Nested axi-symmetric vortex rings
Ronds de fumée concentriques

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Abstract. In an inviscid and incompressible fluid, we prove the existence of nested co-axial vortex rings moving at the same speed.
Résumé. Pour un fluide idéal, nous montrons l'existence de tourbillons annulaires concentriques se déplaçant à la même vitesse.

1 Introduction

In this note we are concerned with steady flows in an ideal fluid (inviscid and with uniform density) consisting of co-axial vortex rings (sets homeomorphic to solid tori) moving along their common axis at the same propagation speed $W \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$. They are nested in the sense that two consecutive rings $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ are such that $\mathcal{R}_{i+1} \subset \mathrm{co}(\mathcal{R}_i)$ (the convex hull of $\mathcal{R}_i$) but $\mathcal{R}_{i+1} \cap \mathcal{R}_i = \emptyset$. In a referential frame attached to the vortices, the equations of motion are given in cylindrical coordinates (r, θ, z) on the domain $\Pi = \{(r, \theta, z) : r > 0, -\pi < \theta \leq \pi, -\infty < z < +\infty\}$ by

$$\begin{cases} \psi \text{ is independent of } \theta, \\ r^{-2} L\psi \text{ is constant on level sets of } \psi, \\ r^{-1}\psi_z \to 0,\ r^{-1}\psi_r \to -W \text{ as } r^2 + z^2 \to \infty, \end{cases} \tag{1}$$

where $L\psi = r(r^{-1}\psi_r)_r + \psi_{zz}$ (see [18, 21, 7, 8]). The so called Stokes stream function ψ is in $C^1(\bar{\Pi}) \cap C^2(\Pi \backslash E(\psi))$, where the set $E(\psi)$ is a finite union of non-degenerate level sets of ψ. In cylindrical coordinates, the velocity field q

is then simply given by $(-r^{-1}\psi_z, 0, r^{-1}\psi_r)$ and the amplitude of the vorticity by $|\mathrm{curl}\ q| = |r^{-1}L\psi|$. As the θ component does not appear in Equation (1), the vortices described in this way are without swirl [20]. The *core* of ψ is the set of points where $L\psi \neq 0$. In what follows, we shall restrict our attention to z-symmetric ψ: $\psi(r,z) = \psi(r,-z)$. The existence of non z-symmetric positive solutions has been ruled out in many situations [16, 3]. Hill discovered in 1894 the following explicit solution propagating at speed $W = 1$:

$$\psi_H(r,z) = \begin{cases} (3/4)r^2\left(1-\rho^2\right), & 0 \le \rho \le 1, \\ (1/2)r^2\left(\rho^{-3}-1\right), & \rho \ge 1, \end{cases}$$

where $\rho^2 = r^2 + z^2$. Its second derivative is discontinuous and

$$r^{-2}L\psi_H(r,z) = \begin{cases} -15/2, & 0 \le \rho \le 1, \\ 0, & \rho \ge 1. \end{cases}$$

The problem admits a simpler formulation [25]. Working in $\mathbb{R}^5$, we set $r^2 = x_1^2 + \ldots + x_4^2$, $z = x_5$, $\rho^2 = r^2 + z^2$ and $\psi(r,z) = (1/2)r^2 v(r,z) - (1/2)Wr^2$. We denote by Δ the usual Laplacian in $\mathbb{R}^5$. We seek v, axi-symmetric in $\mathbb{R}^5$, satisfying on $\{x \in \mathbb{R}^5 : r > 0\}$

$$\begin{cases} \Delta v \text{ constant on level sets of } r^2 v - Wr^2, \\ rv_z \to 0, \ \ v + (1/2)rv_r \to 0 \text{ as } \rho \to \infty. \end{cases} \tag{2}$$

Moreover v is in $C^1(\mathbb{R}^5) \cap C^2(\{x \in \mathbb{R}^5 : r > 0, x \notin E(v)\})$, where $E(v)$ is a finite union of non-degenerate level sets of $r^2 v - Wr^2$. The Hill's vortex is simply expressed as

$$v_H(r,z) = \begin{cases} (5/2) - (3/2)\rho^2, & 0 \le \rho \le 1, \\ \rho^{-3}, & \rho \ge 1, \end{cases}$$

which is radially symmetric in $\mathbb{R}^5$.

