Destabilizing Effects of Small Time-Delays on Feedback Controlled Descriptor Systems*

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Abstract: In the last 15 years the problems of stabilizability and stabilization of descriptor systems have received considerable attention. In this paper it will be shown that if the open-loop system exhibits impulsive behaviour, then the stability of the closed-loop system is extremely sensitive to small delays. More precisely, if F is the feedback which leads to a stable and impulsive-free closed-loop system, then there exist numbers $\varepsilon_j > 0$ and $s_j \in \mathbb{C}$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and $\lim_{j\to\infty} \operatorname{Re} s_j = +\infty$ and such that the delayed closed-loop system obtained by applying the feedback $u(t) = Fx(t-\varepsilon_j)$ has a pole at s_j.

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1. Introduction

Stabilizability and pole-placement problems for linear descriptor systems given by

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

have received a great deal of attention in the last 15 years, see for example Bunse-Gerstner et al. [2, 3], Dai [4], Kautsky et al. [11], Mehrmann [16], Shayman [18] and Shayman and Zhou [19], to mention just a few references. Stabilization (pole assignment) by feedback of the form $u(t) = Fx(t)$ requires not only the closed-loop system to be stable (to have prescribed poles), but also that it be robust, in the sense that closed-loop stability (the configuration of the closed-loop poles) is insensitive to perturbations in the plant and controller data. The problem of robust pole placement for descriptor systems has for example been addressed in [11], where numerical procedures for generating robust feedback systems with prescribed poles are given. The perturbations considered in [11] are of the form

$$E \rightsquigarrow E + \delta E\,, \quad A \rightsquigarrow A + \delta A\,. \quad B \rightsquigarrow B + \delta B\,, \quad F \rightsquigarrow F + \delta F\,.$$

In this paper we consider perturbations which are induced by 'small' time-delays in the feedback loop, symbolically

$$u(t) = Fx(t) \rightsquigarrow u(t) = Fx(t - \varepsilon)\,.$$

It will be shown that if the descriptor system to be controlled exhibits impulsive behaviour, then the stability of the feedback system is extremely sensitive to such perturbations. More precisely, if $u(t) = Fx(t)$ is a feedback control which leads to a stable and impulsive-free closed-loop system, then there exist numbers $\varepsilon_j > 0$ and $s_j \in \mathbb{C}$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and $\lim_{j\to\infty} \operatorname{Re} s_j = +\infty$ and such that the delayed closed-loop system obtained by applying the feedback $u(t) = Fx(t - \varepsilon_j)$ has a pole at s_j.

The phenomenon of destabilization of feedback systems by arbitrarily small delays in the loop is not new and has been thoroughly studied in the context of infinite-diemnsional systems, but to the best of our knowledge has not been investigated for descriptor systems. It seems that the paper [1] by Barman et al. from 1973 is the first one devoted to this topic. More recently, researchers working in control of partial differential equations, 'rediscovered' the destabilizing effect of small delays in various examples involving vibrating systems, see for example Datko [5, 6], Datko et al. [7] and Desch and Wheeler [8]. Whilst these papers are based on partial differential equation and related techniques, a frequency-domain point of view is taken in [1]. The approach developed by Logemann and Rebarber [13] and Logemann et al. [14] is similar in spirit to that in [1], but is not tied to the restrictions imposed in [1] such as the assumption that the open-loop transfer function has at most finitely many poles in the closed right-half plane. Using results from [14], Logemann and Townley [15] have shown that small delays in the feedback loop can also have a destabilizing effect on certain neutral functional differential equations.

The paper is organized as follows. In Section 2 we present some preliminaries on descriptor systems which are needed later in the paper. In Section 3 we state a destabilization result from [13] for systems described by ill-posed (not necessarily rational) transfer function matrices. By 'ill-posed' we mean, that the transfer function matrix is unbounded in any right-half plane of the complex plane. We then show that the crucial assumption in this result is always satisfied for improper rational matrices. In Section 4 we apply the results of Section 3 to descriptor systems with impulsive behaviour which are strongly stabilizable in the sense that there exists a feedback $u(t) = Fx(t)$ which stabilizes the system and simultaneously eliminates its impulsive behaviour. The key idea here is to reformulate the closed-loop system as a system which has been obtained by applying output feedback to a controlled and observed descriptor system with improper transfer matrix. Finally, in Section 5 we consider dynamic feedback. We show that if an improper rational transfer function matrix can be stabilized in an L^2-sense by dynamic feedback, then the closed-loop system can be destabilized by arbitrarily small delays in the feedback loop.

Notation: In the following let $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$ (where $\alpha \in \mathbb{R}$) and let $\mathbb{C}_\alpha^{cl}$ denote the closure of $\mathbb{C}_\alpha$, i.e. $\mathbb{C}_\alpha^{cl} = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s \geq \alpha\}$. The field of all meromorphic functions on $\mathbb{C}_\alpha$ is denoted by M_α, while H_α^∞ denotes the algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$. Finally, if T is a matrix in $\mathbb{C}^{n\times n}$, then $\sigma(T)$ and $r(T)$ denote the spectrum and the spectral radius of T, respectively.

2. Preliminaries on Descriptor Systems

Consider a controlled descriptor system a of the form

$$E\dot{x}(t) = Ax(t) + Bu(t)\,, \tag{2.1}$$

where $E, A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. For the rest of the paper we shall assume that (2.1) is *regular*, i.e.

(A 1) $$\det(sE - A) \not\equiv 0\,. \qquad \diamond$$

As is well-known and as the following example shows, the system $E\dot{x} = Ax$ may have impulsive behaviour.

