HARMONIC TWO-SPHERES IN COMPACT SYMMETRIC SPACES, REVISITED

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Introduction

The purpose of this article is to give a new description of harmonic maps from the two-sphere S^2 to a compact symmetric space G/K, using a method suggested by Morse theory. The method leads to surprisingly short proofs of most of the known results, and also to new results.

The simplest non-trivial examples of such harmonic maps occur when $G/K = S^n$ or $\mathbf{C}P^n$. In 1967, E. Calabi gave a construction of all harmonic maps $S^2 \to S^n$. In the early 1980s, all harmonic maps $S^2 \to \mathbf{C}P^n$ were obtained by a similar construction; this work was carried out independently by various authors, full details appearing in a 1983 paper of J. Eells and J. C. Wood. Motivation for continued interest in this problem had been provided by mathematical physicists, searching for simple analogues of the Yang-Mills equations, and indeed the above constructions turned out to be related to twistor theory. From this point of view, harmonic maps $\phi : S^2 \to S^n$ or $\mathbf{C}P^n$ were characterized as compositions of the form $\phi = \pi \circ \Phi$, where $\Phi : S^2 \to Z$ is a "super-horizontal" holomorphic map into a "twistor space" Z, and $\pi : Z \to S^n$ or $\mathbf{C}P^n$ is a "twistor fibration". These results are surveyed, with complete references, in [Ee-Le]. During the 1980s, various generalizations of the construction $\phi = \pi \circ \Phi$ appeared, for harmonic maps $S^2 \to G/K$ into various compact symmetric spaces G/K. These were refined into a general and purely Lie-theoretic treatment in [Bu-Ra]. We shall refer to this as "the twistor construction".

Except in the original cases $G/K = S^n$ or $\mathbf{C}P^n$, the twistor construction does not produce *all* harmonic maps $S^2 \to G/K$. Even in the case $G/K = Gr_2(\mathbf{C}^4)$ there exist harmonic maps which do not arise via the twistor construction. The search for a more general construction led to several special techniques (see the survey [Ee-Le]), by means of which all harmonic maps $S^2 \to Gr_k(\mathbf{C}^n)$ could, in principle at least, be constructed from "holomorphic data". A quite different procedure was introduced by K. Uhlenbeck in [Uh], for harmonic maps $S^2 \to U_n$ or $Gr_k(\mathbf{C}^n)$, using ideas from another part of mathematical physics, namely the theory of integrable systems. The construction was made more explicit by Wood in [Wo1] and [Wo2]. However, it remained unclear how to generalize these results to other compact Lie groups or symmetric spaces, and even in the case $G = U_n$ or $G/K = Gr_k(\mathbf{C}^n)$ the results were much less explicit than for $G/K = S^n$ or $\mathbf{C}P^n$.

In this paper we shall give a straightforward description of all harmonic maps from S^2 to a compact Lie group G, in terms of the special harmonic maps obtained via the twistor construction. Since any compact symmetric space can be immersed totally geodesically in its group of isometries, this description includes all harmonic maps from S^2 to compact symmetric spaces as well. In the case of an *inner* symmetric space, we shall describe the harmonic maps in full detail.

Our description has two main features. First, we obtain new and very simple proofs of the known results for $G = U_n$ and $G/K = Gr_k(\mathbf{C}^n)$, and we make those results more explicit. Second, our method works equally well for more general G and G/K, where few results were previously known.

The fact that all harmonic maps $S^2 \to S^n$ or $\mathbf{C}P^n$ arise from the twistor construction is an easy consequence of our method. A proof of this fact using a similar strategy has been given by G. Segal in [Se], but our approach shows clearly why S^n and $\mathbf{C}P^n$ are privileged in this way. Other well known results on harmonic maps $S^2 \to G$ or G/K have simple explanations within our framework, such as the existence of various "Bäcklund transformations", or the factorization theorem of [Uh] for harmonic maps $S^2 \to U_n$ or $Gr_k(\mathbf{C}^n)$. We shall give a proof of this factorization theorem which explains Lie-theoretically why there is no straightforward generalization to other Lie groups.

The idea of our method is very simple: given a harmonic map ϕ, we apply a (special) family of "dressing transformations", to obtain a family of harmonic maps $\{\phi^t\}_{t\in(0,\infty)}$, such that $\phi^\infty = \lim_{t\to\infty} \phi^t$ is a harmonic map obtained via the twistor construction. This leads to explicit results because the family has a Morse-theoretic interpretation, namely that ϕ^t is obtained by applying the gradient flow of a certain well known Morse function (to an associated map).

The main technical result is a classification of harmonic maps into types, the types (for each fixed G or G/K) being indexed by a finite number of elements in the Lie algebra $\mathbf{g}$ of G. (Strictly speaking, these types correspond to a finite number of conjugacy classes of parabolic subalgebras of the complexified Lie algebra $\mathbf{g}^{\mathbf{C}}$.) Geometrical properties of harmonic maps are reflected by Lie-algebraic properties of these elements. Our first main application of this is an estimate of the "minimal uniton number" $r(\phi)$ of a harmonic map ϕ. For $G = U_n$, Uhlenbeck showed that the maximal value of $r(\phi)$ is $n-1$. We obtain the maximal values for all G and G/K. The second main application is a "Weierstrass formula", by means of which harmonic maps $S^2 \to G$ or G/K are described explicitly in terms of meromorphic functions on S^2 (i.e. rational functions). This includes the explicit formulae which were known for $G/K = S^n$ or $\mathbf{C}P^n$, and underlies Wood's formulae for $G = U_n$. It generalizes a Weierstrass formulae of R. Bryant for harmonic maps obtained via the twistor construction (see [Br1],[Br2]).

While this method is quite succesful in extending and clarifying the existing theory, we believe that our point of view can be justified further. For example, our Weierstrass formulae are sufficiently more explicit than the existing formulae (even in the case $G = U_n$ or $G/K = Gr_k(\mathbf{C}^n)$) that they can be used to study the space "Harm" of harmonic maps. Similar Morse-theoretic ideas have already been used to study $\mathrm{Harm}(S^2, S^n)$ and $\mathrm{Harm}(S^2, \mathbf{C}P^n)$ in [Gu-Oh] and [Fu-Gu-Ko-Oh], and our methods should permit generalizations to other G and G/K. Secondly, our approach to Weierstrass formulae was influenced by the paper [Do-Pe-Wu], in which a general scheme is suggested for constructing harmonic maps $M \to G$ or G/K in terms of "Weierstrass data", for compact Riemann surfaces M of arbitrary genus g. Our results show how this scheme may be implemented in the case $g = 0$. The case $g = 1$ has received much recent attention, and the scheme seems likely to be implementable also in this case (see [Bu-Fe-Pe-Pi], [Bu], [Mc]).

The paper is arranged as follows. In §1 we review the standard reformulation of harmonic maps in terms of extended solutions, from [Uh]. We also discuss complex extended solutions, a notion suggested in [Do-Pe-Wu]. We give a general definition of uniton number and minimal uniton number. Morse theory enters the picture in the form of the classical energy functional on the loop group ΩG. This is well known from [Bo], and goes back to Morse himself. We use the modern version provided by the theory of loop groups (see [Pr-Se]), and so we review this in §2. The main ingredients are the Birkhoff and Bruhat decompositions, which are most conveniently described in a purely algebraic manner. In §3 we review the twistor construction for harmonic maps into symmetric spaces, following [Bu-Ra]. Such harmonic maps are characterized by the fact that they correspond to S^1-invariant extended solutions, for a certain S^1-action on ΩG. (This relevance of this action for harmonic maps was first pointed out by C.-L. Terng.) These harmonic maps are fundamental in our theory. As explained earlier, our method is to analyse general extended solutions in terms of associated S^1-invariant extended solutions. This is where Morse theory, in the guise of the Bruhat decomposition, enters. Our results are given in §4 for harmonic maps $S^2 \to G$, and in §5 for harmonic maps $S^2 \to G/K$. In fact, the results apply equally well to harmonic maps $M \to G$ or G/K of finite uniton number, where M is any Riemann surface. This includes those maps described in the literature as isotropic, or superminimal. Finally, there are two short appendices. In Appendix A we collect together the special results which apply to harmonic maps of "low uniton number". To some extent this explains the historical development of the subject, in which the fundamental results for S^n and

$\mathbf{C}P^n$ were gradually extended to more complicated symmetric spaces. Appendix B considers the role of the Birkhoff decomposition, and the relation between our method and the "meromorphic potentials" of [Do-Pe-Wu].

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§1 Review of extended solutions

Extended solutions and harmonic maps.

Let G be a connected compact Lie group, with Lie algebra $\mathbf{g}$. Let M be a connected (but not necessarily compact) Riemann surface, without boundary. To study harmonic maps $M \to G$, we use the well known correspondence between harmonic maps and extended solutions, as in [Uh]. The relevant definitions will be summarized briefly in this section.

If X is any manifold, we denote the "free" and "based" loop spaces of X (respectively) by ΛX, ΩX, where

$$\begin{aligned}\Lambda X &= \{\gamma : S^1 \to X \mid \gamma \text{ is smooth}\}\\ \Omega X &= \{\gamma \in \Lambda X \mid \gamma(1) = x\}\end{aligned}$$

where x is a fixed basepoint of X. If X is a group, we refer to these as the free and based *loop groups* of X, and we take $x = e$, the identity element of the group. We consider S^1 here to be the set of unit complex numbers, i.e. $S^1 = \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}$.

Let $G^{\mathbf{C}}$ be the complexification of G, with Lie algebra $\mathbf{g}^{\mathbf{C}}$ (thus $\mathbf{g}^{\mathbf{C}} = \mathbf{g} \otimes \mathbf{C}$).

Definition: A map $\Phi : M \to \Omega G$ is an *extended solution* if it satisfies the equation

$$(E) \qquad \Phi(z,\lambda)^{-1}\Phi_z(z,\lambda) = \tfrac{1}{2}(1 - \tfrac{1}{\lambda})A(z)$$

for some map $A : M \to \mathbf{g}^{\mathbf{C}}$. (Here we write $\Phi(z,\lambda)$ for $\Phi(z)(\lambda)$, with $z \in M$ and $\lambda \in S^1$, and we write Φ_z for the partial derivative of Φ with respect to z.)

Theorem 1.1 [Uh].

(1) If $\Phi : M \to \Omega G$ is an extended solution, then the map $\phi : M \to G$ defined by $\phi(z) = \Phi(z,-1)$ is harmonic.

(2) If $\phi : M \to G$ is harmonic, and if M is simply connected, then there exists an extended solution $\Phi : M \to \Omega G$ such that $\phi(z) = \Phi(z,-1)$. This Φ is unique up to multiplication on the left by an element $\gamma \in \Omega G$ such that $\gamma(-1) = e$. $\square$

It is sometimes convenient to say that an extended solution $\tilde{\Phi} : M \to \Lambda G$ is a map which satisfies the following equation:

$$(\tilde{E}) \qquad \tilde{\Phi}(z,\lambda)^{-1}\tilde{\Phi}_z(z,\lambda) = B(z) + \tfrac{1}{\lambda}C(z)$$

for some maps $B, C : M \to \mathbf{g}^{\mathbf{C}}$. Certainly a solution of (E) is a solution of $(\tilde{E})$; conversely, if $\tilde{\Phi}$ is a solution of $(\tilde{E})$, then it is easy to verify that $\Phi(z,\lambda) = \tilde{\Phi}(z,\lambda)\tilde{\Phi}(z,1)^{-1}$ is a solution of (E).

In this paper we shall be concerned entirely with extended solutions of "finite uniton number". This concept was introduced in [Uh] for the case $G = U_n$. We shall give a definition for general G, which extends the definition of [Uh] in a natural way. First, recall that the adjoint representation is the homomorphism

$$\mathrm{Ad} : G \to O(\mathbf{g}), \quad \mathrm{Ad}(g)\xi = \tfrac{d}{dt} g(\exp\, t\xi)g^{-1}|_0,$$

where $O(\mathbf{g})$ is the orthogonal group of (the vector space) $\mathbf{g}$ with respect to a fixed bi-invariant inner product on $\mathbf{g}$. If G is a matrix group, then $\mathrm{Ad}(g)\xi = g\xi g^{-1}$. The image $\mathrm{Ad}\, G$ is isomorphic to $G/Z(G)$, where $Z(G)$ is the centre of G.

Definition: A loop $\gamma : S^1 \to G$ is *algebraic* if it is the restriction of an algebraic map $\mathbf{C}^* \to G^{\mathbf{C}}$.

The algebraic loops in ΩG form a subgroup, $\Omega_{\mathrm{alg}} G$. If $\theta : G \to U_n$ is a representation of G, and $\gamma \in \Omega_{\mathrm{alg}} G$, then $\theta(\gamma)$ is necessarily a polynomial in λ and λ^{-1}. Taking $\theta = \mathrm{Ad}$, we obtain a filtration

$$\{e\} = \Omega^0_{\mathrm{alg}} G \subseteq \Omega^1_{\mathrm{alg}} G \subseteq \ldots \subseteq \cup_{k \geq 0} \Omega^k_{\mathrm{alg}} G = \Omega_{\mathrm{alg}} G$$

by defining $\Omega^k_{\mathrm{alg}} G$ to be the set of loops γ such that $\mathrm{Ad}(\gamma)$ is of the form $\sum_{|i| \leq k} \lambda^i T_i$.

Definition: Let $\Phi : M \to \Omega G$ be an extended solution. If $\Phi(M) \subseteq \Omega^k_{\mathrm{alg}} G$ and $\Phi(M) \not\subseteq \Omega^{k-1}_{\mathrm{alg}} G$, we say that Φ has *uniton number* k. We write $r(\Phi) = k$.

That this is not a vacuous definition is shown by the next theorem:

Theorem 1.2 [Uh],[Se]. *Let $\Phi : M \to \Omega G$ be an extended solution. If M is compact, then there exists some $\gamma \in \Omega G$ and some $k \geq 0$ such that $\gamma\Phi(M) \subseteq \Omega^k_{\mathrm{alg}} G$.* $\square$

Actually, Uhlenbeck only considers the case $G = U_n$, and proves that if $\Phi : M \to \Omega U_n$ is an extended solution then there is some $\gamma \in \Omega U_n$ such that $\gamma\Phi$ has the form $\sum_{i=0}^k \lambda^i A_i$. It is easy to deduce Theorem 1.2 from this, by viewing $\mathrm{Ad}\, G$ as a subgroup of the unitary group $U(\mathbf{g}^{\mathbf{C}})$. Uhlenbeck goes on to define the uniton number of $\gamma\Phi = \sum_{i=0}^k \lambda^i A_i$ to be k. This definition coincides with ours so long as $A_0 \neq 0$. (Our definition factors out the effect of scalar loops λ^k, which are assigned uniton number 0.)

