Non-smooth dynamical systems and the grazing bifurcation

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Abstract

We study some dynamical systems described by non-smooth maps. which arise naturally in studies of certain physical systems - especially impact oscillators. We identify a new form of bifurcation - the *grazing bifurcation* and show (by example) that this leads to complex dynamics, including chaotic behaviour interspersed with *period adding* windows of periodic behaviour. We then construct a normal form for the grazing bifurcation and classify the dynamics which can be observed. The article concludes with some open problems.

1 Introduction

Many classical treatments of dynamical systems make the underlying assumption that the systems they are describing are smooth. Whilst this assumption is reasonable for most physical problems there are also systems which are best described (at least on macroscopic scales) in terms of non-smooth dynamics. The simplest such example (an an early introduction to dynamical systems for many of us) is the trajectory observed when an elastic ball is dropped on to a hard floor. Many other similar examples exist in mechanical and electrical systems. At first it would appear that a systematic study of such systems is hopeless, after all, the study of smooth dynamical systems is hard enough! However, by making reasonable assumptions about the departure of the system from smoothness, some generic properties emerge and we are able to identify a new form of bifurcation - the *grazing bifurcation*. This bifurcation arises when under a change of parameters the system evolves so that a smooth motion first starts to lose regularity. An example of this occurs when a freely oscillating cantilever beam is brought closer to an obstacle. At some point an impact will occur and remarkably, this simple experiment often leads to complex chaotic motion. The grazing bifurcation describes a mechanism for this transition and may (possibly) account for many other examples of chaotic vibration observed in mechanical systems. The grazing bifurcation has now been observed in many non-smooth systems, and part of the purpose of these notes is to motivate it and to describe some of the recent exciting mathematical results concerning it. We will conclude with some open problems.

2 Examples of non-smooth dynamical systems

We start our discussion by considering some systems which arise naturally and which have non-smooth dynamics. Such problems arise in many fileds but were perhaps first considered by Fermi [F], who was considering the origin of cosmic rays. He proposed that they might originate from the exchange of chrged particles between large gaseous clouds. A greatly simplified model of this comprises a ball in a cylinder accelerated by a moving piston. The ball typically moves to the end wall of the cylinder, impacts and then rebounds, hitting the cylinder again and thus continuing the motion. Thus the motion of the ball is a sequence of smooth motions with instantaneous changes in velocity. This system is Hamiltonian and has received much attention in the theoretical physics literature, where it is shown, for example, that a periodically moving cylinder can lead to chaotic motion of the ball. A similar model has been used by True [Tr] and Knudsen [Kn] to describe the motion of railway bogies. Closely related is the problem of finding the motion of an elastic ball moving under gravity and dropped on to a periodically moving cylinder. This system was introduced by Pustylnikov [Pu] and described in detail by Guckenheimer and Holmes [GH]. Again, a periodic excitation can lead to chaotic motion of the ball. A demonstration of this behaviour has been constructed by Popp [Po] and it is also used as an example of controlling chaotic systems. We can extend this system by considering the ball to be suspended on a spring. If it is struck from below by a moving obstacle then we observe a combination of the free motion of the ball on the spring together with the forced motion due to the impacts. Similarly, if the ball is suspended above a fixed obstacle and oscillated from above, then small motions on the spring will not bring it into contact with the obstacle and hence will be periodic. Larger motions will however result in impacts and it is this transition which is of interest to us. Typically we observe chaotic motion or large amplitude periodic motion with high velocity impacts between the ball and the obstacle. Very closely realted are problems concerning harmonic oscillators with springs that have stiffness that depends discontinuously upon position, for example the motion of the system described by

$$\frac{d^2x}{dt^2} + K(x)x = \cos(\omega t) \tag{1}$$

where

$$K(x) = k_1 \quad \text{if} \quad x < x_0, \quad K(x) = k_2 \quad \text{if} \quad x > x_0 \tag{2}$$

These systems, which have been used by Thompson [TS] to study the motion of ships in a harbour and by Doole & Hogan [HD] to look at the motion of suspension bridges, are analysed in the seminal paper by Shaw & Holmes [SH]. If k_2/k_1 is large and k_1 not small then they have very similar dynamics to the simple impacting model. Problems with $k_2 = 0$ occur in systems with play or backlash between components Kleczka et. al. [K]. Both the ball on a spring and the piecewise linear models above are very closely related to the motion of an elastic beam impacting an obstacle for which some experimental results are presented in Cusumano et. al. [CSK] (see also [Nord3]), or of tubes (in say a heat exchanger) excited by a flow and vibrating against their supports [Goyd]. These systems are more complex, however, as the beam or tube may vibrate in many modes. Another very natural physical system giving rise to discontinuous dynamics is the stick-slip behaviour of systems with friction [PS1,PS2] including the motion of turbine blades [Pf]. In such systems the friction force is always in the direction opposite to that of the motion. If we consider a forced mass on a spring which is in friction contact with a moving obstacle, then a simplified model of the motion is given by:

$$\frac{d^2x}{dt^2} + \mu \mathrm{sgn}\left(\frac{dx}{dt}\right) + x = \phi(t). \tag{3}$$

Where $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$. If $\phi(t) = \sin(t)$ then such an oscillator can exhibit periodic or chaotic motion. (The periodic motion is well known as, for example wheel squeal in railway bogies or in the squeaking of chalk on a black-board.) Again, grazing phenomena are observed when a system just starts to slip as the friction force becomes too weak to maintain equilibrium. Behaviour very similar to grazing also appears to occur in electrical circuits. Indeed the simulations of DC-DC convertors described by Hamill [Ham] are qualitatively very similar to the results obtained from a grazing bifurcation.

