum_{j\in\mathbf{Z}} \mathfrak{g}^j$$

in which $\mathfrak{g}^0 = \mathfrak{g}_0$ and $\mathfrak{g}^j = 0$ for $|j| \geq k$. Explicitly, $\mathfrak{b}$ fixes a basis of simple roots $\alpha_1, \ldots, \alpha_n$ and the $\mathbf{Z}$-grading corresponds to giving each of these weight 1. The space $\mathfrak{g}^j$, for $j \neq 0$, is the sum of all root spaces for roots of weight j. In that case one knows that

$$\mathfrak{g}_j = \mathfrak{g}^j + \mathfrak{g}^{k-j} \quad \text{and} \quad \mathfrak{b} = \sum_{j\geq 0} \mathfrak{g}^j.$$

For any AdT-invariant subspace $\mathfrak{v} \subset \mathfrak{g}$ let $[\mathfrak{v}]$ denote the vector bundle $G \times_T \mathfrak{v}$. Recall that there is a canonical isomorphism $T^{\mathbf{C}}(G/T) \simeq [\mathfrak{p}^{\mathbf{C}}]$ where $\mathfrak{g}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \mathfrak{p}^{\mathbf{C}}$ is the reductive decomposition obtained from the $\mathbf{Z}_k$-grading. This incorporates the isomorphism $T^{(1,0)}(G/T) \simeq [\mathfrak{n}_-]$ where

$$\mathfrak{n}_- = \sum_{j<0} \mathfrak{g}^j.$$

Observe $\mathfrak{n}_-$ is 'opposite to' the nilradical $\mathfrak{n}_+ = [\mathfrak{b}, \mathfrak{b}]$ of $\mathfrak{b}$. A map $\psi : \mathbf{R}^2 \to G/T$ is called primitive when $\partial\psi : T^{(1,0)}\mathbf{R}^2 \to [\mathfrak{p}^{\mathbf{C}}]$ takes values in $[\mathfrak{g}_{-1}]$ and holomorphic when it takes values in $[\mathfrak{n}_-]$. Hence ψ is primitive holomorphic when $\partial\psi$ takes values in $[\mathfrak{g}^{-1}]$ (which is the super-horizontal distribution of Burstall & Rawnsley [4]). Let us fix now a non-zero root vector $e_{-\alpha_i}$ for each root $-\alpha_i$, then $\mathfrak{g}^{-1}$ is spanned by $e_{-\alpha_1}, \ldots, e_{-\alpha_n}$ and from Kostant we learn the following facts

Lemma 1 ([6] p998) *An element of $\mathfrak{g}^{-1}$ is principal nilpotent iff it is of the form $X = \sum_{i=1}^{n} X_i e_{-\alpha_i}$ for $X_i \in \mathbf{C}^*$. These elements form a single $\mathrm{Ad}T^{\mathbf{C}}$-orbit in $\mathfrak{g}^{-1}$. If $X \in \mathfrak{g}^{-1}$ is principal nilpotent then $\mathfrak{g}^X \cap \mathfrak{b} = 0$, where $\mathfrak{g}^X$ is the centraliser of X.*

To present the Symes map construction of primitive maps we must introduce the appropriate loop groups and algebras. Fix $0 < \epsilon < 1$ and let C be the union of circles of radii ϵ and ϵ^{-1} on the λ-plane $\mathbf{C}_\lambda$. We define

$$\Lambda^\epsilon G = \{\text{real-analytic } \ g : C \to G^{\mathbf{C}} | \overline{g(\lambda)} = g(\bar{\lambda}^{-1})\}.$$

The conjugation used here is with respect to the real form G of $G^{\mathbf{C}}$. We will denote the Lie algebra of this group by $\Lambda^\epsilon\mathfrak{g}$. We can think of elements of $\Lambda^\epsilon G$ and $\Lambda^\epsilon\mathfrak{g}$ as pairs of loops e.g. $(g, \bar{g})$ where $\bar{g}$ is shorthand for $\overline{g(1/\bar{\lambda})}$. The loop group which occurs in [3] is the subgroup of τ-equivariant loops in $\Lambda^\epsilon G$ i.e.