When M is both compact and simply connected, i.e. $M = S^2$, Theorems 1.1 and 1.2 say that any harmonic map $M \to G$ corresponds to an extended solution of finite uniton number. Because of this, our results will apply to arbitrary harmonic maps $S^2 \to G$, but only to those harmonic maps $M \to G$ which come from extended solutions of finite uniton number.

Since the correspondence between harmonic maps and extended solutions is not one to one (even when M is simply connected), we need a separate definition for the "uniton number of a harmonic map". Let $\phi : M \to G$ be a harmonic map, and let $\Phi : M \to \Omega G$ be an extended solution with $r(\Phi) < \infty$ such that $\Phi(z, -1) = \phi(z)$. Associated to ϕ we have the non-negative integer

$$\tilde{r}(\phi) = \min\{r(\gamma\Phi) \mid \gamma \in \Omega_{\mathrm{alg}} G\}.$$

Unlike the definition of $r(\Phi)$, however, the definition of $\tilde{r}(\phi)$ depends on the local isomorphism class of G. This is not appropriate for our purposes, as harmonic maps into locally isomorphic Lie groups are (locally) equivalent. So we modify $\tilde{r}(\phi)$ as follows:

Definition: Let $\phi : M \to G$ be a harmonic map. Assume that there exists an extended solution $\Phi : M \to \Omega G$ such that $r(\Phi) < \infty$ and $\Phi(z, -1) = \phi(z)$. We define the *minimal uniton number* of ϕ as the non-negative integer

$$r(\phi) = \min\{r(\gamma \operatorname{Ad} \Phi) \mid \gamma \in \Omega_{\mathrm{alg}} \operatorname{Ad} G\}.$$

The minimal uniton number of ϕ, which depends only on $\operatorname{Ad}\phi$, is a measure of the "complexity" of the harmonic map. In the case $G = U_n$, our definition agrees exactly with Uhlenbeck's definition.

We conclude with a simple example which illustrates the advantages of $r(\phi)$ over $\tilde{r}(\phi)$. Let $\phi : S^2 \to S^2$ be a holomorphic map. Consider the totally geodesic embedding $i : S^2 \cong \mathbf{C}P^1 \to SU_2$, $l \mapsto (p - p^\perp)(\pi_l - \pi_l^\perp)$, where π_l denotes Hermitian projection on l, and p is a fixed Hermitian projection operator of rank 1. The composition $i \circ \phi$ is a harmonic map $S^2 \to SU_2$. A suitable extended solution $\Phi : S^2 \to \Omega SU_2$ is given by $\Phi(z, \lambda) = (p + \frac{1}{\lambda} p^\perp)(\pi_{\phi(z)} + \lambda \pi^\perp_{\phi(z)})$. It is easy to see that $\tilde{r}(\phi) = r(\Phi) = 2$. However, we find that $r(\phi) = 1$, because $\Omega PSU_2 = \Omega SU_2 \sqcup [p + \frac{1}{\lambda} p^\perp]\Omega SU_2$. (We consider ΩSU_2 as the identity component of ΩPSU_2, and $[p + \frac{1}{\lambda} p^\perp]$ as the topologically non-trivial loop in $PSU_2 \cong U_2/Z(U_2)$ which is given by the loop $p + \frac{1}{\lambda} p^\perp$ in U_2.) Thus, it is only by passing from G to $\operatorname{Ad} G$ that we are able to remove the superfluous factor $p + \frac{1}{\lambda} p^\perp$.

Complex extended solutions.

It is well known (see [Pr-Se]) that the loop group ΩG admits the structure of an (infinite dimensional) complex manifold. One way to describe this complex structure comes from the "Iwasawa decomposition" of the loop group $\Lambda G^{\mathbf{C}}$. In the following statement, $\Lambda^+ G^{\mathbf{C}}$ denotes the subgroup of $\Lambda G^{\mathbf{C}}$ consisting of maps $S^1 \to G^{\mathbf{C}}$ which extend holomorphically to the region $D^+ = \{\lambda \mid |\lambda| < 1\}$:

Theorem 1.3 [Pr-Se]. *The product map $\Lambda^+ G^{\mathbf{C}} \times \Omega G \to \Lambda G^{\mathbf{C}}$ is a diffeomorphism. In particular, $\Lambda G^{\mathbf{C}} = \Omega G \, \Lambda^+ G^{\mathbf{C}}$, and any $\gamma \in \Lambda G^{\mathbf{C}}$ may be written uniquely in the form $\gamma = \gamma_u \gamma_+$, where $\gamma_u \in \Omega G$ and $\gamma_+ \in \Lambda^+ G^{\mathbf{C}}$.* $\square$

This result has a number of useful consequences. The first of these concerns the definition of the complex structure on ΩG. Each of the groups $\Lambda G^{\mathbf{C}}$, $\Lambda^+ G^{\mathbf{C}}$ is, in a natural way, a complex Lie group, and so the homogeneous space $\Lambda G^{\mathbf{C}} / \Lambda^+ G^{\mathbf{C}}$ is a complex manifold. By the theorem, we have an identification $\Omega G \cong \Lambda G^{\mathbf{C}} / \Lambda^+ G^{\mathbf{C}}$, so this gives the required complex structure on ΩG. It follows from standard theory of homogeneous spaces that the holomorphic tangent bundle $T^{1,0}\Omega G$ of ΩG may be identified with $\Lambda G^{\mathbf{C}} \times_{\Lambda^+ G^{\mathbf{C}}} (\Lambda \mathbf{g}^{\mathbf{C}} / \Lambda^+ \mathbf{g}^{\mathbf{C}})$.

Second, the theorem shows that there is a natural action of the complex group $\Lambda G^{\mathbf{C}}$ on the complex manifold ΩG, given by the natural action of $\Lambda G^{\mathbf{C}}$ on $\Lambda G^{\mathbf{C}} / \Lambda^+ G^{\mathbf{C}}$. We can express this action in the following way: if $\gamma \in \Lambda G^{\mathbf{C}}$ and $\delta \in \Omega G$, then

$$\gamma \cdot \delta = (\gamma \delta)_u.$$

It turns out that $\Lambda G^{\mathbf{C}}$ acts as a symmetry group of the extended solution equation (E), i.e. if Φ is an extended solution then so is $\gamma \cdot \Phi$. It was shown in [Gu-Oh] that this action is essentially the same as the "dressing action" introduced in [Uh].

The third consequence is that we may reformulate the extended solution equation (E) in terms of the complex loop group. Let us write $\Phi = [\Psi]$, where $\Psi : M \to \Lambda G^{\mathbf{C}}$. (Such a map Ψ exists locally, at least. If M is simply connected, Ψ exists globally.) Note that $[\Psi]$ may also be written Ψ_u.

Proposition 1.4. *The map Φ satisfies (E) if and only if Ψ satisfies*

$$(E^{\mathbf{C}}) \qquad \begin{cases} \operatorname{Im} \lambda \Psi^{-1} \Psi_z & \subseteq \ \Lambda^+ \mathbf{g}^{\mathbf{C}} \\ \operatorname{Im} \Psi^{-1} \Psi_{\bar z} & \subseteq \ \Lambda^+ \mathbf{g}^{\mathbf{C}} \end{cases}$$

*(where "*Im*" denotes "image").*

Proof. By Theorem 1.3 we may write $\Psi = \Phi \Delta$, where $\Delta : M \to \Lambda^+ G^{\mathbf{C}}$. The identities $\Psi^{-1}\Psi_z = \Delta^{-1}\Phi^{-1}\Phi_z \Delta + \Delta^{-1}\Delta_z$, $\Psi^{-1}\Psi_{\bar z} = \Delta^{-1}\Phi^{-1}\Phi_{\bar z}\Delta + \Delta^{-1}\Delta_{\bar z}$ show that $(E^{\mathbf{C}})$

is equivalent to

$$(E') \qquad \begin{cases} \text{Im } \lambda\Phi^{-1}\Phi_z & \subseteq \Lambda^+\mathbf{g}^{\mathbf{C}} \\ \text{Im } \Phi^{-1}\Phi_{\bar{z}} & \subseteq \Lambda^+\mathbf{g}^{\mathbf{C}} \end{cases}.$$

But (E') is equivalent to (E), because if $\Phi^{-1}\Phi_z = \sum \lambda^i A_i$ then $\Phi^{-1}\Phi_{\bar{z}} = \sum \lambda^{-i}\bar{A}_i$, where the bar denotes complex conjugation in $\mathbf{g}^{\mathbf{C}}$ with respect to the real form $\mathbf{g}$. $\square$

Because of the identification $T^{1,0}\Omega G \cong \Lambda G^{\mathbf{C}} \times_{\Lambda^+ G^{\mathbf{C}}} (\Lambda\mathbf{g}^{\mathbf{C}}/\Lambda^+\mathbf{g}^{\mathbf{C}})$, the condition Im $\Psi^{-1}\Psi_{\bar{z}} \subseteq \Lambda^+\mathbf{g}^{\mathbf{C}}$ says that $\Phi = [\Psi] : M \to \Omega G$ is holomorphic. This is the key to yet another reformulation of the extended solution equation. Let $\Lambda^* G^{\mathbf{C}}$ denotes the subgroup of $\Lambda G^{\mathbf{C}}$ consisting of maps $S^1 \to G^{\mathbf{C}}$ which extend holomorphically to the region $D^* = \{\lambda \mid 0 < |\lambda| < 1\}$.

Definition: A holomorphic map $\Psi : M \to \Lambda^* G^{\mathbf{C}}$ is a *complex extended solution* if Im $\lambda\Psi^{-1}\Psi_z \subseteq \Lambda^+\mathbf{g}^{\mathbf{C}}$.

Theorem 1.5 [Do-Pe-Wu].

(1) If $\Psi : M \to \Lambda^* G^{\mathbf{C}}$ is a complex extended solution, then $\Phi = \Psi_u$ is an extended solution.

(2) If $\Phi : M \to \Omega G$ is an extended solution, and z_0 is any point of M, then there exists a neighbourhood M_0 of z_0 and a complex extended solution $\Psi : M_0 \to \Lambda G^{\mathbf{C}}$ such that $\Phi|_{M_0} = \Psi_u$. $\square$

We shall give a proof of Theorem 1.5 in the case of extended solutions of finite uniton number, in §4.

§2 Review of loop groups

Birkhoff and Bruhat decompositions.

The (based) loop group ΩG may be identified with the complex homogeneous space $\Lambda G^{\mathbf{C}}/\Lambda^+ G^{\mathbf{C}}$, as we remarked in §1. This is proved in [Pr-Se] by showing that both spaces may be identified with a certain infinite dimensional complex Grassmannian. The analogy between ΩG and (finite dimensional) Grassmannians $Gr_k(C^n)$ is in fact one of the main themes of [Pr-Se]. In this section we shall review one aspect of this, namely the Birkhoff and Bruhat decompositions of ΩG. These are analogous to the Schubert decomposition of $Gr_k(C^n)$.

Let us choose a maximal torus T of G. The group of homomorphisms $S^1 \to T$ may be identified with the integer lattice $I = (\exp 2\pi)^{-1}(e) \cap \mathbf{t}$ in $\mathbf{t}$, by associating to $\xi \in I$

the homomorphism $\gamma_\xi : \lambda = e^{\sqrt{-1}\,t} \mapsto \exp\, t\xi$. Let us also choose a fundamental Weyl chamber in $\mathbf{t}$. The intersection of I with this will be denoted I'. Whereas I parametrizes homomorphisms $S^1 \to T$, I' parametrizes conjugacy classes of homomorphisms $S^1 \to G$ (or, equivalently, orbits of homomorphisms $S^1 \to T$ under the Weyl group).

Theorem 2.1 (Chapter 8 of [Pr-Se]).

(1) Birkhoff decomposition: $\Lambda G^{\mathbf{C}} = \bigsqcup_{\xi \in I'} \Lambda^- G^{\mathbf{C}}\, \gamma_\xi\, \Lambda^+ G^{\mathbf{C}}$.

(2) Bruhat decomposition: $\Lambda_{\mathrm{alg}} G^{\mathbf{C}} = \bigsqcup_{\xi \in I'} \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}}\, \gamma_\xi\, \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}}$.

(A loop $\gamma \in \Lambda G^{\mathbf{C}}$ is said to be algebraic if both $\mathrm{Ad}\,\gamma$ and $\mathrm{Ad}\,\gamma^{-1}$ are of the form $\sum_{|i| \le k} \lambda^i T_i$; $\Lambda_{\mathrm{alg}} G^{\mathbf{C}}$ denotes the subgroup of algebraic loops, and $\Lambda^+_{\mathrm{alg}} G^{\mathbf{C}}$ denotes $\Lambda_{\mathrm{alg}} G^{\mathbf{C}} \cap \Lambda^+ G^{\mathbf{C}}$.)

Corollary 2.2 (Chapter 8 of [Pr-Se]).

(1) Birkhoff decomposition: $\Omega G = \bigsqcup_{\xi \in I'} \Lambda^- G^{\mathbf{C}} \cdot \gamma_\xi$.

(2) Bruhat decomposition: $\Omega_{\mathrm{alg}} G = \bigsqcup_{\xi \in I'} \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}} \cdot \gamma_\xi$.

The nature of these decompositions is best understood in terms of Morse theory. We shall review this next, following [Pr], [Pr-Se].

Morse theory.