We now introduce a functional setting suited to the case $W = 1$. For $\epsilon > 0$, let M_ϵ be the set of functions $v \in C^1_{sym}(2\bar{B})$ such that $|v - v_H|_{C^1(2\bar{B})} \le \epsilon$, $\bar{B}$ being the closed unit ball in $\mathbb{R}^5$. The notation sym means that we restrict ourselves to functions that are both axi-symmetric and z-symmetric. We endow M_ϵ with the C^1-metric and define $S : 2\bar{B} \times M_\epsilon \times \mathbb{R}_+ \to \mathbb{R}$ by

$$S(x,v,k) = \int_{2\bar{B}} \omega^{-1}|x - \tilde{x}|^{-3} H(v(\tilde{x}) - 1 - k\tilde{r}^{-2})d\tilde{x},$$

where H is the Heaviside function and ω is the volume of $\bar{B}$. The Newtonian potential $-15^{-1}\omega^{-1}|x-\tilde{x}|^{-3}$ is the fundamental solution of Laplace's equation in $\mathbb{R}^5$ (see [22]).

The choice of the functional space is crucial. We can mention C^1 spaces [18] and $W^{1,2}$ Sobolev spaces, preferred in the variational approach [21, 2]. A comment on the respective advantages of these spaces may be found in [4]. Let us just point out that, as a function of v, $S(.,v,k) \in C^1_{sym}(\mathbb{R}^5)$ is continuous on M_ϵ, but not Fréchet differentiable. The constant $k \geq 0$ is related to the flux between the axis of symmetry and the core. The description of the level set $r^2v - r^2 - k = 0$ for $k > 0$ is one of the key issues in the work of Norbury [26]. See Fig. 2 in [4] for a nice picture. Norbury proved that a branch of solutions parametrized by $k > 0$ small emanates from Hill's vortex, their cores being homeomorphic to solid tori. Amick and Fraenkel [4] showed a uniqueness result for Norbury's vortices and Amick and Turner [5] extended Norbury's local result to a global one, working in the Sobolev space $W^{1,2}$.

Let us translate in our framework some essential features. There exist $\epsilon > 0$ and $\tilde{k} > 0$ such that $S(.,v,k)$ is compact as an operator from $M_\epsilon \times [0,\tilde{k}[$ to $C^1_{sym}(\mathbb{R}^5)$. For $x \in 2\bar{B}$, we set $S_0(v)(x) = S(x,v,0)$. By the uniqueness result of Amick and Fraenkel [3], v_H is the unique fixed point of S_0 in M_ϵ and, as shown by Amick and Turner [5], $\deg(I - S_0, v_H, 0) = -1$. We shall sketch the proof in the next Section.

Clearly, for $a > 0$ and $W \in \mathbb{R}^*$, the dilatation of Hill's vortex $v(r,z) = Wv_H(r/a, z/a)$ is a solution of (2) with $-\Delta v = Wa^{-2}15$ on the ball $a\bar{B}$. We can also dilate the whole Norbury's family. Each of the nested vortex rings we shall construct in Section 3 is near some dilatation (here the term "contraction" would be more appropriate) of some Norbury's vortex. The asymptotic speed W of an inner vortex ring should be interpreted as a relative velocity with respect to the flow created by the outer vortex rings. However, on the whole, the vortices move at speed 1. The main point is to check that the constitutive vortex rings do not interact too much. This kind of arguments was introduced for PDE by Angenent in [6], extending previous results by Palmer [27] on ODE. This method is also central in the recent work of Bessi [9] and in [10], both dealing with homoclinic chaos in Hamiltonian systems via variational methods. See also [14] and the references therein. In [9, 10], a non-degenerate "primary" homoclinic solution is found variationally, for example by the mountain-pass lemma, and time-translations give a whole family of homoclinic solutions. The interaction between them is proved to be small if their mutual distances (in the time coordinate) are large enough. Finally degree theory and a homotopy argument allow one to deduce the existence of a whole family of "multibump" solutions. Going back to vortices, Norbury's family plays the role of the primary homoclinic solution and dilatation the role of time-translation. Our result on vortex rings may be regarded as establishing some kind of spatial chaos. The reader interested in homoclinic chaos via variational methods should consult the fundamental paper by Séré [28] and [1, 15, 24] for extensions to PDE. For the definition and basic properties of the degree of a nonlinear mapping in an infinite-dimensional space, see for example Chapter 7 of [17].