As in any dynamical system it is natural to ask questions such as when the system will be in equilibrium, when it will have periodic states, when it will be chaotic and what the transitions will be between the states. We now study these questions by looking at the particular example of the impact oscillator.

3 The impact oscillator and its associated maps

The impact oscillator is the idealisation of the problem comprising the ball on a spring hitting an obstacle. This system has been studied in the Russian and Czech literature [Pet] for some time but has more recently aroused a great deal interest, both for its (obvious) applications and the interesting nature of the mathematics describing it. The simplest impact oscillator is described by the system

$$\frac{d^2x}{dt^2} + \zeta\frac{dx}{dt} + x = \cos(\omega t), \quad x < d, \tag{4}$$

$$\frac{dx}{dt} \to -\lambda\frac{dx}{dt}, \quad x = d, \tag{5}$$

(where $\lambda < 1$ is the coefficient of restitution), so that it has a series of impacts with the obstacle (located at $x = d$) at times t_i. with velocities v_i. Between impacts its motion is smooth. An experimental realisation of this oscillator using gliders on an air-bed is described in [RW].

The impact oscillator may exhibit both periodic and chaotic behaviour depending upon the parameters r,d and ω, with transitions between these states described by both smooth and grazing bifurcations. The simplest such motion is periodic with period

$$T = \frac{2\pi n}{\omega}, n = 1, 2, 3, ..$$

Such a motion typically repeats after m impacts such that

$$v_{i+m} = v_i \quad \text{and} \quad t_{i+m} = t_i + T$$

and is referred to as an (m, n) or an m/n orbit. Of these orbits the $(1, n)$ are the most common, are easiest to analyse, and typically have the largest amplitude and domains of attraction. It is straightforward to write down algebraic conditions for these orbits [SH],[W1],[W2],[BDC] and to analyse their (linear) stability under small perturbations. In many respects they demonstrate the same features as fixed points of a smooth dynamical system in that they have attracting manifolds and may lose stability through period-doubling or saddle-node bifurcations. However, an alternative mechanism for the loss of stability arises when an additional impact occurs at a time t such that $t_i < t < t_i + T$. If this first occurs at a given parameter value say $d = d_0$ (so that the intermediate impact has zero velocity) then the periodic orbit encounters a *grazing bifurcation* at d_0 which typically leads to chaotic or more complex behaviour.

A convenient way to study the impact oscillator is by considering a map P relating one impact to the next. It can be shown that if an impac t occurs with (positive) velocity v_i at a time t_i then the velocity and time (v_{i+1}, t_{i+1}) of the next impact are a wel defined function of v_i and the phase $\phi_i = (t_i)_{\mathrm{mod}(T)}$. This leads us to consider the map

$$P : (\phi_i, v_i) \rightarrow (\phi_{i+1}, v_{i+1}).$$

The structure of this map is described in detail in [BDC], it can be contructed easily numerically and is a natural way to study the dynamics of the impact oscillator. It maps the cylinder to itself, a (m, n) periodic orbit is a fixed point of the iterated map P^m and P is smooth apart from on a one-dimensional (branched) set S. This is the set defined by

$$S = \{(\phi, v) : P(\phi, v) = (\psi, 0)\}.$$

Across this set P is discontinuous and introduces infinite local stretching into the phase space. To consider the action of this map, let S be the set defined above and W the image set $P^2(S)$. The nature of the discontinuity of P is that points on one side of S and close to S are mapped by P to points close to W and on one side of W. In contrast points on the other side of S are first mapped to a neighbourhood of the line $v = 0$ and then are mapped to the other side of W on the second iterate of P. More significant is the stretching that this map introduces. To show this we introduce a new (local) coordinate system (x, y) so that $y = 0$ for a point on S and x, y represent coordinates in directions tnajent to and orthogonal to S. Here we consider the points for *positive* y to be first mapped to a neighbourhood of $v = 0$ and then to a neighbourhood of W. Similarly we introduce x', y' parallel and orthogonal to W. Then, in this new coordinate system Nordmark [Nord1] (see also [BD1]) showed that locally P took the following form:

$$\text{if} \quad y < 0 \quad \text{then} \quad (x', y') = P(x, y) = (ax + by, cy), \tag{6}$$

$$\text{if} \quad y > 0 \quad \text{then} \quad (x', y') = P^2(x, y) = (ax + dy, e\sqrt{y}) \tag{7}$$

where the values of a, b, c, d, e are chosen suitably.

The most important feature of this map is that it has a square-root singularity as $y \rightarrow 0$ which leads to infinite local stretching in this limit. As an immediate consequence of this stretching we see complex chaotic behaviour. For example, if a set K of data intersects the set S then the action of P is to stretch and fold K leading to an action on K closely resembling that of a horseshoe map. Furthermore, if a fixed point of P changes under the variation of a parameter so that it intersects S then the action of the stretching destabilises this point - leading to the grazing bifurcation and complex motion.