Let $E : \Omega G \to \mathbf{R}$ denote the usual energy functional on paths, $E(\gamma) = \int_{S^1} |\gamma'|^2$. This is a Morse-Bott function, i.e. each of its critical manifolds is non-degenerate. The critical points are the geodesics in G which pass through the identity element, i.e. the homomorphisms $S^1 \to G$, and the (connected) critical manifolds are the conjugacy classes of such homomorphisms. We write

$$\Omega_\xi = \{ g\gamma_\xi g^{-1} \mid g \in G \}$$

for the conjugacy class of the homomorphism γ_ξ. We write

$$S_\xi = \{ \gamma \in \Omega G \mid \ \gamma \text{ flows into } \Omega_\xi \}$$
$$U_\xi = \{ \gamma \in \Omega G \mid \ \gamma \text{ flows out of } \Omega_\xi \}$$

for the unstable and stable manifolds of Ω_ξ (where "flow" refers to the flow of the gradient vector field $-\nabla E$). The unstable manifold U_ξ has the structure of a vector bundle over Ω_ξ; we denote the bundle map by $u_\xi : U_\xi \to \Omega_\xi$. The rank of this bundle is the Morse index of γ_ξ, which may be expressed in terms of the roots of G (we shall give the formula shortly). All these facts were discovered by Bott ([Bo]).

The relation with the Birkhoff and Bruhat decompositions is given by the following result of Pressley:

Theorem 2.3 [Pr].

(1) $S_\xi = \Lambda^- G^{\mathbf{C}} \cdot \gamma_\xi$.

(2) $U_\xi = \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}} \cdot \gamma_\xi$.

(A similar theorem holds for the Schubert decomposition of the Grassmannian $Gr_k(\mathbf{C}^n)$, and, in fact, for any generalized flag manifold - see [Pa].)

The theorem is proved in [Pr] by taking a "Hamiltonian" point of view, in contrast to the differential geometric methods of [Bo]. Namely, the energy functional E is viewed as a Hamiltonian function associated to a symplectic action of the group S^1 on ΩG; this action is given by

$$u \cdot \gamma(\lambda) = \gamma(u\lambda)\gamma(u)^{-1}, \quad u \in S^1, \gamma \in \Omega G.$$

The critical points of E are the fixed points of this action. The symplectic structure of ΩG actually comes from a Kähler structure. If the gradient is taken with respect to the Kähler metric, then the flow of $-\nabla E$ can be described *explicitly* in terms of the "complexification" of the S^1-action. Let $\mathbf{C}^*_{\geq 1} = \{\lambda \in \mathbf{C} \mid 1 \leq |\lambda| < \infty\}$. This semigroup acts on ΩG by

$$u \cdot \gamma(\lambda) = \gamma(u\lambda)\Lambda^+ G^{\mathbf{C}} \in \Lambda G^{\mathbf{C}}/\Lambda^+ G^{\mathbf{C}}, \quad u \in \mathbf{C}^*_{\geq 1}, \gamma \in \Omega G.$$

It turns out that the flow line of $-\nabla E$ starting at a point $\gamma \in \Omega G$ is given by the action of the subsemigroup $[1, \infty)$ of $\mathbf{C}^*_{\geq 1}$. The flow line of ∇E starting at γ is defined for all time if and only $\gamma \in \Omega_{\mathrm{alg}} G$, in which case it is given by the action of the subsemigroup $(0, 1]$ of $\mathbf{C}^* = \{\lambda \in \mathbf{C} \mid 0 < |\lambda| < \infty\}$. Note that the formula for the action of $\mathbf{C}^*_{\geq 1}$ on ΩG extends to an action of $\mathbf{C}^*$ on $\Omega_{\mathrm{alg}} G$. It was observed by Terng (see §7 of [Uh]) that this action preserves the extended solution equation (E). (In combination with the earlier action of $\Lambda_{\mathrm{alg}} G^{\mathbf{C}}$, this gives a symmetry group $\mathbf{C}^* \ltimes \Lambda_{\mathrm{alg}} G^{\mathbf{C}}$ of (E)).

The holomorphic vector bundle $u_\xi : U_\xi \to \Omega_\xi$.

We shall need an explicit description of the vector bundle $u_\xi : U_\xi \to \Omega_\xi$. Let Δ be the set of roots of $\mathbf{g}^{\mathbf{C}}$ with respect to the maximal torus T; thus $\Delta \subseteq \sqrt{-1}\,\mathbf{t}^*$. We have already chosen a fundamental Weyl chamber, so we have a decomposition $\Delta = \Delta^+ \sqcup \Delta^-$ of Δ into the subsets of positive and negative roots. As above, we take $\xi \in I'$. It follows that $\alpha(\xi)/\sqrt{-1} \in \mathbf{Z}$ for any $\alpha \in \Delta$, and $\alpha(\xi)/\sqrt{-1} \geq 0$ for any $\alpha \in \Delta^+$. Let $\mathbf{g}_\alpha$ be the root space of α (thus, $\mathrm{ad}(\tau)\eta = \alpha(\tau)\eta$ for all $\tau \in \mathbf{t}$, $\eta \in \mathbf{g}_\alpha$). Let $\mathbf{g}^\xi_i$ be the $\sqrt{-1}\,i$-eigenspace of $\mathrm{ad}\,\xi$. With these definitions, we have:

$$\mathbf{g}^{\mathbf{C}} = \textstyle\bigoplus_i \mathbf{g}^\xi_i, \qquad \mathbf{g}^\xi_i = \textstyle\bigoplus_{\alpha(\xi)=\sqrt{-1}\,i} \mathbf{g}_\alpha.$$

Definition: The *height* of ξ is the non-negative integer $r(\xi)=\max\{i \mid \mathbf{g}_i^\xi \neq 0\}$.

First we shall describe Ω_ξ abstractly, as a complex homogeneous space. Consider the orbit $G^{\mathbf{C}}\cdot\gamma_\xi$ in $\Lambda G^{\mathbf{C}}\cdot\gamma_\xi=\Omega G$. The isotropy subgroup at γ_ξ is the subgroup $P_\xi=G^{\mathbf{C}}\cap\gamma_\xi(\Lambda^+G^{\mathbf{C}})\gamma_\xi^{-1}$ of $G^{\mathbf{C}}$. The Lie algebra of P_ξ is

$$\begin{aligned}\mathbf{p}_\xi &= \mathbf{g}^{\mathbf{C}}\cap \mathrm{Ad}(\gamma_\xi)\Lambda^+\mathbf{g}^{\mathbf{C}}\\ &= \mathbf{t}^{\mathbf{C}}\oplus(\textstyle\bigoplus_{\alpha\in\Delta^-}\mathbf{g}_\alpha)\oplus(\textstyle\bigoplus_{\alpha\in\Delta^+,\alpha(\xi)=0}\mathbf{g}_\alpha)\\ &= \textstyle\bigoplus_{i\le 0}\mathbf{g}_i^\xi.\end{aligned}$$

Here we use the fact that $\mathrm{Ad}\,\gamma_\xi=\mathrm{Ad}\exp t\xi=e^{\mathrm{ad}\,t\xi}$, which is given by multiplication by λ^i on $\mathbf{g}_i^\xi$. Evidently we have $G\cdot\gamma_\xi\subseteq G^{\mathbf{C}}\cdot\gamma_\xi$. It turns out, however, that these are equal. This follows from the Iwasawa decomposition $G^{\mathbf{C}}=GAN$, where A,N are the connected subgroups of $G^{\mathbf{C}}$ corresponding to the Lie algebras $\mathbf{a}=\sqrt{-1}\,\mathbf{t}$, $\mathbf{n}=\bigoplus_{\alpha\in\Delta^-}\mathbf{g}_\alpha$. (Note that $AN\subseteq P_\xi$.) We therefore have an explicit identification

$$\Omega_\xi\cong G^{\mathbf{C}}/P_\xi,$$

and the holomorphic tangent bundle of Ω_ξ is $T^{1,0}\Omega_\xi\cong G^{\mathbf{C}}\times_{P_\xi}(\mathbf{g}^{\mathbf{C}}/\mathbf{p}_\xi)$.

Next, we shall describe U_ξ in a similar fashion. It is by definition a complex homogeneous space of the group $\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}}$, and the isotropy subgroup at γ_ξ is the subgroup $\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}}\cap\gamma_\xi(\Lambda^+G^{\mathbf{C}})\gamma_\xi^{-1}$. The Lie algebra of the isotropy subgroup is

$$\begin{aligned}\Lambda^+_{\mathrm{alg}}\mathbf{g}^{\mathbf{C}}\cap\mathrm{Ad}(\gamma_\xi)\Lambda^+\mathbf{g}^{\mathbf{C}} &= \textstyle\bigoplus_{i\ge 0,\,\alpha(\xi)/\sqrt{-1}\le i}\lambda^i\mathbf{g}_\alpha\\ &= \textstyle\bigoplus_{i\ge 0}\lambda^i(\bigoplus_{j\le i}\mathbf{g}_j^\xi)\\ &= \textstyle\bigoplus_{0\le i\le r(\xi)}\lambda^i\mathbf{f}_i^\xi\end{aligned}$$

where $\mathbf{f}_i^\xi=\bigoplus_{j\le i}\ \mathbf{g}_j^\xi$.

Finally, we claim that the bundle $u_\xi:U_\xi\to\Omega_\xi$ is given simply by the natural map

$$\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}}/\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}}\cap\gamma_\xi(\Lambda^+G^{\mathbf{C}})\gamma_\xi^{-1}\to G^{\mathbf{C}}/P_\xi$$

which is induced by the homomorphism $\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}}\to G^{\mathbf{C}},\quad \gamma\mapsto\gamma(0)$. This follows from the fact that the flow of ∇E is given by the action of $(0,1]$, as described earlier.

We note that this is in fact a *holomorphic* fibre bundle. To see that it is a *vector* bundle (and to calculate its rank), we consider the fibre over γ_ξ. From the above description, this fibre is $\Lambda^+_{e,\mathrm{alg}}G^{\mathbf{C}}\cdot\gamma_\xi$, where $\Lambda^+_{e,\mathrm{alg}}G^{\mathbf{C}}=\{\gamma\in\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}}\mid\gamma(0)=e\}$. It may therefore be identified with the homogeneous space

$$\Lambda^+_{e,\mathrm{alg}}G^{\mathbf{C}}\ /\ \Lambda^+_{e,\mathrm{alg}}G^{\mathbf{C}}\cap\gamma_\xi(\Lambda^+G^{\mathbf{C}})\gamma_\xi^{-1}.$$

This is a complex manifold of dimension $\sum_{\alpha\in\Delta^+,\alpha(\xi)\neq 0}(\alpha(\xi)/\sqrt{-1}-1)$ (i.e. the dimension of $\bigoplus_{0<i<\alpha(\xi)/\sqrt{-1}}\ \lambda^i\mathbf{g}_\alpha$). We can be even more explicit:

Proposition 2.4. *The fibre of $u_\xi : U_\xi \to \Omega_\xi$ over γ_ξ is*

$$\Lambda^+_{e,\mathrm{alg}} G^{\mathbf{C}} \cdot \gamma_\xi = \exp\, \mathbf{u}_\xi \cdot \gamma_\xi \cong \exp\, \mathbf{u}_\xi \cong \mathbf{u}_\xi$$

where $\mathbf{u}_\xi$ is the (finite dimensional) nilpotent subalgebra of $\Lambda^+_{\mathrm{alg}} \mathbf{g}^{\mathbf{C}}$ defined by

$$\mathbf{u}_\xi = \bigoplus\nolimits_{0<i<r(\xi)} \lambda^i (\mathbf{f}^\xi_i)^\perp, \qquad (\mathbf{f}^\xi_i)^\perp = \bigoplus\nolimits_{i<j\le r(\xi)} \mathbf{g}^\xi_j.$$

Proof. By construction, the isotropy subgroup of $\exp\, \mathbf{u}_\xi$ at γ_ξ is trivial, so we have $\exp\, \mathbf{u}_\xi \cdot \gamma_\xi \cong \exp\, \mathbf{u}_\xi$. Since $\mathbf{u}_\xi$ is a finite dimensional nilpotent Lie algebra, its exponential map is biholomorphic, so we have $\exp\, \mathbf{u}_\xi \cong \mathbf{u}_\xi$. It remains to show that $\exp\, \mathbf{u}_\xi \cdot \gamma_\xi = \Lambda^+_{e,\mathrm{alg}} G^{\mathbf{C}} \cdot \gamma_\xi$. Since $\exp\, \mathbf{u}_\xi$ is a subgroup of $\Lambda^+_{e,\mathrm{alg}} G^{\mathbf{C}}$, it suffices to show that both orbits have the same dimension (cf. [Pr], page 558, conditions (A) and (B)). That they do follows from the formula above for the dimension, and the definition of $\mathbf{u}_\xi$. $\square$

We deduce from this that $u_\xi : U_\xi \to \Omega_\xi$ is a holomorphic vector bundle, of rank $\sum_{\alpha\in\Delta^+, \alpha(\xi)\neq 0} (\alpha(\xi)/\sqrt{-1} - 1)$.

For later use, we shall compute explicitly the inclusions of holomorphic tangent spaces

$$T^{1,0}\Omega_\xi \longrightarrow T^{1,0} U_\xi \longrightarrow T^{1,0}\Omega G.$$

It suffices to work over the point γ_ξ. Applying left translation by γ_ξ^{-1}, we have a sequence

$$\mathbf{g}^{\mathbf{C}}/\mathbf{p}_\xi \xrightarrow{\ i_1\ } \Lambda^+_{\mathrm{alg}} \mathbf{g}^{\mathbf{C}} / \Lambda^+_{\mathrm{alg}} \mathbf{g}^{\mathbf{C}} \cap \mathrm{Ad}(\gamma_\xi) \Lambda^+ \mathbf{g}^{\mathbf{C}} \xrightarrow{\ i_2\ } \Lambda \mathbf{g}^{\mathbf{C}} / \Lambda^+ \mathbf{g}^{\mathbf{C}}.$$

Lemma 2.5. *The maps i_1, i_2 are given by*

$$\begin{aligned} i_2 \circ i_1 [\eta] &= [\lambda^{-i} \eta] \\ i_2 [\lambda^j \eta] &= [\lambda^{j-i} \eta] \end{aligned}$$

where $\eta \in \mathbf{g}^\xi_i$.