This work is one among many on the co-existence and multiplicity of vortices. We refer to the book by Lamb [23], and especially to section 164. For a recent account of the phenomenon of leapfrogging of vortices, see the book by Berger [8]. Other results obtained via rearrangement theory (for example vortices in dumb-bell-shaped regions) may be found in the work of Burton [11, 12]. As pointed out to the author by Dr G.R. Burton, explicit steady two-dimensional co-existent vortices are given in Lamb [23], section 165, formula (10) (see also [13]).

2 Some properties of $S(x, v, k)$

The following lemma is intended to state some basic properties of $S(x, v, k)$ which will be used subsequently. For the proof, see [26, 4, 22].

Lemma 1 *There exist $\epsilon, \tilde{k} > 0$ such that for all $v \in M_\epsilon$ and $k \in [0, \tilde{k}[$*

1. $\{x \in 2\bar{B} : v(x) - 1 - kr^{-2} \geq 0\} \subset \{x \in \mathbb{R}^5 : \ \rho < 3/2$ *and* $r > 2\sqrt{k}/3\}$;
2. $\{x \in 2\bar{B} : r > 0$ *and* $v(x) - 1 - kr^{-2} = 0\}$ *is a non-degenerate level set; for* $k > 0$, *its projection on the* (r, z) *half-plane is a Jordan curve;*
3. $S(., v, k) \in C^1(\mathbb{R}^5)$ *and*

$$\partial_1 S(x, v, k) = \int_{2\bar{B}} -3\omega^{-1}|x - \tilde{x}|^{-5}(x - \tilde{x})H(v(\tilde{x}) - 1 - k\tilde{r}^{-2})d\tilde{x};$$

4. *$S(.,v,k)$ is C^∞ at $x\in\mathbb{R}^5$ satisfying $v(x)-1-kr^{-2}\neq 0$; for $|x|\geq 2$, $|\partial_1^2 S(x,v,k)|\leq Const|x|^{-5}$, where the constant is independent of v and k;*

Considered as a mapping from $M_\epsilon\times[0,\tilde{k}[$ to $C^1(\mathbb{R}^5)$ endowed with the norm $\|u\|=\sup\{|u(x)|(1+|x|^3)+|u'(x)|(1+|x|^4):x\in\mathbb{R}^5\}$, S is compact.

The next result is crucial for the third section. Its proof closely follows Amick-Turner [5], the main difference being the functional setting.

Lemma 2 *Hill's vortex v_H (restricted to $2\bar{B}$) is the unique fixed point of S_0 in M_ϵ and $\deg(I-S_0,v_H,0)=-1$.*

