Stokes Waves

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Dedicated to Louis Nirenberg

Contents

1 Introduction

Amongst Professor Nirenberg's many distinguished contributions to free-boundary questions (e.g. [32], [33], [34], [35], [36]) is his joint work with Littman on Levi-Civita's formulation of the classical water-wave problem from nonlinear hydrodynamics. Their treatment of small amplitude periodic waves on water of infinite depth is the one which Stoker included in his influential treatise on water waves ([53], page 522 and [41], page 242.) The present purpose, forty years on, is to give a self-contained analysis of a global continuum of waves from zero up to and including the limiting Stokes wave of greatest height with its well-known contained angle of 120 degrees. The methods involved have been heavily influenced by Professor Nirenberg's work, combining as they do continuation methods with the Maximum Principle and Hopf boundary-point lemma [24], global bifurcation theory which has its foundations in topological degree [48], and hard analytic estimates based upon integral equations. There is now a substantial body of such analytic theory and a review seems timely. But important open questions remain (Section 13): for example, the convexity of the Stokes wave of greatest height is unproven, and there is no rigorous theory of secondary bifurcation.

In order to show the existence of periodic water waves, both Nekrasov [46], [47] and Levi-Civita [39] used conformal mappings to reduce the question to one of existence of a harmonic function satisfying nonlinear Neumann boundary conditions on a fixed domain. (The harmonic function in question is the angle between the velocity vector in the flow and the horizontal, so its maximum is a measure of the amplitude (crest-to-trough) of a wave. However, the fact that it equals the angle of inclination of streamlines is central to an interpretation of the theory.) These nonlinear Neumann problems were in turn formulated as nonlinear integral equations [27], [45], [63], and the question of existence of small, non-zero solutions addressed by power-series methods or iterative procedures [53]. Then, with a thorough understanding of the role of the implicit function theorem in bifurcation theory [18], came recognition that the small amplitude theory for periodic waves is a special case of bifurcation from a simple eigenvalue. (The theory of periodic waves on water of finite depth is similar to that for infinite depth [5], but the equally important question of the existence of solitary waves, which Scott-Russell had discovered empirically on a canal in Scotland, is not quite so straightforward [5], [6], [11], [23].)

The global theory of periodic waves began when Krasovskii [38] recognised that Levi-Civita's formulation of the water-wave problem could be regarded as a nonlinear operator equation for elements of the cone of non-negative functions in a Banach space. He applied Krasnosel'skii's [37] non-constructive topological methods for positive operator equations to prove the existence of a set of waves where the maximum angle to the horizontal takes all values from zero up to, but not including, $\pi/6$. The limiting value $\pi/6$ was particularly suggestive because Stokes [54] had postulated that the form of steady waves is limited by an extreme wave with two separate tangents making an angle of $\pi/6$ with the horizontal at its crest. Also, there was a seemingly natural obstacle (connected with his use of Zygmund's theorem in conjugate function theory, [64], Ch.VII, (2.11)) to an extension of his proof to obtain waves with greater maximum angle to the horizontal. Krasovskii was therefore led to conjecture that his set contained all waves which exist.

With the invention by Rabinowitz [50] of abstract global bifurcation theory, and its refinement for positive operators by Dancer [20] and Turner [60], a further attack on the periodic water-wave problem by Keady and Norbury [31], who also worked in the cone of non-negative functions, was possible. They used Nekrasov's formulation and inferred the existence of a global *connected* set of solutions which contains waves of all maximum angles from zero up to, but not including, $\pi/6$. The Nekrasov formulation and global bifurcation theory led to the further conclusion that this set contains waves with flow speeds at the wave crests, relative to the wave speed, arbitrarily close to zero. The picture was further refined when the existence of a solution of Nekrasov's equation corresponding to a wave with zero relative flow speed at its crest was shown to lie in the closure (in an obvious sense) of the continuum which Keady and Norbury had found [43], [57]. Such waves are said to have stagnation points at their crests and, as was proved in [3], there is a corner at each crest with a contained angle of 120 degrees. Thus Stokes' conjecture about the existence of a wave of extreme form was proven. Earlier McLeod [43] had proved that there are waves with maximum angle to the horizontal strictly greater than $\pi/6$. Thus Krasovskii's conjecture is false.

In this paper we give complete proofs of the main results about the Stokes-wave problem starting with its basic formulation as a free boundary problem for a harmonic function in an unknown domain in the plane. At the outset only the natural smoothness required to state the problem is assumed. (Indeed the boundary is supposed only to be continuously differentiable, so that the usual form of the Hopf boundary-point lemma [22], [25], [49], is not available for the initial analysis; see however [25], page 46.) Lewy's theorem shows that the boundary must be a real-analytic curve and that the complex potential must have an analytic extension across the boundary. With such regularity up to the boundary the equations can be manipulated at will and various estimates on the wave slope and speed emerge from calculations involving the Maximum Principle for harmonic functions in the plane. These matters are dealt with in Sections 2, 3, 4 and 5, where the question of the existence of non-trivial Stokes waves is ignored. (Uniform horizontal flow with any speed c is a trivial solution of the free boundary-value problem.)

To formulate the boundary-value problem in a way which is amenable to existence theory we follow Levi-Civita and Nekrasov who, in the 1920s, used a hodograph transformation to map the unknown domain occupied by the water into a fixed semi-infinite strip in a complex plane where the variable is the complex potential of the fluid flow. As a function of this new independent variable, the velocity field of the fluid is written in polar co-ordinates and the angular variable θ is regarded as an unknown harmonic function on the strip which satisfies nonlinear boundary conditions. In Levi-Civita's treatment the nonlinear boundary conditions involve both θ and its complex conjugate (see **(N3′)** in Section 6), whereas Nekrasov manipulated the equations to eliminate the conjugate function and expressed the problem in terms of θ alone (see **(N3)**, Section 6). In Section 7 the problem is formulated as Nekrasov's integral equation in a cone in a Banach space of continuous functions. The cone involved [5] is a subset of the cone of 2π-periodic, odd functions which are non-negative on $(0,\pi)$, the elements of which enjoy two particular properties: $\theta(t)/\sin(t/2)$ is non-negative and non-increasing of $(0,\pi)$ and θ is decreasing on $(\pi/2,\pi)$. The latter is important because of its implication for the automatic convexity of a portion of the wave's surface, bearing in mind that θ is the angle between the free surface and the horizontal. The former is important for compactness reasons. Helly's Selection Theorem [56] says that any bounded sequence of monotone functions has a subsequence which converges everywhere, and hence is compact in L_p spaces on a bounded interval. The use of the cone K therefore obviates the need for the Ascoli-Arzela theorem to obtain various compactness results. In particular, gradient estimates in some earlier versions of the theory are systematically replaced here by arguments based on membership of K. (It is interesting to observe that the proof that K is invariant under the nonlinear operator which arises in Nekrasov's equation involves the Maximum Principle for a non-self-adjoint elliptic operator different from the Laplacian [22], [25], [49].) A global existence theory for Nekrasov's integral equation in K now follows from classical global bifurcation theory, and the existence of a solution corresponding to a wave with a stagnation point at its crest is a trivial consequence of working in K and Helly's Selection Theorem. (Here the existence theory in K greatly simplifies the proofs in [43], [57].)

The paper then turns to an examination of the behaviour of large amplitude Stokes waves whose existence has been established. We give the complete proofs which McLeod and Amick used to establish that there are waves with maximum angle to the horizontal greater than $\pi/6$ ($=0.5236$) but that the maximum angle is bounded above by 0.5434. In the present treatment of McLeod's lower-bound result there are minor simplifications due to membership of K. In proving the upper-bound results, the role of a particular harmonic function $V(\mu,\theta)$ is taken for granted even though it is is far from obvious and its definition by Amick is at the heart of his deep contribution to this problem. The proof here is further greatly simplified by seeking *a priori* bounds only for solutions in a continuum which bifurcates from the trivial solution. The power of continuation arguments in alliance with Maximum Principles is well illustrated by the technique for obtaining *a priori* bounds for this continuum. (See [24] for an alliance of continuation arguments with the Maximum Principle with entirely different effect.)

Section 12 considers the solution of Nekrasov's equation which gives a wave with a stagnation point at its crest. Armed with Amick's estimates and using the properties derived from membership of K, a proof of the Stokes conjecture which differs only in some details from the original [3] completes this treatment of Stokes waves on deep water.

Although this global theory might seem to offer a good account of steady, periodic water waves, there is strong numerical evidence that the problem is a great deal more complex. The final section of the paper gives a very brief description of how our present understanding fails to predict numerically-observed secondary and subsequent bifurcations, and points to the need for further research.

2 The Water-Wave Problem

In its simplest form the water-wave problem concerns two dimensional motion of a perfect liquid with a free surface, acted on by gravity and surface tension. Suppose, for definiteness, that in Cartesian co-ordinates gravity acts in the negative y-direction and that the liquid at rest occupies the region $\{(x,y) : -\hat{h} < y < 0\}$, for some fixed $\hat{h}$ which may be infinite. In motion suppose for the moment that the liquid surface $S(t)$ at time t is the graph of a function:

$$S(t) = \{(x, h(x;t)) : -\infty < x < \infty\} \ .$$

The velocity field is supposed to be the gradient of a time-dependent velocity potential $\phi(\cdot;t)$ which is required, by Bernoulli's theorem and the natural kinematic boundary conditions, to satisfy the following for $t \in \mathbb{R}$:

$$\begin{aligned}
\triangle\phi(x,y;t) = 0,\ -\hat{h} < y < h(x;t),\ x \in \mathbb{R}\ ;\\
\phi_y(x,-\hat{h};t) = 0 \ \text{ if } \hat{h} < \infty\ ;\\
\nabla\phi(x,y;t) \to (0,0) \ \text{ as } y \to -\infty \ \text{ if } \hat{h} = \infty\ ;\\
h_t(x;t) + h_x(x;t)\phi_x(x,h(x;t);t) - \phi_y(x,h(x,t);t) = 0\ ;\\
\phi_t(x,h(x;t);t) + \frac{1}{2}|\nabla\phi(x,h(x;t);t)| + gh(x;t)\\
-\frac{\sigma h_{xx}(x;t)}{(1+h_x(x;t)^2)^{3/2}} = \ \text{const},\quad x \in \mathbb{R}\ .
\end{aligned}$$

Here g denotes the acceleration due to gravity, σ is the coefficient of surface tension, subscripts, such as ϕ_x, ϕ_{xy}, denote partial derivatives, $\nabla\phi = (\phi_x, \phi_y)$ and $\triangle\phi = (\phi_{xx} + \phi_{yy})$.

A steady wave is one in which the wave profile and the velocity field are both stationary with respect to a frame of reference in uniform horizontal motion. To find such a solution let

$$\phi(x,y;t) = \Phi(x-ct, y) \text{ and } h(x,t) = H(x-ct)\ .$$

Then Φ and H are required to satisfy the following equations:

$$\begin{aligned}
\triangle\Phi(x,y) = 0,\ -\hat{h} < y < H(x),\quad x \in \mathbb{R}\ ;\\
\Phi_y(x,-\hat{h}) = 0 \text{ if } \hat{h} < \infty,\quad x \in \mathbb{R}\ ;\\
\nabla\Phi(x,y) \to (0,0) \text{ as } y \to -\infty \text{ if } \hat{h} = \infty,\quad x \in \mathbb{R}\ ;\\
(\Phi_x(x,H(x)) - c)H'(x) - \Phi_y(x,H(x)) = 0,\quad x \in \mathbb{R}\ ;\\
-c\Phi_x(x,H(x)) + \frac{1}{2}|\nabla\Phi(x,H(x))|^2 + gH(x)\\
-\frac{\sigma H''(x)}{(1+H'(x)^2)^{3/2}} = \ \text{const.}\ ,\quad x \in \mathbb{R}\ .
\end{aligned}$$

Here prime denotes differentiation with respect to the single variable, x.

When $\sigma > 0$ and $g = 0$ this is the *capillary-wave* problem and families of solution are known in closed form [19]. When $\sigma > 0$ and $g > 0$ this is the *capillary-gravity* wave problem and there is a huge and growing literature (see [4]), [12], [15], [28], [29], [30], [55], [59]). The case $\sigma = 0$ and $g > 0$ is the steady *gravity-wave* problem of which Stokes waves are the classical periodic solutions.

3 Stokes Waves on Deep Water

Historically speaking, the theoretically most important steady wave problem is that which corresponds to symmetric, periodic waves travelling on the surface of water which is at rest at infinite depth. Because of important numerical and theoretical evidence about the existence of periodic waves of different types [9], [17], [51], it is important to define carefully at the outset what in the sequel will be referred to as a Stokes wave. Intuitively these are symmetric waves whose profile rises and falls exactly once per wavelength. It is, however, unnecessary to make further assumptions about its form; in particular there is no need to assume *a priori* that the free surface is the graph of a function. In the preceding discussion define a (new) function ϕ by

$$\phi(x, y) = \Phi(x, y) - cx \ .$$

This reduces the problem to one of steady waves on the surface of a liquid moving uniformly at infinite depth from right to left with speed c and the function ϕ will be referred to as the relative-velocity potential. Suppose the waves have period 2λ.

The two components of the Stokes-wave problem are (I) the free surface in non-parametric form and (II) the relative stream function ψ which is a harmonic conjugate of the relative-velocity potential.

I Suppose that $t \to (X(t), Y(t))$ is a continuously differentiable function with the following properties:

(A) $(X(t), Y(t)) = (-X(-t), Y(-t)) = (X(t + 2\lambda) - 2\lambda, Y(t + 2\lambda)), \ t \in \mathbb{R}$;

(B) $X(0) = Y(0) = 0$;

(C) $|X'(t)| + |Y'(t)| > 0$ and $Y'(t) \geq 0, \quad t \in [-\lambda, 0]$;

(D) $(X(t_1), Y(t_1)) = (X(t_2), Y(t_2))$ only if $t_1 = t_2$.

Let

$$S = \{(X(t), Y(t)) : t \in \mathbb{R}\} \ .$$

Such a C^1-curve S has unbounded horizontal and bounded vertical extent and separates the plane into two unbounded components. By **I(A)** each of the lines $\{x = k\lambda\}$, $k \in \mathbb{Z}$, is a line of symmetry of S, and by **I(D)** each of these lines of symmetry intersects S exactly once. Also by **I(D)** and **I(A)**, $X'(0) > 0$. Let Ω denote the component of $\mathbb{R}^2 \backslash S$ which contains all points with $-y$ sufficiently large. The outward normal to Ω at $(X(t), Y(t)), t \in \mathbb{R}$, is therefore $(-Y'(t), X'(t))$, by continuity since this is so at $t = 0$. The domain Ω is then required to accommodate the following elliptic boundary-value which, *per se*, would be over determined if the domain were independently prescribed *a priori*.

II There exists a function ψ such that

(A) $\psi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and $\triangle\psi = 0$ on Ω ;

(B) $\psi \geq 0$ on Ω and $\psi = 0$ on S ;
(C) $\psi(x,y) = \psi(x+2\lambda, y) = \psi(-x,y),\ (x,y) \in \Omega$;
(D) $\frac{1}{2}|\bigtriangledown \psi|^2 + gy = \frac{1}{2}Q^2$ on S, where $|\bigtriangledown \psi(0,0)| = Q > 0$;
(E) $\bigtriangledown\psi(x,y) \to (0,-c)$ as $y \to -\infty$ uniformly for $x \in \mathbb{R}$.

Note that $\psi \in C^1(\bar{\Omega})$ means that the gradient of ψ has a continuous extension to $\bar{\Omega}$ and the limiting values are required to satisfy **II(D)**. Of course, being harmonic, ψ is a real-analytic function in the open set Ω. Note also that from **I(A)-(C)** it follows that $y \leq 0$ on S and hence, by **II(D)**, $|\bigtriangledown \psi| > 0$ on S. This observation enables one to prove in the next Section that S is smoother than merely C^1; it is a real analytic curve and the function ψ can be extended as a harmonic function on an open neighbourhood of $\bar{\Omega}$. (This result is due to Lewy [40].)

Once these results have been proved, further properties of Stokes waves follow quickly using the Maximum Principle and the Hopf boundary-point lemma.

4 Regularity of the Free Surface

Let $(x_0, y_0) \in S$. Now S is a C^1-curve and $\psi \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ is harmonic in Ω. Let ϕ be the harmonic conjugate of ψ in Ω, defined by integrating the Cauchy-Riemann equations such that $\phi + i\psi$ is analytic in Ω, with $\phi(x_0,y_0) = 0$. Then $\phi \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$. By a standard construction (see [21]) and the references therein) there is a disc B centred at (x_0,y_0) and functions $\hat{\phi}, \hat{\psi}$ in $C^1(B)$ such that $\hat{\phi} = \phi, \hat{\psi} = \psi$ in $B \cap \bar{\Omega}$, $\hat{\psi} < 0$ in $B \backslash \bar{\Omega}$ and $(\hat{\phi}, \hat{\psi})(x_0,y_0) = (0,0)$. Furthermore, since $\bigtriangledown\hat{\psi}(x_0,y_0)$ is non-zero, by the inverse function theorem there is a disc D centred at the origin in $\mathbb{R}^2$ and a continuously differentiable function $F : D \to B$ such that $F(0,0) = (x_0,y_0)$ and $(\hat{\phi}(F(\rho,\eta)), \hat{\psi}(F(\rho,\eta)) = (\rho,\eta)$ for all $(\rho,\eta) \in D$. Let $F(\rho,\eta) = (u(\rho,\eta), v(\rho,\eta))$. Then $(u(\rho,\eta), v(\rho,\eta)) \in \Omega$ if and only if $\eta > 0$ and $(u(\rho,0), v(\rho,0))$ is a parametrisation of S in a neighbourhood of (x_0,y_0).