Proof. We have $[\eta] = \frac{d}{dt} (\exp\, t\eta) P_\xi |_{t=0}$, so $i_2 \circ i_1 [\eta] = \frac{d}{dt} \gamma_\xi^{-1} (\exp\, t\eta) \gamma_\xi \Lambda^+ G^{\mathbf{C}} |_{t=0} = [\lambda^{-i}\eta]$. Similarly $[\lambda^j \eta] = \frac{d}{dt} (\exp\, t\lambda^j \eta) \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}} \cap \gamma_\xi (\Lambda^+ G^{\mathbf{C}}) \gamma_\xi^{-1} |_{t=0}$, so $i_2 [\lambda^j \eta]$ is equal to $\frac{d}{dt} \gamma_\xi^{-1} (\exp\, t\lambda^j \eta) \gamma_\xi \Lambda^+ G^{\mathbf{C}} |_{t=0} = [\lambda^{j-i} \eta]$. $\square$

The derivative $Du_\xi : T^{1,0} U_\xi \to T^{1,0} \Omega_\xi$ corresponds to a map

$$\Lambda^+_{\mathrm{alg}} \mathbf{g}^{\mathbf{C}} / \Lambda^+_{\mathrm{alg}} \mathbf{g}^{\mathbf{C}} \cap \mathrm{Ad}(\gamma_\xi) \Lambda^+ \mathbf{g}^{\mathbf{C}} \to \mathbf{g}^{\mathbf{C}}/\mathbf{p}_\xi.$$

Lemma 2.6. *The map Du_ξ is given by*

$$[\lambda^j\eta] \mapsto \begin{cases} 0 & \textit{if } j>0 \\ [\eta] & \textit{if } j=0 \end{cases}$$

where $\eta \in \mathbf{g}_i^\xi$.

Proof. This follows from a calculation similar to that of Lemma 2.5, noting that u_ξ is defined by evaluation at $\lambda=0$. $\square$

We shall also need later the following mild generalization of Proposition 2.4, describing the part of U_ξ which lies over the "big cell" of Ω_ξ. The big cell of Ω_ξ means the subspace

$$N'\cdot\gamma_\xi \ \subseteq\ \Omega_\xi$$

where N' is the "opposite" nilpotent subgroup to N, i.e. the connected subgroup of G with Lie algebra

$$\mathbf{n}' = \bigoplus\nolimits_{\alpha\in\Delta+}\mathbf{g}_\alpha.$$

By a calculation similar to that of Proposition 2.4, $N'\cdot\gamma_\xi$ is diffeomorphic to the vector space $\bigoplus_{\alpha\in\Delta+,\,\alpha(\xi)\neq 0}\mathbf{g}_\alpha$, i.e. $\bigoplus_{i>0}\mathbf{g}_i^\xi$. The part of U_ξ which lies over this "big cell" is

$$u_\xi^{-1}(N'\cdot\gamma_\xi) = \Lambda^+_{N',\mathrm{alg}}G^{\mathbf{C}}\cdot\gamma_\xi$$

where $\Lambda^+_{N',\mathrm{alg}}G^{\mathbf{C}} = \{\gamma\in\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}} \mid \gamma(0)\in N'\}$. Using the method of Proposition 2.4 again, we have:

Proposition 2.7. $\Lambda^+_{N',\mathrm{alg}}G^{\mathbf{C}}\cdot\gamma_\xi = \exp\mathbf{u}_\xi^0\cdot\gamma_\xi \cong \exp\mathbf{u}_\xi^0 \cong \mathbf{u}_\xi^0$, *where $\mathbf{u}_\xi^0$ is the (finite dimensional) subalgebra $\bigoplus_{0\le i<r(\xi)}\lambda^i(\mathbf{f}_i^\xi)^\perp$ of $\Lambda^+_{\mathrm{alg}}\mathbf{g}^{\mathbf{C}}$.* $\square$

§3 The twistor construction

S^1-invariant extended solutions.

A fundamental role in the classification theory is played by the following special extended solutions:

Definition: An *S^1-invariant extended solution* is an extended solution $\Phi: M\to\Omega G$ such that $\mathrm{Im}\,\Phi\subseteq\Omega_\xi$, for some $\xi\in I'$.

Let $\Phi: M\to\Omega_\xi$ be an (S^1-invariant) extended solution. From the definition of Ω_ξ, the harmonic map $\phi=\Phi(\ ,-1): M\to G$ factors through

$$N_\xi = \{g\gamma_\xi(-1)g^{-1} \mid g\in G\},$$

i.e. the conjugacy class of $\gamma_\xi(-1)$. This is a symmetric space; it is diffeomorphic to $G/C(\gamma_\xi(-1))$, where $C(g)$ denotes the centralizer of g. The inclusion of N_ξ in G is known to be totally geodesic, with respect to the natural Riemannian metrics on N_ξ and G constructed from a bi-invariant inner product on $\mathbf{g}$. It follows from this that a harmonic map $M \to N_\xi$ is the same thing as a harmonic map $M \to G$ which factors through N_ξ. Hence, ϕ is a harmonic map into N_ξ. Observe that $\phi = \pi_\xi \circ \Phi$, where the map

$$\pi_\xi : \Omega_\xi \to N_\xi$$

is given by $g\gamma_\xi g^{-1} \mapsto g\gamma_\xi(-1)g^{-1}$.

The extended solution equation admits the following interpretation in this case. We define a subbundle

$$H^{1,0}\Omega_\xi = G^{\mathbf{C}} \times_{P_\xi} (\mathbf{f}_1^\xi/\mathbf{f}_0^\xi)$$

of the holomorphic tangent bundle $T^{1,0}\Omega_\xi \cong G^{\mathbf{C}} \times_{P_\xi} (\mathbf{g}^{\mathbf{C}}/\mathbf{f}_0^\xi)$, and we say that a holomorphic map $\Phi : M \to \Omega_\xi$ is *super-horizontal* (with respect to π_ξ) if and only if Φ_z takes values in $H^{1,0}\Omega_\xi$. Then we have:

Proposition 3.1. *A holomorphic map $\Phi : M \to \Omega_\xi$ is an extended solution if and only if it is super-horizontal.*

Proof. This follows immediately from Proposition 1.4 and Lemma 2.5. $\square$

Hence, if $\Phi : M \to \Omega_\xi$ is holomorphic and super-horizontal, then $\pi_\xi \circ \Phi : M \to N_\xi$ is harmonic. In the terminology of [Bu-Ra], $\pi_\xi : \Omega_\xi \to N_\xi$ is a *twistor fibration.*

If $\Phi = [\Psi\gamma_\xi]$, where $\Psi : M \to G^{\mathbf{C}}$, then the condition for Φ to be super-horizontal and holomorphic may be written more explicitly as

$$\operatorname{Im} \Psi^{-1}\Psi_z \subseteq \mathbf{f}_1^\xi$$
$$\operatorname{Im} \Psi^{-1}\Psi_{\bar z} \subseteq \mathbf{f}_0^\xi.$$

This is a special case of Proposition 1.4.

The uniton number of an S^1-invariant extended solution is given by:

Proposition 3.2. *Let $\Phi : M \to \Omega_\xi$ be an extended solution. Then the uniton number of Φ is $r(\xi)$.*

Proof. It suffices to prove that $\operatorname{Ad}(g\gamma_\xi)$ is of the form $\sum_{|i| \le r(\xi)} \lambda^i T_i$, with $T_{r(\xi)} \neq 0$, for any $g \in G$. In fact, it suffices to prove this when $g = e$. We know that $\operatorname{Ad}\gamma_\xi$ is given by multiplication by λ^i on $\mathbf{g}_i^\xi$. Since $r(\xi) = \max\{i \mid \mathbf{g}_i^\xi \neq 0\}$, the result follows. $\square$

Canonical twistor fibrations.

Harmonic maps into symmetric spaces which correspond to S^1-invariant extended solutions have been studied intensively. It turns out that they arise from a distinguished subset of the twistor fibrations π_ξ, the so called canonical twistor fibrations. These fibrations may be described without reference to the loop group of G. We shall review this theory, following [Bu-Ra].

Let $\mathbf{g}^{\mathbf{C}}$ be a complex semisimple Lie algebra, with (compact) real form $\mathbf{g}$, Cartan subalgebra $\mathbf{t}^{\mathbf{C}}$, and simple roots $\alpha_1, \dots, \alpha_l$. Any subset $\mathcal{P} = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ of $\{\alpha_1, \dots, \alpha_l\}$ defines a subalgebra $\mathbf{p}$ of $\mathbf{g}^{\mathbf{C}}$, via the formula

$$\mathbf{p} = \mathbf{t}^{\mathbf{C}} \oplus (\bigoplus_{\alpha \in \Delta^-} \mathbf{g}_\alpha) \oplus (\bigoplus_{\alpha \in \Delta'} \mathbf{g}_\alpha)$$

where $\Delta' = \{\alpha \in \Delta^+ \mid \alpha = \sum_{i \notin \mathcal{P}} n_i \alpha_i, n_i \geq 0\}$. It is well known that this gives a one to one correspondence between subsets of the simple roots and (conjugacy classes of) parabolic subalgebras of $\mathbf{g}^{\mathbf{C}}$.

Let $\xi_1, \dots, \xi_l \in \mathbf{t}$ be dual to $\alpha_1, \dots, \alpha_l$, in the sense that $\alpha_i(\xi_j) = \sqrt{-1}\,\delta_{ij}$. Let G be the compact connected Lie group with *trivial centre* corresponding to the Lie algebra $\mathbf{g}$. The lattice $\mathbf{Z}\xi_1 \oplus \cdots \oplus \mathbf{Z}\xi_l$ is the integer lattice I of G. To the subset $\mathcal{P} = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ we associate the element $\xi = \xi_{i_1} + \cdots + \xi_{i_k}$ of I. In the notation of §2, we then have

$$\mathbf{p} = \bigoplus_{i \leq 0} \mathbf{g}_i^\xi = \mathbf{p}_\xi.$$

Let $\mathbf{h} = \mathbf{p} \cap \mathbf{g}$, and let H be the connected subgroup of G with Lie algebra $\mathbf{h}$. Evidently we have $\mathbf{h}^{\mathbf{C}} = \mathbf{g}_0^\xi$. The natural inclusion $G/H \to G^{\mathbf{C}}/P$ is a diffeomorphism (for dimensional reasons).

We define a subalgebra $\mathbf{k}^{\mathbf{C}}$ of $\mathbf{g}^{\mathbf{C}}$ by

$$\mathbf{k}^{\mathbf{C}} = \bigoplus_{i\,\mathrm{even}} \mathbf{g}_i^\xi.$$

Let $\mathbf{k} = \mathbf{k}^{\mathbf{C}} \cap \mathbf{g}$, and let K be the connected subgroup of G with Lie algebra $\mathbf{k}$. The homogeneous space G/K is a symmetric space. Since $K \subseteq H$, there is a natural map $\pi : G/H \to G/K$. The map π is called the *canonical twistor fibration* associated to $\mathcal{P}$.

Proposition 3.3. *Let $\xi = \xi_{i_1} + \cdots + \xi_{i_k}$, as above. Then the map $\pi : G/H \to G/K$ coincides with the map $\pi_\xi : \Omega_\xi \to N_\xi$.* □

Proof. We have already noted that $\mathbf{p} = \mathbf{p}_\xi$. Since N_ξ is the conjugacy class of $\gamma_\xi(-1)$, it suffices to prove that the Lie algebra of the centralizer of $\gamma_\xi(-1)$ is $\mathbf{k}$. This is elementary. □

In view of this, we make the following definition:

Definition: Let $\xi \in I'$. We say that ξ is *canonical* if $\xi = \xi_{i_1} + \cdots + \xi_{i_k}$ for some $i_1, \ldots, i_k$ (equivalently, if each simple root takes the value 0 or $\sqrt{-1}$ on ξ). We call γ_ξ a *canonical geodesic.*

In general, if $\xi \in I'$, then we have $\xi = \sum_{j=1}^k n_{i_j} \xi_{i_j}$ for some $n_{i_1}, \ldots, n_{i_k} \in \mathbf{N}$. So ξ is canonical if and only if each n_{i_j} is 1. We shall make use of the following technical result later on.

Lemma 3.4 [Bu-Ra]. *Let $r = r(\xi) = \max\{i \mid \mathbf{g}_i^\xi \neq 0\}$.*

(1) If ξ is canonical, then $\mathbf{g}_i^\xi \neq 0$ if and only if $-r \leq i \leq r$.

(2) Let $\xi = \sum_{j=1}^k n_{i_j} \xi_{i_j}$. We define ξ_{can} by $\xi_{\mathrm{can}} = \sum_{j=1}^k \xi_{i_j}$. Then $\mathbf{g}_0^\xi = \mathbf{g}_0^{\xi_{\mathrm{can}}}$ and $\mathbf{f}_0^\xi = \mathbf{f}_0^{\xi_{\mathrm{can}}}$. $\square$

§4 Harmonic maps into Lie groups

Classification of extended solutions in terms of canonical elements.

The basis for our analysis of extended solutions of finite uniton number is the following simple observation:

Proposition 4.1. *Let $\Phi : M \to \Omega^k_{\mathrm{alg}} G$ be an extended solution. Then there exists some $\xi \in I'$, and some discrete subset D of M, such that $\Phi(M - D) \subseteq U_\xi$.*

Proof. This is a consequence of three facts: (i) Φ is holomorphic, (ii) $\dim_{\mathbf{C}} M = 1$, and (iii) the closures of the pieces of the Bruhat decomposition give algebraic subvarieties of the (finite dimensional) complex projective algebraic variety $\Omega^k_{\mathrm{alg}} G$. $\square$

Recall (from §2) that we have a holomorphic vector bundle $u_\xi : U_\xi \to \Omega_\xi$.

Proposition 4.2. *If $\Phi : M - D \to U_\xi$ is an extended solution, then $u_\xi \circ \Phi : M - D \to \Omega_\xi$ is an extended solution.*

Proof. Certainly $u_\xi \circ \Phi$ is holomorphic. By the formula for Du_ξ of Lemma 2.6, $u_\xi \circ \Phi$ is an extended solution. $\square$

Proposition 4.3. *If $\Phi : M - D \to U_\xi$ is an extended solution, then the uniton number of Φ is equal to the uniton number of $u_\xi \circ \Phi$, namely $r(\xi)$.*

Proof. This is similar to the proof of Proposition 3.2. If $\gamma \in \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}}$, we have $\gamma \cdot \gamma_\xi = \gamma \gamma_\xi \delta$ for some $\delta \in \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}}$. In particular, we have $\gamma(0), \delta(0) \in G^{\mathbf{C}}$. By the argument of Proposition 3.2, we see that $\mathrm{Ad}(\gamma\cdot\gamma_\xi)$ is of the form $\sum_{i\geq -r(\xi)}\lambda^i T_i$, with $T_{-r(\xi)}\neq 0$. Since $\mathrm{Ad}(\gamma\cdot\gamma_\xi)\in O(\mathbf{g}^{\mathbf{C}})$, we must have $T_i=\bar{T}_{-i}$ for all i, so the result follows. □

Thus, to any extended solution Φ (of finite uniton number) we may associate an S^1-invariant extended solution $u_\xi\circ\Phi$, with the same uniton number. Geometrically, $u_\xi\circ\Phi$ is obtained from Φ by applying the gradient flow of the Morse-Bott function $E:\Omega G\to\mathbf{R}$. From the description of this flow in §2, we have $u_\xi\circ\Phi=\lim_{t\to\infty}\Phi^t$, where Φ^t is the extended solution $e^{-t}\cdot\Phi$. This gives a "classification" of extended solutions in terms of S^1-invariant extended solutions, which is finer than the classification by uniton number.