We now give two examples of the motion at some specific parameter values which demonstrate some of the dynamics possible.

Resonant motion, period doubling and grazing.

If $d = 0, r = 0.8$ and $\zeta = 0$, then a unique $(1,1)$ orbit is stable for ω in a neighbourhood of $\omega = 2$ and has a large (resonant) amplitude and impact velocity v_1. For certain (narrowly defined) parameter values, this orbit co-exists with a $(3,3)$ or with a (chattering) $(\infty, 6)$ orbit [BD2]. Due to the stretching associated with the map P, the domains of attraction of these more complex orbits can be large, complex and tangled with that of the $(1,1)$ orbit. In a typical experiment, where ω is increased steadily and the behaviour of the system observed, these coexisting orbits might appear to be a thin band of noise superimposed on the simpler $(1,1)$ orbit. Possible behaviour of this nature has been observed by Hamill [Ham]. As ω increases (or decreases) the simple $(1,1)$ orbit loses stability at a smooth super-critical period-doubling bifurcation to a $(2,2)$ orbit with impact velocities v_1 and v_2 The $(2,2)$ orbit period doubles again as ω increases, to a $(4,4)$ orbit with impact velocities v_1, v_2, v_3, v_4. Chaotic behaviour (with an associated strange attractor) is observed when $\omega = 2.8$. As ω is increased further, the $(2,2)$ orbit restabilises and is globally attracting in a range of ω centred upon $\omega = 3$. The orbit is then destroyed at a grazing bifurcation at $\omega = 3.01$ when an intervening impact occurs between the existing two impacts. There then follows a period of chaotic motion before a simple globally attracting $(1,2)$ orbit arises at $\omega = 3.35$. This behaviour is presented in the following figure which shows the values of the impact velocity v_i that occur after 1500 impacts with an initial start at $\phi = 0.5, v = -1.5$. A detailed analysis of the transition from the $(1,1)$ to $(2,2)$ orbit and the consequent restabilisation of the $(2,2)$ orbit is given in [BL].

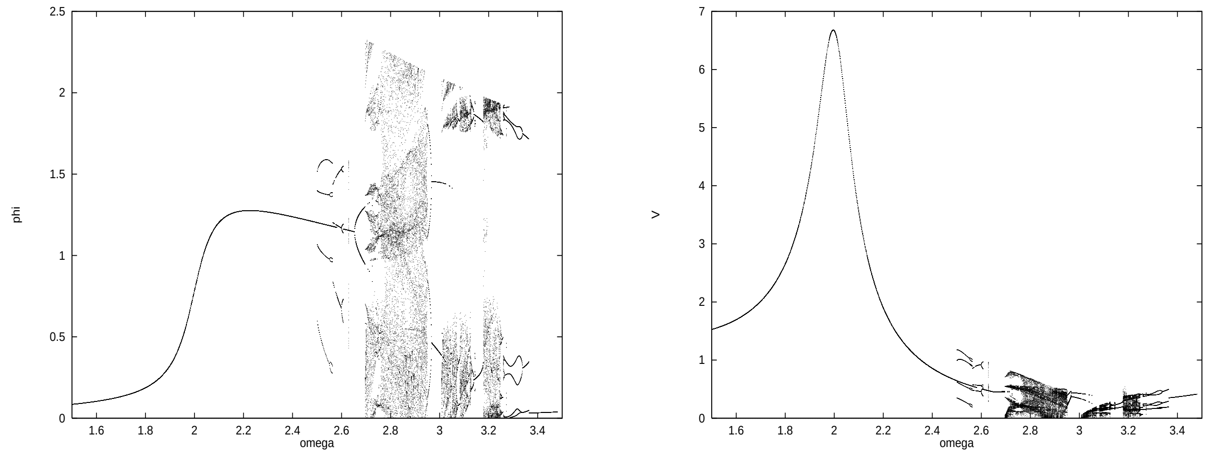


Figure 1: Bifurcation diagram showing impact phase (ϕ) and velocity (V) with $d = 0, \lambda = 0.8$, $\phi_0 = 0.5$, $v_0 = -1.5$ and 3000 initial transients.

First impacting phenomena.

If $d > 1$ and we take the same parameter values as above, then for small values of ω the forced motion of the oscillator will be too small for any impacting to occur. However, as ω is increased toward the natural frequency of the oscillator, then the amplitude of the forced motion increases and there is a first value of $\omega = \omega_1$ given by the smaller root of the equation

$$1/d = \sqrt{(\zeta\omega)^2 + (\omega^2 - 1)^2}, \tag{8}$$

(so that $\omega_1 \approx \sqrt{1 - 1/d}$ when ζ is small), when an impact of zero velocity first occurs. For subsequent values of ω the motion is most complex and the transition from the simple non-impacting motion to this complex motion is described by the grazing bifurcation [Fo,FB1,FB2]. Interestingly, for larger values of ω the motion stabilises to a simple $(1,1)$ periodic orbit which is globally attracting, but which also has regions of parameter values for which it coexists with other more complex orbits. There is then a subsequent value of $\omega = \omega_{SN} \approx 1.45$ where the $(1,1)$ orbit experiences a simple saddle-node bifurcation and loses stability. In this case the oscillator rweturns to a non-impacting state. If ω is now *reduced* then there is a further value of ω given by the larger root of 8 so that if ζ is small

$$\omega_2 \approx \sqrt{1 + 1/d}, \quad \text{where} \quad \omega_1 < 1 < \omega_2 < \omega_{SN},$$

when a zero velocity impact takes place and the motion of the oscillator is again complex. There is thus a natural hysteresis in the behaviour of the impact oscillator as a function of ω with complex motion generated by grazing.