Let $\hat{D} = \{(\rho,\eta) \in D : \eta > 0\}$. Then $(u+iv)|_{\hat{D}}$ is the inverse of $(\phi + i\psi)|_{B\cap\Omega}$ and hence u and v are harmonic functions on $\hat{D}$. Also

$$u_\eta = -v_\rho = \frac{\psi_x}{|\bigtriangledown \psi|^2} \text{ and } v_\eta = u_\rho = \frac{\psi_y}{|\bigtriangledown \psi|^2} \ ,$$

and u and v have C^1-extensions onto $\eta = 0$ in $\overline{\hat{D}}$. Moreover, the boundary condition **II(D)** implies that

$$\frac{1}{2|\bigtriangledown v(\rho,0)|^2} + gv(\rho,0) = \frac{1}{2}Q^2 \text{ when } (\rho,0) \in \overline{\hat{D}} \ .$$

If $v_\eta(0,0) \neq 0$ then

$$\frac{\partial v}{\partial \eta}(\rho,0) = A(v(\rho,0), \frac{\partial v}{\partial \rho}(\rho,0)), \quad (\rho,0) \in \overline{\hat{D}} \ ,$$

where A is an analytic function defined in a neighbourhood of $(v(0,0), v_\rho(0,0))$ by

$$A(a,b) = \pm\left\{\frac{1}{Q^2 - 2ga} - b^2\right\}^{\frac{1}{2}} \ .$$

(Note $Q^2 - 2v(0,0) \neq v_\rho(0,0)^{-2}$ since $v_\eta(0,0) \neq 0$, and the choice of $\pm$ is determined by the sign of $v_\eta(0,0)$.) On the other hand, when $v_\eta(0,0) = 0$ (which occurs when the tangent to the free surface is vertical)

$$\frac{\partial u}{\partial \eta}(\rho, 0) = A\left(v(\rho,0), \frac{\partial u}{\partial \rho}(\rho,0)\right), \ (\rho, 0) \in \overline{\hat{D}}$$

where now the real-analytic function A is defined in a neighbourhood of $(v(0,0), v_\eta(0,0))$.

The following theorem of Lewy [40] shows that there is a disc $\tilde{D}$ centred at $(0,0)$ and a complex analytic function $\tilde{u} + i\tilde{v}$ defined on $\tilde{D}$ which coincides with $u + iv$ on $\overline{\hat{D}} \cap \tilde{D}$. Thus $(u(\rho,0), v(\rho,0))$ is a real-analytic function of ρ in an open neighbourhood of 0. Since $(u(\rho,0), v(\rho,0))$ is a local parametrisation of S in a neighbourhood of (x_0, y_0) it follows that S is a real-analytic curve. The harmonic extension of ϕ, ψ across S may now be obtained by taking the inverse of $\tilde{u} + i\tilde{v}$ in a neighbourhood of (x_0, y_0). It remains to state and prove the result upon which all this depends.

Theorem 1 (LEWY) *Let D denote a half-disc $\{(x,y) : |x|^2 + |y|^2 < r^2, \ y > 0\}$. Suppose that $F = U + iV$ is a complex analytic function on D such that both $U, V \in C^1(\bar{D})$ and suppose that*

$$U_y(x,0) = \mathcal{A}(U(x,0), V(x,0), U_x(x,0)), \quad |x| < r \ ,$$

where $\mathcal{A}$ is a complex-valued analytic function of all its arguments in a neighbourhood of $(U(0,0), V(0,0), U_x(0,0))$ in $\mathbb{C}^3$ which is real when its arguments are real. Then there exists a disc $\tilde{D}$ centred at $(0,0)$, and an analytic function $\tilde{U} + i\tilde{V} : \tilde{D} \to \mathbb{C}$ such that

$$U + iV = \tilde{U} + i\tilde{V} \ \text{ on } D \cap \tilde{D} \ .$$

Proof. Let

$$U(0,0) = \alpha_0, \ U_y(0,0) + iU_x(0,0) = \beta_0, \ iU_x(0,0) = \gamma_0, \ iV(0,0) = \delta_0$$

and let

$$\mathcal{F}(\alpha, \beta, \gamma, \delta) = \gamma - \beta + \mathcal{A}(\alpha, -i\delta, -i\gamma) \ .$$

Then

$$\mathcal{F}(\alpha_0, \beta_0, \gamma_0, \delta_0) = 0$$

and

$$\text{Real } \frac{\partial \mathcal{F}}{\partial \gamma}(\alpha_0, \beta_0, \gamma_0, \delta_0) = 1$$

since $\mathcal{A}$ is real when its argument is real. Therefore, by the implicit function theorem, the equation $\mathcal{F}(\alpha, \beta, \gamma, \delta) = 0$ may be solved to give γ uniquely in a neighbourhood of γ_0 in $\mathbb{C}$ as an analytic function of (α, β, δ) in a neighbourhood of $(\alpha_0, \beta_0, \delta_0)$ in $\mathbb{C}^3$. Let us denote that functional dependence by

$$\gamma = \Gamma(\alpha, \beta, \delta) \text{ where } \gamma_0 = \Gamma(\alpha_0, \beta_0, \delta_0) \ .$$

Now use Γ to construct a complex-valued function W in a rectangle $[-\hat{\delta},\hat{\delta}]\times[0,\hat{\delta}]$, for some $\hat{\delta}>0$, as follows. First let $W(0,y), y\in[0,\hat{\delta}]$ for some $\hat{\delta}>0$ sufficiently small, be the solution of the initial-value problem

$$\begin{aligned}\frac{d}{dy}(W(0,y)) &= \Gamma(W(0,y),F_y(0,y),F(0,y)-W(0,y))\\ W(0,0) &= U(0,0)\ ,\end{aligned}$$

where $F(x,y)=U(x,y)+iV(x,y)$. Then let $W(x,y), x\in[-\hat{\delta},\hat{\delta}]$ be given by the solution of the initial-value problem (for fixed $y\in[0,\hat{\delta}]$)

$$\frac{d}{dx}(W(x,y)) = -i\Gamma(W(x,y),iF_x(x,y),F(x,y)-W(x,y))\ .$$

It is clear, by the standard theory of initial-value problems, that this construction defines a continuous function W on a closed rectangle $[-\hat{\delta},\hat{\delta}]\times[0,\hat{\delta}]$ for some $\hat{\delta}>0$ sufficiently small. Now let $\hat{y}=\hat{\delta}/2$ and consider the complex initial-value problem

$$\begin{aligned}\frac{dw}{dz} &= -i\Gamma\left(w(z),iF'(z),\ F(z)-w(z)\right)\\ w(i\hat{y}) &= W(0,\hat{y})\ .\end{aligned}$$

This equation is locally solvable. Also, restricted to the line $x=0$ it coincides with the equation for $W(0,y)$ and restricted to a line $y=c\in[0,\hat{\delta}]$ it coincides with the equation for $W(x,c)$. Hence $w(x+iy)$ is defined for $-\hat{\delta}<x<\hat{\delta},\ 0<y<\hat{\delta}$ and coincides with W there. Hence w has a continuous extension to the line segment $-\hat{\delta}<x<\hat{\delta},\ y=0$.

Now by hypothesis,

$$0=\mathcal{A}(U(x,0),-i(F(x,0)-U(x,0)),-i(iU_x(x,0)))-iF_x(x,0)+iU_x(x,0)$$

and hence

$$iU_x(x,0)=\Gamma(U(x,0),iF_x(x,0),F(x,0)-U(x,0)),\ \ x\in(-\hat{\delta},\hat{\delta})$$

for $\hat{\delta}>0$ sufficiently small. Since $W(0,0)=U(0,0)$ it follows that

$$W(x,0)=U(x,0)\text{ for }x\in(-\hat{\delta},\hat{\delta})$$

and, in particular, the continuous extension of the complex analytic function w to the line $\{y=0,x\in(-\hat{\delta},\hat{\delta})\}$ is real and coincides with U on $y=0$.

For $(x,y)\in(-\hat{\delta},\hat{\delta})\times(-\hat{\delta},0)$ let

$$\tilde{F}(x+iy)=2\overline{w(x-iy)}-\overline{F(x-iy)}\ .$$

Then $\tilde{F}$ is a complex analytic function on its domain of definition and it has a continuous extension onto the line $(-\hat{\delta},\hat{\delta})\times\{0\}$, namely

$$\tilde{F}(x+i0)=U(x,0)+iV(x,0)\ .$$

By Morera's theorem $\tilde{F}$ is the analytic continuation of F across the line $y=0$, and the proof is complete. $\square$

5 Properties of Stokes Waves

With the knowledge that S is real-analytic we may assume henceforth that **I (A)-(D)** are satisfied by real-analytic functions X and Y and that ψ belongs to $C^\infty(\bar{\Omega})$. The primary goal now is to show that $X'(t) \neq 0,\ t \in \mathbb{R}$, and $Y'(t) = 0$ if, and only if, $t = k\lambda,\ k \in \mathbb{Z}$, and hence that the free surface S is the graph of a function. This result, which is due to Speilvogel [52], follows from the fact that the pressure P is a sub-harmonic function. Here

$$P(x, y) = \frac{1}{2}|\nabla \psi(x, y)|^2 + gy - \frac{1}{2}Q^2,\ (x, y) \in \Omega\ .$$

Then $P = 0$ on S, by **II(D)**, $P \in C^\infty(\bar{\Omega})$ by Lewy's Theorem, and

$$\triangle P(x, y) = |\nabla \psi_x|^2 + |\nabla \psi_y|^2 \geq 0 \text{ on } \Omega\ .$$

Since

$$P(x, y) \to -\infty \text{ as } y \to \infty$$

by **II(E)**, the Maximum Principle gives that $P < 0$ on Ω and

$$(-Y'(t), X'(t)).\nabla P(X(t), Y(t)) = c(t) > 0, \quad t \in \mathbb{R}\ , \tag{1}$$

by the Hopf boundary-point lemma. Since P is zero on S

$$(X'(t), Y'(t)).\nabla P(X(t), Y(t)) = 0,\ t \in \mathbb{R} \tag{2}$$

and hence

$$(X'(t)^2 + Y'(t)^2)P_x(X(t), Y(t)) = -c(t)Y'(t), \quad t \in \mathbb{R}\ .$$

Moreover ψ is harmonic,

$$X'(t)\psi_x(X(t), Y(t)) + Y'(t)\psi_y(X(t), Y(t)) = 0, \quad t \in \mathbb{R}\ , \tag{3}$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\psi_y^2(X(t), Y(t))\right) & (4)\\
= \;& \psi_y(X(t), Y(t))\left\{\psi_{xy}(X(t), Y(t))X'(t) + \psi_{yy}(X(t), Y(t))Y'(t)\right\}\\
= \;& X'(t)\{\psi_y(X(t), Y(t))\psi_{xy}(X(t), Y(t)) + \psi_x(X(t), Y(t))\psi_{xx}(X(t), Y(t))\}\\
= \;& X'(t)\frac{\partial}{\partial x}\left(\frac{1}{2}|\nabla \psi|^2\right)\Big|_{(X(t), Y(t))} = X'(t)P_x(X(t), Y(t))\\
= \;& \frac{-X'(t)Y'(t)c(t)}{X'(t)^2 + Y'(t)^2}, \quad t \in \mathbb{R}\ ,
\end{aligned}$$

where $c > 0$ is a real-analytic function.

Now **I(A)**,**(C)** give $Y'(0) = 0$ and $X'(0) > 0$, and $\psi_y(0, 0) = \psi_y(X(0), Y(0)) < 0$ by the Hopf boundary-point lemma. Let $(a, 0]$ be the maximal sub-interval of $(-\lambda, 0]$

upon which $X'(t) > 0$. Therefore, since $Y'(t) \geq 0$ on $[-\lambda, 0]$ by **I(C)**, it follows from (4) that

$$\psi_y^2(X(a), Y(a)) \geq \psi_y^2(X(0), Y(0)) > 0 \ .$$

If $a > -\lambda$ then $X'(a) = 0$ and $Y'(a) \neq 0$ because of the definition of a and **I(C)**. But (3) holds because ψ is constant on S. Evaluated at $t = a$ this gives a contradiction. We conclude that $X'(t) > 0$ on $(-\lambda, 0]$. Since $X'(-\lambda) \neq 0$ because $Y'(-\lambda) = 0$, this shows $X' \neq 0$ on $[-\lambda, 0]$. That $X'(t) \neq 0$ for all $t \in \mathbb{R}$ is then immediate from **I(A)**. Hence

$$X'(t) > 0, \quad t \in \mathbb{R} \ . \tag{5}$$

Since ψ is harmonic and has its minimum on $\bar{\Omega}$ at every point of S the Hopf boundary-point lemma and the fact that $X'(t) > 0, \quad t \in \mathbb{R}$, together give that

$$\psi_y(X(t), Y(t)) < 0, \quad t \in \mathbb{R} \ . \tag{6}$$

Hence, by (3) and **I(C)**,

$$\psi_x(X(t), Y(t)) \geq 0, \quad t \in (-\lambda, 0) \ . \tag{7}$$

Now let

$$\Omega_\lambda = \{(x, y) \in \Omega : -\lambda < x < 0\} \ .$$

Then ψ_x is harmonic in Ω_λ, $\psi_x \to 0$ as $y \to -\infty$ and $\psi_x \geq 0$ on $\partial\Omega_\lambda$ by **II(C)**. Hence

$$\psi_x > 0 \text{ on } \Omega_\lambda \ . \tag{8}$$

Now suppose that $\psi_x(X(t_0), Y(t_0)) = 0, \ \ t_0 \in (-\lambda, 0)$. Then, by the Hopf boundary-point lemma, $\psi_{xy}(X(t_0), Y(t_0)) < 0$. Note from (3) and (6) that $Y'(t_0) = 0$ and hence from **I(C)** and (2) that

$$0 = P_x(X(t_0), Y(t_0)) = \psi_y(X(t_0), Y(t_0))\psi_{xy}(X(t_0), Y(t_0)) \ .$$

This is a contraction. Hence

$$\psi_x(X(t), Y(t) > 0 \text{ and } Y'(t) > 0, \quad t \in (-\lambda, 0) \ . \tag{9}$$

From the regularity of the wave profile and the inequalities (5) and (9), there results that the following version of the Stokes-wave problem is simpler than, and equivalent to, the original.

S(a) There is a 2λ-periodic, real-analytic, even function h such that

$$\begin{aligned} h(x) &< h(0) = 0 \text{ if } x \neq 0 \bmod (\lambda) \ ; \\ h'(x) &> 0 \text{ for } x \in (-\lambda, 0) \ . \end{aligned}$$

Thus $|h(-\lambda)|$ is the *amplitude of the wave.* Let

$$\begin{aligned} S = \{(x, h(x)) : x \in \mathbb{R}\} \quad &\text{and} \quad S_\lambda = \{(x, h(x)) : -\lambda < x < 0\} \ , \\ \Omega = \{(x, y) : y < h(x) : \ x \in \mathbb{R}\} \quad &\text{and} \quad \Omega_\lambda = \{(x, y) \in \Omega : -\lambda < x < 0\} \ . \end{aligned}$$

Then there exists a function ψ as follows:

S(b) $\triangle\psi = 0$ in Ω;

S(c) $\psi \in C^\infty(\bar{\Omega})$;

S(d) $\psi > 0$ in Ω, $\quad \psi = 0$ on $\partial\Omega$;

S(e) $\psi(-x,y) = \psi(x+2\lambda,y) = \psi(x,y),\ (x,y) \in \bar{\Omega}$;

S(f) $\frac{1}{2}|\bigtriangledown\psi(x,h(x))|^2 + gh(x) = \frac{1}{2}Q^2, \quad x \in \mathbb{R}$, where $Q = |\bigtriangledown\psi(0,0)| > 0$;

S(g) $\bigtriangledown\psi(x,y) \to (0,-c)$ as $y \to -\infty$, uniformly for $x \in \mathbb{R}$.

We summarise established properties of the solution and infer some others.

(i) $\psi_x > 0$ in $\Omega_\lambda \cup S_\lambda$, by (8) and (9);

(ii) $\psi_y < 0$ on $\bar{\Omega}$. This is true on S by (6) and therefore on Ω by the Maximum Principle for the harmonic function ψ_y on Ω ($\psi_y \to -c$ as $y \to -\infty$).

(iii)
$$\begin{aligned} 0 < \psi_{xx}(\pm\lambda,y) &= -\psi_{yy}(\pm\lambda,y),\ (\pm\lambda,y) \in \Omega\ , \\ 0 > \psi_{xx}(0,y) &= -\psi_{yy}(0,y),\ (0,y) \in \Omega\ , \end{aligned}$$

by **(i)** and the Hopf boundary-point lemma. (Note that $\psi_x(\pm\lambda,y) = \psi_x(0,y) = 0$ if $(\pm\lambda,y)$ or $(0,y) \in \Omega$, by evenness and periodicity of ψ.)