As a first step in using this classification, we note a useful reformulation of the extended solution equation. For any (smooth) map $\Phi: M-D\to U_\xi$, and any point $z_0\in M-D$, we may write

$$\Phi|_{M_0}=A\cdot\gamma_\xi,\quad A:M_0\to\Lambda^+_{\mathrm{alg}}G^{\mathbf{C}}$$

where M_0 is some neighbourhood of z_0 in $M-D$. In other words, we represent $\Phi: M_0\to\Omega G\cong\Lambda G^{\mathbf{C}}/\Lambda^+G^{\mathbf{C}}$ as $[A\gamma_\xi]$, where $A\gamma_\xi: M_0\to\Lambda_{\mathrm{alg}}G^{\mathbf{C}}$. By Proposition 1.4, the extended solution equation is equivalent to the conditions

$$\begin{aligned}\mathrm{Im}\ \lambda\,\mathrm{Ad}(\gamma_\xi^{-1})A^{-1}A_z&\subseteq\Lambda^+\mathbf{g}^{\mathbf{C}}\\ \mathrm{Im}\ \mathrm{Ad}(\gamma_\xi^{-1})A^{-1}A_{\bar z}&\subseteq\Lambda^+\mathbf{g}^{\mathbf{C}}.\end{aligned}$$

Let us write

$$A^{-1}A_z=\sum_{i\geq 0}\lambda^i A_i',\quad A^{-1}A_{\bar z}=\sum_{i\geq 0}\lambda^i A_i''.$$

Since $\mathrm{Ad}(\gamma_\xi^{-1})\eta=\lambda^{-i}\eta$ if $\eta\in\mathbf{g}_i^\xi$, we have:

Proposition 4.4. *The extended solution equation for $\Phi=[A\gamma_\xi]:M_0\to U_\xi$ is equivalent to the conditions*

$$\begin{aligned}\mathrm{Im}\ A_i'&\subseteq\mathbf{f}_{i+1}^\xi\quad \textit{for } 0\leq i\leq r(\xi)-2\\ \mathrm{Im}\ A_i''&\subseteq\mathbf{f}_i^\xi\quad\ \ \ \textit{for } 0\leq i\leq r(\xi)-1.\end{aligned}$$

(For higher values of i, these conditions are vacuous.) □

Estimates of the minimal uniton number.

The possible values of the minimal uniton number of a harmonic map are severely restricted by the next theorem.

Theorem 4.5. *Assume that G is semisimple, with trivial centre. Let $\Phi : M \to \Omega^k_{\mathrm{alg}} G$ be an extended solution (of finite uniton number). Then there exists some* canonical *$\xi \in I'$, some $\gamma \in \Omega_{\mathrm{alg}} G$, and some discrete subset D of M, such that $\gamma\Phi(M - D) \subseteq U_\xi$.*

Proof. As explained above, we may represent $\Phi|_{M-D}$ as $A \cdot \gamma_\xi$ for some $\xi \in I'$, some discrete subset D of M, and some $A : M - D \to \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}}$ satisfying the conditions of Proposition 4.4. Since G is semisimple, with trivial centre, we may write $\xi = \sum_{j=1}^k n_{i_j} \xi_{i_j}$, with $n_{i_j} \geq 1$, where $\xi_1, \dots, \xi_l$ are dual to the simple roots $\alpha_1, \dots, \alpha_l$ (as in §3). If ξ happens to be canonical, i.e. $n_{i_j} = 1$ for all j, we are done. If not, then $\hat\xi = \xi - \xi_{\mathrm{can}} = \sum_{j=1}^k (n_{i_j} - 1)\xi_{i_j}$ is a non-zero element of I'.

We claim that $\mathbf{f}^\xi_{i+1} \subseteq \mathbf{f}^{\hat\xi}_i$ for $i \geq 0$, and $\mathbf{f}^\xi_0 \subseteq \mathbf{f}^{\hat\xi}_0$. Assuming this for the moment, we deduce from Proposition 4.4 that A satisfies the conditions

$$\operatorname{Im} A'_i \subseteq \mathbf{f}^{\hat\xi}_i$$
$$\operatorname{Im} A''_i \subseteq \mathbf{f}^{\hat\xi}_i$$

for $0 \leq i \leq r(\xi) - 1$. These conditions say that $A \cdot \gamma_{\hat\xi}$ is both holomorphic and anti-holomorphic, hence constant (in z). In other words, it defines an element of ΩG, which we shall call γ^{-1}. Thus, $\gamma^{-1} = A \cdot \gamma_{\hat\xi} = A\gamma_{\hat\xi} B$, for some map $B : M - D \to \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}}$. We then have $\gamma\Phi \cdot \gamma_\xi = B^{-1}\gamma_{\hat\xi}^{-1} A^{-1} A \cdot \gamma_\xi = B^{-1} \cdot \gamma_{\xi_{\mathrm{can}}}$, which is the desired conclusion.

It remains to prove the above claim. Since $\operatorname{Ad}\gamma_\xi$, $\operatorname{Ad}\gamma_{\xi_{\mathrm{can}}}$ and $\operatorname{Ad}\gamma_{\hat\xi}$ are simultaneously diagonalized on the root space $\mathbf{g}_\alpha$, it follows that

$$\mathbf{g}^\xi_k = \bigoplus\nolimits_{0 \leq i \leq k} \left(\mathbf{g}^{\hat\xi}_i \cap \mathbf{g}^{\xi_{\mathrm{can}}}_{k-i} \right).$$

From this and Lemma 3.4 (2), $\mathbf{f}^\xi_i \subseteq \mathbf{f}^{\hat\xi}_i$ for all $i \geq 0$. To show that $\mathbf{f}^\xi_{i+1} \subseteq \mathbf{f}^{\hat\xi}_i$ for $i \geq 0$, it suffices to show that $\mathbf{g}^\xi_{i+1} \cap (\mathbf{g}^{\hat\xi}_{i+1} \cap \mathbf{g}^{\xi_{\mathrm{can}}}_0) = 0$. From Lemma 3.4 (2) again, we have $\mathbf{g}^\xi_0 = \mathbf{g}^{\xi_{\mathrm{can}}}_0$, so $\mathbf{g}^\xi_{i+1} \cap \mathbf{g}^{\xi_{\mathrm{can}}}_0 = \mathbf{g}^\xi_{i+1} \cap \mathbf{g}^\xi_0 = 0$. $\square$

The key point of this theorem is that we reduce to the situation of a *canonical* ξ. (This reduction is analogous to the concept of "normalization" in [Uh] and [Se], and to the geometrical idea of reducing to "full" harmonic maps in earlier works on this subject.) As a consequence, we obtain estimates for the minimal uniton number of a harmonic map:

Corollary 4.6.

(1) Let G be any compact simple Lie group. Let $\phi : M \to G$ be a harmonic map. Assume that ϕ arises from an extended solution $\Phi : M \to \Omega G$ of finite uniton number.

(If $M = S^2$, this assumption holds automatically.) Then the minimal uniton number of ϕ is not greater than $r(G) = \sum_{i=1}^{l} n_i$, where $\alpha = \sum_{i=1}^{l} n_i\alpha_i$ is the expression for the highest root of G in terms of the simple roots $\alpha_1, \dots, \alpha_l$.

(2) Let G be any compact Lie group, and let the simple factors in the universal cover of G be $G_1, \dots, G_t$. Let ϕ be as in (1). Then the minimal uniton number of ϕ is not greater than $\max\{r(G_1), \dots, r(G_t)\}$.

Proof. (1) Since the minimal uniton number depends only on $\operatorname{Ad}\phi : M \to \operatorname{Ad} G$, it suffices to prove the statement when G has trivial centre. Using the notation of §2 and §3, we have

$$\begin{aligned}
&\max\{r(\xi) \mid \xi \text{ canonical}\} \\
&= \max\left\{ \alpha\Big(\sum_{i\in\mathcal{P}} \xi_i\Big)/\sqrt{-1} \;\middle|\; \mathcal{P} \subseteq \{\alpha_1, \dots, \alpha_l\},\ \alpha = \sum_{i=1}^{l} m_i\alpha_i \in \Delta^+ \right\} \\
&= \Big(\sum_{i=1}^{l} n_i\alpha_i\Big)\Big(\sum_{i=1}^{l} \xi_i\Big)/\sqrt{-1} = \sum_{i=1}^{l} n_i.
\end{aligned}$$

By Proposition 3.2 (and the above theorem), this is an upper bound for the minimal uniton number. (2) As in part (1), it suffices to prove the statement for $\operatorname{Ad} G_1 \times \cdots \times \operatorname{Ad} G_t$. But the statement in this case follows immediately from (1). $\square$

We list below the values of $r(G)$ for (representatives of local isomorphism classes of) the compact simple Lie groups.

G	$r(G)$	$d(G) - 1$
SU_n	$n - 1$	$n - 1$
SO_{2n+1}	$2n - 1$	$2n$
Sp_n	$2n - 1$	$2n - 1$
SO_{2n}	$2n - 3$	$2n - 1$
G_2	5	6
F_4	11	25
E_6	11	26
E_7	17	55
E_8	29	247

The fact that $r(SU_n) = n - 1$ was proved in [Uh] and [Se]. If $\theta : G \to SU_n$ is a faithful representation, then $n - 1$ is an upper bound for the minimal uniton number of a harmonic map into G. By taking $n = d(G)$, the dimension of the smallest faithful representation of G, we obtain the upper bounds $d(G)-1$ in the third column. Our method shows that these upper bounds can be sharpened, especially for the exceptional groups. The question arises as to whether these upper bounds are optimal; it turns out that they are:

Proposition 4.7. *For any compact simple Lie group G, there exists a harmonic map $\phi : S^2 \to G$ such that $r(\phi) = r(G)$.*

Proof. It suffices to construct ϕ when G has trivial centre. Let $\xi_1, \ldots, \xi_l$ be as above, and let $\xi = \xi_1 + \cdots + \xi_l$. We shall construct a holomorphic super-horizontal map $\Phi : S^2 \to \Omega_\xi \cong G^{\mathbf{C}}/P_\xi$ of minimal uniton number $r(G)$.

For each $i = 1, \ldots, l$, let X_i be a non-zero vector in $\mathbf{g}_{\alpha_i}$, and let $X = X_1 + \cdots + X_l \in \mathbf{g}_1^\xi$. By [Ko], there exists some $Y \in \mathbf{g}_{-1}^\xi$ such that $\mathrm{Span}\{\xi, X, Y\}$ is a subalgebra isomorphic to $\mathbf{sl}_2\mathbf{C}$. Let S_ξ be the corresponding subgroup (isomorphic to $SL_2\mathbf{C}$). The induced map $\Phi : S_\xi / S_\xi \cap P_\xi \to G^{\mathbf{C}}/P_\xi$ is holomorphic and super-horizontal (see §3). To prove the proposition, we will show that $r(\gamma\Phi) \geq r(\xi)$ for all $\gamma \in \Omega G$.

Let us assume that $r(\gamma\Phi) < r(\xi)$, for some γ. It is a property of $SL_2\mathbf{C}$ that there exists some $g \in S_\xi$ such that $\mathrm{Ad}(g)\xi = -\xi$, hence for this g we have $\Phi([g]) = \Phi([e])^{-1}$. It follows that $\Phi([e])^2 = \Phi([g])^{-1}\Phi([e]) = (\gamma\Phi)([g])^{-1}(\gamma\Phi)([e])$. If $r(\gamma\Phi) < r(\xi)$, this is a contradiction. $\square$

As a consequence of Theorem 4.5, we can elucidate the structure of harmonic maps ϕ with $r(\phi) = 0$, 1, or 2. It is clear that $r(\phi) = 0$ if and only if ϕ is constant. The situation for $r(\phi) = 1$ is also easy (and well known):

Corollary 4.8. *Let $\phi : M \to G$ be harmonic. Then $r(\phi) = 1$ if and only if ϕ is (up to left translation by an element of G) the composition of a holomorphic map $M \to G/K$, for some Hermitian symmetric space G/K, with a totally geodesic embedding of G/K in G.*

Proof. Suppose $r(\phi) = 1$. For $r(\xi) = 1$, the formula for the rank of the bundle $U_\xi \to \Omega_\xi$ shows that $U_\xi = \Omega_\xi$ (i.e. the Morse index is zero). Moreover, from §3 we see that the twistor fibration $\Omega_\xi \to N_\xi$ is the identity map. The symmetric space N_ξ is Hermitian ($\mathrm{ad}\,\xi$ provides an invariant complex structure), so ϕ is of the stated form. The converse statement follows directly from the existence of a canonical (trivial) twistor fibration for any Hermitian symmetric space (see [Bu-Ra]). $\square$

For $r(\phi) = 2$, we have:

Corollary 4.9. *Let $\xi \in I'$ with $r(\xi) \leq 2$. A holomorphic map $\Phi : M - D \to U_\xi$ is an* *extended solution if and only if $u_\xi \circ \Phi : M - D \to \Omega_\xi$ is super-horizontal.*

Proof. The condition for Φ to be an extended solution is $\operatorname{Im} A_0' \subseteq \mathbf{f}_1^\xi$. This is the condition for Φ to be super-horizontal. (The conditions on A'' are satisfied because Φ is holomorphic.) $\square$

This shows that harmonic maps with $r(\phi) = 2$ correspond to pairs (Φ, σ), where Φ is a holomorphic super-horizontal map into Ω_ξ and σ is a meromorphic section of the holomorphic vector bundle $\Phi^* U_\xi$. For example, all harmonic maps $S^2 \to SU_3$ or U_3 are of this form.