Example 2.1 Let us consider the following single-input system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)\,, \quad t > 0\,. \tag{2.2}$$

If $u(t) \equiv 0$, then we obtain for $t > 0$

$$\dot{x}_2(t) = x_1(t) \qquad \text{and} \qquad x_2(t) = 0\,.$$

Hence $x_1(t) = -x_2(0-)\delta(t)$, and we see that system (2.2) shows impulsive behaviour. $\diamond$

Since in the following we will be interested in descriptor systems with impulsive behaviour it is important to have "nice" characterizations for the existence of impulsive behaviour. Since (2.1) is regular there exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$QEP = \begin{pmatrix} I_{n_1} & 0 \\ 0 & N \end{pmatrix}, \qquad QAP = \begin{pmatrix} J & 0 \\ 0 & I_{n_2} \end{pmatrix}, \tag{2.3}$$

where $n_1+n_2 = n$, $N \in \mathbb{R}^{n_2\times n_2}$ and $J \in \mathbb{R}^{n_1\times n_1}$ are in Jordan form and N is nilpotent, see Dai [4], p. 7 or Gantmacher [9], p. 28. Sometimes, (2.3) is called the *Kronecker canonical form* of the pencil $sE - A$. The *index* of the pencil, denoted by $\operatorname{ind}(E, A)$, is defined to be the nilpotency degree ν of N (i.e. $N^\nu = 0$ and $N^{\nu-1} \neq 0$). If $n_2 = 0$, then we set $\operatorname{ind}(E, A) = 0$. Of course, $\operatorname{ind}(E, A) = 0$ if and only if the matrix E is non-singular.

Using the Kronecker canonical form it easy to see that the system $E\dot{x} = Ax$ admits impulsive solutions if and only if $\operatorname{ind}(E, A) > 1$. Moreover, the following simple result holds true.

Proposition 2.2 *If* (**A 1**) *is satisfied, then the following statements are equivalent*

(i) $\operatorname{ind}(E, A) > 1$,

(ii) *The rational matrix* $(sE - A)^{-1}$ *is not proper,*

(iii) *The rational matrix* $(sE - A)^{-1}E$ *is not strictly proper.*

Proof: **(i) ⇒ (ii):** Suppose that $\nu = \operatorname{ind}(E, A) > 1$ and consider the Kronecker canonical form (2.3). Since

$$s(sN - I)^{-1}N - I = (sN - I)^{-1} \tag{2.4}$$

and since N is nilpotent with nilpotency degree ν it follows that

$$(sN - I)^{-1}N^{\nu-1} = -N^{\nu-1} \neq 0\,.$$

Therefore we may conclude that the rational matrix $(sN - I)^{-1}N$ is not strictly proper, which, by (2.4) implies that $(sN - I)^{-1}$ is not proper. Thus, using (2.3), the rational matrix $(sE - A)^{-1}$ is not proper.

(ii) ⇒ (iii): Assume that (ii) holds. Let $s_0 \in \mathbb{C}$ be such that

$$\det(s_0E - A) \neq 0\,. \tag{2.5}$$

Setting $\mathbf{H}(s) := [(s + s_0)E - A]^{-1}(s_0E - A)$ it follows from the identity

$$s[(s + s_0)E - A]^{-1}E + [(s + s_0)E - A]^{-1}(s_0E - A) = I\,,$$

that

$$[(s + s_0)E - A]^{-1}E = \frac{1}{s}(I - \mathbf{H}(s))\,. \tag{2.6}$$

Using (2.5) and the hypothesis we see that $\mathbf{H}(s)$ is not proper. Consequently the right-hand side of (2.6) is not strictly proper and so $(sE-A)^{-1}E$ is not strictly proper.

(iii)$\Rightarrow$(i): Suppose that $(sE-A)^{-1}E$ is not strictly proper. Now

$$(sE-A)^{-1}A = s(sE-A)^{-1}E - I\,,$$

and so $(sE-A)^{-1}A$ is not proper, implying that $(sE-A)^{-1}$ is not proper either. Hence $(sN-I)^{-1}$ is not proper by (2.3). This yields that $N \neq 0$, which in turn gives $\operatorname{ind}(E,A) > 1$. □

System (2.1) is called *strongly stabilizable* if there exists a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that the closed-loop system

$$E\dot{x}(t) = (A+BF)x(t)$$

obtained from (2.1) by applying the feedback control $u(t) = Fx(t)$ is stable and impulsive free, i.e. $\det(sE-A-BF) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$ and $\operatorname{ind}(E, A+BF) \leq 1$.

It is easy to see that the system in Example 2.1 is strongly stabilizable and moreover the feedback $F = (f_1, f_2)$ is strongly stabilizing if and only if $f_1(1+f_2) > 0$. Although we shall not use it, we state the following theorem which gives a necessary and sufficient algebraic condition for strong stabilizability. For the proof of this result which is reminiscent of the Hautus criterion for state-space systems see Bunse-Gerstner et al. [2].

Theorem 2.3 *The system (2.1) is strongly stabilizable if and only if*

$$\operatorname{rank}(sE-A, B) = n \quad \forall\, s \in \mathbb{C}_0^{cl} \qquad \text{and} \qquad \operatorname{rank}(E, AS, B) = n\,,$$

where the columns of the real matrix S span $\ker E \cap \ker B^T$ and $S^TS = I$.

3. Destabilization by Small Delays in the Frequency Domain

Let $\Omega \subset \mathbb{C}$. A function $\mathbf{H} : \Omega \to \mathbb{C}^{m\times m}$ is called a ($\mathbb{C}^{m\times m}$-valued) *transfer function* if there exists $\alpha \in \mathbb{R}$ such that $\mathbb{C}_\alpha \subset \Omega$ and $\mathbf{H}|_{\mathbb{C}_\alpha} \in M_\alpha^{m\times m}$.