We illustrate the corresponding dynamics when $d = 2, \lambda = 0.8\zeta = 0.1$ in Figure 2. This figure also includes other orbits (particularly in the case $\omega < \omega_1$) where other types of impacting motion can occur.

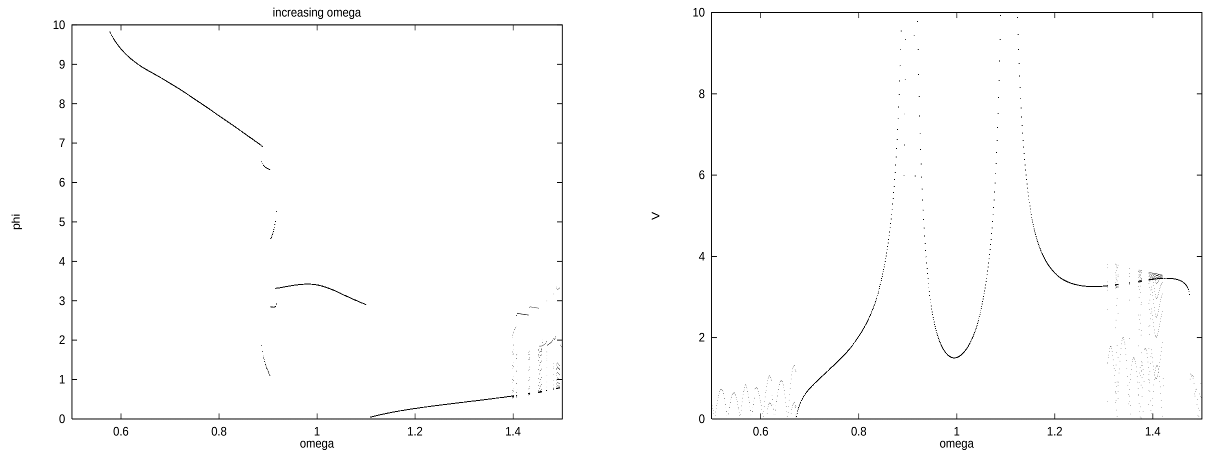


Figure 2: Bifurcation diagram showing impact phase (ϕ) and velocity (V) with $d = 2, \lambda = 0.8,\ \phi_0 = 0.5,\ v_0 = -1.5$ and 3000 initial transients.

Other grazing bifurcations can be observed if ω is fixed and d varies. For example, if $\omega = 2$ and d is reduced from zero, then a grazing bifurcation is observed close to $d = -1/3$. A description of this is given in [BD1].

4 Grazing and the grazing bifurcation

The examples described above naturally lead to the following important question, namely what behaviour is observed when the system parameters vary from a periodic state with no impacts (or high velocity impacts) to one where an impact first occurs (or there is an additional low velocity impact between those of high velocity)? Similar questions arise in studies of the bouncing ball or in friction oscillators. As remarked in Section 3, this corresponds to a fixed point (or orbit) of the map P intersecting the set S as a parameter is varied. The stretching of the phase space close to S gives rise to new bifurcation phenomena which appear to be universal to many systems. The corresponding bifurcation is called a *grazing bifurcation* as it is associated with a low velocity *grazing* impact. This bifurcation was first observed by Paterka [Pat] and discovered independently by Whiston [W1],[W2],[W3] and Nordmark [Nord1],[Nord2],[Nord4]. Subsequently it has been investigated analytically by Foale and Bishop [FB1,FB2], Budd and Dux [BD1], Lamba and Budd [LB] and in a series of papers by the group at Maryland [NOY], [CONG1], [CONG2]. Some experimental results have also been reported in [FB1,FB2] and in the recent work by Mullin [Mull]. In this presentation we follow that of [Nord4], and [NOY] quite closely.

Suppose now that for some parameter d (which may for example be the clearance of the system) a $(1, n)$ orbit exists and is linearly stable and that this orbit has a local maximum between impacts. (Or suppose similarly that an orbit of small amplitude without impact exists.) As d is changed, there will typically be a first value (without loss of generality at $d = 0$) at which a grazing impact first occurs. Typically such an impact introduces a destabilising effect on the $(1, n)$ orbit due to the large stretching of the phase space, and a different motion results. If the $(1, n)$ orbit is stable for $d < 0$ and unstable for $d > 0$ then such a bifurcation is *supercritical* with the resulting motion confined to a neighbourhood of $\sqrt{d}$ of the original orbit [Nord1]. Depending upon other parameters in the system, four types of behaviour are then observed.