(iv)
$$\begin{aligned} \psi_y(\pm\lambda,y) &< -c, \quad (\pm\lambda,y) \in \bar{\Omega}\ , \\ 0 > \psi_y(0,y) &> -c, \quad (\pm\lambda,0) \in \bar{\Omega}\ , \\ |\psi_y(0,0)| &= Q < c\ . \end{aligned}$$

This is immediate from **(ii)**, **(iii)** and **S(g)**.

(v) The convergence of $\bigtriangledown\psi$ to $(0,-c,)$ is (uniformly in x) exponentially fast as $y \to -\infty$.

To see this let $A > 0$ and define a harmonic function W on $[-\lambda,0]\times(-\infty,h(-\lambda)]$ by
$$W(x,y) = A\sin\left(\frac{x}{\lambda}\right)e^{y/\lambda} - \psi_x(x,y)\ .$$
Choose A sufficiently large that
$$W(x,y) > 0 \text{ on } (-\lambda,0)\times\{h(-\lambda)\}\ .$$
Since
$$W(-\lambda,y) = 0 = W(0,y), \quad y < h(-\lambda)$$
and
$$W(x,y) \to 0 \text{ as } y \to -\infty$$
uniformly in x, the Maximum Principle gives
$$W(x,y) > 0 \text{ in } (-\lambda,0)\times(-\infty,h(-\lambda))\ .$$
Hence, by **(i)**
$$0 < \psi_x(x,y) < A\sin\left(\frac{x}{\lambda}\right)e^{y/\lambda} \text{ on } (-\lambda,0)\times(-\infty,h(-\lambda))\ .$$

Now it follows from classical elliptic estimates [25] that

$$|\psi_{yy}| = |\psi_{xx}| \leq \text{ const. } e^{y/\lambda}$$

as $y \to -\infty$, uniformly in x. Therefore, by $\mathbf{S(g)}$, $(\psi_x, \psi_y) \to (0, -c)$ (uniformly in x) exponentially in y as $y \to -\infty$.

To interpret the next result, first note that the origin of Cartesian co-ordinates has been specified uniquely (up to horizontal translations through the wave period) by being located at one of the wave crests. Note also that ψ has been normalised so that $\psi(0,0) = 0$.

(vi) There is a constant d such that

$$\frac{c}{2g}(c^2 - Q^2) \leq d < h(-\lambda)\psi_y(-\lambda, h(-\lambda)) = |h(-\lambda)|\ |\nabla\psi(-\lambda, h(-\lambda))|$$

and

$$\psi(x,y) + cy + d \to 0 \text{ as } y \to -\infty$$

uniformly in x and exponentially in y.

Since $\psi_y(x,y) + c \to 0$ and $\psi_x(x,y) \to 0$ exponentially as $y \to -\infty$ it follows that for some constant d

$$\psi(x,y) + cy + d \to 0$$

exponentially as $y \to -\infty$. Now by **(iii)**

$$\begin{aligned} 0 &< \int_{-\infty}^{h(-\lambda)} y \frac{\partial^2}{\partial y^2}(\psi(-\lambda, y) + cy + d)dy \\ &= h(-\lambda)\psi_y(-\lambda, h(-\lambda)) - d \\ &= |h(-\lambda)|\ |\nabla \psi(-\lambda, h(-\lambda))| - d\ . \end{aligned}$$

Let $P = \frac{1}{2}|\nabla \psi|^2 + gy - \frac{1}{2}Q^2$ and for $\epsilon > 0$ note that

$$\begin{aligned} \triangle\left(P + \frac{(g-\epsilon)\psi}{c}\right) &\geq 0 \text{ in } \Omega\ , \\ P + \frac{(g-\epsilon)\psi}{c} &= 0 \text{ on } \partial\Omega \end{aligned}$$

and

$$P(x,y) + \frac{(g-\epsilon)}{c}\psi(x,y) \to -\infty \text{ as } y \to -\infty\ .$$

Hence, for each $\epsilon > 0$, $P + \dfrac{(g-\epsilon)}{c} < 0$ on Ω and so

$$P(x,y) + \frac{g\psi(x,y)}{c} < 0 \text{ on } \Omega \tag{10}$$

and

$$P(x,y) + \frac{g\psi(x,y)}{c} = 0 \text{ on } \partial\Omega\ , \tag{11}$$

by the Maximum Principle. Hence, by (10) and **(v)**

$$\frac{1}{2}(c^2 - Q^2) - \frac{gd}{c} \leq 0 \ ,$$

which proves **(vi)**.

(vii) $$\frac{d}{dx}(\psi_y^2(x, h(x))) < 0, \quad x \in (-\lambda, 0) \ .$$

This is a re-iteration of (4) in the light of (5), (9) and **S(a)**. Since $\psi_y(x, h(x))$, which is the horizontal relative velocity field at the free surface, is negative, this says that the relative horizontal speed of the flow at the free surface is *decreasing* from the trough at $-\lambda$ to the crest at 0. Note also that $h'(x) > 0$ on $(-\lambda, 0)$ gives

(viii) $$\frac{d}{dx}| \bigtriangledown \psi(x, h(x))| < 0 \ , \quad x \in (-\lambda, 0) \ .$$

This is immediate from **S(f)** and **S(a)**.

(ix) $$\frac{d}{dx}\left(\psi_y(x, h(x)) - \frac{g}{c}h(x)\right) > 0 \ .$$

This follows since, by (10), (11) and the Hopf boundary-point lemma,

$$P_x(x, h(x)) + \left(\frac{g}{c}\right)\psi_x(x, h(x)) < 0, \quad x \in (-\lambda, 0) \ .$$

Therefore, by (4),

$$\frac{d}{dx}\left(\frac{1}{2}\psi_y^2(x, h(x))\right) = P_x(x, h(x))$$

$$< -\left(\frac{g}{c}\right)\psi_x(x, h(x)) = \left(\frac{g}{c}\right)\psi_y(x, h(x))h'(x) \ .$$

Since $\psi_y(x, h(x)) < 0, \quad x \in (-\lambda, 0)$, the result follows.

Since $h(0) = 0$ and $\psi_y(0, 0) = -Q$ it now follows that

$$\psi_y(x, h(x)) - \frac{g}{c}h(x) < -Q, \quad x \in [-\lambda, 0) \ . \tag{12}$$

(x) $$\max_{x \in \mathbb{R}} | \bigtriangledown \psi(x, h(x))| \leq 2c - Q \ .$$

Note first that by **S(f)**,

$$\max_{x \in \mathbb{R}} | \bigtriangledown \psi(x, h(x))| = | \bigtriangledown \psi(-\lambda, h(-\lambda)| = |\psi_y(-\lambda, h(-\lambda))| \ .$$

Also by **S(f)** and (12), for $x \in (-\lambda, 0)$,

$$\frac{1}{2}|(\psi_y(x, h(x))|^2 - \frac{1}{2}Q^2 \leq -gh(x) < -cQ - c\psi_y(x, h(x)) \ ,$$

whence

$$\frac{1}{2}|\psi_y(x, h(x))|^2 - c|\psi_y(x, h(x))| < \frac{1}{2}Q^2 - cQ \ .$$

Since $Q < |\psi_y(x, h(x))|$, an inspection of the graph of the quadratic $\frac{1}{2}Q^2 - cQ$ yields that $|\psi_y(x, h(x)| < 2c - Q$. Hence $|\psi_y(-\lambda, h(-\lambda))| < 2c - Q$ as required and

(xi) $$\frac{1}{2g}(c^2 - Q^2) < |h(-\lambda)| < \frac{2c}{g}(c - Q) .$$

The left-hand inequality follows from **(iv)** and **S(f)**.

(xii) It is appropriate to anticipate the observation that the angle between the free surface and the horizontal is bounded by 0.5434 radians. This is shown in general in [1] and in our Section 11 for waves on a connected set which bifurcates from the trivial flow. Also, we will see in Theorem 6(e), Section 8, that, for fixed λ, the possible speeds c of such waves are bounded above and below by positive constant proportional to $\sqrt{\lambda}$.

6 The Hodograph Transformation

We now turn to establishing the existence of solutions of the Stokes-wave problem **S(a)-(g)**. This involves a transformation of variables. Suppose that ψ solves **S(a)-(g)** and that $\phi + i\psi$ is analytic in Ω with $\phi(0,0) = 0$. Then, by the Cauchy-Riemann equations, $\phi(\pm\lambda, y)$ is independent of $y < h(-\lambda)$ and hence

$$\phi(\lambda, y) - \phi(-\lambda, y) = \int_{-\lambda}^{\lambda} \phi_x(x, y)dy = \int_{-\lambda}^{\lambda} \psi_y(x, y)dy = -2\lambda c,$$

by **S(g)**. Since $\phi(0, y)$ is independent of $y < 0$, $\phi(0, y) = 0$ for all $y < 0$. Let

$$\tilde{\Omega}_\lambda = \{x + iy, -\lambda < x < \lambda,\ y < h(x)\}$$

and let $f : \tilde{\Omega}_\lambda \to \mathbb{C}$ be given by

$$f(x + iy) = \frac{-\pi}{\lambda c}(\phi(x, y) + i\psi(x, y)) .$$

(Here, and elsewhere where it is convenient to do so, we exercise the right to identify points of the plane either by their Cartesian co-ordinates or as complex numbers described in Cartesian or polar co-ordinates. With this convention, subscripts denote partial derivatives of real-valued functions with respect to real co-ordinates of the independent variable in whatever co-ordinate system is indicated by the notation.) Then $f(\tilde{\Omega}_\lambda) = R$ where

$$R = \{\rho + i\eta : -\pi < \rho < \pi,\ \eta < 0\} .$$

Since $\phi_x = \psi_y < 0$ on $\tilde{\Omega}_\lambda$, it is clear that $f : \tilde{\Omega}_\lambda \to R$ is a conformal bijection. Let $\xi : R \to \tilde{\Omega}_\lambda$ be the inverse of f and let

$$(\tilde{\tau} + i\tilde{\theta})(\rho + i\eta) = \log f'(\xi(\rho + i\eta))$$

where the logarithm is defined on the complex plane cut along the negative real axis. Then it follows easily that

$$e^{\tilde{\tau}}(\cos\tilde{\theta} + i\sin\tilde{\theta})(\rho + i\eta) = \frac{-\pi}{\lambda c}(\phi_x(\xi(\rho + i\eta)) + i\psi_x(\xi(\rho + i\eta))) ,$$

whence

$$\begin{aligned}
e^{\tilde{\tau}(\rho+i\eta)} &= \frac{\pi}{\lambda c}|\nabla\phi(\xi(\rho+i\eta))| = \frac{\pi}{\lambda c}|\nabla\psi(\xi(\rho+i\eta))| \;; \\
e^{\tilde{\tau}(\rho+i\eta)} &\to \frac{\pi}{\lambda} \text{ as } \eta \to -\infty \;; \\
\tilde{\theta}(i\eta) &= \tilde{\theta}(\pi \pm i\eta) = 0, \quad \eta < 0 \;; \\
0 &< \theta(\rho) \; < \; \pi/2, \quad 0 < \rho < \pi \;; \\
-\tilde{\theta}(-\rho) &= \tilde{\theta}(\rho), \quad 0 < \rho < \pi \;.
\end{aligned}$$

It is important to note that $\tan\tilde{\theta}(\xi(\rho)) = -h'(\xi(\rho))$, $\rho \in (0,\pi)$ and so $\tilde{\theta}$ is *the acute angle (measured positive clockwise) between the free surface and the horizontal direction* $(-1,0)$ *at the points* $\xi(\rho) \in S$. The boundary condition $\mathbf{S(f)}$ transforms as follows:

$$\frac{1}{2}|\nabla\psi(\xi(\rho))|^2 = \frac{\lambda^2 c^2}{2\pi^2} e^{2\tilde{\tau}(\rho)}$$

and, by definition,

$$h(\text{Real } \xi(\rho)) = \text{ Imag } (\xi(\rho)) \;.$$

Hence, since $f(\xi(\rho)) = \rho$,

$$h(\text{Real } \xi(\rho)) = \text{Imag } \int_0^\rho (\xi'(\nu))d\nu$$

$$= \text{Imag } \int_0^\rho \frac{1}{f'(\xi(\nu))}\,d\nu = -\int_0^\rho e^{-\tilde{\tau}(\nu)} \sin\tilde{\theta}(\nu)d\nu \;.$$

Hence in the transformed variables, the nonlinear boundary condition $\mathbf{S(f)}$ becomes

$$\frac{\lambda^2 c^2}{2\pi^2}\, e^{2\tilde{\tau}(\rho)} - g\int_0^\rho e^{-\tilde{\tau}(\nu)} \sin\tilde{\theta}(\nu)d\nu = \frac{1}{2}Q^2$$

where $Q = \lambda c e^{\tilde{\tau}(0)}/\pi$. A differentiation with respect to ρ gives

$$\frac{\lambda^2 c^2}{\pi^2} e^{2\tilde{\tau}(\rho)}\tilde{\tau}_\rho(\rho) - g e^{-\tilde{\tau}(\rho)}\sin\tilde{\theta}(\rho) = 0, \;\; \rho \in (-\pi,\pi) \;,$$

and hence

$$\frac{1}{3}\frac{\lambda^2 c^2}{\pi^2 g} e^{3\tilde{\tau}(\rho)} = \frac{1}{3}\frac{\pi Q^3}{\lambda c g} + \int_0^\rho \sin\tilde{\theta}(\nu)d\nu \;.$$

A substitution in the previous expression now gives, since $\tilde{\tau} + i\tilde{\theta}$ is analytic,

$$\begin{aligned}
\tilde{\theta}_\eta(\rho + i0) &= \tilde{\tau}_\rho(\rho + i0) = \frac{g\pi^2}{\lambda^2 c^2} e^{-3\tilde{\tau}(\rho)} \sin\tilde{\theta}(\rho) \\
&= \frac{\sin\tilde{\theta}(\rho)}{3\left\{\frac{\pi Q^3}{3\lambda g c} + \int_0^\rho \sin\tilde{\theta}(\nu)d\nu\right\}} \;.
\end{aligned}$$

This argument is reversible. So suppose that $\tilde{\theta} \in C^1(\bar{R}) \cap C^2(R)$ is such that

$$\begin{aligned} \triangle\tilde{\theta} &= 0 \text{ on } R\,, \\ \tilde{\theta}(\rho + i\eta) &= -\tilde{\theta}(-\rho + i\eta) \text{ on } R\,, \\ \tilde{\theta}(\pm\pi + i\eta) &= 0, \quad \eta < 0\,, \\ \frac{\partial\tilde{\theta}}{\partial\eta}\Big|_{(\rho+i0)} &= \frac{1}{3}\,\frac{\sin\tilde{\theta}(\rho)}{\beta + \int_0^\rho \sin\tilde{\theta}(\nu)d\nu}\,, \quad \rho \in (-\pi,\pi)\,, \end{aligned}$$

for some $\beta > 0$. Then if λ and c are such that

$$\left\{\frac{3g\lambda}{\pi c^2}\right\}^{1/3} = \frac{1}{\pi}\int_0^\pi \frac{\cos\tilde{\theta}(\rho)}{\{\beta + \int_0^\rho \sin\tilde{\theta}(\nu)d\nu\}^{1/3}}d\nu \tag{13}$$

then there exists a Stokes wave of wavelength λ and speed c on a flow of infinite depth, and the speed of the flow at the crest is given by

$$Q = \left(\frac{3g\lambda c\beta}{\pi}\right)^{1/3}\,. \tag{14}$$

It follows easily that the function $\tilde{\theta} : R \to \mathbb{R}$ has a real-analytic extension as a harmonic function on the lower half-plane $H = \{(\rho,\eta) : \eta < 0\}$ which is odd and 2π-periodic in the ρ-direction. (This follows because the extension by reflection and periodicity is weakly harmonic and therefore classically harmonic on the lower half plane.)

For calculations later it is slightly more convenient to map R onto the unit disc. In polar co-ordinates let

$$D = \{re^{is} : 0 \le r < 1,\ -\pi < s \le \pi\}$$

and let

$$(\tau + i\theta)(re^{is}) = -(\tilde{\tau} + i\tilde{\theta})(-s + i\log r)\,, \quad re^{is} \in D\,.$$

Then

(N1) $\triangle\theta = 0$ in D.
(N2) $\theta \in C^1(\bar{D}) \cap C^\infty(D)$.
(N3) $\displaystyle \frac{\partial\theta}{\partial r}\Big|_{e^{is}} = \frac{\sin\theta(e^{is})}{3(\beta + \int_0^s \sin\theta(e^{i\nu})d\nu)}\,, \quad s \in (-\pi,\pi]$.

We mention again the need for a relaxed attitude to the independent variable. At times it is convenient to regard $\theta(e^{is})$ simply as an odd function of $s \in \mathbb{R}$ which is zero at π, and a switch to the notation $\theta(s)$, as in the next Section, is appropriate. However, later functions on S^1 are extended as harmonic functions on the unit disc, in which case $\theta(re^{is})$ is a better notation.

Note that, in terms of the conjugate operator $\mathbf{C}$ on the unit circle S^1 ([10], [64])

$$\mathbf{C}(\theta)(e^{it}) = \frac{1}{3}\left\{\log(\beta + \int_0^t \sin\theta(e^{i\nu})\,d\nu) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(\beta + \int_0^t \sin\theta(e^{i\nu})\,d\nu)\,dt\right\}$$

and hence

(N3′) $\quad \dfrac{\partial \theta}{\partial r}\Big|_{e^{is}} = \dfrac{\alpha}{3} e^{-3(\mathbf{C}\theta)(e^{is})} \sin\theta(e^{is}),$

where

$$\alpha = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(\beta + \int_0^t \sin\theta(e^{i\nu})d\nu)\, dt\right\}.$$

The alternative **(N3′)** was the cornerstone of Krasovskii's theory [38], which was the first account of large amplitude Stokes waves and of Levi-Civita's [39] theory of small amplitude waves. But existence theory follows more easily from **(N3)** which Keady and Norbury [31] used in their version of Stokes-wave existence theory. What follows is a substantial refinement of that theory.