Let $\mathbf{H}$ be a transfer function and consider the feedback system shown in Figure 3.1, where u is the input function, y is the output function and the block with transfer function $e^{-\varepsilon s}$ represents a delay of length $\varepsilon \geq 0$. Delayed state feedback for descriptor systems is captured by the configuration shown in Fig. 3.1 (see Section 4).

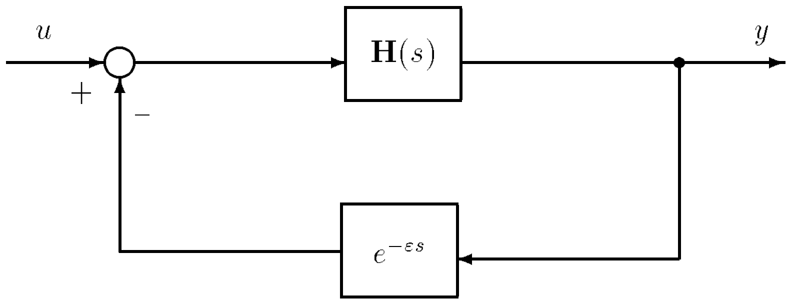

Figure 3.1: Feedback system with delay

If $\det(I + e^{-\varepsilon s}\mathbf{H}(s)) \not\equiv 0$, then the function $\mathbf{G}_\varepsilon$ defined by

$$\mathbf{G}_\varepsilon(s) := \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1} \tag{3.1}$$

is a transfer function, the so-called closed-loop transfer function of the feedback system shown in Figure 3.1.

Definition 3.1 Let $\mathbf{H}$ be a transfer function. $\mathbf{H}$ is called *well-posed* if $\mathbf{H} \in (H^\infty_\alpha)^{m\times m}$ for some $\alpha \in \mathbb{R}$. If $\mathbf{H}$ is not well-posed, then it is called *ill-posed.* $\diamond$

The terminology introduced in the above definition has its origin in the theory of abstract infinite-dimensional control systems. Roughly speaking, a well-posed transfer function can be realized by a "well-posed" state-space system and vice versa, see Salamon [17] and Weiss [20]. In many cases the transfer function of a descriptor system is an improper rational matrix, and hence ill-posed in the above sense.

The next result shows that for a large class of ill-posed transfer functions arbitarily small delays lead to closed-loop poles with arbitrarily large real parts. In order to state the theorem we introduce some more notation. For $\delta \in (0, \pi]$ define the open sector $\mathcal{S}(\delta)$ by

$$\mathcal{S}(\delta) := \{\lambda e^{i\varphi} \mid \lambda \in (0,\infty),\ \varphi \in (-\delta,\delta)\}\,,$$

and for $\psi \in [-\pi, \pi)$ set

$$\mathcal{S}(\psi,\delta) := e^{i\psi}\mathcal{S}(\delta)\,.$$

Theorem 3.2 *Let* $\mathbf{H}$ *be a transfer function and asssume that the following conditions hold:*

(i) *There exist* $\theta \in (0, \pi/2)$ *and* $\alpha > 0$ *such that* $\mathbf{H}(s)$ *is holomorphic in* $\mathcal{S}(\theta) \cap \mathbb{C}_\alpha$.

(ii) *There exist numbers $\rho > \alpha$, $\mu > 0$, $\psi \in [-\pi, \pi)$, $\delta \in (0, \theta)$ and $\eta \in (0, \pi/2)$ such that*

$$\lim_{|s|\to\infty,\, s\in\mathbb{S}(\delta)} r(\mathbf{H}(s)) = \infty\,, \tag{3.2}$$

$$r(\mathbf{H}(s)) \le |s|^\mu \qquad \text{for all } s \in \mathbb{S}(\delta)\cap\mathbb{C}_\rho\,, \tag{3.3}$$

$$\sigma(\mathbf{H}(s)) \subset \mathbb{C}\setminus\mathbb{S}(\psi,\eta) \qquad \text{for all } s \in \mathbb{S}(\delta)\cap\mathbb{C}_\rho\,. \tag{3.4}$$

Then there exist sequences (ε_j) and (s_j) with

$$\varepsilon_j > 0\,, \quad \varepsilon_j \to 0\,, \quad s_j \in \mathbb{C}_0\,, \quad \operatorname{Im} s_j \to \infty\,, \quad \operatorname{Re} s_j \to \infty$$

and such that for any $j \in \mathbb{N}$, s_j is a pole of $\mathbf{G}_{\varepsilon_j}$.

The proof of Theorem 3.2 can be found in Logemann and Rebarber [13]. Condition (3.2) guarantees that $\mathbf{H}$ is ill-posed, while condition (3.4) says that the spectrum of $\mathbf{H}(s)$ does not spiral around the origin as s moves in a sector of sufficiently small angle.

In order to apply Theorem 3.2 to descriptor systems we need the following technical result.