- *Case A.* (Large coefficient of restitution). There is an immediate creation of an intermittent chaotic motion together with a large set of unstable periodic orbits. There is a non-zero interval $0 < d < d'$ in which there are no periodic windows. The chaotic motion lies on an attracting set of size proportional to $\sqrt{d}$. It is shown in [LB] and [BD1] that the number of unstable peridic orbits and the reciprocal of the Lyapunov exponents both scale as $\log(d)$.
- *Case B.* (Intermediate coefficient of restitution). There are an infinite sequence of periodic windows alternating with chaotic bands without windows. The windows are period *adding* so that if there is a window at some value of d containing an n-impact orbit, then there is a next window (as d decreases) containing an (n+1)-impact orbit. The width (and location) of the windows decreases geometrically as $d \to 0$.
- *Case C.* (Low coefficient of restitution). No chaotic motion exists, but the period-adding windows do exist and overlap giving multiple periodic states and an infinite sequence of periodic motions.
- *Case D.* The $(1, n)$ orbit and an unstable orbit coalesce at $d = 0$.

An example of a bifurcation taking the form of Case B is presented in Figure 3 and of Case C in Figure 4.

It is most interesting to compare this behaviour with that of the bifurcation diagram typical of the logistic map and described by Sarkovski's theorem. In this case there are no intervals of purely chaotic motion which do not contain periodic windows. Furthermore the relation between the periodic orbits in the windows is somewhat more complex than period-adding. Indeed, it is period-adding which is the strongest indicator of a grazing bifurcation.

We now show briefly, why this behaviour occurs. To do this we follow the elegant treatment given in [Nord4] and consider the bifurcation associated with a one-dimensional map which inherits many of the features of the impact map. A similar approach is taken in [FB1,FB2]. In [BD1] and in [NOY] these arguments are generalised to the true two-dimensional map. However, this generalisation is relatively straightforward as close to the bifurcation the important action of the map P is largley confined to the one-dimensional stable manifold of the fixed point (where we note that the manifold persists even if the fixed point itself is destroyed in the graze).

Accordingly, motivated by the local form of the map P close to S given in 6,7, we look at the dynamics of the map $f(x)$ defined in [Nord4] by

$$f(x) = \sqrt{d-x} + \lambda x, \quad x \leq d, \tag{9}$$

$$f(x) = \lambda x, \quad x \geq d, \tag{10}$$

where $\lambda < 1$ is equivalent to the coefficient of restitution. The most important feature of this map is the square root singularity which introduces locally unbounded stretching. For this map, d plays the role of a bifurcation parameter.

If $d < 0$ then $f(x)$ has a single (stable) fixed point at $x = 0$. This loses stability if $d > 0$ and for small $d > 0$ we observe motion consistent with the first three of the cases above, with Case A if $2/3 < \lambda < 1$, Case B if $1/4 < \lambda < 2/3$ and Case C if $0 < \lambda < 1/4$.

We now show how this behaviour arises when $d > 0$. It can be shown easily, that if $\lambda d < x_0 < d$ and if d is sufficiently small, then due to the stretching associated with the square root, the first iterate $x_1 = f(x_0)$ of $f(x)$ will lie in the region $x_1 > d$ and that there will be subsequently a (possibly large) number $m(x_0, d)$ of iterates x_n of $f(x)$ with $x_n > d$ if $n \leq m$, such that a first iterate x_{m+1} reenters the interval $I \equiv \{x : \lambda d < x < d\}$. Thus the set I acts as a trapping region for the flow. The value of $m(x_0, d)$ takes its maximum value $M(d)$ when $x_0 = \lambda$ and decreases monotonically, as $x_)$ increases so that it is constant on subintervals of I. In a typical motion an iteration lies in I once and in the range $x > d$ more frequntly. A periodic orbit with this property is refered to as a *maximal orbit* in [NOY].

Motivated by this discussion we introduce a map $F : I \to I$ from the interval $[\lambda d, d]$ to itself which is defined by

$$F(x_0) = x_{m+1}$$

Figure 3: A grazing bifurcation with period adding windows separated by bands of chaotic behaviour