The situation for $r(\phi) \geq 3$ is inevitably more complicated. However, we shall see that a rather explicit description is possible even in this case.

Weierstrass formulae for extended solutions.

Further information may be obtained by making use of the (Zariski) open subset $u_\xi^{-1}(N' \cdot \gamma_\xi) = \exp \mathbf{u}_\xi^0 \cdot \gamma_\xi$ of U_ξ, from Proposition 2.7. Since an extended solution $\Phi : M - D \to U_\xi$ is holomorphic, and the complement of $u_\xi^{-1}(N' \cdot \gamma_\xi)$ in U_ξ is a proper algebraic subvariety, there exists a discrete subset D' of M, with $D \subseteq D' \subseteq M$, such that $\Phi(M - D') \subseteq \exp \mathbf{u}_\xi^0 \cdot \gamma_\xi$. (After left translation by a constant loop if necessary, the image of Φ will not be entirely contained in the complement of $u_\xi^{-1}(N' \cdot \gamma_\xi)$.) We can therefore write

$$\Phi|_{M-D'} = \exp C \cdot \gamma_\xi$$

where $C : M - D' \to \mathbf{u}_\xi^0$ is a (vector-valued) *holomorphic function.* The map $\Psi = (\exp C)\gamma_\xi$ is a complex extended solution in the sense of §1, and what we have just said constitutes a proof of Theorem 1.5, for extended solutions of finite uniton number. Since $\Omega^k_{\mathrm{alg}} G$ is a projective algebraic variety, C is a meromorphic function from M to $\mathbf{u}_\xi^0$.

Conversely, if $C : M \to \mathbf{u}_\xi^0$ is meromorphic, the condition (from Proposition 4.4) for $\Phi = \exp C \cdot \gamma_\xi$ to be an extended solution is

$$\operatorname{Im} A_i' \subseteq \mathbf{f}_{i+1}^\xi, \quad 0 \leq i \leq r(\xi) - 2$$

where $(\exp C)^{-1}(\exp C)_z = \sum_{0 \leq i \leq r(\xi)-1} \lambda^i A_i'$. We shall investigate this condition more closely. To do so, let us write $r = r(\xi)$ and

$$C = C_0 + \lambda C_1 + \cdots + \lambda^{r-1} C_{r-1}, \quad C_i = c_i^{i+1} + c_i^{i+2} + \cdots + c_i^r$$

where each function $c_i^j : M \to \mathbf{g}_j^\xi$ is meromorphic. We shall make essential use of the fact that $[\lambda^i \mathbf{g}_{i+k}^\xi, \lambda^j \mathbf{g}_{j+l}^\xi] \subseteq \lambda^{i+j} \mathbf{g}_{i+j+k+l}^\xi$.

By the well known formula for the derivative of the exponential map ([He], Chapter 2, Theorem 1.7), we have

$$
\begin{aligned}
(\exp\, C)^{-1}(\exp\, C)_z &= \tfrac{I-e^{-\operatorname{ad} C}}{\operatorname{ad} C} C_z \\
&= C_z - \tfrac{1}{2!}(\operatorname{ad} C)C_z + \tfrac{1}{3!}(\operatorname{ad} C)^2 C_z - \tfrac{1}{4!}(\operatorname{ad} C)^3 C_z + \dots
\end{aligned}
$$

The condition for an extended solution is that the coefficient of λ^i in this expression should have zero component in each of $\mathbf{g}^\xi_{i+2}, \dots, \mathbf{g}^\xi_r$.

For $i = 0$, this means that

$$
(C_0)_z - \tfrac{1}{2!}(\operatorname{ad} C_0)(C_0)_z + \tfrac{1}{3!}(\operatorname{ad} C_0)^2(C_0)_z - \tfrac{1}{4!}(\operatorname{ad} C_0)^3(C_0)_z + \dots
$$

should have zero component in each of $\mathbf{g}^\xi_2, \dots, \mathbf{g}^\xi_r$. We obtain equations (for $j = 2, \dots, r$) of the form "$(c_0^j)_z$ = terms involving $c_0^1, \dots, c_0^{j-1}$ and their z-derivatives". There is no condition at all on c_0^1, which may be taken to be any $\mathbf{g}^\xi_1$-valued meromorphic function on M. Each of $c_0^2, \dots, c_0^r$ may then be determined (locally) by integration.

For the coefficient of λ^i, when $i > 0$, we obtain a similar system of equations for C_i, i.e. for $c_i^{i+2}, \dots, c_i^r$. For each $j = i+2, \dots, r$, we have an equation of the form "$(c_i^j)_z$ = terms involving c_l^k for $k < i+2, l < i$ and their z-derivatives". Therefore we may choose c_i^{i+1} to be any $\mathbf{g}^\xi_{i+1}$-valued meromorphic function on M, after which $c_i^{i+2}, \dots, c_i^r$ may be determined (locally) by integration.

Thus, any choice of meromorphic functions $c_0^1, c_1^2, \dots, c_{r-1}^r$ gives rise — locally — to an extended solution of uniton number (at most) r. It is not guaranteed that the extended solution is defined on the whole of M. To determine when this happens is a more delicate matter (see [Br1] for an example). Nevertheless, we can conclude at least that all harmonic maps of S^2 are "algebraic" in the following sense:

Theorem 4.10. *Every harmonic map $S^2 \to G$ arises from an extended solution which may be obtained explicitly by choosing a finite number of rational functions and then performing a finite number of algebraic operations and integrations.* $\square$

Weierstrass was the first to give a procedure of this type, in the context of minimal surfaces in $\mathbf{R}^3$. Bryant ([Br1]) used the above method to give explicit constructions of harmonic maps from Riemann surfaces into S^4, from meromorphic functions. In later work on the twistor construction ([Br2]), he gave essentially the above method for holomorphic super-horizontal maps into Ω_ξ. For harmonic maps from S^2 into U_n, Wood ([Wo2]) proved a version of Theorem 4.10, by making a detailed analysis of Uhlenbeck's results.

Example: harmonic maps $S^2 \to U_n$.

With standard conventions for the maximal torus, integer lattice, and fundamental Weyl chamber, the canonical geodesics $\gamma_\xi : S^1 \to U_n$ are the homomorphisms of the form $\gamma_{k_1,k_2,\dots,k_n}(\lambda) = \mathrm{diag}(\lambda^{k_1}, \lambda^{k_2}, \dots \lambda^{k_n})$, with $k_1 \geq k_2 \geq \dots \geq k_n$, and $|k_i - k_{i-1}| = 0$ or 1 for all i. We may assume that $k_n = 0$; this corresponds to fixing representatives of the canonical geodesics for $PU_n = U_n/Z(U_n)$, under the surjection $\pi_1 U_n \cong \mathbf{Z} \to \mathbf{Z}/(n-1)\mathbf{Z} \cong \pi_1 PU_n$. There are 2^{n-1} canonical geodesics of this type.

Let us consider the Weierstrass representation for the most complicated type of harmonic map, namely where the canonical geodesic is $\gamma_\xi(\lambda) = (\lambda^{n-1}, \lambda^{n-2}, \dots, 1)$. (The minimal uniton number takes its greatest possible value here, $n-1$.) We have $\mathbf{g}^{\mathbf{C}} = \mathbf{g}^\xi_{n-1} \oplus \dots \oplus \mathbf{g}^\xi_0 \oplus \dots \oplus \mathbf{g}^\xi_{n-1}$, where $\mathbf{g}^\xi_s$ is the set of $n \times n$ complex matrices (x_{ij}) with $x_{ij} = 0$ unless $j - i = s$.

The nature of the equations for $C = C_0 + \lambda C_1 + \dots + \lambda^{n-1} C_{n-1}$ will be more transparent if we write them out for $n = 4$. In this case, $C = C_0 + \lambda C_1 + \lambda^2 C_2$, and this is of the form

$$C = \begin{pmatrix} 0 & a_1 & b_1 & c_1 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & d_1 & e_1 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 & 0 & f_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are no conditions on $c_0^1 = (a_1, a_2, a_3)$, $c_1^2 = (d_1, d_2)$ and $c_2^3 = f_1$. The conditions on the remaining components of C_0 and C_1 are as follows:

$$(C_0)_z - \frac{1}{2}[C_0, (C_0)_z] + \frac{1}{6}[C_0, [C_0, (C_0)_z]] \text{ has zero component in } \mathbf{g}^\xi_2, \mathbf{g}^\xi_3$$
$$(C_1)_z - \frac{1}{2}([C_0, (C_1)_z] + [C_1, (C_0)_z]) \text{ has zero component in } \mathbf{g}^\xi_3.$$

The first equation is simply the condition that $\exp C_0 \cdot \mathrm{diag}(\lambda^3, \lambda^2, \lambda, 1)$ be super-horizontal. The conjugacy class Ω of the canonical geodesic $\mathrm{diag}(\lambda^{n-1}, \lambda^{n-2}, \dots, 1)$ is the full flag manifold $F_{1,2,\dots,n-1}(\mathbf{C}^n)$, and it is easy to see that a super-horizontal holomorphic map $\Phi : S^2 \to F_{1,2,\dots,n-1}(\mathbf{C}^n)$ may be represented on the complement of a finite set D by a "holomorphic frame" of the form

$$A = \begin{pmatrix} | & & | & | \\ f^{(n-1)} & \dots & f' & f \\ | & & | & | \end{pmatrix}$$

where $f : S^2 - D \to \mathbf{C}^n$ is holomorphic. By our assumptions,

$$A \cdot \mathrm{diag}(\lambda^3, \lambda^2, \lambda, 1) = \exp C_0 \cdot \mathrm{diag}(\lambda^3, \lambda^2, \lambda, 1).$$

From this it is easy to verify that $\exp\, C_0$ must be of the form

$$\exp\, C_0 = \begin{pmatrix} 1 & \delta & \alpha'/\gamma' & \alpha \\ 0 & 1 & \beta'/\gamma' & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta = (\alpha'/\gamma')'/(\beta'/\gamma')'$$

where α, β, γ are rational functions (and prime denotes z-derivative). Conversely, for any α, β, γ , the above formula gives a solution C_0 of the first equation.

The second equation reduces to the differential equation

$$e_1' = \frac{1}{2}(a_1 d_2' - a_1' d_2 + d_1 a_3' - d_1' a_3),$$

from which e_1 may be determined by integration.

Thus, any harmonic map $S^2 \to U_4$ of this type corresponds to six rational functions.

For $n = 3$ there are no differential equations to solve (beyond C_0), as predicted by Corollary 4.9. The most general harmonic map with $r(\phi) = 2$ arises from an extended solution $\Phi = \exp\, C \cdot \operatorname{diag}(\lambda^2, \lambda, 1)$, where

$$C = \begin{pmatrix} 1 & \alpha'/\beta' & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and α, β, γ are arbitrary rational functions. This description is equivalent to a description of harmonic maps $S^2 \to U_3$ given in [Wo2], although the latter was expressed somewhat differently, in terms of "uniton factorizations". The role of such factorizations will be considered next.

Transforms and factorizations.

We have seen that the Bruhat decomposition of ΩG provides a straightforward approach to describing harmonic maps of finite uniton number, particularly when the domain is S^2. We shall indicate briefly how the existing theory of such maps can be understood from our point of view.

Initial work on harmonic maps from Riemann surfaces to symmetric spaces was based on the idea of "Bäcklund transformations", whereby a given harmonic map ϕ_1 is transformed into a new harmonic map ϕ_2. Such transformations appear in our theory in the following way. Consider an extended solution

$$\Phi = A \cdot \gamma_\xi : M - D \to U_\xi, \quad \xi = \xi_{i_1} + \cdots + \xi_{i_k}.$$

For any non-empty subset $J = \{j_1, \ldots, j_m\}$ of $\{i_1, \ldots, i_k\}$, we write $\xi_J = \xi_{j_1} + \cdots + \xi_{j_m}$.

Proposition 4.11. *If $A\cdot\gamma_\xi$ is an extended solution, then so is $A\cdot\gamma_{\xi_J}$.*

Proof. It is obvious that $\mathbf{f}_i^{\xi_J}\subseteq\mathbf{f}_i^{\xi}$ for all $i\ge 0$, so the assertion follows from Proposition 4.4. $\square$

This gives 2^{k-1} transforms of our original extended solution. For example, let us take $\Phi=A\cdot\gamma_\xi$, where

$$A=\begin{pmatrix} | & & | & | \\ f^{(n-1)} & \dots & f' & f \\ | & & | & | \end{pmatrix},\quad \gamma_\xi=\operatorname{diag}(\lambda^{n-1},\lambda^{n-2},\dots,1)$$

in the case $G=U_n$. Then we obtain a new extended solution $A\cdot\gamma_{\xi_J}$ by taking $\gamma_{\xi_J}=\operatorname{diag}(\lambda^2,\dots,\lambda^2,\lambda,1,\dots,1)$, where λ appears as the $(n-i)$-th diagonal element in γ_{ξ_J}. In fact, $A\cdot\gamma_{\xi_J}$ represents the "i-th harmonic transform" of the holomorphic map $[f]:S^2\to\mathbf{C}P^{n-1}$, namely the map $\operatorname{Span}\{f,f',\dots,f^{(i-1)}\}^\perp\cap\operatorname{Span}\{f,f',\dots,f^{(i)}\}$.

In [Uh], Uhlenbeck considered harmonic maps into complex Grassmannians as examples of harmonic maps into U_n, and introduced a transform procedure for such maps. This was defined as follows: if Φ_1,Φ_2 are extended solutions corresponding to ϕ_1,ϕ_2, then Φ_2 is obtained from Φ_1 by multiplication: $\Phi_2=U\Phi_1$, where U is a certain map into a Grassmannian (necessarily a solution to a first order system analogous to the Cauchy-Riemann equations). The new extended solution Φ_2 is said to be obtained from the old extended solution Φ_1 by "adding a uniton". To show that any harmonic map can be constructed by transforming a constant map finitely many times is equivalent to showing that any extended solution can be factored in the form $\Phi=U_1\dots U_r$. In [Uh] and [Se], such a factorization was given for harmonic maps from S^2 to $G=U_n$ or $G/K=Gr_k(\mathbf{C}^n)$. This was subsequently generalized in [Bu-Ra] to the case of a group G of "type H", i.e. a group whose universal cover contains no factors locally isomorphic to G_2, F_4, or E_8.