Proposition 3.3 *If $\mathbf{H}$ is a rational matrix of size $m \times m$, then there exist constants $\delta, \rho > 0$, $\psi \in [-\pi, \pi)$ and $\eta \in (0, \pi/2)$ such that condition (3.4) is satisfied.*

Proof: Define $\tilde{\mathbf{H}}(s) := s^\nu\,\mathbf{H}(1/s)$, where we choose $\nu \in \mathbb{N}$ such that $\tilde{\mathbf{H}}(0) := \lim_{s\to 0}\tilde{\mathbf{H}}(s) = 0$. We want to study the behaviour of $\sigma(\tilde{\mathbf{H}}(s))$ as $s \to 0$ in $\mathbb{S}(\delta)$ for some small $\delta > 0$. Consider

$$\mathbf{Q}(w,s) := \det(wI - \tilde{\mathbf{H}}(s)) = w^m + q_1(s)w^{m-1} + \ldots + q_m(s)\,,$$

where the $q_1(s), \ldots, q_m(s)$ are are real-rational functions. Write $q_i(s) = n_i(s)/d_i(s)$, where $n_i(s)$ and $d_i(s)$ are coprime real polynomials, and let $p_0(s)$ denote the lowest common multiple of $d_1(s), \ldots, d_m(s)$. Introducing the polynomials $p_i(s) = p_0(s)q_i(s)$ we define

$$\mathbf{P}(w,s) := p_0(s)\mathbf{Q}(s,w) = p_0(s)w^m + p_1(s)w^{m-1} + \ldots + p_m(s)\,,$$

which is a real polynomial in the variables s and w. Since $\tilde{\mathbf{H}}(s)$ is holomorphic at 0, it follows that $p_0(0) \neq 0$. Hence there exists an open neighbourhood U of 0 such that $p_0(s) \neq 0$ for all $s \in U$. As a consequence, if $s_0 \in U$, then a complex number w_0 is an eigenvalue of $\tilde{\mathbf{H}}(s_0)$ if and only if w_0 is a root of $\mathbf{P}(w, s_0) = 0$ and the multiplicities are the same.

CASE 1: Suppose that $\mathbf{P}(w,s)$ is irreducible. It then follows from algebraic function theory (see e.g. Hille [10], pp. 93 or Knopp [12], pp. 110) that there exists a multivalued analytic function $\mathbf{w}(s)$ satisfying $\mathbf{P}(\mathbf{w}(s), s) \equiv 0$. Notice that $\mathbf{w}(0) = \{0\}$, since $\sigma(\tilde{\mathbf{H}}(0)) = \sigma(0) = \{0\}$. Let $w_1(s), w_2(s), \ldots, w_m(s)$ denote the branches of $\mathbf{w}(s)$. There exists a neighbourhood $V_k \subset U$ of 0 such that for all $s \in V_k$, $w_k(s)$ can be represented by a Puiseux series of the form

$$w_k(s) = \sum_{j=1}^{\infty} \gamma_{kj} s^{j/p_k}\,, \qquad \text{for some } p_k \in \{1, 2, \ldots, m\}\,.$$

For each $k \in \{1, 2, \ldots, m\}$ let ℓ_k be the smallest integer such that $\gamma_{k\ell_k} \neq 0$. Then for any $\delta \in (0, \pi/2)$ we have that

$$\lim_{s \to 0,\, s \in \mathbb{S}(\delta)} \frac{w_k(s)}{\gamma_{k\ell_k} s^{\ell_k/p_k}} = 1\,.$$

Therefore we may write for all $s \in \mathbb{S}(\delta)$ of sufficiently small modulus

$$w_k(s) \approx \gamma_{k\ell_k} |s|^{\ell_k/p_k} e^{i[\arg s + 2\pi q_k]\ell_k/p_k}\,, \qquad \text{for some } q_k \in \{0, 1, \ldots, p_k - 1\}\,.$$

Setting $\varphi_k := \arg \gamma_{k\ell_k} + 2\pi q_k \ell_k/p_k$, we see that for any $\varepsilon > 0$

$$w_k(s) \in \mathbb{S}(\varphi_k, \ell_k\delta/p_k + \varepsilon)\,, \tag{3.5}$$

provided $s \in \mathbb{S}(\delta)$ and $|s|$ is sufficiently small. Consequently, for a sufficiently small $\delta_0 > 0$ there exist numbers $\psi \in [-\pi, \pi)$ and $\eta_0 \in (0, \pi/2)$ such that for all $s \in \mathbb{S}(\delta_0)$ of sufficiently small modulus, $\mathbf{w}(s) \subset \mathbb{C} \setminus \mathbb{S}(\psi, \eta_0)$, and hence

$$\sigma(\tilde{\mathbf{H}}(s)) \subset \mathbb{C} \setminus \mathbb{S}(\psi, \eta_0)\,.$$

Therefore, there exist numbers $\delta \in (0, \delta_0)$ and $\eta \in (0, \eta_0)$ such that for all $s \in \mathbb{S}(\delta)$ of sufficiently small modulus

$$\sigma[(1/s^\nu)\tilde{\mathbf{H}}(s)] \subset \mathbb{C} \setminus \mathbb{S}(\psi, \eta)\,.$$

Now recall that $\mathbf{H}(1/s) = (1/s^\nu)\tilde{\mathbf{H}}(s)$, and so it follows that for all $s \in \mathbb{S}(\delta)$ of sufficiently large modulus

$$\sigma(\mathbf{H}(s)) \subset \mathbb{C} \setminus \mathbb{S}(\psi, \eta)\,.$$

Hence by choosing $\rho > 0$ sufficiently large we have that

$$\sigma(\mathbf{H}(s)) \subset \mathbb{C} \setminus \mathbb{S}(\psi, \eta) \qquad \forall\, s \in \mathbb{S}(\delta) \cap \mathbb{C}_\rho\,,$$

which is (3.4).