$$\Lambda^\epsilon G_\tau = \{g \in \Lambda^\epsilon G : \tau g(\lambda) = g(\omega\lambda)\},$$

where ω is a primitive k-the root of unity: its Lie algebra will be denoted by $\Lambda^\epsilon\mathfrak{g}_\tau$. We also require subgroups of $\Lambda^\epsilon G_\tau$ determined by the annulus $E = \{\epsilon < |\lambda| < 1/\epsilon\}$ and the union of discs $I = \{|\lambda| < \epsilon\} \cup \{|\lambda| > 1/\epsilon\}$ on the λ-sphere. These subgroups are

$$\begin{aligned}
\Lambda^\epsilon_E G_\tau &= \{g \in \Lambda^\epsilon G_\tau : g \text{ extends holomorphically to } g : E \to G^{\mathbf{C}}\} \\
\Lambda^\epsilon_I G_\tau &= \{g \in \Lambda^\epsilon G_\tau : g \text{ extends holomorphically to } g : I \to G^{\mathbf{C}}\} \\
\Lambda^\epsilon_{I,A} G_\tau &= \{g \in \Lambda^\epsilon_I G_\tau : g(0) \in A\}
\end{aligned}$$

where $A = \exp(i\mathfrak{t})$ is the solvable factor in the Iwasawa decomposition of $T^{\mathbf{C}}$ (observe that every $g(\lambda) \in \Lambda^\epsilon_I G_\tau$ has $g(0) \in T^{\mathbf{C}}$). Recall that the complex group $\Lambda^\epsilon G_\tau$ has its own Iwasawa-type decomposition (see [7]) which says that every $g \in \Lambda^\epsilon G_\tau$ has a unique factorisation of the form

$$g = g_E g_I \quad \text{where} \quad g_E \in \Lambda^\epsilon_E G_\tau,\ g_I \in \Lambda^\epsilon_{I,A} G_\tau. \tag{1}$$

The Symes map of [3] is a map from the subspace

$$\Lambda^\epsilon_{-1,\infty} = \{(\lambda^{-1}X, \lambda\bar{X}) : X \in \mathfrak{g}_{-1}\} + \Lambda^\epsilon_I\mathfrak{g}_\tau \subset \Lambda^\epsilon\mathfrak{g}_\tau$$

to the space of based primitive maps

$$\mathcal{H} = \{\psi : \mathbf{R}^2 \to G/T |\ \psi \text{ is primitive with } \psi(0) = T\}.$$

To define it let us first agree that whenever $(\eta, \bar{\eta}) \in \Lambda^\epsilon\mathfrak{g}_\tau$, and with z the usual complex coordinate on $\mathbf{R}^2$, $\exp(z\eta)$ will denote the real 2-parameter subgroup of

$\Lambda^\epsilon G_\tau$ obtained by exponentiating $(z\eta, \bar{z}\bar{\eta})$. From now on we will identify elements of $\Lambda^\epsilon\mathfrak{g}_\tau$ and $\Lambda^\epsilon G_\tau$ with their restriction to $|\lambda| = \epsilon$ without comment. The Symes map is defined to be

$$\Lambda^\epsilon_{-1,\infty} \to \mathcal{H} \quad ; \quad \eta \to [F^\eta] \tag{2}$$

where $F^\eta : \mathbf{R}^2 \to \Lambda^\epsilon_E G_\tau$ is the extended frame defined by $F^\eta = \exp(z\eta)_E$, using the factorisation (1). Here $[F^\eta]$ denotes the gauge-equivalence class of based extended framings corresponding to a based primitive map. If we write the Fourier expansion for η as

$$\eta = \eta_{-1}\lambda^{-1} + \ldots$$

then one knows from [3] that for $\psi = [F^\eta]$ the differential $\partial\psi$ takes values in the $G^{\mathbf{C}}$-orbit of η_{-1}. Consequently, we will be mostly interested in the restriction of the Symes map (2) to the subspace

$$\Lambda^{-1,\infty} = \{(\lambda^{-1}X, \lambda\bar{X}) : X \in \mathfrak{g}^{-1}\} + \Lambda^\epsilon_I\mathfrak{g}_\tau$$

so that we obtain a map

$$\Lambda^{-1,\infty} \to \mathcal{S} = \{\psi = [F^\eta] : \psi \text{ is primitive holomorphic}\}. \tag{3}$$

Note in particular that we have defined $\mathcal{S}$ so that this map is surjective.