7 Nekrasov's Integral Equation

To reduce **(N1)-(N3)** to an integral equation consider the general Neumann problem

$$\triangle\, u = 0, \quad x \in D\,, \tag{15}$$

$$\lim_{(r,s)\to(1,s_0)} \frac{\partial}{\partial r} u(re^{is}) = f(e^{is_0}),\ s \in (-\pi, \pi]\,, \tag{16}$$

where $f \in C(S^1)$ and $f(e^{is})$ is an odd function of s. (Here $f \in C(S^1)$ means that f is continuous on the unit circle.) Then it is classical (it follows from [64], Ch.III, (6.13) and the Hopf boundary-point lemma) that (15) and (16) have a unique solution $u \in C^\infty(D)\cap C(\bar{\Omega})$, with $u(re^{is})$ odd in s, and the solution is given in closed form in the formula

$$u(re^{is}) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty} \frac{r^k \sin ks \sin kt}{k}\right) f(e^{it})\ dt\ ,$$

where on S^1 the formula

$$u(e^{is}) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k}\right) f(e^{it})\ dt$$

is valid. For $r \in [0,1]$, let

$$K_r(s,t) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{r^k \sin ks \sin kt}{k}\ .$$

Then

$$K_1(s,t) = \frac{1}{2\pi}\log\left|\frac{\sin \frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|\ ,\ (s,t) \in [-\pi,\pi]\times[-\pi,\pi]\ ,\ s \neq t\ .$$

Then if $f \in C(S^1)$ and $f(e^{is})$ is odd in s, and if

$$u(re^{is}) = \int_{-\pi}^{\pi} K_r(s,t) f(e^{it}) dt\ , \quad r \in [0,1]\ , \tag{17}$$

then u satisfies (15) and (16).

To find Stokes waves of all amplitudes it is therefore necessary and sufficient to find a function $\theta : [-\pi, \pi] \to \mathbb{R}$ and $\beta > 0$ with the following properties:

$$\theta \text{ is continuous and odd on } [-\pi, \pi] \; ; \tag{18}$$

$$\theta(0) = \theta(\pi) = 0; \tag{19}$$

$$\pi/2 > \theta(t) \geq 0, \quad t \in [0, \pi] \; ; \tag{20}$$

$$\theta(s) = \frac{1}{3} T\mathcal{N}(\beta, \theta)(s) \tag{21}$$

where

$$\mathcal{N}(\theta, \beta)(t) = \frac{\sin \theta(t)}{\beta + \int_0^t \sin \theta(\nu) d\nu}$$

and

$$Tf(s) = \int_{-\pi}^{\pi} K_1(s,t) f(t) dt \; ,$$

for odd, continuous functions f with $f(0) = f(\pi) = 0$. Note that if $f \in L_2(-\pi, \pi)$ is odd and has Fourier sine coefficients $\{a_n\}$ then the sine coefficients of Tf are $\{a_n/n\}$. Thus T maps the space of odd functions in $L_2(-\pi, \pi)$ compactly into the space of odd continuous functions on $[-\pi, \pi]$ with the supremum norm. A bootstrap argument then yields that every solution of (18)-(21) is a 2π-periodic, odd, infinitely differentiable function and hence, by Lewy's theorem and the connection with Stokes waves, it is real-analytic. The Stokes-wave problem is reduced to finding solutions of (18)-(21). Equation (21) is called Nekrasov's Integral Equation. Since similar nonlinear equations in continuum mechanics are known to have exact periodic solutions [7], [58], it is intriguing to speculate that Nekrasov's equation might also have an exact solution, at least for some values of β. (Note that if in Nekrasov's equation the term $1/3$ which multiplies the right-hand side is replaced by 1, then the odd function which is $(\pi - s)/2$ on $(0, \pi)$ is an exact solution of the altered equation with $\beta = 0$. See Lemma 5.) However, no exact solution of Nekrasov's equation is known, and further progress requires an existence theory. Happily a rather complete theory follows using cones in Banach spaces and global bifurcation theory, [20], [50].

8 Invariant Cones

Let

$$X = \{f : [-\pi, \pi] \to \mathbb{R} : f \text{ is continuous, odd, } f(0) = f(\pi) = 0\} \; ,$$

a Banach space with the supremum norm. Then $T : X \to X$ is a compact linear operator, as was seen at the end of Section 7.

Let

$$\begin{aligned} \hat{K} &= \{f \in X : f \geq 0 \text{ on } [0, \pi]\} \; ; \\ \tilde{K} &= \{f \in \hat{K} : f(t) \, / \, (\sin t/2) \text{ is non-increasing on } [0, \pi] \; ; \\ K &= \{f \in \tilde{K} : f(t) \leq f(s) \text{ for all } s \in [\pi - t, t], \text{ when } t \in [\pi/2, \pi]\} \; . \end{aligned}$$

Each of these sets is a closed, convex cone in the Banach space X. The following observations about elements of K will be useful in the sequel: for all $f \in K$

$$f \text{ is odd on } (-\pi, \pi) \; ; \tag{22}$$

$$f \geq 0 \text{ on } (0, \pi) \; ; \tag{23}$$

$$f \text{ is non-increasing on } (\pi/2, \pi) \; ; \tag{24}$$

$$f(t)/t \text{ is non-increasing on } (0, \pi) \; . \tag{25}$$

The following result is elementary.

Lemma 2 *Every sequence in $\tilde{K}$ which is bounded in X is compact in $L_p(-\pi, \pi)$, $1 \leq p < \infty$.*

Proof. Let $\{u_n\} \subset \tilde{K}$ be bounded in X and let $v_n(t) = u_n(t) / \sin(t/2)$. Then v_n is even and $\{v_n|_{[a,\pi]}\}$ is a sequence of bounded monotone functions for each $a > 0$. Therefore there is a subsequence $\{v_{n_k}\}$ which converges everywhere on $[a, \pi]$. (See e.g. Helly's Selection Theorem [56].) Now a diagonalisation argument gives a subsequence $\{u_{n_\ell}\}$ of $\{u_n\}$ which converges everywhere on $[0, \pi]$. Since $\{u_{n_\ell}\}$ is bounded the required result follows from the Dominated Convergence Theorem and the oddness of u_n. □

The goal is a proof that Nekrasov's equation has solutions in K; a first step is a proof that $T : K \to K$. We use the Maximum Principle although a direct proof based on the kernel K_1 is also possible.

Let $D^+ = \{re^{it} : 0 < r < 1, \quad 0 < t < \pi\}$, the open half-disc. If $f \in X$, then

$$u(re^{is}) = 2 \int_0^\pi K_r(s,t) f(t) \, dt \tag{26}$$

defines a function which is harmonic in D^+, continuous on $\overline{D^+}$, zero when $t = 0$ and π, and

$$\lim_{(r,s) \to (1,s_0)} \frac{\partial u}{\partial r}(re^{is}) = f(s_0), \;\; s_0 \in [0, \pi] \; . \tag{27}$$

Note also that

$$Tf(s_0) = \lim_{(r,s) \to (1,s_0)} u(re^{is}) \; . \tag{28}$$

Theorem 3 *The operator T on X maps $\hat{K}$ to $\hat{K}$, $\tilde{K}$ to $\tilde{K}$ and K to K.*

Proof. Let $f \in \hat{K}$. By (28) it suffices to show that $u \geq 0$ in D^+, where u is given by (26). If $u < 0$ at a point of D^+, then u has a negative minimum at some point e^{it}, $0 < t < \pi$, by the Maximum Principle. By the Hopf boundary-point lemma $(\partial u/\partial r)(e^{it_0}) < 0$, which contradicts (27) since $f(t_0) \geq 0$. Thus $Tf \geq 0$ on $[0, \pi]$ and hence $T : \hat{K} \to \hat{K}$.

To show that $T : \tilde{K} \to \tilde{K}$ it suffices to show that $Tf \in \tilde{K}$ when $f \in \tilde{K}$ and f is smooth. A simple approximation argument then completes the proof. So assume

$f \in \tilde{K} \subset \hat{K}$ is smooth and let u be defined by (26). As already noted, $u \geq 0$ on D^+, and hence

$$\frac{\partial}{\partial t} u(re^{it})|_{t=\pi} \leq 0 \ , \quad 0 < r \leq 1 \ . \tag{29}$$

Let $w : \overline{D^+} \to \mathbb{R}$ be defined by

$$w(re^{it}) = 2(\sin(t/2)) u_t(re^{it}) - (\cos(t/2)) u(re^{it}) \ .$$

Then w is smooth on $\overline{D^+}$,

$$w_t(re^{it}) = (\sin(t/2))\{2u_{tt}(re^{it}) + \frac{1}{2} u(re^{it})\}$$

and hence

$$\triangle\, w(re^{it}) - \frac{1}{r^2}(\cot(t/2)) w_t(re^{it}) - \frac{1}{4r^2} w(re^{it}) = 0 \ . \tag{30}$$

Also

$$w(re^{i0}) = 0 \text{ and } w(re^{i\pi}) \leq 0 \text{ by (29)} \ . \tag{31}$$

Now (30) is an elliptic equation which w satisfies and hence, by the Maximum Principle applied to balls interior to D^+, the maximum and minimum of w on $\overline{D^+}$ are attained on the boundary of D^+. By (31) the maximum is non-negative. Suppose it is positive. Then it is attained at e^{it_0}, $0 < t_0 < \pi$, because of (31). The Hopf boundary-point lemma then gives

$$\begin{aligned} 0 < \frac{\partial w}{\partial r}(e^{it_0}) &= 2 \sin\left(\frac{t_0}{2}\right) u_{rt}(e^{it_0}) - \cos\left(\frac{t_0}{2}\right) u_r(e^{it_0}) \\ &= 2 \sin\left(\frac{t_0}{2}\right) f'(t_0) - \cos\left(\frac{t_0}{2}\right) f(t_0) \ , \end{aligned} \tag{32}$$

by the smoothness of u and (27). Here prime denotes differentiation with respect to $t \in (0, \pi)$. However, for $t \in (0, \pi)$, the fact that $f \in \tilde{K}$ gives,

$$0 \geq \frac{d}{dt}\left\{\frac{f(t)}{\sin(t/2)}\right\} = \frac{2 \sin\left(\frac{t}{2}\right) f'(t) - \cos\left(\frac{t}{2}\right) f(t)}{2 \sin^2(t/2)} \ .$$

This contradicts (32). Hence $w \leq 0$ in D^+ and, in particular,

$$2 \sin(t/2) u_t(e^{it}) - (\cos(t/2)) u(e^{it}) \leq 0 \ .$$

Thus $Tf \in \tilde{K}$ when $f \in \tilde{K}$ is smooth, by (28), which is what we set out to prove.

Finally, suppose $f \in K$, let u be defined as previously and let w_α be defined on a segment D_α of the unit disc as follows:

$$w_\alpha(re^{it}) = u(re^{it}) - u(re^{i(2\alpha - t)})$$

where $\pi/2 < \alpha < \pi$ and

$$D_\alpha = \{re^{it} : 0 < r < 1,\ 2\alpha - \pi < t < \alpha\} \ .$$

Then w_α is harmonic on D_α,

$$w_\alpha(re^{\alpha i}) = 0 \tag{33}$$

and, since $f \in \hat{K}$

$$w_\alpha(re^{(2\alpha-\pi)i}) = u(re^{(2\alpha-\pi)i} \geq 0 \ . \tag{34}$$

Since w_α is continuous on $\overline{D_\alpha}$ it follows that it attains its minimum on the boundary. If the minimum is negative, then by (33) and (34) it must be attained at a point e^{it_0}, $2\alpha - \pi < t_0 < \alpha$ at which point $\partial w_\alpha / \partial r < 0$, by the Hopf boundary-point lemma. Therefore

$$\begin{aligned} 0 > \frac{\partial w_\alpha}{\partial r}\left(e^{it_0}\right) &= \frac{\partial u}{\partial r}\left(e^{it_0}\right) - \frac{\partial u}{\partial r}\left(e^{i(2\alpha - t_0)}\right) \\ &= f(t_0) - f(2\alpha - t_0) \ . \end{aligned}$$

However, since $\alpha \in (\pi/2, \pi)$,

$$t_0 \in [\pi - (2\alpha - t_0), 2\alpha - t_0] \text{ and } 2\alpha - t_0 \in [\pi/2, \pi] \ ,$$

and therefore

$$f(t_0) - f(2\alpha - t_0) \geq 0$$

since $f \in K$. This is a contradiction and we conclude that $w_\alpha \geq 0$ on D_α when $f \in K$. Therefore for any $\alpha \in (\pi/2, \pi)$

$$Tf(t) - Tf(2\alpha - t) \geq 0, \quad 2\alpha - \pi < t < \alpha \ .$$

Hence $Tf \in K$ if $f \in K$ and the proof is complete. □

To complete the demonstration that Nekrasov's equation can be dealt with in the cone K let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(u) = \begin{cases} 1 & \text{if} \quad u > \pi/2 \\ \sin u & \text{if} \quad -\pi/2 < u < \pi/2 \\ -1 & \text{if} \quad u < -\pi/2 \end{cases}$$

and let $N : (0, \infty) \times \hat{K} \to \hat{K}$ be defined by

$$N(\mu, u) = \frac{\mu F(u(t))}{1 + \mu \int_0^t F(u(\nu)) d\nu} \ , \quad t \in [-\pi, \pi] \ .$$

Since F is odd, it is clear that the denominator is an even function of t which is non-negative on $[0, \pi]$ if $u \in \hat{K}$. Thus $N : (0, \infty) \times \hat{K} \to \hat{K}$ as required. However, even more is true.

Theorem 4 *The operator N maps $(0, \infty) \times \tilde{K}$ into $\tilde{K}$ and $(0, \infty) \times K$ into K, and in both contexts it is continuous.*

Proof. To see that $N:(0,\infty)\times\tilde{K}\to\tilde{K}$ it suffices to show that $N(\mu,u)\in\tilde{K}$ when $u\in\tilde{K}$ and u is smooth. In this case note that

$$\frac{d}{dt}\left(\frac{F(u(t))}{\sin(t/2)}\right)=\frac{2F'(u(t))u'(t)\sin(t/2)-F(u(t))\cos(t/2)}{2\sin^2(t/2)}$$

$$=\frac{F'(u(t))\{2u'(t)\sin(t/2)-u(t)\cos(t/2)\}}{2\sin^2(t/2)}+\frac{\cos(t/2)\{u(t)F'(u(t))-F(u(t))\}}{2\sin^2(t/2)}$$

$$\leq 0 \text{ for } t\in(0,\pi)$$

since F is non-decreasing, $F(u)/u$ is decreasing on $(0,\infty)$ and $u\in\tilde{K}$. Therefore for $\mu>0$, and $u\in\tilde{K}$ smooth, and for $0<s<t<\pi$,

$$\frac{N(\mu,u)(t)}{\sin(t/2)}-\frac{N(\mu,u)(s)}{\sin(s/2)}$$

$$=\frac{\mu}{\{1+\mu\int_0^t F(u(\nu))d\nu\}}\left\{\frac{F(u(t))}{\sin(t/2)}-\frac{F(u(s))}{\sin(s/2)}\right\}$$

$$+\frac{\mu F(u(s))}{\sin(s/2)}\left\{\frac{1}{1+\mu\int_0^t F(u(\nu))d\nu}-\frac{1}{1+\mu\int_0^s F(u(\nu))d\nu}\right\}$$

$$\leq 0\ .$$

Therefore $N(\mu,u)\in\tilde{K}$ when $\mu>0$ and $u\in\tilde{K}$.

Finally, to see that $N:(0,\infty)\times K\to K$ it suffices to note that $1+\mu\int_0^t F(u(\nu))d\nu$ is a non-decreasing positive function of t when $u\in K$ and $\mu>0$ and F is monotone non-decreasing on $(0,\infty)$. Thus, for $s\in[\pi-t,t]$ and $t\in[\pi/2,\pi]$,

$$\begin{aligned} N(\mu,u(t)) \quad - \quad N(\mu,u(s)) \quad &= \quad \frac{\mu(F(u(t))-F(u(s)))}{1+\mu\int_0^t F(u(\nu))d\nu} \\ &+ \quad \mu F(u(s))\left\{\frac{1}{1+\mu\int_0^t F(u(\nu))d\nu}-\frac{1}{1+\mu\int_0^s F(u(\nu))d\nu}\right\} \\ &\leq \quad 0 \text{ if } u\in K\ . \end{aligned}$$

Hence $N:(0,\infty)\times K\to K$ and the proof is complete. $\square$

Now we need the result of an explicit calculation.