Proof. The unique solution in $C^1(\mathbb{R}^5)$ of $v=S(.,v,0)$ vanishing at infinity is v_H [3]. For $\rho_0\in]0,\infty]$, we introduce the Green function of the Laplacian on the ball $\rho_0\bar{B}$ with Dirichlet conditions at the boundary (see section 2.5 in [22]):

$$G_{\rho_0}(x,\tilde{x})=|x-\tilde{x}|^{-3}-(\rho_0^{-2}|x|^2|\tilde{x}|^2+\rho_0^2-2x\tilde{x})^{-3/2}$$

and $G_\infty(x,\tilde{x})=|x-\tilde{x}|^{-3}$, $\tilde{x}\neq x$. Note the inequality

$$\rho_0^{-2}|x|^2|\tilde{x}|^2+\rho_0^2-2x\tilde{x}\geq -4+\rho_0^2-4\rho_0\geq \rho_0^2/2$$

for $|x|\leq\rho_0$, $\tilde{x}\in 2\bar{B}$ and ρ_0 large enough. For $\delta>0$, let $f_\delta\in C(\mathbb{R})$ be the increasing function such that $f_\delta(x)=0$ if $x\leq 0$, $f_\delta(x)=x/\delta$ if $0\leq x\leq\delta$ and $f_\delta(x)=1$ if $x\geq\delta$. Set $f_0=H$ and

$$S_{\delta,\rho_0}(v)(x)=\int_{2\bar{B}}\omega^{-1}G_{\rho_0}(x,\tilde{x})f_\delta(v(\tilde{x})-1)d\tilde{x},$$

$v\in M_\epsilon$, $\delta,\rho_0^{-1}\geq 0$. The previous S_0 is now identified with $S_{0,\infty}$. We have

$$\partial_x S_{\delta,\rho_0}(v)(x)=\int_{2\bar{B}}\omega^{-1}\partial_1 G_{\rho_0}(x,\tilde{x})f_\delta(v(\tilde{x})-1)d\tilde{x}.$$

Again S_{δ,ρ_0} is a compact operator from M_ϵ to $C^1(2\bar{B})$ for δ,ρ_0^{-1} small and the dependence of $S_{\delta,\rho_0}\in C(M_\epsilon,C^1(2\bar{B}))$ with respect to $\delta,\rho_0^{-1}\geq 0$ is continuous (in the uniform topology). Moreover, for $\delta>0$, S_{δ,ρ_0} is C^1 with respect to v and the derivative at any v is the compact operator

$$\left(S'_{\delta,\rho_0}(v)u\right)(x)=\int_{2\bar{B}}\omega^{-1}G_{\rho_0}(x,\tilde{x})f'_\delta(v(\tilde{x})-1)u(\tilde{x})d\tilde{x}.$$

This can be obtained by approximating f_δ by smooth functions.

For $\delta, \rho_0^{-1} > 0$, any fixed point $v \in M_\epsilon$ of S_{δ,ρ_0} may be extended to all $\rho_0 \bar{B}$ such that

$$-\Delta v = 15 f_\delta(v-1) \text{ on } \rho_0 \bar{B},\ v = 0 \text{ on } \partial(\rho_0 \bar{B}). \tag{3}$$

It is proved in [5], Lemma 2.2, that for some $\epsilon > 0$ and all small $\delta, \rho_0^{-1} > 0$, equation (3) has a unique solution v_{δ,ρ_0} near v_H in $C^1(\rho_0 \bar{B})$, the estimates being uniform in δ and ρ_0. The eigenvalue problem for $S'_{\delta,\rho_0}(v|_{2\bar{B}})$ with $v = v_{\delta,\rho_0}$ is equivalent to the equation

$$-\lambda \Delta u = 15 f'_\delta(v_{\delta,\rho_0} - 1)u \text{ a.e.},\ u \in C^1_{sym}(\rho_0 \bar{B}) \cap H^2(\rho_0 B),\ u = 0 \text{ on } \partial(\rho_0 \bar{B}),\ \lambda \in \mathbb{R}. \tag{4}$$

In Theorem 2.3 of [5], it is shown that for a set of $\delta, \rho_0^{-1} > 0$ whose closure contains $\delta = \rho_0^{-1} = 0$, equation (4) has only one eigenvalue λ larger then 1 (counting multiplicity), the others being smaller then 1. As a consequence, $\deg(I - S_{\delta,\rho_0}, M_\epsilon, 0) = -1$ and, by homotopy, $\deg(I - S_0, v_H, 0) = -1$. ■