At this point it is useful to state a result from [3] concerning the fibres of the Symes map:

Lemma 2 ([3] lemma 3.1) *Let $\eta, \xi \in \Lambda^\epsilon_{-1,\infty}$. Then $[F^\eta] = [F^\xi]$ if and only if $\exp(-\mathrm{ad}\ z\eta)\cdot\xi \in \eta + \Lambda^\epsilon_I\mathfrak{g}_\tau$, if and only if $\xi_{-1} = \eta_{-1}$ and $(\mathrm{ad}\ \eta)^m \cdot \xi \in \Lambda^\epsilon_I\mathfrak{g}_\tau$ for all $m \geq 1$.*

Now let us recall the dressing action of $\Lambda^\epsilon_I G_\tau$ on primitive maps. For our purposes it suffices to describe this for maps in the image of (2). For any $\eta \in \Lambda^\epsilon_{-1,\infty}$ and any $g \in \Lambda^\epsilon_I G_\tau$ we define

$$g\#[F^\eta] = [(gF^\eta)_E].$$

It is shown in [3] that this is a group action and the Symes map intertwines the adjoint action of $\Lambda^\epsilon_I G_\tau$ on $\Lambda^\epsilon_{-1,\infty}$ with the dressing action. Since $\mathfrak{g}^{-1}$ is preserved by $T^{\mathbf{C}}$, the adjoint action of $\Lambda^\epsilon_I G_\tau$ preserves the smaller space $\Lambda^{-1,\infty}$ and therefore the dressing action restricts to $\mathcal{S}$.

Now we come to an observation which plays the principal role in obtaining our results: $\Lambda^\epsilon G_\tau$ is isomorphic to $\Lambda^\delta G$ where $\delta = \epsilon^k$. For our purposes the most convenient way to exhibit this is at the Lie algebra level. We define $\iota : \Lambda^\delta\mathfrak{g} \to \Lambda^\epsilon\mathfrak{g}_\tau$ to be the isomorphism characterised by

$$\iota(\lambda^m X) = \lambda^{km+j}X \quad \text{for } X \in \mathfrak{g}_j,\ m \in \mathbf{Z}. \tag{4}$$

In particular, one readily checks that if $\Lambda_+\mathfrak{g} = \{\eta \in \Lambda^\delta_I\mathfrak{g} \text{ with } \eta(0) = 0\}$ and $\Lambda_0\mathfrak{b}$ denotes the constant loops with values (on $|\lambda| = \delta$) in $\mathfrak{b}$ then we obtain an isomorphism $\iota : \Lambda^\delta_0\mathfrak{b} + \Lambda^\delta_+\mathfrak{g} \to \Lambda^\epsilon_I\mathfrak{g}_\tau$ by restriction. We will write the resultant decomposition of $\Lambda^\epsilon_I\mathfrak{g}_\tau$ as

$$\Lambda^\epsilon_I\mathfrak{g}_\tau = \Lambda_0\mathfrak{b} + \Lambda_+\mathfrak{g} \tag{5}$$

Corresponding to this there is a decomposition of $\Lambda_I^\epsilon G_\tau$ which we will write as $\Lambda_I^\epsilon G_\tau = \Lambda_0 B.\Lambda_+ G$. Observe that $\Lambda_0 B \simeq B$, its elements are polynomial loops and its adjoint action on $\Lambda_I^\epsilon \mathfrak{g}_\tau$ preserves both summands in (5). Finally, observe that $\Lambda^{-1,\infty} \subset \iota(\Lambda_I^\delta \mathfrak{g})$ so that if we let $\Lambda_0\mathfrak{g}$ denote the image under ι of the subalgebra of constant loops in $\Lambda^\delta \mathfrak{g}$ then we may write

$$\Lambda^{-1,\infty} = \Lambda_{(0)}^{-1,\infty} + \Lambda_+^{-1,\infty}, \tag{6}$$

where $\Lambda_{(0)}^{-1,\infty} = \Lambda^{-1,\infty} \cap \Lambda_0\mathfrak{g}$ and $\Lambda_+^{-1,\infty} = \Lambda^{-1,\infty} \cap \Lambda_+\mathfrak{g}$. Consequently each $\eta \in \Lambda^{-1,\infty}$ has a unique expression as

$$\eta = \eta_{(0)} + \eta_+, \quad \text{for } \eta_{(0)} \in \Lambda_0\mathfrak{g},\ \eta_+ \in \Lambda_+\mathfrak{g}.$$

We obtain $\eta_{(0)}$ by passing to $\Lambda^\delta \mathfrak{g}$ and evaluating at $\lambda = 0$. The adjoint action of $\Lambda_0 B$ on $\Lambda^{-1,\infty}$ also preserves both the summands in (6).