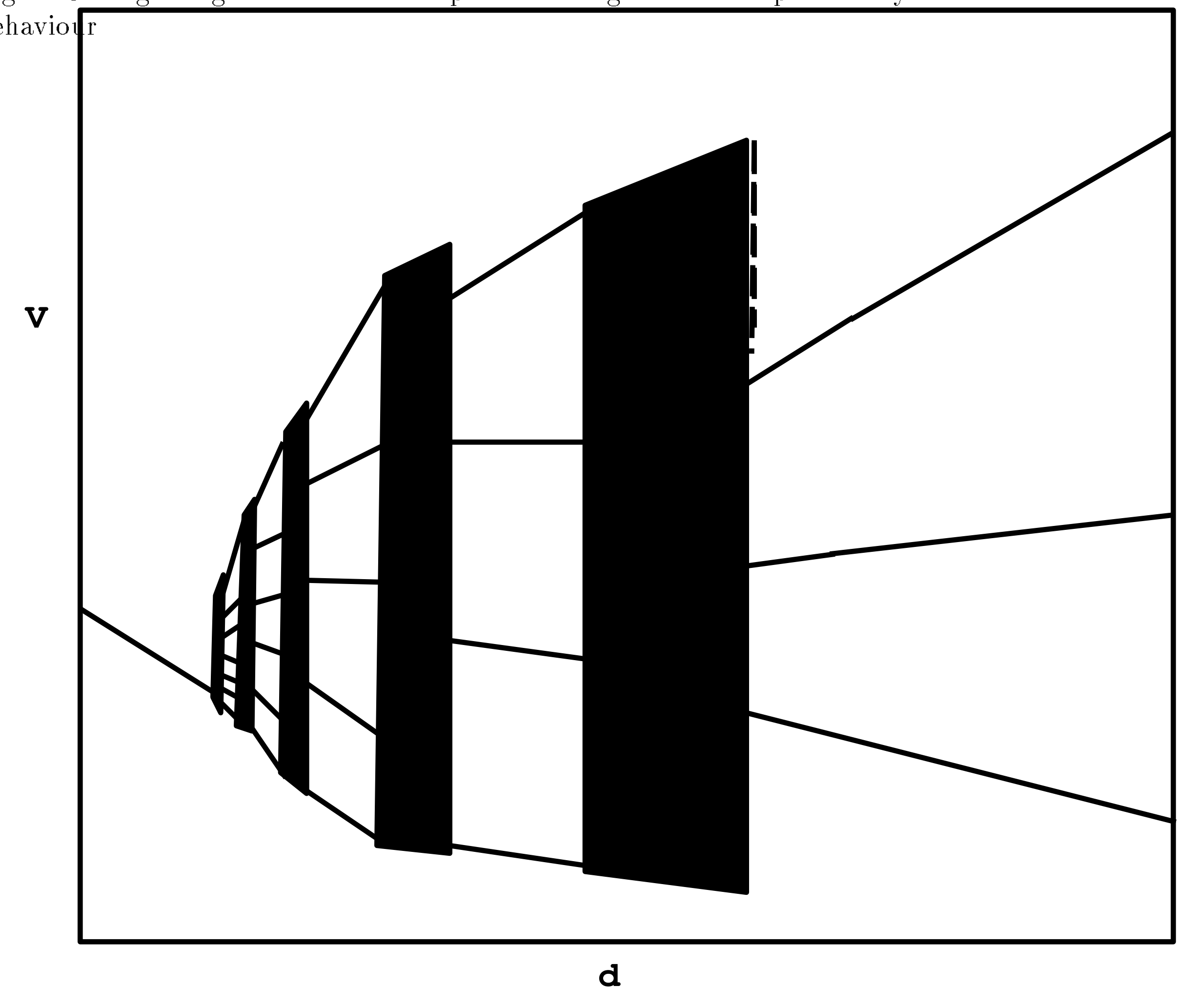
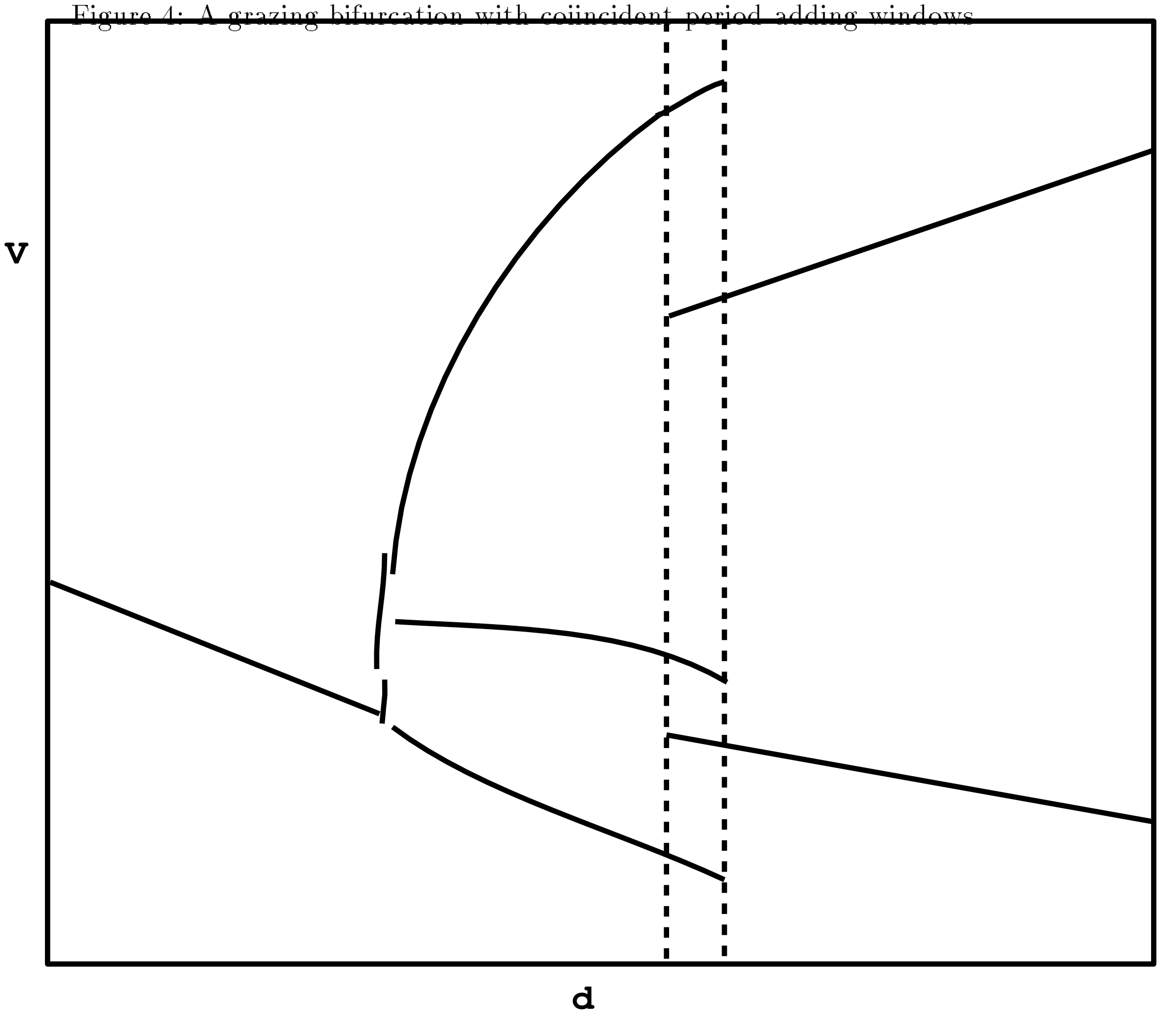