Case 2: If $\mathbf{P}(w, s)$ is not irrecducible, we can write

$$\mathbf{P}(w, s) = \prod_{j=1}^{\iota} \mathbf{P}_j(w, s)\,,$$

where the polynomials $\mathbf{P}_1(w, s), \ldots, \mathbf{P}_\iota(w, s)$ are irrecducible. Then each of the equations $\mathbf{P}_i(w, s) = 0$ defines an algebraic function $\mathbf{w}_i(s)$ whose branches satisfy a condition similar to (3.5). Consequently, we can invoke the same argument as in Case 1 to prove the claim. $\square$

4. Descriptor Systems with Delayed Feedback Control

In this section we need an extra assumption in addition to (**A 1**)

(**A 2**) System (2.1) is strongly stabilizable. $\diamond$

In order to apply Theorem 3.2 to state-feedback controlled descriptor systems we need the following lemma.

Lemma 4.1 *Suppose that* (**A 1**) *and* (**A 2**) *are satisfied and let* $F \in \mathbb{R}^{m\times n}$ *be a strongly stabilizing feedback matrix for (2.1). If* $\operatorname{ind}(E, A) > 1$*, then the rational matrix* $F(sE - A)^{-1}B$ *is improper.*

Proof: Set $\mathbf{X}(s) := (sE - A - BF)^{-1}$. Then $\mathbf{X}(s)$ is a rational matrix and since F is strongly stabilizing, it follows from Proposition 2.2 that

$$\mathbf{X}(s) \text{ is proper}. \tag{4.1}$$

Now $(sE - A)\mathbf{X}(s) - BF\mathbf{X}(s) = I$, and so

$$\mathbf{X}(s) - (sE - A)^{-1}BF\mathbf{X}(s) = (sE - A)^{-1}. \tag{4.2}$$

By hypothesis, $\operatorname{ind}(E, A) > 1$, and thus using Proposition 2.2 we have that $(sE - A)^{-1}$ is improper. Combining (4.1) and (4.2) we see that

$$(sE - A)^{-1}B \text{ is improper}. \tag{4.3}$$

On the other hand, we obtain from $\mathbf{X}(s)(sE - A) - \mathbf{X}(s)BF = I$ that

$$\mathbf{X}(s)B - \mathbf{X}(s)BF(sE - A)^{-1}B = (sE - A)^{-1}B. \tag{4.4}$$

Combining (4.1), (4.3) and (4.4) we see that $F(sE - A)^{-1}B$ is improper. $\square$

Consider the descriptor system (2.1). An application of the feedback control

$$u(t) = v(t) + Fx(t - \varepsilon), \qquad \text{where} \quad \varepsilon \geq 0$$

leads to the closed-loop system

$$E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon) + Bv(t). \tag{4.5}$$

Here $v(t)$ denotes the input into the feedback system. In order to apply the input-output results in Section 3, we introduce the controlled and observed descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = -Fx(t). \tag{4.6}$$

By an application of the output-feedback law

$$u(t) = v(t) - y(t-\varepsilon)$$

to system (4.6) we obtain the same closed-loop system (4.5). Since the transfer function of (4.6) is given by

$$\mathbf{H}(s) = -F(sE - A)^{-1}B\,, \tag{4.7}$$

it follows that

$$\mathbf{G}_\varepsilon(s) = \mathbf{H}(s)(I + e^{-\varepsilon s}\mathbf{H}(s))^{-1} = -F(sE - A - e^{-\varepsilon s}BF)^{-1}B$$

is the transfer function of the closed-loop system (4.5).

Proposition 4.2 *Suppose that* (**A 1**) *and* (**A 2**) *are satisfied and let* $F \in \mathbb{R}^{m\times n}$ *be a strongly stabilizing feedback matrix for (2.1). If* $\operatorname{ind}(E, A) > 1$*, then* $\lim_{|s|\to\infty} r(F(sE - A)^{-1}B) = \infty$.

Proof: Defining $\mathbf{H}(s)$ by (4.7), it follows from Lemma 4.1 that

$$\lim_{|s|\to\infty} \|\mathbf{H}(s)\| = \infty\,, \tag{4.8}$$

where $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{C}^m$. By hypothesis, $\operatorname{ind}(E, A + BF) \leq 1$, and so $(sE - A - BF)^{-1}$ is proper (by Proposition 2.2). Therefore, we obtain that the rational matrix

$$\mathbf{G}_0(s) = \mathbf{H}(s)(I + \mathbf{H}(s))^{-1} = -F(sE - A - BF)^{-1}B \quad \text{is proper}. \tag{4.9}$$

It follows from the identity $\mathbf{H} = \mathbf{G}_0(I - \mathbf{G}_0)^{-1}$ via (4.8) and (4.9) that $\lim_{|s|\to\infty} \det(I - \mathbf{G}_0(s)) = 0$. Setting $D := \lim_{|s|\to\infty} \mathbf{G}_0(s)$, we obtain that $\det(I - D) = 0$, and thus

$$1 \in \sigma(D)\,. \tag{4.10}$$

Let (s_j) be a sequence of complex numbers such that $\lim_{j\to\infty} |s_j| = \infty$. For sufficiently large j, s_j is not a pole of either $\mathbf{H}$ or $\mathbf{G}_0$. Thus $-1 \notin \sigma(\mathbf{H}(s_j))$, because otherwise s_j would be a pole of $(I + \mathbf{H})^{-1}$, and hence of $\mathbf{G}_0 = \mathbf{H}(I + \mathbf{H})^{-1} = I - (I + \mathbf{H})^{-1}$. Consequently, if $\xi_j \in \sigma(\mathbf{H}(s_j))$, then $\lambda_j := \xi_j/(1 + \xi_j)$ is a well-defined complex number and $\lambda_j \in \mathbf{G}_0(s_j)$. Moreover, any $\lambda_j \in \sigma(\mathbf{G}_0(s_j))$, can be written as $\lambda_j = \xi_j/(1+\xi_j)$, where $\xi_j \in \sigma(\mathbf{H}(s_j))$. It therefore follows from (4.10) that there exist numbers $\xi_j \in \sigma(\mathbf{H}(s_j))$ such that

$$\lim_{j\to\infty} \frac{\xi_j}{1 + \xi_j} = 1\,.$$

But this implies that $\lim_{j\to\infty} |\xi_j| = \infty$, and so $\lim_{j\to\infty} r(\mathbf{H}(s_j)) = \infty$. Since this is true for any sequence with $\lim_{j\to\infty} |s_j| = \infty$, it follows that $\lim_{|s|\to\infty} r(\mathbf{H}(s)) = \infty$. □