3 The image of the Symes map.

In this section I will prove:

Proposition 1 *For any $\eta, \xi \in \Lambda^{-1,\infty}$ with η_{-1} principal nilpotent, $[F^\eta] = [F^\xi]$ iff $\eta_{(0)} = \xi_{(0)}$. Therefore the Symes map (3) restricts to give a bijection between the finite dimensional variety $\{\eta \in \Lambda_{(0)}^{-1,\infty} : \eta_{-1}$ is principal nilpotent$\}$ and the subspace $\mathcal{S}_{\mathrm{full}} \subset \mathcal{S}$ of full maps.*

The proof consists of two lemmas.

Lemma 3 *For any $\eta \in \Lambda^{-1,\infty}$ we have $[F^\eta] = [F^{\eta_{(0)}}]$.*

Proof. From lemma 2 this happens if

$$\exp(-\mathrm{ad}\ z\eta_{(0)}) \cdot \eta \in \eta_{(0)} + \Lambda_I^\epsilon \mathfrak{g}_\tau, \text{ for all } z.$$

Write $\gamma(z) = \exp(z\eta_{(0)})$, then

$$\mathrm{Ad}\ \gamma(z)^{-1} \cdot \eta = \mathrm{Ad}\ \gamma(z)^{-1} \cdot \eta_{(0)} + \mathrm{Ad}\ \gamma(z)^{-1} \cdot \eta_+.$$

The first term is $\eta_{(0)}$ and the second lies in $\Lambda_+\mathfrak{g} \subset \Lambda_I^\epsilon \mathfrak{g}_\tau$ since $\gamma : \mathbf{C} \to \Lambda_0 G$. Therefore the condition above is satisfied.■

Remark. The only properties we have used here are consequences of the isomorphism ι in (4) and the compatibility of the $\mathbf{Z}$ and $\mathbf{Z}_k$-gradings. It is not hard to show that the same situation is present for any flag manifold, so the lemma above also holds for primitive holomorphic maps into these spaces. Consequently we can say that the Symes map produces only a finite dimensional family of primitive holomorphic maps into any flag manifold, given by the image of $\Lambda_{(0)}^{-1,\infty}$.

It follows from this lemma that we are only interested in the subspace $\Lambda_{(0)}^{-1,\infty} \subset \Lambda_0\mathfrak{g}$. When we identify $\mathfrak{g}^{\mathbf{C}}$ with the algebra $\Lambda_0\mathfrak{g}$ of constant loops we identify $\mathfrak{g}^{-1} + \mathfrak{b}$ with $\Lambda_{(0)}^{-1,\infty}$. Therefore proposition 1 will follow from lemma 2 and the next lemma. In it we will write any $X \in \mathfrak{g}$ as a sum of its homogeneous components in the $\mathbf{Z}$-grading: $X = \sum X_j$.

Lemma 4 *Let $X, Y \in \mathfrak{g}^{-1} + \mathfrak{b}$ have $X_{-1} = Y_{-1}$ principal nilpotent. Then $(\text{ad } X)^m \cdot Y \in \mathfrak{b}$ for all $m \geq 1$ implies $Y = X$.*

Proof. We will show by induction that $X_j = Y_j$ for every $j = -1, \dots, k-1$. Since $X_{-1} = Y_{-1}$ already, we assume $X_i = Y_i$ for $i < j$ for some fixed $j \geq 0$.

We are given that $(\text{ad } X)^j[X, Y] \in \mathfrak{b}$, therefore in particular $(\text{ad } X)^j[X, Y]_{j-1} \in \mathfrak{b}$. Moreover, this would have $\mathfrak{g}^{-1}$-component $(\text{ad } X_{-1})^j[X, Y]_{j-1}$, so this expression must vanish. Now

$$[X, Y]_{j-1} = \sum_{i=-1}^{j} [X_i, Y_{j-i-1}] = [X_{-1}, Y_j - X_j]$$

using the assumption that $X_i = Y_i$ for $i < j$. Therefore

$$[X_{-1}, (\text{ad } X_{-1})^j (Y_j - X_j)] = 0$$

and since the second term in the bracket belongs to $\mathfrak{b}$ (indeed, to $\mathfrak{g}^0$) it must vanish by lemma 1. It follows recursively that $(\text{ad } X_{-1})^i (Y_j - X_j) = 0$ for all $0 \leq i \leq j$ and therefore $Y_j = X_j$.■