The factorization amounts to a reorganization of the earlier Bäcklund transformation approach, and can be obtained from our approach in a similar way. (We shall give the factorization theorem for U_n here, postponing the Grassmannian case to the next section.) Namely, given $A\cdot\gamma_\xi$, we consider the sequence

$$\xi_{i_1},\ \xi_{i_1}+\xi_{i_2},\ \dots,\ \xi_{i_1}+\dots+\xi_{i_k}.$$

From this we obtain a sequence of extended solutions

$$\Phi_1=A\cdot\gamma_{i_1},\ \Phi_2=A\cdot\gamma_{i_1}\gamma_{i_2},\ \dots,\ \Phi_k=A\cdot\gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_k},$$

where, to simplify notation, we write γ_i for γ_{ξ_i}. When $G=U_n$, we claim that the factorization

$$\Phi=\Phi_1(\Phi_1^{-1}\Phi_2)(\Phi_2^{-1}\Phi_3)\dots(\Phi_{k-1}^{-1}\Phi_k)$$

is a "uniton factorization" in the sense of [Uh] or [Se], i.e. that $\Phi_j^{-1}\Phi_{j+1}$ is a map into a Grassmannian. To see this, we write $\Phi_j = A \cdot \gamma_{i_1} \dots \gamma_{i_j} = A\gamma_{i_1} \dots \gamma_{i_j} B_j$ for some $\Lambda^+ G^{\mathbf{C}}$-valued maps B_j. Then $\Phi_j^{-1}\Phi_{j+1} = B_j^{-1}\gamma_{i_{j+1}} B_{i_{j+1}} = B_j^{-1} \cdot \gamma_{i_{j+1}}$, which is a map into $\Lambda^+ G^{\mathbf{C}} \cdot \gamma_{i_{j+1}}$. For $G = U_n$ (see Lemma 4.12 below), it turns out that $\Lambda^+ G^{\mathbf{C}} \cdot \gamma_{i_{j+1}} = G^{\mathbf{C}} \cdot \gamma_{i_{j+1}} = Gr_k(\mathbf{C}^n)$ for some k, so this completes the proof of our claim.

What we have shown is that a factorization of the extended solution $\Phi = A \cdot \gamma_\xi$ results from a choice of factorization of the canonical geodesic γ_ξ. For example, when $G = U_n$ and $\gamma_\xi = \mathrm{diag}(\lambda^{n-1}, \lambda^{n-2}, \dots, 1)$, we may choose the factorization

$$\mathrm{diag}(\lambda^{n-1}, \lambda^{n-2}, \dots, 1) = \mathrm{diag}(\lambda, \dots, \lambda, 1) \ \ \mathrm{diag}(\lambda, \dots, \lambda, 1, 1) \ \ \dots \ \ \mathrm{diag}(\lambda, 1, \dots, 1).$$

In order to generalize this to the case of other Lie groups, we need:

Lemma 4.12. *Assume that G is simple. Assume that α_i is a miniscule simple root, i.e. that the coefficient of α_i in the highest root of G is one. Then γ_i has Morse index zero (as a critical point of the energy functional), and the critical manifold $G^{\mathbf{C}} \cdot \gamma_i$ is a Hermitian symmetric space.*

Proof. The assumption on α_i means that $r(\xi_i) = 1$, in our earlier notation. The conclusion follows as in the proof of Corollary 4.8. □

When $G = SU_n$, all simple roots are miniscule, and all Hermitian symmetric spaces G/K are Grassmannians; this is the fact that we used earlier. For general G, not all simple roots are miniscule — in fact, the groups G_2, F_4 and E_8 have no miniscule simple roots at all. In the general case, the argument above gives:

Proposition 4.13. *Any extended solution $\Phi : M \to \Omega_{\mathrm{alg}} G$ admits a factorization $\Phi = \Phi_1 \Phi_2 \dots \Phi_k$, where each Φ_i is a map into the (closure of the) unstable manifold U_{ξ_j} corresponding to a simple root α_j. Each sub-product $\Phi_1 \Phi_2 \dots \Phi_i$ is an extended solution.* □

A refinement of such a factorization into "linear factors" was given in [Bu-Ra], for maps of S^2 into groups of type H, by exploiting further the global geometry of S^2.

§5 Harmonic maps into symmetric spaces

Our theory for Lie groups may be modified to deal with symmetric spaces, if we replace loop groups by "twisted" loop groups. As in the case of Lie groups, we obtain short proofs of the known results, as well as new results. We shall only consider *inner* symmetric spaces, however. An inner symmetric space is a homogeneous space of the form G/K where $C(g)_0 \subseteq K \subseteq C(g)$ for some $g \in G$, where $C(g)_0$ is the identity component of $C(g)$ (and $C(g) = \{x \in G \mid xg = gx\}$). We shall use the following convenient characterization up to local isomorphism of such spaces.

Proposition 5.1.

(1) Let G be a compact (connected) Lie group. Then each component of $\sqrt{e} = \{g \in G \mid g^2 = e\}$ is a compact inner symmetric space.

(2) Conversely, any compact (connected) inner symmetric space may be immersed in such a Lie group G as a connected component of $\sqrt{e}$. Moreover, it may be assumed that G has trivial centre.

Proof. If $x \in \sqrt{e}$, then the conjugacy class of x is an inner symmetric space, isomorphic to $G/C(g)$. There are at most a finite number of such conjugacy classes (as one sees by considering the intersection of $\sqrt{e}$ and a maximal torus of G). Hence $\sqrt{e}$ consists of a finite number of conjugacy classes, each of which is a connected component. (2) This is a consequence of Proposition 4.5 of [Bu-Ra]. The assertion concerning the centre follows from the fact that the centre of G is contained in the identity component of $C(x)$, for any $x \in G$. $\square$

For example, if $G = U_n$, then the connected components of $\sqrt{e}$ are the symmetric spaces $Gr_k(\mathbf{C}^n)$ for $k = 0, 1, \ldots, n$.

It is well known that the embedding of each component of $\sqrt{e}$ in G, in Proposition 5.1, is totally geodesic. Harmonic maps into inner symmetric spaces may therefore be viewed as special harmonic maps into G. As in [Uh] and [Se], we may characterize the corresponding special extended solutions in terms of the involution

$$T : \Omega G \to \Omega G, \quad T(\gamma)(\lambda) = \gamma(-\lambda)\gamma(-1)^{-1}.$$

We write

$$(\Omega G)_T = \{\gamma \in \Omega G \mid T(\gamma) = \gamma\}$$

for the fixed set of T. If $\Phi : M \to \Omega G$ is an extended solution, it is easy to verify that $T(\Phi)$ is also an extended solution.

Proposition 5.2.

(1) Let $\Phi : M \to (\Omega G)_T$ be an extended solution. Then $\phi(z) = \Phi(z, -1)$ defines a harmonic map from M into (a connected component of) $\sqrt{e}$.

(2) Let $\phi : M \to \sqrt{e}$ be harmonic. Assume that there exists an extended solution $\tilde{\Phi} : M \to \Omega G$ such that $\phi(z) = \Phi(z,-1)$. Then there exists an an extended solution $\Phi : M \to (\Omega G)_T$ such that $\phi(z) = \Phi(z,-1)$.

Proof. Statement (1) is obvious. For statement (2), we use the fact (from §1) that the set of extended solutions corresponding to ϕ is $\{\gamma\tilde{\Phi} \mid \gamma \in \Omega G, \gamma(-1) = e\}$. We must find a γ such that $\Phi = \gamma\tilde{\Phi}$ takes values in $(\Omega G)_T$. To do this, choose some $z_0 \in M$, and choose some $\xi \in \mathbf{g}$ with $\exp 2\pi\xi = e$ and $\phi(z_0) = \exp \pi\xi$. (This is possible as exp is surjective and $\phi(z_0)^2 = e$.) Then let $\gamma = \gamma_\xi \tilde{\Phi}(z_0)^{-1}$, so that $\Phi(z_0) = \gamma_\xi$. We claim that Φ takes values in $(\Omega G)_T$. Both Φ and $T(\Phi)$ are extended solutions corresponding to ϕ, so we must have $\Phi = \delta T(\Phi)$ for some $\delta \in \Omega G$. But $\Phi(z_0) = T(\Phi)(z_0) = \gamma_\xi$, so $\delta = e$, and $\Phi = T(\Phi)$, as required. $\square$

The involution T of ΩG is holomorphic, as it is induced by the complex involution $T : \Lambda G^{\mathbf{C}} \to \Lambda G^{\mathbf{C}}$ defined by $T(\gamma)(\lambda) = \gamma(-\lambda)$. The action of $\mathbf{C}^*_{\geq 1}$ on ΩG commutes with the action of T (since $-1 \in \mathbf{C}^*_{\geq 1}$). Hence, the Bruhat decomposition of $\Omega_{\mathrm{alg}} G$ (described in §2) induces a natural decomposition of $(\Omega_{\mathrm{alg}} G)_T$. For each $\xi \in I'$ (the intersection of the integer lattice with a fundamental Weyl chamber) we obtain a holomorphic vector bundle $(U_\xi)_T \to (\Omega_\xi)_T$, where $(U_\xi)_T = U_\xi \cap (\Omega G)_T$ and $(\Omega_\xi)_T = \Omega_\xi \cap (\Omega G)_T = \Omega_\xi$. From Proposition 2.4 we deduce:

Proposition 5.3. *The fibre of $(U_\xi)_T \to \Omega_\xi$ over γ_ξ is*

$$(\Lambda^+_{e,\mathrm{alg}} G^{\mathbf{C}})_T \cdot \gamma_\xi = \exp(\mathbf{u}_\xi)_T \cdot \gamma_\xi \cong \exp(\mathbf{u}_\xi)_T \cong (\mathbf{u}_\xi)_T$$

where $(\mathbf{u}_\xi)_T = \bigoplus_{0<2i<r(\xi)} \lambda^{2i} (\mathbf{f}^\xi_{2i})^\perp$. $\square$

As in Proposition 2.7, we have a similar description of a Zariski open subspace

$$(\Lambda^+_{N',\mathrm{alg}} G^{\mathbf{C}})_T \cdot \gamma_\xi = \exp(\mathbf{u}^0_\xi)_T \cdot \gamma_\xi \cong \exp(\mathbf{u}^0_\xi)_T \cong (\mathbf{u}^0_\xi)_T$$

of $(U_\xi)_T$, where $(\mathbf{u}^0_\xi)_T = \bigoplus_{0\leq 2i<r(\xi)} \lambda^{2i} (\mathbf{f}^\xi_{2i})^\perp$.

We may now proceed as in §4. If $\Phi : M \to (\Omega^k_{\mathrm{alg}} G)_T$ is an extended solution, then there exists some $\xi \in I'$, and some discrete subset D of M, such that $\Phi(M - D) \subseteq (U_\xi)_T$. Moreover, by taking a larger D if necessary, we may assume that $\Phi(M - D) \subseteq \exp(\mathbf{u}^0_\xi)_T \cdot \gamma_\xi$, and so

$$\Phi|_{M-D} = A \cdot \gamma_\xi, \quad A = \exp C$$

where $C : M \to (\mathbf{u}^0_\xi)_T$ is meromorphic. If $\phi : M \to \sqrt{e}$ is the corresponding harmonic map, i.e. $\phi(z) = \Phi(z,-1)$, then the connected component of $\sqrt{e}$ containing the image of ϕ must be the symmetric space N_ξ.

In Theorem 4.5 we demonstrated the existence of an element $\gamma \in \Omega G$ such that $\gamma\Phi(M-D) \subseteq U_{\xi_{\mathrm{can}}}$, where $\xi = \sum_t n_t \xi_t$ and $\xi_{\mathrm{can}} = \sum_t \xi_t$. (As in §4, we write $\xi_1, \dots, \xi_l$ for the elements of I' which are dual to the simple roots $\alpha_1, \dots, \alpha_l$.) In the presence of the additional condition $T(\Phi) = \Phi$, we can make a stronger statement:

Theorem 5.4. *Assume that G is semisimple, with trivial centre. Let $\Phi : M \to (\Omega^k_{\mathrm{alg}} G)_T$ be an extended solution (of finite uniton number). As explained above, we have $\Phi(M-D) \subseteq (U_\xi)_T$ for some $\xi = \sum_t n_t \xi_t$. Then there exists some $\gamma \in \Omega^k_{\mathrm{alg}} G$ with $\gamma(-1) = e$, such that $\gamma\Phi(M-D) \subseteq (U_{\xi'_{\mathrm{can}}})_T$, where $\xi'_{\mathrm{can}} = \sum_{n_t \text{ odd}} \xi_t$.*

Proof. Let $\hat\xi = \xi - \xi'_{\mathrm{can}}$. We shall prove that $A \cdot \gamma_{\hat\xi}$ is both holomorphic and anti-holomorphic, hence independent of z. The statement concerning $\gamma\Phi$ will then follow, exactly as in Theorem 4.5. Since A contains only even powers of λ, the same is true of $A^{-1}A_z = \sum_{i \ge 0} \lambda^i A'_i$, i.e. we have $A'_i = 0$ for all odd i. As in the proof of Theorem 4.5, we have to show that $\mathrm{Im}\, A'_i \subseteq \mathbf{f}_i^{\hat\xi}$ for all $i \ge 0$. The extended solution equation (Proposition 4.4) gives $\mathrm{Im}\, A'_i \subseteq \mathbf{f}^\xi_{i+1}$ for all $i \ge 0$. Thus, it suffices to show that $\mathbf{f}^\xi_{2j+1} \subseteq \mathbf{f}^{\hat\xi}_{2j}$ for all $j \ge 0$. We have $\mathbf{f}^\xi_{2j+1} \subseteq \mathbf{f}^{\hat\xi}_{2j+1}$ as usual. Since $\mathbf{f}^{\hat\xi}_{2j+1} = \mathbf{f}^{\hat\xi}_{2j} \oplus \mathbf{g}^{\hat\xi}_{2j+1}$, we need to prove that $\mathbf{g}^{\hat\xi}_{2j+1} = 0$. But this is clear, as $\alpha(\hat\xi) \in 2\sqrt{-1}\,\mathbf{Z}$ for any positive root α, because each coefficient of ξ_i in $\hat\xi$ is (by construction) even. The required γ is given by $\gamma^{-1} = A \cdot \gamma_{\hat\xi}$. Finally, to prove that $\gamma(-1) = e$, we observe that the (constant) extended solution $\gamma^{-1} : M \to (U_{\hat\xi})_T$ gives a (constant) harmonic map $\gamma^{-1}(-1) : M \to N_{\hat\xi} \subseteq G$, and $N_{\hat\xi} = \{g(\exp \pi\hat\xi)g^{-1} \mid g \in G\} = \{e\}$. □

It should be noted that $N_\xi = N_{\xi'_{\mathrm{can}}}$ here. In view of Proposition 5.2 (2), we obtain the following upper bound on the minimal uniton number:

Corollary 5.5. *Assume that G is semisimple, with trivial centre. Let N be a (compact) inner symmetric space, embedded in G as a connected component of $\sqrt{e}$. Let $\phi : M \to N$ be a harmonic map of finite uniton number. Then the minimal uniton number of ϕ is not greater than $r(N) = \max\{r(\xi) \mid \xi \text{ canonical}, N_\xi = N\}$.* □

In the terminology of [Bu-Ra], $r(N)$ is the maximum height of any flag manifold which fibres canonically over N. The computation of $r(N)$ is a purely Lie-algebraic matter (see part C of Chapter 4 of [Bu-Ra]). The following list contains $r(N)$ for all compact irreducible inner symmetric spaces of classical type.