Figure 4: A grazing bifurcation with coiincident period adding windows



and which is continuous on those subintervals for which m is constant, and is discontinuous across the interval boundaries. Thus F has M continuous branches, and on each such branch it takes the form:

$$F(z) = \frac{\lambda^m}{\sqrt{d}}\sqrt{1-z} + \lambda^{m+1} z$$

where $z = x/d$. (See [BD1] for a very similar description of a corresponding two-dimensional map). It is shown in [Nord1] (and can be verified easily) that decreasing d by a factor of λ^2 increases M by one and adds an extra branch to F but otherwise the form (and magnitude of F) does not change. To formalise this Nordmark introduced a new parameter $\mu \in (\lambda^2, 1]$ such that

$$\mu \equiv \left(\frac{\lambda}{F(\lambda)}\right)^2$$

then for large M

$$\mu \approx \frac{d}{\lambda^{2(M-1)}(1-\lambda)} \tag{11}$$

Under this change of variables, and putting $k = M - m$ we get

$$F(z) = \frac{\lambda\sqrt{1-\lambda}}{\sqrt{1-\lambda}\sqrt{\mu}\lambda^k} + \frac{\lambda^{M+1}}{\lambda^k}\left(z - \lambda\frac{\sqrt{1-z}}{\sqrt{1-\lambda}}\right) \approx G(z)$$

where

$$G(z,\mu) = \frac{\lambda\sqrt{1-z}}{\sqrt{1-\lambda}\sqrt{\mu}\lambda^k}. \tag{12}$$

The map $G(z,\mu)$ is illustrated in Figure 5.

We see from this result that if μ is constant, then F is well approximated by the universal function G. Now, from (11) it is evident that reducing d by a factor of λ^2 will leave μ the same provided that M is increased by one. The dynamics of the map F will be essentially the same in these two cases and will be described by the dynamics of the map G although M is increased by one. This is the mechanism behind the period-adding phenomena.

The map G is continuous on the intervals I_k defined by

$$I_0 = [\lambda, B_0], \quad I_k = (B_{k-1}, B_k], \quad k = 1, 2, 3, ..$$

where

$$B_k = 1 - \mu\lambda^{2k}(1-\lambda).$$

The map G is essentially independent of the function f, provided that this function has a square root singularity and gives a *normal form* for the grazing bifurcation and this normal form extends to higher dimensions. We now look at the dynamics of its iterates.