4 Characterisation of the dressing orbits.

In proposition 1 we established the bijection between $\mathcal{S}_{\text{full}}$ and the variety $\{\eta \in \Lambda_0^{-1,\infty} : \eta_{-1} \text{ is principal nilpotent}\}$. The latter is canonically identifiable with the open subvariety

$$V = \cup\{X + \mathfrak{b} : X \in \mathfrak{g}^{-1} \text{ is principal nilpotent}\} \subset \mathfrak{g}^{-1} + \mathfrak{b}$$

which is preserved by the adjoint action of B.

Lemma 5 *The set of dressing orbits in $\mathcal{S}_{\text{full}}$ is canonically identifiable with V/B.*

Proof. Clearly the bijection $V \to \mathcal{S}_{\text{full}}$ intertwines the adjoint action of B on V with the dressing action of $\Lambda_0 B \simeq B$ on $\mathcal{S}_{\text{full}}$. Now, for any map $[F^{\eta_{(0)}}] \in \mathcal{S}_{\text{full}}$ we see that $\Lambda_+ G$ stabilises $[F^{\eta_{(0)}}]$. For if $g \in \Lambda_+ G$ then

$$(\text{Ad } g \cdot \eta_{(0)})_{(0)} = \eta_{(0)}$$

(to see this pass to $\Lambda^\epsilon \mathfrak{g}$ and evaluate at $\lambda = 0$). Therefore the $\Lambda_I^\epsilon G_\tau$-action collapses to the $\Lambda_0 B$-action.■

Now let us fix a principal nilpotent $e_- \in \mathfrak{g}^{-1}$, defined by

$$e_- = \sum_{i=1}^{n} e_{-\alpha_i}.$$

From lemma 1 and the fact that no non-trivial element of $T^{\mathbf{C}}$ stabilises e_- we deduce that

$$V/T^{\mathbf{C}} \simeq e_- + \mathfrak{b}.$$

Therefore, using the fact that $B = T^{\mathbf{C}}.N_+$ for $N_+ = \exp(\mathfrak{n}_+)$, we obtain the identification

$$V/B \simeq (e_- + \mathfrak{b})/N_+.$$

Now it only requires an application of some results of Kostant [6] to describe the latter quotient space. Let us recall the relevant results from [6].

Given e_- there is a unique principal nilpotent $e_+ \in \mathfrak{b}$ for which $\{[e_+, e_-], e_+, e_-\}$ is an S-triple (see [5] p996). In that case the centraliser $\mathfrak{g}^{e_+}$ of e_+ has the following properties.

Lemma 6 ([6] lemmas 10,12, theorem 8) *(i)* $\mathfrak{b} = \mathfrak{g}^{e_+} + [e_-, \mathfrak{n}_+]$, *(ii) every element of* $e_- + \mathfrak{b}$ *is quasi-regular, (iii) every* $G^{\mathbf{C}}$*-orbit of a quasi-regular element in* $\mathfrak{g}^{\mathbf{C}}$ *intersects* $e_- + \mathfrak{g}^{e_+}$ *in precisely one place.*

Using these facts we can prove:

Proposition 2 *Every* N_+*-orbit in* $e_- + \mathfrak{b}$ *intersects* $e_- + \mathfrak{g}^{e_+}$ *in precisely one place. Consequently, every* B*-orbit in* V *intersects* $e_- + \mathfrak{g}^{e_+}$ *in precisely one place.*

Proof. In light of the lemma above, it suffices to show that every N_+-orbit in $e_- + \mathfrak{b}$ intersects $e_- + \mathfrak{g}^{e_+}$ somewhere. So we will show that for every $Y \in \mathfrak{b}$ there exists an $n \in \mathfrak{n}_+$ with

$$\exp(\mathrm{ad}\ n)(e_- + Y) \in e_- + \mathfrak{g}^{e_+}.$$

The argument simply uses the $\mathbf{Z}$-grading. As before we write

$$Y = \sum_0^{k-1} Y_j, \quad n = \sum_1^{k-1} n_j,$$

and expand $X = \exp(\mathrm{ad}\ n)(e_- + Y)$ similarly. We find, for $j = 1, \ldots, k-1$,

$$X_{j-1} = [n_j, e_-] + f(e_-, n_i, Y_i)$$

where $f(e_-, n_i, Y_i) \in \mathfrak{b}$ is an expression only involving terms with $i < j$. Therefore we can find at each stage an $n_j \in \mathfrak{n}_+$ for which $X_{j-1} \in \mathfrak{g}^{e_+}$ by virtue of the first statement of the previous lemma (note in particular $\mathfrak{g}^{k-1} \subset \mathfrak{g}^{e_+}$).■