A solution $x(t)$ of equation (4.5) with $v(t) \equiv 0$ is called a *mode* of (4.5) if

$$x(t) = e^{s_0 t} x_0 \,, \qquad \text{where} \quad s_0 \in \mathbb{C}, \;\; x_0 \in \mathbb{C}^n \setminus \{0\} \,.$$

The number s_0 is called the *exponent* of the mode. Setting

$$\mathbf{\Delta}_\varepsilon(s) := sE - A - e^{-\varepsilon s} BF \,,$$

it is trivial to prove that (4.5) has a mode with exponent s_0 if and only if $\det \mathbf{\Delta}_\varepsilon(s_0) = 0$.

We are now in the position to state and prove the main result of this paper.

Theorem 4.3 *Suppose that* (**A 1**) *and* (**A 2**) *are satisfied and let* $F \in \mathbb{R}^{m \times n}$ *be a strongly stabilizing feedback matrix for (2.1). If* $\mathrm{ind}(E, A) > 1$, *then there exist sequences* (ε_j) *and* (s_j) *with*

$$\varepsilon_j > 0 \,, \quad \varepsilon_j \to 0 \,, \quad s_j \in \mathbb{C}_0 \,, \quad \operatorname{Im} s_j \to \infty \,, \quad \operatorname{Re} s_j \to \infty$$

and such that for any $j \in \mathbb{N}$, *the delayed closed-loop system*

$$E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon_j) + Bv(t)$$

has a mode with exponent s_j.

Proof: Define $\mathbf{H}(s)$ by (4.7). Since $\mathbf{H}(s)$ is a rational matrix, there clearly exists a number $\mu > 0$ such that $r(\mathbf{H}(s)) \leq |s|^\mu$ for $|s|$ sufficiently large. Combining this with propositions 3.3 and 4.2 it follows that $\mathbf{H}(s)$ satisfies the conditions (i) and (ii) in Theorem 3.2. Hence there there exist sequences (ε_j) and (s_j) with

$$\varepsilon_j > 0 \,, \quad \varepsilon_j \to 0 \,, \quad s_j \in \mathbb{C}_0 \,, \quad \operatorname{Im} s_j \to \infty \,, \quad \operatorname{Re} s_j \to \infty$$

and such that s_j is a pole of

$$\mathbf{G}_{\varepsilon_j}(s) = \mathbf{H}(s)(I + e^{-\varepsilon_j s}\mathbf{H}(s))^{-1} = -F(sE - A - e^{-\varepsilon_j s} BF)^{-1} B \,.$$

Hence s_j is a pole of $\mathbf{\Delta}_{\varepsilon_j}^{-1}(s) = (sE - A - e^{-\varepsilon_j s} BF)^{-1}$, and so $\det \mathbf{\Delta}_{\varepsilon_j}(s_j) = 0$. It follows that the closed-loop system $E\dot{x}(t) = Ax(t) + BFx(t - \varepsilon_j) + Bv(t)$ has a mode with exponent s_j. $\square$

The system in Example 2.1 satisfies the assumptions of the above theorem. Moreover, as already mentioned the feedback $F = (f_1, f_2)$ is strongly stabilizing if and only if $f_1(1 + f_2) > 0$. However, Theorem 4.3 shows that for any such feedback there exist delays $\varepsilon_j > 0$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and complex numbers s_j with $\lim_{j\to\infty} \operatorname{Re} s_j = \infty$ and such that the closed-loop system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} 1 & 0 \\ f_1 & 1 + f_2 \end{pmatrix} x(t - \varepsilon_j)$$

has a mode with exponent s_j.

Remark 4.4 In Lemma 4.1, Proposition 4.2 and Theorem 4.3 we have assumed that (**A 2**) holds and that the feedback matrix F is strongly stabilizing. However, an inspection of the proofs in this section shows that we have never used the property that F is stabilizing in the sense that the exponents of the closed-loop modes have negative real parts. What actually has been used is that F renders the closed-loop system (4.5) impulsive-free. Therefore the results in this section remain true if instead of (**A 2**) we merely assume that any impulsive behaviour of $E\dot{x} = Ax + Bu$ can be eliminated by some suitable feedback F (i.e. there exists a matrix F such that $\mathrm{ind}(E, A + BF) \leq 1$). $\diamond$

5. Delayed Dynamic Feedback for Descriptor Systems: A Frequency-Domain Point of View

In Section 4 we have considered static feedback. Several authors have investigated more general feedback schemes for descriptor systems, such as proportional and derivative feedback or dynamic compensation, see for example Bunse-Gerstner et al. [3], Shayman [18] and Shayman and Zhou [19]. In this section we sketch a frequency-domain approach which shows that these more complicated controllers do not exhibit better robustness properties with respect to small delays in the feedback loop than static proportional feedback.