Lemma 5

$$\int_0^\pi K_1(s,t)\tan\left(\frac{t}{2}\right)dt=\frac{s}{2}\ ,\quad s\in(0,\pi)\ ,$$

and hence

$$\frac{1}{3}\int_0^\pi K_1(s,t)\cot\left(\frac{t}{4}\right)dt<\frac{\pi}{3}\ ,\quad s\in(0,\pi)\ .$$

Proof. For each $k \in \mathbb{N}$ it follows by Cauchy's Integral formula (and a limiting argument) that for $k \in \mathbb{N}$

$$\begin{aligned} 0 &= \int_{S^1} z^{k-1} \log(1+z) dz = i \int_{-\pi}^{\pi} e^{ikt} \log(1+e^{it}) dt \\ &= i \int_{-\pi}^{\pi} e^{ikt} \{\log(2\cos t/2) + it/2\} \ . \end{aligned}$$

Hence

$$\int_{-\pi}^{\pi} \tan\left(\frac{t}{2}\right) \frac{\sin kt}{k} dt = \int_{-\pi}^{\pi} 2\log\left(2\cos\frac{t}{2}\right) \cos kt \ dt = \int_{-\pi}^{\pi} t \sin kt \ dt \ .$$

Thus

$$\int_0^\pi K_1(s,t) \tan\left(\frac{t}{2}\right) \ dt = \frac{s}{2}, s \in (0,\pi)$$

and a change of variables gives

$$\int_0^\pi K_1(s,t) \cot\left(\frac{t}{2}\right) dt = \frac{\pi - s}{2}, \quad s \in (0,\pi) \ .$$

Since

$$\cot\frac{t}{4} \leq 2\left(\cot\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right)$$

the result follows. $\square$

Theorem 6 *Suppose that $(\mu, \theta) \in (0,\infty) \times K$, $\theta \not\equiv 0$ and $\theta = \frac{1}{3} TN(\mu,\theta)$. Then*

(a) $0 < \theta(t) < \pi/3$ *for all* $s \in (0,\pi)$*;*
(b) $\mu > 3$*;*
(c)

$$\frac{1}{\mu} + \int_0^s \sin\theta(\nu)\, d\nu \geq \gamma s, \quad s \in [0,\pi] \ ;$$

where $\gamma > 0$ is independent of μ and θ;
(d)

$$\left| \int_0^\pi \frac{\cos 3\theta(t)}{\frac{1}{\mu} + \int_0^t \sin\theta(\nu) d\nu} \ dt \right| \leq \Gamma \ ,$$

where Γ is independent of $\mu > 3$ and θ.

In particular, θ satisfies Nekrasov's equation with $\beta = 1/\mu$.
(e) The Stokes waves to which these solutions correspond have

$$m \leq \frac{g\lambda}{c^2} < M$$

for absolute constants m and M. (See (13).)

Proof. If $\theta \in K$, then $F(\theta) \in K$, by the preceding proof and therefore

$$\begin{aligned}
\int_0^t F(\theta(\nu))d\nu &= \int_0^t \sin\left(\frac{\nu}{2}\right)\left\{F(\theta(\nu))/\sin\left(\frac{\nu}{2}\right)\right\}\, d\nu \\
&\geq \frac{F(\theta(t))}{\sin\left(\frac{t}{2}\right)} \int_0^t \sin\left(\frac{\nu}{2}\right) d\nu = 2F(\theta(t))\left(\frac{1-\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right) \\
&= 2F(\theta(t))\tan(t/4)\ .
\end{aligned}$$

Hence for any $\mu > 0$ and $\theta \in K$,

$$N(\mu,\theta) = \frac{\mu F(\theta(t))}{1+\mu\int_0^t F(\theta(\nu))d\nu} \leq \frac{1}{2}\cot\left(\frac{t}{4}\right)\ . \tag{35}$$

Hence, since the kernel K_1 of the linear operator T is non-negative on $[0,\pi]\times[0,\pi]$ and odd in s and t separately,

$$\begin{aligned}
\theta(s) &= \frac{2}{3}\int_0^\pi K_1(s,t)N(\mu,\theta)(t)\ dt \\
&< \frac{1}{3}\int_0^\pi K_1(s,t)\cot\left(\frac{t}{4}\right)\ dt\ ,\ \text{by (35)} \\
&< \frac{\pi}{3}\ ,\ \text{for } s\in(0,\pi)\ .
\end{aligned}$$

Also, if $\theta \not\equiv 0$, then $N(\mu,\theta) \geq 0$ and the inequality is strict on a set of positive measures. Hence $\theta > 0$ on $(0,\pi)$ since for $s \in (0,\pi)$, $K_1(s,t) > 0$ almost everywhere $\theta > 0$ on $(0,\pi)$.

Since $F(u) = \sin u$ on $[-\pi/3,\pi/3]$, it follows that $(1/\mu,\theta)$ is a solution of Nekrasov's equation.

Now for $s \in [0,\pi]$,

$$\begin{aligned}
\theta(s) &= \frac{2}{3}\int_0^\pi K_1(s,t)\frac{\mu\sin\theta(t)}{1+\mu\int_0^t \sin\theta(\nu)d\nu}\ dt \\
&\geq \frac{2}{3}\left(\frac{\mu}{1+\mu\int_0^s \sin\theta(\nu)d\nu}\right)\int_0^s tK_1(s,t)\left(\frac{\sin\theta(t)}{t}\right)\ dt \\
&\geq \frac{2}{3}\left(\frac{\mu\sin\theta(s)}{1+\mu\int_0^s \sin\theta(\nu)d\nu}\right)\int_0^s \frac{t}{s}K_1(s,t)\ dt\ ,
\end{aligned}$$

since $\theta \in K \subset \tilde{K}$.

Now for $0 < t < s < \pi$

$$1 \geq \frac{\sin\left(\frac{s-t}{2}\right)}{\left(\frac{s-t}{2}\right)} \geq \frac{\sin\left(\frac{s+t}{2}\right)}{\left(\frac{s+t}{2}\right)} \geq \frac{\sin s}{s}\ .$$

Therefore

$$\frac{\sin\left(\frac{s+t}{2}\right)}{\sin\left(\frac{s-t}{2}\right)} \geq \left(\frac{\sin s}{s}\right)\left(\frac{s+t}{s-t}\right)$$

and so

$$K_1(s,t) \geq \frac{1}{2\pi}\left\{\log\left(\frac{\sin s}{s}\right) + \log\left(\frac{s+t}{s-t}\right)\right\} \ .$$

Hence

$$\begin{aligned}
\int_0^s \frac{t}{s} K_1(s,t)\ dt \quad &\geq \quad \frac{s}{4\pi}\log\left(\frac{\sin s}{s}\right) + \frac{1}{2\pi}\int_0^s \frac{t}{s}\log\left(\frac{s+t}{s-t}\right)\ dt \\
&= \quad \frac{s}{4\pi}\log\left(\frac{\sin s}{s}\right) + \frac{s}{2\pi}\int_0^1 u\log\left(\frac{1+u}{1-u}\right)\ ds \\
&= \quad \frac{s}{2\pi}\left\{\frac{1}{2}\log\left(\frac{\sin s}{s}\right) + 1\right\} \ .
\end{aligned}$$

Since $0 \leq \theta < \pi/3$, $\theta(s)/\sin\theta(s) \leq 2\pi/3\sqrt{3}$, and hence

$$\frac{2\pi^2}{\sqrt{3}}\left(\frac{1}{\mu} + \int_0^s \sin\theta(\nu)d\nu\right) \geq s\left(1 + \frac{1}{2}\log\left(\frac{\sin s}{s}\right)\right) \ .$$

Hence there is an interval $[0,a]$, independent of (μ,θ) such that

$$\frac{2\pi^2}{\sqrt{3}}\left(\frac{1}{\mu} + \int_0^s \sin\theta(\nu)d\nu\right) \geq \frac{1}{2}s, \quad s \in [0,a] \ .$$

Thus

$$\frac{1}{\mu} + \int_0^s \sin\theta(\nu)d\nu \geq \frac{\sqrt{3}a}{4\pi^3}s, \quad s \in [0,\pi] \ ,$$

since $\sin\theta \geq 0$ on $[0,\pi]$.

Now suppose that $\tau + i\theta$ is analytic on the unit disc and that τ has zero mean on S^1, i.e. $-\tau|_{S^1}$ is the conjugate of $\theta|_{S^1}$. (It is important to distinguish this τ from the one mentioned briefly at the end of Section 6. They differ by a constant, as will be seen presently.) Then

$$-\tau_t(e^{it}) = \theta_r(e^{it}) = \frac{\sin\theta(e^{it})}{3\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu\right)}$$

whence

$$\tau(e^{it}) = -\frac{1}{3}\log\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu\right) + a \ ,$$

where

$$a = \frac{1}{6\pi}\int_{-\pi}^{\pi}\log\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu\right)$$

since $\tau|_{S^1}$ has mean zero. Now $(\tau + i\theta)(0) = 0$ and so Cauchy's integral formula gives

$$\begin{aligned}
1 \quad &= \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{3\tau(e^{it})}\cos 3\theta(e^{it})dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{3a}\cos 3\theta(e^{it})}{\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu\right)}\ dt \\
&= \quad \frac{\alpha(\theta,\mu)}{\pi}\int_0^{\pi}\frac{\cos 3\theta(e^{it})dt}{\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})dv\right)}
\end{aligned}$$

where, by part (c),

$$\alpha(\theta,\mu) = \exp\left\{\frac{1}{\pi}\int_0^\pi \log\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})dv\right)dt\right\}$$

$$\geq \exp\left\{\frac{1}{\pi}\int_0^\pi \log(\gamma t)dt\right\} = \exp\{\log\gamma + \log\pi - 1\}\ .$$

The number on the right-hand side is independent of θ and μ. Since $\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu \leq \left(\frac{1}{3}+\pi\right)$, we have shown that

$$\left|\int_0^\pi \frac{\cos 3\theta(e^{it})}{\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu}dt\right| = \left|\frac{\pi}{\alpha(\theta,\mu)}\right| \geq \Gamma$$

for some constant Γ independent of μ and θ, as required. An inspection of (13) in the light of (a), (b) and (c) yields (e).

This completes the proof. □

9 Global Existence Theory

The purpose here is to describe how Rabinowitz's global bifurcation theory [50] as formulated for cones by Dancer [20] immediately leads to the existence of solutions of Nekrasov's equation. Without further ado, we state the result we need in an abstract form due to Dancer.

Suppose that Y is a real Banach space and $C \subset Y$ is a closed convex cone. Suppose that

(i) $A : [0,\infty) \times C \to C$ is continuous and maps bounded sets to relatively compact sets.
(ii) $A(0,y) = A(\lambda,0) = 0$ for all $\lambda \neq [0,\infty)$ and $y \in K$.
(iii) $A(\lambda,y) = \lambda By + R(\lambda,y)$ where $B : Y \to Y$ is a compact linear operator and $\|R(\lambda,y)\| / \|y\| \to 0$ as $\|y\| \to 0,\ y \in C$, uniformly for λ in compact intervals of $[0,\infty]$.
(iv) B has an eigenvector $y_0 \in C\backslash\{0\}$ corresponding to an eigenvalue $\lambda_0 > 0$ of B which is at least as large as the modulus of any eigenvalue of B, and B has no other eigenvalue with eigenvectors in C.

Let

$$\mathcal{S} = \{(\lambda,y) \in [0,\infty) \times C : A(\lambda,y) = y,\ y \neq 0\} \cup \{(\lambda_0^{-1},0)\}\ .$$

Theorem 7 *If **(i)–(iv)** hold then the $\mathcal{C}$ component of $\mathcal{S}$ which contains $(\lambda_0^{-1},0)$ is unbounded in $[0,\infty) \times Y$.* □

This result seems tailored for Nekrasov's equation in the cone K defined in Section 8. Let Y and C above be given by X and K of Section 8, and let

$$A(\mu,\theta) = \frac{1}{3}TN(\mu,\theta),\ (\mu,\theta) \in [0,\infty) \times X$$

where T is defined at the end of Section 7. Finally identify B here with $\frac{1}{3}T$. We have seen that $T : X \to X$ is compact (Section 7) and since N is clearly bounded and continuous, **(i)** and **(ii)** are satisfied. It is obvious from the formula

$$Tf(t) = \frac{1}{3\pi} \int_{-\pi}^{\pi} \left(\sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k} \right) f(t)dt$$

that the eigenvalues of T are $1/3k,\ k \in \mathbb{N}$, and that the corresponding eigenfunctions are $\sin kt$. Since $\sin t \in K$ and $\sin kt \notin K$ for $k > 1$, the hypotheses of the theorem are satisfied. Since, in the light of Theorem 6, every solution $(\mu, \theta) \in [0, \infty) \times K$ of $A(\mu, \theta) = \theta$ gives a solution of Nekrasov's equation we can deduce the following result. A continuum is a maximal closed connected set.

Theorem 8 *There exists an unbounded continuum $\mathcal{C}$ in $[0, \infty) \times K$ of solutions of Nekrasov's equation*

$$\theta(s) = \frac{1}{3} \int_{-\pi}^{\pi} K_1(s,t) \frac{\mu \sin \theta(t)}{1 + \mu \int_0^t \sin \theta(\nu) d\nu}\ dt$$

such that the following hold.

(i) $(\mu, 0) \in \mathcal{C}$ *if and only if* $\mu = 3$.
(ii) *if* $(\mu, \theta) \in \mathcal{C}$ *and* $\theta \neq 0$, *then* $\mu > 3$ *and* $0 < \theta(s) < \pi/3$ *on* $(0, \pi)$.
(iii) $\theta'(s) < 0,\ s \in [\pi/2, \pi]$.
(iv) $\theta(s)/s$ *is non-increasing on* $(0, \pi)$.

Proof. The existence of $\mathcal{C}$ is a direct corollary of the abstract theorem. The estimate (ii) was established in the preceding section and (iii) and (iv) are important corollaries of membership of K. $\square$

10 The Behaviour as $\mu \to \infty$

We have seen that for all $\mu > 3$ there is a non-zero solution, θ_μ say, of Nekrasov's equation with $0 < \theta < \pi/3$. The question of the behaviour of these solutions as $\mu \to \infty$ is of great interest. The following simple argument shows that

$$\liminf_{\mu \to \infty} \ \|\theta_\mu\| \geq \pi/6\ , \tag{36}$$

where $\|\theta\| = \sup\{|\theta(t)|; t \in [0, \pi]\}$ for $\theta \in X$.

Let $\mu_\ell \to \infty$ as $\ell \to \infty$ and let θ_ℓ denote θ_{μ_ℓ}. Then by Lemma 2 we may suppose, without loss of generality, that $\theta_\ell \to \theta^*$ pointwise and in $L_2(-\pi, \pi)$. By Theorem 6 (c) $\theta^* \neq 0$. Now suppose that for some $\epsilon > 0$, $\|\theta_\ell\| \leq \pi/6 - \epsilon$ for all ℓ. Then Theorem 6 (d) and Fatou's lemma gives

$$\left| \int_0^{\pi} \frac{\cos 3\theta^*(t)}{\int_0^t \sin \theta^*(\nu) d\nu} dt \right| \leq \Gamma < \infty\ ,$$

which is clearly false since $|\theta^*(t)| \leq \pi/6 - \epsilon$ almost everywhere. This proves (36).

Presently we shall give a proof of McLeod's result [43] that

$$\limsup_{\mu\to\infty} \|\theta_\mu\| > \frac{\pi}{6} \tag{37}$$

but before doing so we observe that in any case the function θ^* satisfies the limiting equation

$$\theta^*(s) = \frac{1}{3}\int_{-\pi}^{\pi} K_1(s,t)\frac{\sin\theta^*(t)}{\int_0^t \sin\theta^*(w)}dw \tag{38}$$

where $|\theta^*(s)| \leq \pi/3$ and $\int_0^t \sin\theta^*(w)dw \geq \gamma t$, $t \in (0,\pi)$. To see this, note that for each ℓ,

$$\left|\frac{\sin\theta_\ell(t)}{\frac{1}{\mu_\ell} + \int_0^t \sin\theta_\ell(w)dw}\right| \leq \frac{1}{\gamma t}\ ,$$

by Theorem 6 (c). Now for each $s \in (0,\pi)$, $K_1(s,t) \geq 0$, is odd in t and

$$\int_0^\pi K_1(s,t)\frac{1}{t}dt < \infty$$

by Lemma 5. Hence θ^* satisfies the limiting equation by the Dominated Convergence Theorem. We shall return to the limiting equation later, but for now we concentrate on obtaining (37).

Suppose that $(\mu,\theta_\mu) \in (3,\infty)\times K$ is a solution of Nekrasov's Integral Equation. Note that $\theta_\mu(t)/t$ is non-increasing on $[0,\pi]$, and even, because $\theta_\mu \in K$. Let

$$\hat\theta_\mu(s) = \theta_\mu(s/\mu) \text{ for all } s \in [0,\mu\pi]\ .$$

Then $\hat\theta_\mu(s)/s$ is even and non-increasing for $s \in [0,\mu\pi]$. If $\{\mu_\ell\}$ is an arbitrary sequence with $\mu_\ell \to \infty$ we may (as in the proof of Lemma 2) extract a subsequence such that $\hat\theta_{\mu_\ell}(s)$ converges pointwise on $[0,\infty)$ to a function $\hat\theta$, where $|\hat\theta(t)| \leq \pi/3,\ t \in [0,\infty)$.