N	$r(N)$
$SU_n/S(U_m \times U_{n-m}),\ 1 \le m < n/2$	$2m$

$SU_n/S(U_m \times U_{n-m}),\ 1 \le m = n/2$	$2m-1$
$SO_{2n+1}/SO_m \times SO_{2n+1-m},\ 1 \le m < n$	$2m$
$SO_{2n+1}/SO_m \times SO_{2n+1-m},\ 1 \le m = n$	$2m-1$
$Sp_n/Sp_m \times Sp_{n-m},\ 1 \le m < n/2$	$4m$
$Sp_n/Sp_m \times Sp_{n-m},\ 1 \le m = n/2$	$4m-2$
$Sp_n/U_n,\ n \ge 1$	$2n-1$
$SO_{2n}/SO_{2m} \times SO_{2n-2m},\ 2 \le 2m \le n-2$	$4m$
$SO_{2n}/SO_{2m} \times SO_{2n-2m},\ 2 \le 2m = n-1$	$4m-1$
$SO_{2n}/SO_{2m} \times SO_{2n-2m},\ 2 \le 2m = n$	$4m-3$
$SO_{2n}/U_n,\ n \ge 3$	$2n-4$

Notable examples are the sphere S^{2n} ($r = 2$ if $n \ge 2$), complex projective space $\mathbf{C}P^n$ ($r = 2$ if $n \ge 2$), quaternionic projective space $\mathbf{H}P^n$ ($r = 4$ if $n \ge 2$), and the complex quadric $Q_n = SO_{n+2}/SO_2 \times SO_n$ ($r = 4$ if $n \ge 5$). The result for the Grassmannian $Gr_m(\mathbf{C}^n) = SU_n/S(U_m \times U_{n-m})$ was conjectured by Uhlenbeck (Problem 9 of [Uh]), and independent proofs have been given recently in Chapter 20 of [Gu] and in [Do-Sh].

In §4, we saw that harmonic maps into Lie groups with $r(\phi) \le 2$ are particularly easy to describe. For maps into symmetric spaces, we have similar results for $r(\phi) \le 3$. We begin with $r(\phi) = 2$.

Corollary 5.6. *If $\Phi : M \to (\Omega^k_{\mathrm{alg}} G)_T$ is an extended solution with $r(\Phi) \le 2$, then Φ is an S^1-invariant extended solution.*

Proof. By Proposition 5.3, we have $(U_\xi)_T = \Omega_\xi$ if $r(\xi) \le 2$. □

In particular, if N is an inner symmetric space with $r(N) = 2$, then every harmonic map $\phi : S^2 \to N$ is given by the twistor construction. As we have noted, this is the case for $N = S^{2n}$ or $\mathbf{C}P^n$. Therefore, Corollary 5.7 gives the well known theorem of Calabi (for S^{2n}) and Eells and Wood (for $\mathbf{C}P^n$).

For $r(\phi) = 3$, we have the following analogue of Corollary 4.9:

Corollary 5.7. *Let $\xi \in I'$ with $r(\xi) \le 3$. A holomorphic map $\Phi : M - D \to (U_\xi)_T$ is an extended solution if and only if $u_\xi \circ \Phi : M - D \to \Omega_\xi$ is super-horizontal.*

Proof. The conditions for Φ to be an extended solution are $\operatorname{Im} A_0' \subseteq \mathbf{f}_1^\xi$, $\operatorname{Im} A_1' \subseteq \mathbf{f}_2^\xi$. The first of these is the condition for Φ to be super-horizontal; the second is vacuous as $A_i' = 0$ for all odd i. □

As in Corollary 4.9, we conclude that harmonic maps into symmetric spaces with $r(\phi) = 3$ correspond to pairs (Φ, σ), where Φ is a holomorphic super-horizontal map into Ω_ξ and σ is a meromorphic section of the holomorphic vector bundle $\Phi^*(U_\xi)_T$.

With obvious modifications, the remarks in §4 on Weierstrass representations, transforms, and factorizations apply equally to harmonic maps into symmetric spaces.

Appendix A: Harmonic maps of low uniton number

We summarize here the various special cases of our results on harmonic maps from S^2 to a Lie group G or symmetric space N, which we encountered in §4 and §5. There are three essentially different kinds of behaviour.

a) Twistor construction

The twistor construction of [Bu-Gu] gives all harmonic maps which arise from S^1-invariant extended solutions; the latter are simply holomorphic super-horizontal maps into generalized flag manifolds. This includes maps with $r(\phi) = 0$ or 1 (constant maps or holomorphic maps into Hermitian symmetric spaces, respectively). It also includes all harmonic maps $S^2 \to N$ with $r(\phi) = 2$ (Corollary 5.6). For $N = S^{2n}$ or $\mathbf{C}P^n$, this means all harmonic maps.

b) Twistor construction + meromorphic section

Any harmonic map $S^2 \to G$ with $r(\phi) \leq 2$, or any harmonic map $S^2 \to N$ with $r(\phi) \leq 3$, is given by a pair (θ, σ), where θ is a harmonic map obtained via the twistor construction, and σ is a meromorphic section of a holomorphic vector bundle (over a generalized flag manifold). For precise statements, see Corollaries 4.9 and 5.7. For $G = SU_3$, $N = Sp_2/U_2 \cong Q_3$ or $Gr_2(\mathbf{C}^4) \cong Q_4$, all harmonic maps arise this way.

c) Twistor construction + meromorphic section(s) satisfying first order differential equations

This is the general case. The differential equations are solvable, locally, by integration. The example $G = SU_4$, which we gave in §4, shows how simple these equations can be, however. This is typical of the situation for harmonic maps $S^2 \to G$ with $r(\phi) = 3$, or harmonic maps $S^2 \to N$ with $r(\phi) = 4$ — i.e. the first level beyond b). All harmonic maps are of this form for the following families of symmetric spaces: $Gr_2(\mathbf{C}^n)$ for $n \geq 5$; Q_n for $n \geq 5$; $\mathbf{H}P^n$ for $n \geq 2$. The same is true for the isolated groups $G = SU_4$, SO_5, Sp_2, or SO_6 (and the symmetric space $N = SO_8/U_4 \cong Q_6$).

Appendix B: The Birkhoff decomposition

So far, we have not made any use of the Birkhoff decomposition of ΩG. For extended solutions of finite uniton number, it is the (finite dimensional) Bruhat manifolds which seem to be most relevant. On the other hand, a useful consequence of the Birkhoff decomposition is the existence of a "big cell", namely the Birkhoff manifold $\Lambda^- G^{\mathbf{C}} \cdot e$, which is an open dense subspace of (the identity component of) ΩG. This gives a natural coordinate chart on ΩG. For example, in [Do-Pe-Wu], a special role is played by extended solutions $\Phi : M \to \Omega G$ whose image lies in the big cell. In addition to the extended solution condition Im $\lambda\Phi^{-1}\Phi_z \subseteq \Lambda^+\mathbf{g}^{\mathbf{C}}$, we also have in this case the condition Im $\Phi^{-1}\Phi_z \subseteq \Lambda^-\mathbf{g}^{\mathbf{C}}$, so we conclude that $\Phi^{-1}\Phi_z = \frac{1}{\lambda}V$ for some meromorphic function $V : M \to \mathbf{g}^{\mathbf{C}}$. (More precisely, we conclude that $\Phi^{-1}\Phi_z$ is of the form $U + \frac{1}{\lambda}V$; but U is necessarily zero because $\Phi(z,1) = e$ for all z.) This V is called the "Weierstrass data" in [Do-Pe-Wu].

After translation by a suitable element of ΩG, any extended solution has this special form, away from a discrete subset. In the case of the Weierstrass representation of §4 and §5, this may be accomplished explicitly because of the following fact:

Proposition B1. *Let $U_\xi = \Lambda^+_{\mathrm{alg}} G^{\mathbf{C}} \cdot \gamma_\xi$ (as in §4). Then a dense open subset of $\gamma_\xi^{-1} U_\xi$ is contained in the big cell of ΩG.*

Proof. Recall (from Proposition 2.7) that we have a dense open subset $\exp \mathbf{u}^0_\xi \cdot \gamma_\xi$ of U_ξ. An element C of $\mathbf{u}^0_\xi$ is of the form $C = C_0 + \lambda C_1 + \cdots + \lambda^{r-1} C_{r-1}$, where $C_i = c_i^{i+1} + c_i^{i+2} + \cdots + c_i^r$, and $c_i^j \in \mathbf{g}^\xi_j$. We have $\mathrm{Ad}\,\gamma_\xi^{-1} c_i^j = \lambda^{-j} c_i^j$. Hence $\gamma_\xi^{-1}(\exp \mathbf{u}^0_\xi \cdot \gamma_\xi) = \exp \mathrm{Ad}\,\gamma_\xi^{-1} \mathbf{u}^0_\xi \cdot e \subseteq \Lambda^- G^{\mathbf{C}} \cdot e$, as required $\square$

For example, consider the extended solution $\Phi = \exp C \cdot \gamma$ (for a harmonic map $S^2 \to U_4$) from §4, where

$$C = \begin{pmatrix} 0 & a_1 & b_1 & c_1 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & d_1 & e_1 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 & 0 & f_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \lambda^3 & & & \\ & \lambda^2 & & \\ & & \lambda & \\ & & & 1 \end{pmatrix}.$$

A calculation shows that

$$\Phi^{-1}\Phi_z = \frac{1}{\lambda} \begin{pmatrix} 0 & a_1' & d_1' & f_1' \\ 0 & 0 & a_2' & d_2' \\ 0 & 0 & 0 & a_3' \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in this case. Thus, the Weierstrass data consists precisely of the derivatives of the rational functions $a_1, a_2, a_3, d_1, d_2, f_1$. We saw in §4 that these rational functions (locally) parametrize extended solutions of this type.

References

[Bo] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–281.

[Br1] R.L. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Diff. Geom. **17** (1982), 455–473.

[Br2] R.L. Bryant, *Lie groups and twistor spaces*, Duke Math. J. **52** (1985), 223–261.

[Bu] F.E. Burstall, *Harmonic tori in spheres and complex projective spaces*, J. reine angew. Math. **469** (1995), 149–177.

[Bu-Fe-Pe-Pi] F.E. Burstall, D. Ferus, F. Pedit and U. Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Annals of Math. **138** (1993), 173–212.

[Bu-Ra] F.E. Burstall and J.H. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, Lecture Notes in Math. 1424, 1990, Springer.

[Do-Pe-Wu] J. Dorfmeister, F. Pedit, and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, preprint.

[Do-Sh] Y.-X. Dong and Y.-B. Shen, *On factorization theorems of harmonic maps into $U(N)$ and minimal uniton numbers*, preprint, Hangzhou University.

[Ee-Le] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. Lond. Math. Soc. **20** (1988), 385–524.

[Ee-Wo] J. Eells and J.C. Wood, *Harmonic maps from surfaces into complex projective spaces*, Adv. in Math. **49** (1983), 217–263.

[Fu-Gu-Ko-Oh] M. Furuta, M.A. Guest, M. Kotani, and Y. Ohnita, *On the fundamental group of the space of harmonic 2-spheres in the n-sphere*, Math. Zeit. **215** (1994), 503–518.

[Gu] M.A. Guest, *Harmonic Maps, Loop Groups, and Integrable Systems*, to appear, Cambridge Univ. Press.

[Gu-Oh] M.A. Guest and Y. Ohnita, *Group actions and deformations for harmonic maps*, J. Math. Soc. Japan. **45** (1993), 671–704.

[He] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, 1978, Academic Press.

[Ko] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[Mc] I. McIntosh, *A construction of all non-isotropic harmonic tori in complex projective space*, preprint.

[Pa] G.D. Parker, *Morse theory on Kähler homogeneous spaces*, Proc. Amer. Math. Soc. **34** (1972), 586–590.

[Po] K. Pohlmeyer, *Integrable Hamiltonian systems and interactions through constraints*, Commun. Math. Phys. **46** (1976), 207–221.

[Pr] A.N. Pressley, *The energy flow on the loop space of a compact Lie group*, J. Lond. Math. Soc. **26** (1982), 557–566.

[Pr-Se] A.N. Pressley and G.B. Segal, *Loop Groups*, 1986, Oxford Univ. Press.

[Se] G.B. Segal, *Loop groups and harmonic maps*, Advances in Homotopy Theory, London Math. Soc. Lecture Notes 139, Cambridge Univ. Press, 1989, pp. 153–164.

[Uh] K.K. Uhlenbeck, *Harmonic maps into Lie groups (Classical solutions of the chiral model)*, J. Diff. Geom. **30** (1989), 1–50.

[Wo1] J.C. Wood, *The explicit construction and parametrization of all harmonic maps from the two-sphere to a complex Grassmannian*, J. reine angew. Math. **386** (1988), 1–31.

[Wo2] J.C. Wood, *Explicit construction and parametrization of harmonic two-spheres in the unitary group*, Proc. Lond. Math. Soc. **59** (1989), 608–624.

[Za-Mi] V.E. Zakharov and A.V. Mikhailov, *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Sov. Phys. JETP **47** (1978), 1017–1027.

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