In the following let $\mathcal{R} = \mathbb{R}(s)$ denote the field of real-rational functions. The transfer function matrix $\mathbf{P}(s)$ of an m-input, p-output regular descriptor system of the form

$$E\dot{x}(t) = Ax(t) + Bu(t)\,, \tag{5.1a}$$
$$y(t) = Cx(t) + Du(t)\,, \tag{5.1b}$$

is given by $\mathbf{P}(s) = C(sE - A)^{-1}B + D$ and is an element in $\mathcal{R}^{p\times m}$. Conversely, any matrix $\mathbf{P} \in \mathcal{R}^{p\times m}$ can be realized by a regular descriptor system. A fairly general class of dynamic feedback compensators is given by

$$N\dot{z}(t) = Fz(t) + Gy(t) + H\dot{y}(t)\,, \tag{5.2a}$$
$$u(t) = Kz(t) + Ly(t) + M\dot{y}(t)\,, \tag{5.2b}$$

where we assume that $\det(sN - F) \not\equiv 0$. The transfer function of (5.2) can then be expressed as $\mathbf{K}(s) = K(sN - F)^{-1}(G + sH) + L + sM$.

Definition 5.1 If $\mathbf{P} \in \mathcal{R}^{p\times m}$ and $\mathbf{K} \in \mathcal{R}^{m\times p}$, we say that $\mathbf{K}$ *stabilizes* $\mathbf{P}$ if $\det(I + \mathbf{P}(s)\mathbf{K}(s)) \not\equiv 0$ and the transfer function

$$F(\mathbf{P},\mathbf{K}) = \begin{pmatrix} \mathbf{K}(I+\mathbf{PK})^{-1} & -\mathbf{KP}(I+\mathbf{KP})^{-1} \\ \mathbf{PK}(I+\mathbf{PK})^{-1} & \mathbf{P}(I+\mathbf{KP})^{-1} \end{pmatrix} \tag{5.3}$$

is bounded in $\mathbb{C}_0$, i.e. $F(\mathbf{P},\mathbf{K}) \in (H_0^\infty)^{(m+p)\times(m+p)}$. $\diamond$

Note that $F(\mathbf{P},\mathbf{K})$ is the transfer function from $(\hat{u}_1,\hat{u}_2)$ to $(\hat{y}_1,\hat{y}_2)$ of the feedback system shown in Fig. 5.1 (where the superscript ˆ denotes Laplace transformation) if we take there $\varepsilon = 0$. In the time-domain, Definition 5.1 means that L^2-inputs (u_1,u_2) are mapped boundedly to L^2-outputs (y_1,y_2). In the following the transfer function of the feedback system in Fig. 5.1 is denoted by $F_\varepsilon(\mathbf{P},\mathbf{K})$. Clearly, $F_0(\mathbf{P},\mathbf{K}) = F(\mathbf{P},\mathbf{K})$. Moreover, setting $\mathbf{K}_\varepsilon(s) = e^{-\varepsilon s}\mathbf{K}(s)$, we see that

$$F_\varepsilon(\mathbf{P},\mathbf{K}) = \begin{pmatrix} \mathbf{K}(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1} & -\mathbf{K}_\varepsilon\mathbf{P}(I+\mathbf{K}_\varepsilon\mathbf{P})^{-1} \\ \mathbf{P}\mathbf{K}(I+\mathbf{P}\mathbf{K}_\varepsilon)^{-1} & \mathbf{P}(I+\mathbf{K}_\varepsilon\mathbf{P})^{-1} \end{pmatrix}. \tag{5.4}$$

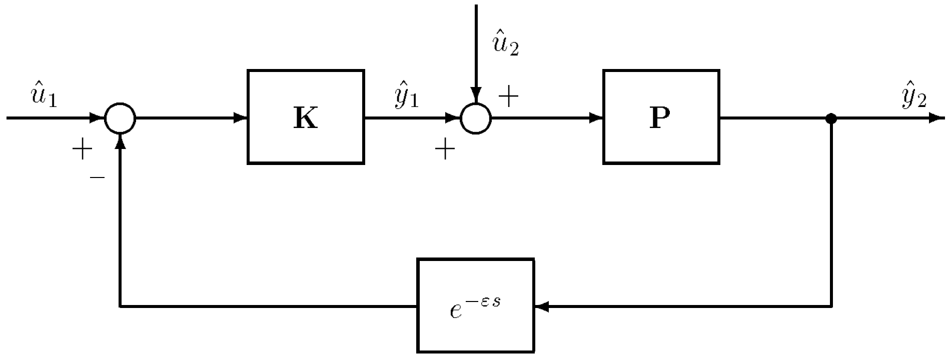

Figure 5.1: Feedback system with plant, compensator and delay

Lemma 5.2 *Let $\mathbf{P} \in \mathcal{R}^{p\times m}$ and $\mathbf{K} \in \mathcal{R}^{m\times p}$ and suppose that $\det(I+\mathbf{P}(s)\mathbf{K}(s)) \not\equiv 0$ and that $F(\mathbf{P},\mathbf{K})$ is proper. If $\mathbf{P}$ or $\mathbf{K}$ is improper, then*

$$\lim_{|s|\to\infty} r(\mathbf{P}(s)\mathbf{K}(s)) = \infty\,, \tag{5.5}$$

Proof: Using similar arguments as in the proof of lemma 8.3 in [14] it follows that the rational matrix $\mathbf{PK}$ is improper. An inspection of the proof of Proposition 4.2 (with $\mathbf{H} = \mathbf{PK}$) then shows that (5.5) is true. □

Combining Theorem 3.2, Proposition 3.3 and Lemma 5.2 we obtain the following destabilization result.