Now note that the equation for $\hat\theta_\mu$ is

$$\hat\theta_\mu(s) = \frac{2}{3}\int_0^{\mu\pi} K_1\left(\frac{s}{\mu},\frac{t}{\mu}\right)\frac{\sin\hat\theta_\mu(t)}{1+\int_0^t \sin\hat\theta_\mu(w)dw}dt\ .$$

By Theorem 6 (c), for $t \in (0,\pi)$

$$\gamma t \leq \frac{1}{\mu} + \int_0^t \sin\hat\theta_\mu(w)dw = \frac{1}{\mu}\left(1 + \int_0^{\mu t}\sin\hat\theta_\mu(w)dw\right)\ ,$$

whence

$$\gamma t \leq 1 + \int_0^t \sin\hat\theta_\mu(w)dw, \quad t \in (0,\mu\pi)\ . \tag{39}$$

Also, for $s,t \in [0,\mu\pi]$,

$$K_1\left(\frac{s}{\mu},\frac{t}{\mu}\right) = \frac{2}{\pi}\log\left|\left\{\frac{\sin\left(\frac{s+t}{2\mu}\right)}{\sin\left(\frac{s-t}{2\mu}\right)}\right\}\right| \leq \frac{2}{\pi}\log\left|\frac{s+t}{s-t}\right|\ ,$$

and for fixed $s \in (0,\infty)$,

$$\frac{1}{\pi}\int_0^\infty \frac{1}{t}\log\left|\frac{s+t}{s-t}\right| dt = \frac{1}{\pi}\int_0^\infty \frac{1}{t}\log\left|\frac{1+t}{1-t}\right| dt = \frac{\pi}{2}\ . \tag{40}$$

Hence, by the Dominated Convergence Theorem, $\hat{\theta} : (0,\infty) \to \mathbb{R}$, satisfies the equation

$$\hat{\theta}(s) = \frac{2}{3\pi}\int_0^\infty \log\left|\frac{s+t}{s-t}\right| \frac{\sin\hat{\theta}(t)}{1+\int_0^t \sin\hat{\theta}(w)dw} dt,\ \ s \in (0,\pi)\ . \tag{41}$$

Since $\theta_\mu(0) = 0$ and the right-hand side of (41) is zero when $s = 0$, the limiting function satisfies (41) on $[0,\infty)$. Since the right-hand side of (41) clearly defines a continuous function on $[0,\infty)$, since $0 \le \hat{\theta} \le \pi/3$ on $[0,\infty)$, we conclude that $\hat{\theta}$ is continuous on $[0,\infty)$. A bootstrap argument now gives that $\hat{\theta}$ is infinitely differentiable. The following observations lead to a proof of (37).

Lemma 9 *Suppose that $\|\theta_{\mu_\ell}\| \le \pi/6$ for all $\ell \in \mathbb{N}$. Then $\hat{\theta}$ has the following properties:*

(i) $1 + \int_0^t \sin\hat{\theta}(w)dw \ge \gamma t, \quad t \in (0,\infty).$

(ii) $\displaystyle\left|\int_0^\infty \frac{\cos 3\hat{\theta}(t)}{1+\int_0^t \sin\hat{\theta}(w)dw} dt\right| < \infty.$

(iii) $\hat{\theta}(t) \to \pi/6$ *as* $t \to \infty$.

Proof. Part (i) is immediate from (39) in the limit as $\mu_\ell \to \infty$. Since $0 \le \theta_{\mu_\ell} \le \pi/6$ on $[0,\pi]$, by assumption, it follows by Fatou's lemma and Lemma 2 that

$$\begin{aligned} 0 \ &\le\ \int_0^\infty \frac{\cos 3\hat{\theta}(t)}{1+\int_0^t \sin\hat{\theta}(w)dw} dt \le \liminf_{\ell\to\infty} \int_0^{\mu\pi} \frac{\cos 3\hat{\theta}_{\mu_\ell}(t)}{1+\int_0^t \sin\hat{\theta}_{\mu_\ell}(w)dw}\ dt \\ &=\ \liminf_{\ell\to\infty} \int_0^\pi \frac{\cos 3\theta_{\mu_\ell}(t)}{\frac{1}{\mu}+\int_0^t \sin\theta_{\mu_\ell}(w)dw}\ dt \le \Gamma\ , \end{aligned}$$

by Theorem 6 (d). Since the integrand is non-negative, the result (ii) follows.
(iii) For $1 < s_1 < s_2 < \infty$,

$$\int_{s_1}^{s_2} \frac{\cos 3\hat{\theta}(t)}{1+\int_0^t \sin\hat{\theta}(w)dw} dt \ge \int_{s_1}^{s_2} \frac{\cos 3\hat{\theta}(t)}{2t}\ dt \ge \int_{s_1}^{s_2} \left(\frac{1-6\hat{\theta}(t)/\pi}{2t}\right)\ dt\ ,$$

since $\cos x \ge 1 - 2x/\pi,\ x \in [0,\pi/2]$,

$$\ge \frac{1}{2}\left\{\log\left(\frac{s_2}{s_1}\right) - \frac{6}{\pi}\frac{\hat{\theta}(s_1)}{s_1}(s_2 - s_1)\right\}$$

since $\hat{\theta}(s)/s$ is non-increasing,

$$= \frac{1}{2}\left\{\log\left(1 + \frac{s_2 - s_1}{s_1}\right) - \frac{6}{\pi}\hat{\theta}(s_1)\left(\frac{s_2 - s_1}{s_1}\right)\right\}\ .$$

Now let $\alpha \in (0, \pi/6)$ and choose $a > 0, \delta > 0$ such that

$$\log(1+a) \geq \frac{6}{\pi}\left(\frac{\pi}{6} - \alpha\right) a + \delta \ .$$

It follows that if $\hat{\theta}(s_1) = \frac{\pi}{6} - \alpha$ and $s_2 = s_1(1+a)$ then

$$\int_{s_1}^{s_2} \frac{\cos 3\hat{\theta}(t)}{1 + \int_0^t \sin\hat{\theta}(w)dw} dt \geq \frac{1}{2}\delta \ .$$

Hence, for each $\alpha \in (0, \pi/6)$ the set $\{s : \theta(s) = \pi/6 - s\}$ is bounded, for otherwise the integral in (ii) would be infinite. Thus

$$\limsup_{t\to\infty} \hat{\theta}(t) = \liminf_{t\to\infty} = \lim_{t\to\infty} \hat{\theta}(t) \ .$$

Since the integral in (ii) is finite, $\hat{\theta}(t) \to \pi/6$ as $t \to \infty$.

This completes the proof. □

The key step in McLeod's proof of (37) is the following.

Lemma 10 *Under the hypotheses of Lemma 9,*

$$1 + \int_0^t \sin\hat{\theta}(t)dt \geq \frac{1}{2}t \quad , \ t \in [0, \infty) \ .$$

Proof. Since (40) holds,

$$\begin{aligned}
\hat{\theta}(s) - \frac{\pi}{6} &= \frac{1}{3\pi}\int_0^\infty \log\left|\frac{s+t}{s-t}\right| \left\{\frac{\sin\hat{\theta}(t)}{1 + \int_0^t \sin\hat{\theta}(\nu)d\nu} - \frac{1}{t}\right\} dt \\
&= \frac{s}{3\pi}\int_0^\infty \log\left|\frac{1+w}{1-w}\right| \left\{\frac{\sin\hat{\theta}(sw)}{1 + \int_0^{sw} \sin\hat{\theta}(\nu)d\nu} - \frac{1}{sw}\right\} dw \ .
\end{aligned}$$

By Lemma 9 (ii), $(\hat{\theta}(s) - \pi/6)/s$ is integrable on $[u, \infty)$ for each $u > 0$ and so

$$\begin{aligned}
\int_u^\infty \frac{\hat{\theta}(s) - \frac{\pi}{6}}{s} ds &= \frac{1}{3\pi}\int_0^\infty \left\{\left(\frac{1}{w}\log\left|\frac{1+w}{1-w}\right|\right) \int_{wu}^\infty \left(\frac{\sin\hat{\theta}(t)}{1 + \int_0^t \sin\hat{\theta}(\nu)d\nu} - \frac{1}{t}\right) dt\right\} dw \\
&= \frac{-1}{3\pi}\int_0^\infty \left\{\left(\frac{1}{w}\log\left|\frac{1+w}{1-w}\right|\right) \log\left(\frac{1 + \int_0^{wu} \sin\hat{\theta}(\nu)d\nu}{\frac{1}{2}wu}\right)\right\} dw \ ,
\end{aligned}$$

since $\sin\hat{\theta}(\nu) \to \frac{1}{2}$ as $\nu \to \infty$,

$$= -\frac{1}{3\pi}\int_0^\infty \frac{1}{s}\log\left|\frac{u+s}{u-s}\right| \log\left\{\frac{1 + \int_0^s \sin\hat{\theta}(\nu)d\nu}{\frac{1}{2}s}\right\} ds \ .$$

To prove the result we suppose that, on the contrary, for some $\hat{s} > 0, \ \delta > 0$,

$$1 + \int_0^{\hat{s}} \sin\hat{\theta}(\nu)d\nu \leq \frac{1}{2}\hat{s} - \delta \ .$$

Then $\hat{s} \geq 2$ and this inequality holds also for all $s \geq \hat{s}$, since $\sin\hat{\theta} \leq 1/2$ by assumption. Hence

$$\log\left\{\frac{1+\int_0^s \sin\hat{\theta}(\nu)d\nu}{\frac{1}{2}s}\right\} \leq \log\left(1-\frac{2\delta}{s}\right) \leq \frac{-2\delta}{s}, \quad s \geq \hat{s}\ .$$

Also for $s \in (0, \hat{s}]$

$$\frac{2}{s} \leq \frac{1+\int_0^s \sin\hat{\theta}(\nu)d\nu}{\frac{1}{2}s} \leq 1+\frac{2}{s} \leq \frac{\tilde{s}+2}{s}$$

and so

$$\left|\log\left\{\frac{1+\int_0^s \sin\hat{\theta}(\nu)d\nu}{\frac{1}{2}s}\right\}\right| \leq \log(2+\tilde{s}) + |\log s|\ .$$

Therefore

$$\begin{aligned}
\int_u^\infty \frac{\hat{\theta}(s)-\frac{\pi}{6}}{s}\, ds &= -\frac{1}{3\pi}\int_0^{\hat{s}} \frac{1}{s}\log\left|\frac{s+u}{s-u}\right| \log\left\{\frac{1+\int_0^s \sin\hat{\theta}(\nu)d\nu}{\frac{1}{2}s}\right\}\ ds \\
&\quad -\frac{1}{3\pi}\int_{\hat{s}}^\infty \frac{1}{s}\log\left|\frac{s+u}{s-u}\right| \log\left\{\frac{1+\int_0^s \sin\hat{\theta}(\nu)d\nu}{\frac{1}{2}s}\right\}\ ds \\
&\geq -\frac{1}{3\pi}\int_0^{\hat{s}} \frac{1}{s}\log\left|\frac{s+u}{s-u}\right| \{|\log s| + \log(2+\tilde{s})\}\ ds \\
&\quad + \frac{2\delta}{3\pi}\int_{\hat{s}}^\infty \frac{1}{s^2}\log\left|\frac{s+u}{s-u}\right|\ ds \\
&\geq -\frac{1}{3\pi}\int_0^{\hat{s}} \frac{1}{s}\log\left(1+\frac{2s/u}{1-s/u}\right) \{|\log s| + \log(2+\tilde{s})\}\ ds \\
&\quad + \frac{2\delta}{3\pi}\int_{\hat{s}}^{u/3} \frac{1}{s^2}\log\left|\frac{s+u}{s-u}\right|\ ds\ , \qquad \text{if } u \geq 3\hat{s}\ .
\end{aligned}$$

Clearly the first term on the right-hand side is $O(1/u)$ as $u \to \infty$. However for $u \geq 3\hat{s}$

$$\begin{aligned}
\int_{\hat{s}}^{u/3} \frac{1}{s^2}\log\left|\frac{s+u}{s-u}\right|\ ds &= \frac{1}{u}\int_{\hat{s}/u}^{1/3} \frac{1}{t^2}\log\left(1+\frac{2t}{1-t}\right)\ dt \\
&\geq \frac{2\log 2}{u}\int_{\hat{s}/u}^{1/3}\frac{dt}{t} \geq \text{ const. } \frac{\log u}{u}
\end{aligned}$$

as $u \to \infty$ for some positive constant. Therefore

$$\int_u^\infty \frac{\hat{\theta}(s)-\pi/6}{s}\ ds > 0$$

for u sufficiently large. Since $\hat{\theta} \leq \pi/6$, this is false, and the required result has been established. □

For θ in the interval $[0, \pi/6]$,

$$\frac{1}{2} - \sin\theta \geq \frac{1}{2}(\pi/6 - \theta) \geq 0$$

and hence the preceding result has, as a corollary that under its hypotheses

$$\int_0^\infty \left(\frac{\pi}{6} - \hat{\theta}(s)\right) ds \leq 2\int_0^\infty \left(\frac{1}{2} - \sin\hat{\theta}(s)\right) ds \leq 2. \tag{42}$$

Theorem 11 (MCLEOD)

$$\limsup_{\ell\to\infty} \|\theta_\ell\| > \pi/6 \ .$$

Proof. For $\alpha \in (-1, 0)$, (42) ensures that

$$\int_0^\infty \frac{\hat{\theta}(s) - \pi/6}{s^{1+\alpha}}\ ds$$

is finite and converges to a finite limit as $\alpha \to 0$, by the Dominated Convergence Theorem. But, by (41)

$$\begin{aligned}\int_0^\infty \frac{\hat{\theta}(s) - \pi/6}{s^{1+\alpha}}\quad ds & \\ &= \frac{1}{3\pi}\int_0^\infty \left\{\left(\int_0^\infty \frac{1}{s^{1+\alpha}}\log\left|\frac{s+t}{s-t}\right| ds\right)\left(\frac{\sin\hat{\theta}(t)}{1+\int_0^t \sin\hat{\theta}(t)} - \frac{1}{t}\right)\right\}\ dt \\ &= \frac{1}{3}\tan\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{1}{t^{1+\alpha}}\log\left(\frac{1+\int_0^t \sin\hat{\theta}(t)}{t/2}\right)\ dt\ ,\end{aligned}$$

where we have used the fact that

$$\int_0^\infty \frac{1}{w^{1+\alpha}}\log\left|\frac{1+w}{1-w}\right| dw = \frac{\pi}{\alpha}\tan\left(\frac{\pi\alpha}{2}\right)$$

and integrated by parts. As $\alpha \to -1$, $\tan(\pi\alpha/2) \to -\infty$. Since the left-hand side is bounded we conclude that

$$\int_0^\infty \frac{1}{t^{1+\alpha}}\log\left(\frac{1+\int_0^t \sin\hat{\theta}(\nu)d\nu}{t/2}\right)\ dt \to 0\ .$$

But this is false since the integrand is non-negative and does not converge pointwise to zero as $\alpha \to 0$. This contradiction proves that

$$\limsup_{\mu\to\infty} \|\hat{\theta}_\mu\| > \pi/6\ ,\ \text{ as required}\ .$$

□

11 A Priori Bounds on the Slope of Stokes Waves

This section is an account of Amick's results on the maximum norm $\|\theta\|$ when $(\mu, \theta) \in \mathcal{C}$, the continuum of solutions of Nekrasov's equation in $\mathbb{R} \times K$ whose existence has been established. These bounds are easier to obtain for elements of a continuum in $\mathbb{R} \times K$ which contains the bifurcation point $(3, 0)$ than for the general solutions in $\mathbb{R} \times \hat{K}$ which his paper treats. (It is an interesting open question whether there are solutions in $\mathbb{R} \times \hat{K}$ which do not lie in $\mathcal{C}$.) When combined with McLeod's result, Amick's estimates yield that

$$\frac{\pi}{6} < \sup\{\|\theta\| : (\mu, \theta) \in \mathcal{C}\} < (1.086)\pi/6 \ . \tag{43}$$

(It is remarkable how close is (43) to the numerically calculated upper bound of $(1.012)\pi/6$, calculated by Longuet-Higgins and Fox [42].) The proof of the right-hand inequality which follows is a simple application of the Maximum Principle and the Hopf boundary-point lemma to a very cleverly devised harmonic function on the disc. Here we give a complete proof, for elements of $\mathcal{C}$, of (43) using Amick's choice of harmonic function. (Amick's paper goes some way to explaining the reasoning and limitations governing this choice of harmonic function.)