A simple calculation shows that

$$G'(z) < G'(\lambda) = -\frac{\lambda}{2(1-\lambda)\sqrt{(\mu)}} \tag{13}$$

and that G gas an infinite number of fixed points z_k with

$$G'(z_k) = \frac{-1}{2}\frac{z_k}{1-z_k}.$$

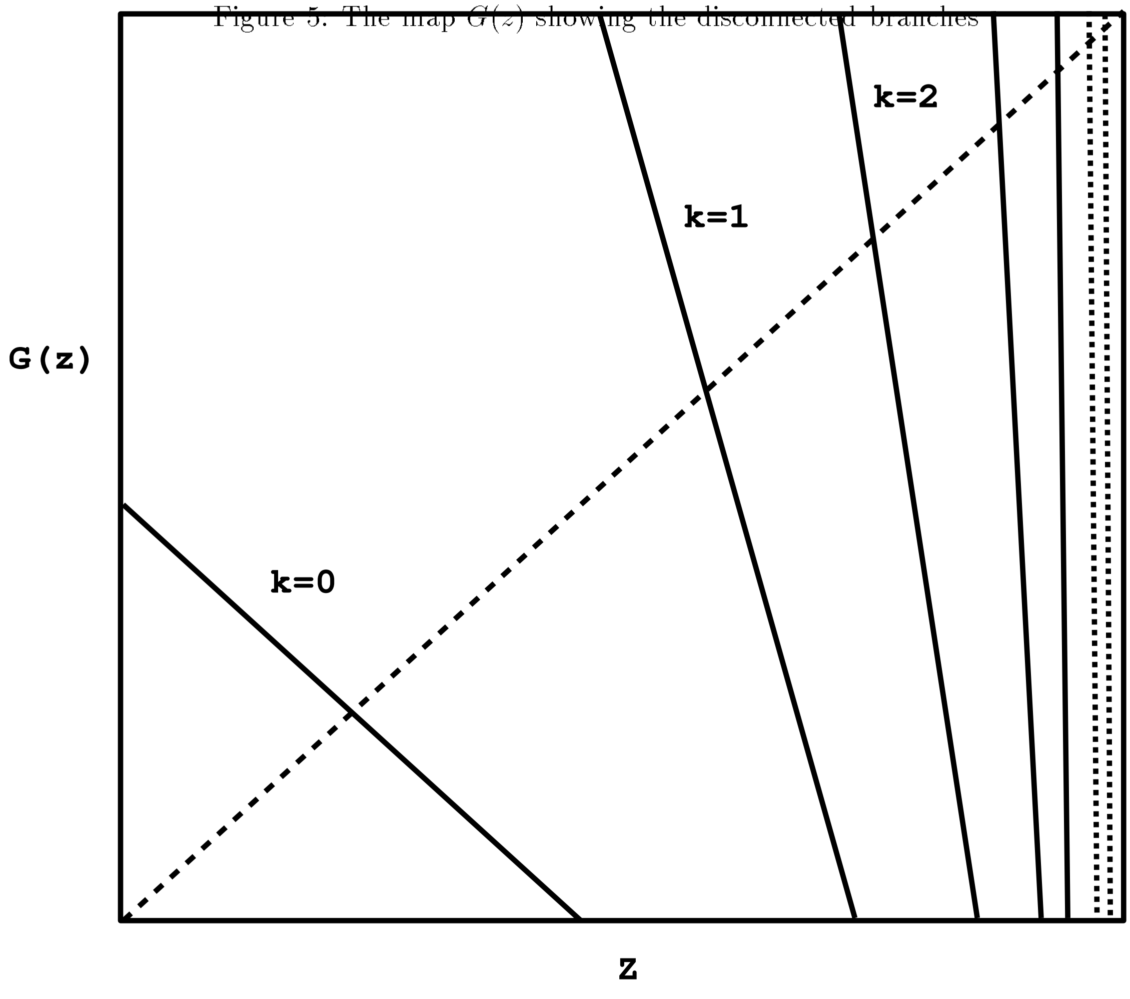


Figure 5: The map $G(z)$ showing the disconnected branches

As $\mu \in (\lambda^2, 1]$ it follows from (13) that if $\lambda > 2/3$ then $G'(z) < -1$ so that no periodic orbits exist and all motion is chaotic.

For each fixed M then, if $\lambda < 2/3$ then it is possible for z_0 to be stable. In particular it exists if $\lambda < 2/3$ and if μ is close to one. For the map f such a fixed point corresponds to a so called *maximal* $M-$*orbit* which has one iterate in the region where the map takes square root form an M iterates in the region where the map is linear. This orbit will then exist if $d < d_n$ where d_n is given by setting $\mu = 1$ in (11) so that

$$d_n(M) \approx (1-\lambda)\lambda^{2(M-1)}.$$

For $d > d_n$ the maximal M-orbit does not exist. and it vanishes discontinuoulsly as d increases. Now, if $1/4 < \lambda < 2/3$ it follows from (13) that the periodic orbit has a super-critical period doubling bifurcation when $\mu = \mu_f$ given by

$$\mu_f = \frac{3\lambda^2}{4(1-\lambda)}.$$

with the corresponding value of

$$d_f(M) \approx \frac{3}{4}\lambda^2\lambda^{2(M-1)}.$$

Thus, for each fixed M we will see an $M-$maximal orbit in the interval $d_f(M) < d < d_n(M)$ and we note that

$$d_f(M+1)/d_f(M) \approx d_n(M+1)/d_n(M) \approx \lambda^2.$$

If $1/4 < \lambda < 2/3$ then Nordmark shows that no periodic orbits of $G(z), G(G(z))$ or of higher iterates exist and consequently the dynamics is completely chaotic in this interval. This picture then explains the existence of period adding windows separated by intervals of chaos.

We now finally look at the case of $\lambda < 1/4$. In this case there are no perid doubling bifurcations and the $M-$orbit persists as d is reduced. Furthermore, the second fixed point z_1 of the map G becomes stable so that we see a coiincidence of an $M-$orbit and an $M+1-$orbit. Thus in this case we no longer have periodic windows separated by chaotic bands, but instead see only periodic behaviour as d is reduced to zero.

5 Conclusions

We have shown that complex dynamic behaviour can be found at a grazing bifurcation and that a normal form of this bifurcation can be constructed to determine the resulting dynamics. These results lead to several interesting open questions. For example, will grazing bifurcations be observed in all of the systems discussed in Section 2.2. There are strong indications of very similar behaviour to grazing in the friction oscillators described in [PS] and in the DC-DC convertors in [Ham] but much work needs to be done to see whether grazing is the fundamental cause of the instabilities observed. A useful study could also be made of grazing phenomena in control systems. Part of this investigation will involve the study of the robustness (and consequent generality) of the grazing bifurcation under changes in the system. As an example of this - we could investigate the grazing phenomena as a global limit of smooth bifurcations (for example if the rigid impact is replaced by a smooth one.) Other open questions are a rigorous proof of the globally attracting nature of the periodic orbits in the period-adding windows and a more subtle study of the way that these lose stability. Much remains to be done!

6 Acknowledgements

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