Theorem 5.3 *Let $\mathbf{P} \in \mathcal{R}^{p\times m}$ and $\mathbf{K} \in \mathcal{R}^{m\times p}$ and suppose suppose that $\mathbf{K}$ stabilizes $\mathbf{P}$. If $\mathbf{P}$ or $\mathbf{K}$ is improper, then there exist sequences (ε_j) and (s_j) with*

$$\varepsilon_j > 0\,, \quad \varepsilon_j \to 0\,, \quad s_j \in \mathbb{C}_0\,, \quad \mathrm{Im}\, s_j \to \infty\,, \quad \mathrm{Re}\, s_j \to \infty$$

and such that for any $j \in \mathbb{N}$, s_j is a pole of $\mathbf{P}(s)\mathbf{K}(s)(I+e^{-\varepsilon_j s}\mathbf{P}(s)\mathbf{K}(s))^{-1}$, and hence of the overall closed-loop transfer function $F_{\varepsilon_j}(\mathbf{P},\mathbf{K})$ given by (5.4).

The above theorem shows that L^2-stability which is robust with respect to small delays cannot be achieved if the system to be controlled or the controller is improper.

Example 5.4 Consider the single-input system (2.2) with observation given by

$$y(t) = (1,0)x(t) = x_1(t)\,.$$

This system has the transfer function $\mathbf{P}(s) = -s$, which is improper. Hence, by Theorem 5.3, there is no compensator of the form (5.2) which achieves robust closed-loop stability. $\diamond$

Remark 5.5 (i) Theorem 5.3 remains true if we replace the assumption that $\mathbf{K}$ stabilizes $\mathbf{P}$ by the weaker assumption that the rational matrix $F(\mathbf{P},\mathbf{K})$ is proper (cf. Remark 4.4).

(ii) Suppose that system (5.1) is strongly stabilizable and strongly detectable†. Then it is not difficult to see that $\mathrm{ind}(E,A) > 1$ if and only if the transfer function matrix $\mathbf{P}(s) = C(sE-A)^{-1}B + D$ is improper. Thus, if $\mathrm{ind}(E,A) > 1$ and if $\mathbf{K}$ stabilizes $\mathbf{P}$, then Theorem 5.3 applies.

(iii) Theorem 4.3 can be re-derived by combining Proposition 4.2 and Theorem 5.3. $\diamond$

References

[1] J.F. Barman, F.M. Callier and C.A. Desoer. L^2-stability and L^2-instability of linear time-invariant distributed feedback systems perturbed by a small delay in the loop, *IEEE Trans. Auto. Control* **18** (1973), 479–484.

[2] A. Bunse-Gerstner, V. Mehrmann and N.K. Nichols. Derivative feedback for descriptor systems, FSP Mathematisierung, Universität Bielefeld, Materialien LVIII, 1989.

[3] A. Bunse-Gerstner, V. Mehrmann and N.K. Nichols. Regularization of descriptor systems by derivative and proportional state feedback, *SIAM J. Matrix Anal. Appl.* **13** (1992), 46–67.

[4] L. Dai. *Singular Control Systems*, Springer-Verlag, Berlin, 1989.

[5] R. Datko. Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, *SIAM J. Control & Opt.* **26** (1988), 697–713.

[6] R. Datko. Two examples of ill-posedness with respect to small time delays in stabilized elastic systems, *IEEE Trans. Auto. Control* **38** (1993), 163–166.

[7] R. Datko, J. Lagnese and M.P. Polis. An example of the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control & Opt.* **24** (1986), 152–156.

[8] W. Desch and R.L. Wheeler. Destabilization due to delay in one-dimensional feedback, pp. 61–83 in *Control and Estimation of Distributed Parameter Systems* edited by F. Kappel, K. Kunisch and W. Schappacher, Birkhäuser Verlag, Boston, 1989.

† The concept of *strong detectability* is obtained by dualizing the notion of strong stabilizability defined in Section 2.

[9] F.R. Gantmacher. *The Theory of Matrices, Vol. II*, Chelsea Publishing Company, New York, 1959.

[10] E. Hille. *Analytic Function Theory, Vol. II*, Chelsea Publishing Company, New York, 1962.

[11] J. Kautsky, N.K. Nichols and E. K.-W. Chu. Robust pole assignment in singular control systems, *Linear Algebra and its Applications* **121** (1989), 9–37.

[12] K. Knopp. *Funktionentheorie, Bd. II* (12. Aufl.), Walter de Gruyter & Co., Berlin, 1971.

[13] H. Logemann and R. Rebarber. The effect of small time-delays on the stability of feedback systems described by partial differential equations, Mathematics Preprint 95/11, University of Bath, 1995 † (submitted for publication).

[14] H. Logemann, R. Rebarber and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop, *SIAM J. Control & Opt.* **34** (1996), 572–600.

[15] H. Logemann and S. Townley. The effect of small delays in the feedback loop on the stability of neutral systems, Mathematics Preprint 95/21, University of Bath, 1995 † (to appear in *Systems & Control Letters*).

[16] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem: Theory and Numerical Solution* Springer-Verlag, Berlin, 1991.

[17] D. Salamon. Realization theory in Hilbert space, *Math. Systems Theory* **21** (1989), 147–164.

[18] M.A. Shayman. Pole placement by dynamic compensation for descriptor systems. *Automatica* **24** (1988), 279–282.

[19] M.A. Shayman and Z. Zhou. Feedback control and classification of generalized linear systems. *IEEE Trans. Auto. Control* **32** (1987), 483–494.

[20] G. Weiss. Transfer functions of regular linear systems, part I: characterization of regularity, *Trans. Amer. Math. Soc.* **342** (1994), 827–854.

† Preprints in this series are available by anonymous ftp from `ftp.maths.bath.ac.uk` in the directory `/pub/preprints`.