Suppose that $(\mu, \theta) \in \mathcal{C}$ and let $\theta(re^{it}), 0 \le r < 1, -\pi < t \le \pi$, be the harmonic extension of θ to the unit disc. Let τ be the function defined in Section 8 such that $\tau + i\theta$ is analytic on the unit disc and τ has zero mean on the unit circle. Then (see Section 8)

$$-\tau_t(e^{it}) = \theta_r(e^{it}) = \frac{\sin\theta(e^{it})}{3\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu\right)} \tag{44}$$

and

$$\tau(e^{it}) = a - \frac{1}{3}\log\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu\right) \tag{45}$$

where

$$a = \frac{1}{3\pi}\int_0^\pi \log\left(\frac{1}{\mu} + \int_0^t \sin\theta(e^{i\nu})d\nu\right) dt \ . \tag{46}$$

Let f be the analytic function on the unit disc defined by

$$f = \tau + i\theta - a \ ,$$

and for $\alpha > 4$ and $A \in (0, 1)$ (to be chosen later) let $V(\mu, \theta)$ be the harmonic function defined on the unit disc by

$$V(\mu, \theta)(z) = -\mathrm{Imag}\ \left\{e^{-\alpha f(z)}\left(zf'(z) + \frac{Ae^{3f(z)}}{3(3-\alpha)}\right)\right\} \tag{47}$$

where f' is the complex derivative of f. For $\phi \in (0, \pi/\alpha)$ let $F : [0, \pi/\alpha] \to \mathbb{R}$ be given by

$$F(\phi) = \cos(\alpha\phi) + \frac{A\sin((3-\alpha)\phi)}{(3-\alpha)\sin\phi} \ . \tag{48}$$

Since $zf'(z) = \theta_t + ir\theta_r, V(\mu, \theta)$ is defined in the unit disc by

$$V(\mu, \theta) = e^{\alpha(a-\tau)}\left[\theta_t \sin(\alpha\theta) - r\theta_r\cos(\alpha\theta) - \frac{Ae^{3(\tau-a)}\sin((3-\alpha)\theta)}{3(3-\alpha)}\right] \ .$$

On S^1, $\theta_r = \left(e^{3(\tau-a)}\sin\theta\right)/3$ if $(\mu,\theta)\in\mathcal{C}$ and hence

$$\begin{aligned}
& V(\mu,\theta)(e^{it}) \qquad (49)\\
= \; & e^{\alpha(a-\tau)}\left[\theta_t\sin(\alpha\theta)-\theta_r\left(\cos(\alpha\theta)+\frac{A\sin((3-\alpha)\theta)}{(3-\alpha)\sin\theta}\right)\right]\Bigg|_{e^{it}}\\
= \; & e^{\alpha(a-\tau)}\left[\theta_t\sin(\alpha\theta)-\theta_r F(\theta)\right]\Bigg|_{e^{it}}, -\pi < t \le \pi \; .
\end{aligned}$$

Note that $F(0)>0$ and $F(\pi/\alpha)<0$ since $\alpha>4$ and $A\in(0,1)$. Let ϕ^* denote the smallest zero of F in $(0,\pi/\alpha)$.

Theorem 12 (AMICK) *If* $V(\mu,\theta)(e^{it}) < 0, t \in (0,\pi)$*, then* $\|\theta\| < \phi^*$*. In particular, if* $V(\mu,\theta)(e^{it}) < 0$*,* $t\in(0,\pi)$ *when* $A = 0.96626$ *and* $\alpha = 4.8$*, then* $\|\theta\| < 0.5434$*.*

Proof. Suppose that the hypotheses hold and that $\|\theta\| > \phi^*$. Since $\theta(e^{i0}) = 0$, there is a smallest value of $t_0\in[0,\pi]$ at which $\theta(e^{it_0}) = \phi^*$. Since $V(\mu,\theta)(e^{it}) < 0$ on $(0,\pi)$ we conclude from (44) and (49) that $\theta_t\sin(\alpha\theta) < 0$ at e^{it_0}. It follows that $\sin(\alpha\theta(e^{it_0})) < 0$, $\alpha\theta(e^{it_0}) > \pi$. But $\alpha\theta(e^{it_0}) = \alpha\phi^* < \pi$. This contradiction proves the general case. The particular case follows from a calculation of the roots of F. $\square$

Note that $V(\cdot,\cdot)(\cdot)$ is a continuous function on $\mathcal{C}\times S^1$, since convergence of $(\mu_n,\theta_n)\in\mathcal{C}$ to $(\mu,\theta)\in\mathcal{C}$ in $\mathbb{R}\times X$ implies uniform convergence of all the derivatives of θ_n to the corresponding derivative of θ, because of the regularising properties of the operator in Nekrasov's equation.

It remains to prove that for all $(\mu,\theta)\in\mathcal{C}, V(\mu,\theta) < 0$ at every point e^{it} on the unit circle with $0 < t < \pi$ when $\alpha = 4.8$ and $A = 0.96626$. This is done by a continuation argument using the Maximum Principle. Let

$$\mathcal{T} = \{(\mu,\theta)\in\mathcal{C}\backslash\{(3,0)\} : V(\mu,\theta)(e^{it}) < 0, \quad t\in(0,\pi)\} \; .$$

Our purpose is to show that $\mathcal{T}$ is non-empty and both open and closed in $\mathcal{C}\backslash\{(3,0)\}$. This will prove that $\mathcal{T} = \mathcal{C}\backslash\{(3,0)\}$ and the *a priori* bounds follow by Amick's Theorem.

First observe that $\mathcal{T}$ is not empty because there is a neighbourhood N of $(3,0)$ in $\mathbb{R}\times X$ such that $N\cap(\mathcal{C}\backslash\{(3,0)\})\subset\mathcal{T}$. It suffices to note that for $(\|\theta\| + |3-\mu|)$ sufficiently small and $(\mu,\theta)\in\mathcal{C}$,

$$[\theta_t\sin(\alpha\theta) - \theta_r\cos(\alpha\theta)]_{e^{it}} < 0, \quad t\in(0,\pi)$$

by (44), since $\|\theta_t\|_\infty \to 0$ as $\|\theta\|\to 0$ for solutions of Nekrasov's equation. That $(\mu,\theta)\in\mathcal{T}$ when $(\mu,\theta)\in(\mathcal{C}\backslash\{(3,0)\})\cap N$ is now immediate by (49).

To show that $\mathcal{T}$ is open and closed in $\mathcal{C}\backslash\{(3,0)\}$, let $H:[0,\pi/\alpha)\to\mathbb{R}$ be defined by

$$H(\phi) = f(\phi) + Ag(\phi) + A^2h(\phi)$$

where, with A and α given above,

$$\begin{aligned}
f(\phi) &= \alpha - 3 + \frac{1}{\tan(\alpha\phi)}\left\{\frac{1}{\tan(\phi)} - \frac{\alpha}{\tan(\alpha\phi)}\right\} ; \\
g(\phi) &= \frac{(3-\alpha)\cos((3+\alpha)\phi) + (\cot(\phi) - 2\alpha\cot(\alpha\phi))\sin((3-\alpha)\phi)}{(3-\alpha)\sin(\alpha\phi)\sin\phi} ; \\
h(\phi) &= \left(\frac{\sin((3-\alpha)\phi)}{(3-\alpha)\sin\phi}\right)^2 \left\{-3 - \alpha\cot^2(\alpha\phi) + (3-\alpha)\cot(\alpha\phi)\cot((3-\alpha)\phi)\right\} .
\end{aligned}$$

It is readily shown, by numerical calculation or symbol manipulation algebra that

$$H(\phi) < 0, \quad \phi \in [0, \pi/\alpha) .$$

Remark. It is not true that with $A = 0.966$ the same function H is negative. In fact $H(0.513) > 0$ for that choice of A, contrary to what is suggested in [1]. This makes no material difference to the estimate obtained. It may be worth offering reassurance that the $(3+\alpha)$ term in the expression for g is not a mis-print for $(3-\alpha)$.

Lemma 13 *$\mathcal{T}$ is closed in $\mathcal{C}\backslash\{(3,0)\}$.*

Proof. Let $(\mu_n, \theta_n) \in \mathcal{C}$ and $(\mu_n, \theta_n) \to (\mu, \theta) \in \mathcal{C}\backslash\{(3,0)\}$ as $n \to \infty$. Then $0 < \theta \le \pi/\alpha$ and $V(\mu,\theta)(e^{it}) \le 0$ on $(0,\pi)$. It will suffice to show that $V(\mu,\theta)(e^{it}) < 0$, $t \in (0,\pi)$; that $0 < \theta < \pi/\alpha$ then follows from Theorem 12. Suppose that this is false and that for $s \in (0,\pi)$, $V(\mu,\theta)(e^{is}) = 0$.

Observe that for $0 \le r < 1$

$$V(\mu,\theta)(re^{i0}) = 0 = V(\mu,\theta)(re^{i\pi}) ,$$

and therefore, by the Maximum Principle, $V(\mu,\theta) < 0$ in the upper half disc $D^+ = \{re^{it} : 0 < r < 1,\ 0 < t < \pi\}$ and hence $V(\mu,\theta)$ has a maximum on $\overline{D^+}$ at e^{is}. At this point the Hopf boundary-point lemma gives that the outward normal derivative of $V(\mu,\theta)$ is strictly positive and the tangential derivative of $V(\mu,\theta)$ on the unit circle at e^{is} is zero. Therefore, since $V(\mu,\theta)(e^{is}) = 0$,

$$\begin{aligned}
0 &< \frac{\partial}{\partial r} V(\mu,\theta)(e^{is}) \qquad (50) \\
&= e^{\alpha(a-\tau)} \Big[\theta_{rt}\sin(\alpha\theta) \\
&\quad + \alpha\theta_r\{\theta_t\cos(\alpha\theta) + \theta_r\sin(\alpha\theta)\} + \theta_{tt}\cos(\alpha\theta) \\
&\quad - \frac{3A\theta_r}{\sin\theta}\left(\frac{\theta_t\sin((3-\alpha)\theta)}{(3-\alpha)} + \frac{\theta_r\cos((3-\alpha)\theta)}{3}\right)\Big]_{e^{is}}
\end{aligned}$$

where we have used the Cauchy-Riemann equations and the identity

$$\theta_r(e^{it}) = \frac{1}{3} e^{3(\tau(e^{it})-a)} \sin\theta(e^{it})$$

to eliminate $e^{3(\tau-a)}$. Differentiation of this identity at $t=s$ gives

$$\theta_{rt}(e^{is}) = \{\cot(\theta)\theta_r\theta_t - 3\theta_r^2\}\Big|_{e^{is}} \ . \tag{51}$$

Also since $(\partial/\partial t)V(\mu,\theta)|_{e^{is}} = 0$ and $V(\mu,\theta)(e^{is}) = 0$, we find that at e^{is}

$$\begin{aligned} \theta_{tt} &= -\left\{\alpha\theta_t^2\cot(\alpha\theta) - \theta_{rt}\left(\cot(\alpha\theta) + \frac{A\sin((3-\alpha)\theta)}{(3-\alpha)\sin(\alpha\theta)\sin\theta}\right)\right. \\ &+ \left.\theta_r\theta_t\left(\alpha - \frac{A\cos((3-\alpha)\theta)}{\sin\theta\sin(\alpha\theta)} + \frac{A\sin((3-\alpha)\theta)\cos\theta}{(3-\alpha)\sin^2\theta\sin(\alpha\theta)}\right)\right\} \ . \end{aligned} \tag{52}$$

Furthermore, observe that because $V(\mu,\theta)(e^{is}) = 0$

$$\begin{aligned} \theta_t(e^{is}) &= \left\{\cot(\alpha\theta) + \frac{A\sin((3-\alpha)\theta)}{(3-\alpha)\sin\theta\sin(\alpha\theta)}\right\}\theta_r\Bigg|_{e^{is}} \\ &= \Gamma\theta_r(e^{is}) \text{ , say .} \end{aligned} \tag{53}$$

The equations (51), (52) and (53) enable all derivatives in (50), except θ_r^2, to be eliminated to yield

$$\begin{aligned} 0 &< e^{\alpha(a-\tau)}\theta_r^2\left\{\sin(\alpha\theta)[\Gamma\cot(\theta) - 3] + \alpha[\Gamma\cos(\alpha\theta) + \sin(\alpha\theta)]\right. \\ &- \cos(\alpha\theta)\left[\Gamma^2(\alpha\cot(\alpha\theta) - \cot\theta) + 3\Gamma\right. \\ &+ \left.\Gamma\left(\alpha - \frac{A\cos((3-\alpha)\theta)}{\sin\theta\sin(\alpha\theta)} + \frac{A\sin((3-\alpha)\theta)\cos\theta}{(3-\alpha)\sin^2\theta\sin(\alpha\theta)}\right)\right] \\ &- \left.\frac{3A}{\sin\theta}\left[\frac{\Gamma\sin((3-\alpha)\theta)}{(3-\alpha)} + \frac{\cos((3-\alpha)\theta)}{3}\right]\right\}\Bigg|_{e^{is}} \ . \end{aligned} \tag{54}$$

However, a *very* long though elementary trigonometric calculation yields that the right-hand side of (54) is

$$\left(e^{\alpha(a-\tau)}\frac{\theta_r^2}{\sin(\alpha\theta)}\right)\Bigg|_{e^{is}} H(\theta(e^{is})) < 0 \tag{55}$$

because of the choice of A and α. This contradiction completes the proof that $\mathcal{T}$ is closed in $\mathcal{C}\backslash\{(3,0)\}$. $\square$

Lemma 14 *$\mathcal{T}$ is open in $\mathcal{C}\backslash\{(3,0)\}$.*

Proof. Note first that if $(\mu,\theta) \in \mathcal{C}\backslash\{(3,0)\}$ then $\theta \in K$ and hence $\theta_t(e^{it}) \le 0,\ t \in [\pi/2,\pi]$. Hence $V(\mu,\theta)(e^{it}) < 0$ for $t \in [\pi/2,\pi)$ for all $(\mu,\theta) \in \mathcal{C}\backslash\{(3,0)\}$.

Now suppose that $\mathcal{T}$ is not open in $\mathcal{C}\backslash\{(3,0)\}$. Then there exists a sequence $\{(\mu_n,\theta_n)\} \subset \mathcal{C}\backslash\mathcal{T}$ such that $(\mu_n,\theta_n) \to (\mu,\theta) \in \mathcal{T}$ in $\mathbb{R}\times X$ as $n \to \infty$. Let $t_n \in (0,\pi/2)$ be such that

$$0 < V(\mu_n,\theta_n)(e^{it_n}) = \max\{V(\mu_n,\theta_n)(e^{it}) : t \in [0,\pi]\}$$

and, without loss of generality, suppose that $t_n \to t \in [0, \pi/2]$ as $n \to \infty$. If $t \in (0, \pi/2]$, then $V(\mu, \theta)(e^{it}) = 0$ which is impossible since $(\mu, \theta) \in \mathcal{C}$. Hence $t_n \to 0$ as $n \to \infty$.

Since $V(\mu_n, \theta_n)$ is harmonic in the upper half disc, it follows that

$$\frac{\partial}{\partial t} V(\mu_n, \theta_n)\Big|_{e^{itn}} = 0 \text{ and } \frac{\partial}{\partial r} V(\mu_n, \theta_n)\Big|_{e^{itn}} > 0$$

and

$$(\theta_n)_t(e^{it_n}) = ((\epsilon_n + 1)\Gamma_n)(\theta_n)_r(e^{it_n}) \tag{56}$$

where $\epsilon_n > 0$ and

$$\Gamma_n = \left\{ \cot(\alpha\theta_n) + \frac{A \sin((3-\alpha)\theta_n)}{(3-\alpha)\sin\theta_n \sin(\alpha\theta_n)} \right\}\Big|_{e^{itn}} ,$$

as in (53). To see that $\epsilon_n \to 0$ note from (44) and (56) that, since $t_n \to 0$ and $\mu_n \to \mu > 3$,

$$\frac{1}{1+\epsilon_n} = \frac{\Gamma_n(\theta_n)_r}{(\theta_n)_t}(e^{it_n}) \to \mu(A+1)/(3\alpha\theta_t(e^{i0})) .$$

Since $(\mu, \theta) \in \mathcal{T}$, $V(\mu, \theta)(e^{it}) < 0,\ t \in (0, \pi)$ and hence, with Γ defined in (53),

$$1 < \frac{\Gamma\theta_r}{\theta_t}(e^{it}) \to \mu(A+1)/(3\alpha\theta_t(e^{i0})) .$$

Hence $\mu(1+A)/3\alpha\theta_t(e^{i0}) = 1$ and $\epsilon_n \to 0$ as $n \to \infty$. Now (54) holds with Γ, θ and s replaced by $(1+\epsilon_n)\Gamma_n$, θ_n and t_n. Since

$$\Gamma_n \le \frac{\text{const.}}{\sin(\alpha\theta_n(e^{it_n}))} \text{ and } \epsilon_n \to 0 ,$$

it follows from a careful inspection of (54) and (55) that $H(0) = \lim_{n\to\infty} H(\theta(e^{it_n})) \ge 0$ for all n sufficiently large.

However, $H(0) < 0$. This contradiction shows that $\mathcal{T}$ is open in $\mathcal{C}\backslash\{(3,0)\}$. $\square$

12 The Wave of Greatest Height

In the mathematical formulation of the Stokes-wave problem (Section 2) it is required by **II(D)** that $|\bigtriangledown \psi(0,0)| \neq 0$. In other words, the relative velocity at a crest of the wave is non-zero. This led to the conclusion that the free surface is the graph of a real-analytic function (Section 4) and the re-formulation **S(a)-(g)** in Section 5. In the context of Nekrasov's equation, (14) means that solutions with $\beta = 1/\mu > 0$ correspond to such smooth solutions of the Stokes-wave problem. However, in Section 10 it is shown that there exists a solution θ^* of (38), Nekrasov's equation with $1/\mu = 0$, which arises as the pointwise limit of solutions of Nekrasov's equation with $\mu \to \infty$. In particular,

$$0 < \theta^*(s) < 0.5434,\ \ s \in (0, \pi) , \tag{57}$$

$$\int_0^t \sin\theta^*(\nu)d\nu > \gamma t \ , \tag{58}$$

$$\theta^*(t)/t \text{ is decreasing on } (0,\pi) \ . \tag{59}$$

These are all consequences of the fact that θ^* is the limit of a sequence θ_{μ_n} where $(\mu_n, \theta_{\mu_n}) \in \mathcal{C}$.