Remark. As Kostant points out in [6] Remark 19′, $e_- + \mathfrak{g}^{e_+}$ is not the only choice of subspace which has the properties we require. For example, for $\mathfrak{g}^{\mathbf{C}} = \mathfrak{sl}_{n+1}$ we can take instead the vector space of 'companion matrices'.

We deduce from this that each map in $\mathcal{S}_{\mathrm{full}}$ has a canonical form given by

$$b_\lambda \# [F^\xi] : \mathbf{C} \to b \exp(zX) \cdot B \tag{7}$$

where the identifications $B \simeq \Lambda_0 B$ and $e_- + \mathfrak{g}^{e_+} \hookrightarrow \Lambda_0^{-1,\infty}$ map $b \mapsto b_\lambda$ and $X \mapsto \xi$. We see that the dressing group acts as the group of base-point preserving holomorphic automorphisms of $G^{\mathbf{C}}/B$. Observe that the extended frame $b_\lambda F^\xi$ has finite uniton number i.e. it is a Laurent polynomial in λ.

To finish this section let us look at a way of determining when two maps $[F^\eta], [F^\xi] \in \mathcal{S}_{\mathrm{full}}$ lie in the same dressing orbit which does not require reducing them to canonical form. Let R be the ring of Ad-invariant polynomials on $\mathfrak{g}^{\mathbf{C}}$ and let $J:\mathfrak{g}^{\mathbf{C}}\to\mathbf{C}^n$ be the morphism of affine varieties obtained by fixing n homogeneous generators for R. For any $\eta\in\Lambda^{-1,\infty}$ we have $\tau\eta(\lambda)=\eta(\omega\lambda)$ from which we deduce that $J(\eta(\lambda))$ must be a function of λ^k (since τ is inner). It follows that $J(\eta(\lambda))$ is holomorphic in λ^k and we may therefore define a map

$$J_0:\Lambda^{-1,\infty}\to\mathbf{C}^n\quad;\quad\eta\mapsto J(\eta)|_{\lambda=0}.$$

It is not hard to see that $J_0(\eta)=J_0(\eta_{(0)})=J(\eta_{(0)})$. Moreover, from [6] theorem 7 we learn that $J:e_-+\mathfrak{g}^{e_+}\to\mathbf{C}^n$ is an isomorphism and therefore we deduce

Corollary 1 *Two maps $[F^\eta],[F^\xi]\in\mathcal{S}_{\text{full}}$ lie in the same dressing orbit if and only if $J_0(\eta)=J_0(\xi)$.*

5 Maps of finite type.

In this section I will show that every map in $\mathcal{S}_{\text{full}}$ is of finite type in the sense of [2], which we now recall.

For each $d\in\mathbb{N}$ define Λ_d to be the vector subspace of $\Lambda_E^\epsilon\mathfrak{g}_\tau$ consisting of Laurent polynomials of degree $\le d$. Whenever $d\equiv 1 \bmod k$ we have a pair of commuting real flows on this space determined by the o.d.e's implicit in the equation

$$\partial\xi/\partial z=[\xi,\lambda^{-1}\xi_{-d}+\frac{1}{2}\xi_{1-d}]\tag{8}$$

for $\xi:\mathbf{R}^2\to\Lambda_d$. A harmonic map is of finite type if it possesses an extended frame F whose left invariant z-derivative satisfies

$$F^{-1}.\partial F/\partial z=\lambda^{-1}\xi_{-d}+\frac{1}{2}\xi_{1-d}$$

for some solution ξ of (8). The Symes map provides a construction for all maps of finite type.