Such a solution of Nekrasov's equation, which as we will see cannot be continuous on $[0,\pi]$ and zero at zero, corresponds to a solution of **S(a)-(g)** with $|\bigtriangledown\psi(0,0)| = 0$. In hydrodynamic terminology there is a stagnation point at the wave crest. In a celebrated paper, Stokes [54] conjectured that waves with stagnation points at their crests exist, but that at the crest the wave profile has a *corner* containing an angle of $2\pi/3$. Here we give the proof [3] of this Stokes conjecture by proving that if θ^* is the function which satisfies (38) and arises as the pointwise limit of θ_{μ_n}, where $(\mu_n, \theta_n) \in \mathcal{C}$ and $\mu_n \to \infty$, then

$$\lim_{s\searrow 0} \theta^*(s) = \pi/6 \ . \tag{60}$$

The proof is by contradiction, and is reduced to a uniqueness question by the following lemma.

Lemma 15 *Suppose that*

$$\theta^*(s) = \frac{2}{3}\int_0^\pi K_1(s,t)\frac{\sin\theta^*(t)}{\int_0^t \sin\theta^*(w)dw}dt, \quad s \in (0,\pi) \ , \tag{61}$$

θ^* satisfies (57) – (59) and there exists a sequence $\alpha_n \searrow 0$ such that $\theta^*(\alpha_n) \to a$. Then there exists a solution ϕ of the equation

$$\phi(x) = \frac{1}{3\pi}\int_0^\infty \log\left(\frac{x+y}{|x-y|}\right)\frac{\sin\phi(y)}{\int_0^y \sin\phi(w)dw}dy, \quad x \in (0,\infty) \ , \tag{62}$$

with ϕ continuous, $\phi(x)/x$ non-increasing, $0 < \phi(x) < 0.544$ on $(0,\infty)$ and $\phi(1) = a$.

Proof. For each n, let

$$\phi_n(x) = \phi^*(\alpha_n x) \ , \quad x \in (0, \pi/\alpha_n) \ .$$

Then

$$\phi_n(x) = \frac{1}{3\pi}\int_0^{\pi/\alpha_n} \log\left(\frac{\sin\frac{\alpha_n}{2}(x+y)}{|\sin\frac{\alpha_n}{2}(x-y)|}\right)\frac{\sin\phi_n(y)}{\int_0^y \sin\phi_n(w)dw}\ dy \ ,$$

and $\displaystyle\int_0^y \sin\phi_n(w)dw \ \geq \ \gamma y, \quad y \in (0,\pi/\alpha_n)$. Moreover, by (59) and Helly's Selection Theorem [56], there is no loss of generality in assuming that $\phi_n(x) \to \phi(x)$ everywhere on $(0,\infty)$ as $n \to \infty$. A repeat of the argument for (41) now gives that ϕ satisfies (62) for all $x \in (0,\infty)$. Since $\int_0^y \sin\phi(w)dw \geq \gamma y$, and the integrand is non-negative and non-zero, the right-hand side of (60) defines a continuous positive function on $(0,\infty)$,

$$a = \lim_{n\to\infty}\theta^*(\alpha_n) = \lim_{n\to\infty}\phi_n(1) = \phi(1) \ .$$

That $\phi(x)/x$ is non-increasing on $(0,\infty)$ is a consequence of the same property of ϕ_n on $(0,\pi/\alpha_n)$ for each n. This completes the proof. □

The proof of (60) therefore follows from a proof of the uniqueness theorem which says that the only solution of (62) is $\theta \equiv \pi/6$ (see (40)).

Suppose henceforth that

$$\phi : (0,\infty) \to \mathbb{R} \text{ is continuous and satisfies (62),} \tag{63}$$

$$0 < \phi(x) < 0.5434, \quad x \in (0,\infty)\ , \tag{64}$$

$$\phi(x)/x \text{ is a non-increasing function of } x \in (0,\infty)\ . \tag{65}$$

Lemma 16

$$\int_0^x \sin\phi(y)dy \geq x/10, \quad x \in (0,\infty)\ .$$

Proof. By (63) and (65), for $x > 0$

$$\begin{aligned}
\phi(x) &\geq \left(\frac{1}{3\pi\int_0^x \sin\phi(w)dw}\right)\int_0^x y\log\left(\frac{x+y}{x-y}\right)\left(\frac{\sin\phi(y)}{y}\right)\,dy \\
&\geq \frac{\sin\phi(x)}{3\pi\int_0^x \sin\phi(w)dw}\int_0^x\left(\frac{y}{x}\right)\log\left(\frac{x+y}{x-y}\right)\,dy \\
&= \frac{x\sin\phi(x)}{3\pi\int_0^x \sin\phi(w)dw}\int_0^1 u\log\left(\frac{1+u}{1-u}\right)\,du \\
&= \frac{x\sin\phi(x)}{3\pi\int_0^\pi \sin\phi(w)dw}\ .
\end{aligned}$$

Hence

$$\frac{1}{x}\int_0^x \sin\phi(y)dy \geq 0.1$$

since $\phi \leq 0.544$ and $\sin\theta/\theta$ is a decreasing function of $\theta \in (0, 0.544)$. □

Let

$$p = \sup_{x\in(0,\infty)} \frac{1}{x}\left|\int_0^x \left(\phi(y) - \frac{\pi}{6}\right)dy\right| \text{ and } m = \sup_{x\in(0,\infty)}\left\{\phi(x) - \frac{\pi}{6}\right\}\ , \tag{66}$$

and note that by Jensen's inequality

$$\frac{1}{x}\int_0^x \sin\phi(y)dy \leq \sin\left(\frac{1}{x}\int_0^x \phi(y)dy\right) \leq \sin\left(\frac{\pi}{6}+p\right)\ , \quad x \in (0,\infty)\ , \tag{67}$$

and since $0 < \phi < 0.544$

$$\begin{aligned}
\frac{1}{x}\int_0^x \sin\phi(y)dy &\geq \frac{\sin(\pi/6+m)}{\pi/6+m}\frac{1}{x}\int_0^x \phi(y)dy \\
&\geq \frac{\sin(\pi/6+m)}{\pi/6+m}\left(\frac{\pi}{6}-p\right) \geq (.95)\left(\frac{\pi}{6}-p\right)\ .
\end{aligned} \tag{68}$$

Let

$$J = \inf_{x\in(0,\infty)}\frac{1}{x}\int_0^x \sin\phi(w)dw \text{ and } K = \sup_{x\in(0,\infty)}\int_0^x \sin\phi(w)dw\ . \tag{69}$$

Lemma 17 *(i)* $p \leq \frac{2}{3\pi}\log(1+\sqrt{2})\log\left(\frac{K}{J}\right)$; *(ii)* $p \leq .00304$.

Proof. By (40),

$$\int_0^x \left(\phi(z) - \frac{\pi}{6}\right) dz = \frac{1}{3}\int_0^\infty q(x,y)\frac{d}{dy}\left\{\log\frac{J}{y}\int_0^y \sin\phi(w)dw\right\} dy \ ,$$

where

$$q(x,y) = \frac{1}{\pi}\left\{x\log\frac{x+y}{|x-y|} + y\log\frac{|x^2-y^2|}{y^2}\right\}$$

and

$$q_y(x,y) = \frac{1}{\pi}\log\frac{|x^2-y^2|}{y^2} \ .$$

Now for $x \in (0,\infty)$, $q(x,y) = O(y\log y)$ as $y \to 0$ and $q(x,y) = O(1/y)$ as $y \to \infty$. Also, by definition of J,

$$\log\frac{J}{y}\int_0^y \sin\phi(w)dw \leq 0 \ ,$$

and since $(x-\sqrt{2}y)q_y(x,y) > 0$ for $y \in (0,\infty)\backslash\{x\}$, we find, upon integrating by parts, that

$$\int_0^x \left(\phi(z) - \frac{\pi}{6}\right) dz \geq -\frac{1}{3}\int_0^{x/\sqrt{2}} q_y(x,y)\log\left(\frac{K}{J}\right) dy$$

$$= -\frac{1}{3}q\left(x, \frac{x}{\sqrt{2}}\right)\log\left(\frac{K}{J}\right) = -\frac{2x}{3\pi}\log(1+\sqrt{2})\log\left(\frac{K}{J}\right) \ .$$

A similar integration on $(x/\sqrt{2},\infty)$ gives

$$\int_0^x \left(\phi(z) - \frac{\pi}{6}\right) dz \leq \frac{2x}{3\pi}\log(1+\sqrt{2})\log\left(\frac{K}{J}\right) \ .$$

Therefore, by Lemma 16 and (67)

$$p \leq \frac{2}{3\pi}\log(1+\sqrt{2})\log\left(10\sin\left(\frac{\pi}{6}+p\right)\right) \ .$$

Since $p \leq \pi/6$ it follows that $p \leq 0.387$. Also, by (67) and (68),

$$p \leq \frac{2}{3\pi}\log(1+\sqrt{2})\log\left(\frac{20}{19}\frac{\sin\left(\frac{\pi}{6}+p\right)}{\left(\frac{\pi}{6}-p\right)}\right) \ .$$

In the range $(0,\pi/6)$, equality holds in this inequality twice, at $p = 0.00304$ and $p = 0.44209$. Since the right-hand side is positive at 0 we concluded that $p \leq 0.00304$. □

Since .00304 radians represents $0.17°$, this simple estimate brings us tantalisingly close to the goal that $p = 0$. To close the gap we exploit the natural connection between equation (62) and another equation which involves the Hilbert transform. As was observed previously,

$$
\begin{aligned}
\phi(x) - \frac{\pi}{6} &= \frac{1}{3\pi}\int_0^\infty \log\left(\frac{x+y}{|x-y|}\right)\frac{d}{dy}\left\{\log\left(\frac{1}{y}\int_0^y \sin\phi(w)dw\right)\right\}dy \\
&= -\frac{1}{3\pi}\mathcal{P}\int_0^\infty \left(\frac{1}{y+x} - \frac{1}{y-x}\right)\log\left(\frac{1}{y}\int_0^y \sin\phi(w)dw\right)dy \\
&= \frac{1}{3\pi}\int_0^\infty \frac{1}{y}\log\left\{\frac{|x-y|}{x+y}\frac{\int_0^{x+y}\sin\phi(w)dw}{\int_0^{|x-y|}\sin\phi(w)dw}\right\}dy \\
&= \frac{1}{3\pi}\int_0^\infty \frac{1}{u}\log\left\{\frac{|1-u|}{1+u}\frac{\int_0^{x(1+u)}\sin\phi(w)dw}{\int_0^{x|1-u|}\sin\phi(w)dw}\right\}du\ , \\
&= \frac{1}{3\pi}\int_0^\infty \frac{1}{u}\log\left\{\frac{|1-u|}{1+u}\left(1+\frac{\int_{x|1-u|}^{x(1+u)}\sin\phi(w)dw}{\int_0^{x|1-u|}\sin\phi(w)dw}\right)\right\}du \\
&= \frac{1}{3\pi}\int_0^\infty \frac{\ell(u)}{u}du\ ,\ \text{say}\ . \qquad (70)
\end{aligned}
$$

Now for all $u > 0$

$$\int_{x|1-u|}^{x(1+u)} \sin\phi(w)dw \le 2xu\sin\left(\frac{\pi}{6}+m\right)$$

and

$$\int_0^{x|1-u|}\sin\phi(w)dw \ge x|1-u|J\ .$$

Hence

$$
\ell(u) \le \begin{cases} \dfrac{2u}{1+u}\left\{\dfrac{\sin(\pi/6+m)}{J}-1\right\} &,\quad 0 < u \le 1\ , \\ \dfrac{2}{1+u}\left\{\dfrac{\sin(\pi/6+m)}{J}-1\right\} &,\quad u > 1\ . \end{cases}
$$

A substitution of this estimate in the (70) now gives

$$\phi(x) - \frac{\pi}{6} \le \frac{4\log 2}{3\pi}\left\{\frac{\sin(\pi/6+m)}{J}-1\right\}\ , \qquad (71)$$

and in particular by (68)

$$m \le \frac{4\log 2}{3\pi}\left\{\frac{\pi/6+m}{\pi/6-p}-1\right\}\ .$$

Therefore,

$$
\begin{aligned}
m &\le \frac{4\log 2}{3\pi}\left(\frac{p}{\frac{\pi}{6}-p-(4/3\pi)\log 2}\right) \\
&\le (1.302)p = Qp\ ,\ \text{say}\ , \qquad (72)
\end{aligned}
$$

since $p \le 0.00304$.

Theorem 18 ([3]) *$p = 0$ and $\phi \equiv \pi/6$.*

Proof. According to Lemma 17 (i), (67) and (68),

$$\begin{aligned} p &\leq \frac{2}{3\pi}\log(1+\sqrt{2})\log\left\{\frac{\left(\frac{\pi}{6}+m\right)\sin(\pi/6+p)}{\sin\left(\frac{\pi}{6}+m\right)(\pi/-p)}\right\} \\ &\leq \frac{2}{3\pi}\log(1+\sqrt{2})\log\left\{\frac{(\pi/6+Qp)\sin(\pi/6+p)}{(\pi/6-p)\sin(\pi/6+Qp)}\right\} \end{aligned}$$

since $\theta/\sin\theta$ is increasing on $(0,\pi/2)$ and (72) holds. Since this is false for $0 < p \leq 0.418$ we conclude that $p = 0$. Thus

$$\int_0^x \left(\phi(x) - \frac{\pi}{6}\right) dx = 0 \text{ for all } x > 0$$

and hence $\phi \equiv \pi/6$, by differentiation. This completes the proof. □

It has been shown formally by Grant [26] and rigorously by Amick, Fraenkel [2] and McLeod [44] that

$$\theta^*(s) - \pi/6 \sim -As^\beta \text{ as } s \searrow 0$$

where $A > 0$ and $\beta \sim 0.802679$ is a root of the equation $\sqrt{3}(1+\beta) = \tan(\pi\beta/2)$. Indeed, modulo some number-theoretic conjectures, Amick and Fraenkel give an asymptotic expansion for $\theta^*(s)$, $s > 0$. Thus it is known that θ^* is decreasing on $(0,\epsilon)$ for some $\epsilon > 0$. Also θ^* is decreasing on $[\pi/2, \pi]$ because it arises as a pointwise limit of elements of the cone K.

13 Open Questions

A most pressing question is whether θ^* is monotone on $(0,\pi)$, as was conjectured when Stokes mused

> *"whether in the limiting form the inclination of the wave to the horizontal continually increases from the trough to the summit ... is one that I cannot certainly decide."*

This question of the convexity of Stokes wave of greatest height is a tantalising open question and, though the numerical evidence points strongly in its favour (see [16], [61], [62], for some relevant studies), an analytic proof has so far eluded us. The uniqueness of θ^* is also open. The operator defined by the right-hand side of equation (61) behaves numerically like a contraction mapping with contraction constant somewhat less than $\frac{1}{2}$, and iterations converge rapidly to a solution [16]. However, there is no mathematical explanation for this rapid convergence.

We finish with a brief discussion of the question of secondary bifurcation. The theory of bifurcation from a simple eigenvalue means that the maximal connected set $\mathcal{C}$ is a smooth curve in a neighbourhood of the bifurcation point $(3,0)$. Numerical evidence [17], [51] suggests that globally $\mathcal{C}$ is a curve, but there is no proof. A further set of open questions concerns the behaviour of θ when $(\mu, \theta) \in \mathcal{C}$ as $\mu \to \infty$. McLeod [43] has argued formally that the number of local maxima and minima of θ increases without bound as $\mu \to \infty$ but admits the lack of a watertight proof. Various other oscillatory phenomena which have been observed numerically [16] also lack mathematical justification.

Now, for each $n \in \mathbb{N}$, let

$$\mathcal{C}_n = \{(n\mu, \theta(nx)) : (\mu, \theta) \in \mathcal{C}\} .$$

A simple change of variables shows that $\mathcal{C}_n$ is a continuum of solutions of Nekrasov's equation which bifurcates from $(3n, 0)$, each element of which yields a steady water wave of minimal period $2\lambda/n$. Let $\mathcal{D}_n$ denote the maximal connected set of non-trivial solutions of Nekrasov's equation which contains $\mathcal{C}_n \backslash \{(3n, 0)\}$. Obviously $\mathcal{C}_1 = \mathcal{D}_1$ since $\mathcal{C}_1 = \mathcal{C}$ and $\mathcal{C}$ is maximal, by definition. But is $\mathcal{C}_n = \mathcal{D}_n$ for $n > 1$? The numerical evidence [8], [17], [51] strongly indicates that the answer is no: there are secondary bifurcation points on each continuum $\mathcal{C}_n$, $n \geq 2$. (Note that the elements of $\mathcal{D}_n$, $n \geq 2$, are not Stokes waves as they have more than one crest and trough per wavelength.) Furthermore, the evidence [8] suggests that there is a neighbourhood U of 0 in X such that no point $(\mu, \theta) \in [0, \infty) \times U$ is a secondary bifurcation point on $\mathcal{D}_n$ for any $n \in \mathbb{N}$. (Each $\mathcal{D}_n$ is a curve in a neighbourhood $(3n, 0)$; the point here is that U seems to be independent of n.)

A highly stimulating explanation for these secondary bifurcation phenomena has been offered in terms of the generic bifurcations of Hamiltonian dynamical systems in an important recent paper by Baesens and MacKay, [9]. They give a new and thought-provoking explanation of the huge complexity which has been uncovered numerically in the steady periodic water-wave problem. However, there is no rigorous theory so far.

Our mathematical understanding of steady waves and their stability properties in the wider context of unsteady flow theory is in its infancy, but there has been significant recent progress, [13], [14].

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