Theorem 1 ([2] theorem 4.3) *For any $\xi\in\Lambda_d$ with $d\equiv 1 \bmod k$ the map $[F^{\lambda^{d-1}\xi}]$ is of finite type and all such maps of finite type arise this way.*

Corollary 2 *Every map in $\mathcal{S}_{\text{full}}$ is of finite type. Therefore $\mathcal{S}_{\text{full}}$ is the space of all full primitive holomorphic maps of finite type in G/T.*

Proof. For any $\eta\in\Lambda_{(0)}^{-1,\infty}$ we observe η has degree in λ at most $k-1$. Set $\xi=\lambda^{-k}\eta+\lambda^k\bar\eta$, then $\xi\in\Lambda_{k+1}$ so that the map $[F^{\lambda^k\xi}]$ is of finite type. But clearly $(\lambda^k\xi)_{(0)}=\eta$ so that, when η_{-1} is principal nilpotent, $[F^{\lambda^k\xi}]=[F^\eta]$ by proposition 1.■

5.1 Maps of the 2-sphere.

Proposition 3 *The map $\psi : \mathbf{C} \to G^{\mathbf{C}}/B$ given by (7) extends rationally to* $\mathbf{C} \cup \{\infty\}$ *if and only if $X = e_-$.*

Consequently only one dressing orbit provides us with primitive holomorphic maps of the 2-sphere into G/T. In the case $G = SU_{n+1}$ this orbit was described in [7] §4.2: it is the dressing orbit of the primitive lift of the Veronese map $S^2 \to \mathbf{CP}^n$.

Proof. From [6] §4.5 we learn that to every $X \in e_- + \mathfrak{g}^{e+}$ there corresponds an $H \in \mathfrak{t}^{\mathbf{C}}$ and an $n \in N_+$ such that $e_- + H = \mathrm{Ad}\ n \cdot X$. Consequently we need only examine when the map $\varphi(z) = \exp(ze_- + zH) \cdot B$ is polynomial in z for some affine open subset of $G^{\mathbf{C}}/B$. But $\gamma(z) = \exp(ze_- + zH)$ is clearly a one parameter subgroup of $N_-.T^{\mathbf{C}}$ and hence φ takes values in the big cell (i.e. the N_--orbit of B) which is affine and open. Moreover

$$\beta(z) = \gamma(z)\exp(-zH)$$

has $\varphi(z) = \beta(z) \cdot B$ and $\beta^{-1}d\beta/dz = \mathrm{Ad}\ \exp(zH) \cdot e_-$, therefore β takes values in N_-. So it suffices to show that $\beta^{-1}d\beta/dz$ is polynomial if and only if $H = 0$. But

$$\mathrm{Ad}\ \exp(zH) \cdot e_- = \sum_{i=1}^{n} e^{-z\alpha_i(H)} e_{-\alpha_i}$$

which demonstrates the result.■

It follows at once that this describes all full primitive holomorphic 2-spheres of finite type in G/T. Now recall (from, for example, [1]) that every harmonic 2-sphere in $\mathbf{CP}^n$ has a unique primitive holomorphic lift into SU_{n+1}/T which is the Frenet frame for a holomorphic 2-sphere in $\mathbf{CP}^n$. Thus the dressing orbit above consists of the Frenet frames for all (base-point preserving) projective linear transformations of the Veronese map. Let us agree to say a harmonic 2-sphere in $\mathbf{CP}^n$ is of finite type whenever its primitive lift is, then we deduce:

Corollary 3 *A full harmonic 2-sphere in $\mathbf{CP}^n$ is of finite type if and only if it lies in the Frenet frame of a rational normal curve of degree n.*

Remark. One expects every harmonic map of finite type to possesses a spectral curve, so it is natural to ask what the spectral curve is for these harmonic 2-spheres in $\mathbf{CP}^n$. Without going into the details, I claim we can take (for every 2-sphere) the real rational curve obtained from $\mathbf{C}_\lambda \cup \{\infty\}$, with the real involution $\lambda \mapsto \bar{\lambda}^{-1}$, by placing a cusp of order n at 0 and ∞ (recall from [8] that the spectral curve for a harmonic torus in $\mathbf{CP}^n$ has two distinguished *smooth* points in real involution). But it appears we can also take real curves of any genus with a pair of cusps of order n in real involution, so that the same harmonic 2-spheres occur in the solution space Λ_d of (8) for every $d \equiv 1 \bmod k$, $d > 1$. As testimony to this, observe that in the proof of corollary 2 we could replace $k+1$ by any appropriate d.

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