3 Nested vortices

We are now ready to construct $n \geq 1$ nested axi-symmetric vortices moving at the same speed. The ith vortex ring $\mathcal{R}_i$ is strongly affected by the flow generated by the outer rings $\mathcal{R}_1, \ldots, \mathcal{R}_{i-1}$, which is described, up to some scaling, by G_i introduced below. However, if $\mathcal{R}_i$ is small enough, its strength increases. This is the reason why we have to take account of G_i only near the axis of symmetry, where we try to fit $\mathcal{R}_i$. As $\mathcal{R}_i$ is to move at speed 1, its relative velocity W_i with respect to the flow G_i near the axis of symmetry is in principle entirely determined. On the other hand, if $\mathcal{R}_i$ is small enough, its influence on the outer rings is insignificant.

In the sequel, ϵ and k are supposed to be sufficiently small positive constants. First we introduce n positive characteristic lengths $a_1 = 1, a_2, \ldots, a_n$. The fact that the vortex rings are nested is reflected in the condition

$$a_{i+1} \leq (1/4) a_i \sqrt{k} \text{ if } i + 1 \leq n.$$

We define recursively $G_i(x, v_1, \ldots, v_{i-1})$ and $W_i(v_1, \ldots, v_{i-1}) \in \mathbb{R}^*$ for $1 \leq$

$i \leq n$, with $x \in 2\bar{B}$ and $v_j \in M_\epsilon$ $(1 \leq j < i)$. We set $G_1(x) \equiv 0$, $W_1 = 1$ and

$$G_i(x, v_1, \ldots, v_{i-1}) = \sum_{j=1}^{i-1} W_j(v_1, \ldots, v_{j-1}) S(a_i x / a_j, v_j, k),$$

$$W_i(v_1, \ldots, v_{i-1}) = 1 - G_i(0, v_1, \ldots, v_{i-1}).$$

Note that $(5/4)^{i-1} \leq (-1)^{i-1} W_i \leq 2^{i-1}$ for all $v_1, \ldots, v_{i-1} \in M_\epsilon$. This is easily proved by induction once we have observed that $S(0, v, k)$ is near $5/2 = v_H(0)$ for $v \in M_\epsilon$ and ϵ, k small enough (independently of n, i, a_j).

We have in mind to solve the system

$$v_i = T_i(., v_1, \ldots, v_n), \ i = 1, \ldots, n, \tag{5}$$

where

$$\begin{aligned} & T_i(x, v_1, \ldots, v_n) = & (6)\\ & W_i(v_1, \ldots, v_{i-1})^{-1} \left(G_i(x, v_1, \ldots, v_{i-1}) - G_i(0, \ldots) \right) & (7)\\ + \ & S(x, v_i, k) & (8)\\ + \ & \xi(|x|) W_i(v_1, \ldots, v_{i-1})^{-1} \sum_{j=i+1}^{n} W_j(\ldots) S(a_i x / a_j, v_j, k). & (9) \end{aligned}$$

The function ξ is smooth and such that $\xi(x) = 1$ for $(2/3)\sqrt{k} \leq x \leq 2$ and $\xi(x) = 0$ for $x \leq (1/2)\sqrt{k}$.

Theorem 3 *Equation (5) admits a solution $(v_{n,1}, \ldots, v_{n,n})$ in $(M_\epsilon)^n$ if the a_j/a_i, $j > i$, are small enough.*

PROOF. As $|(\partial/\partial x) S(a_i x/a_j, v_j, k_j)| \leq a_i/a_j |S(., v_j, k_j)|_{C^1(2\bar{B})} \leq Const\ a_i/a_j$ for $x \in 2\bar{B}$, where the constant is independent of x, a_i, a_j, we easily deduce that (7) goes to 0 in the $C^1(2\bar{B})$ topology, uniformly in $v_1, \ldots, v_{i-1}$, when the a_i/a_j, $i > j$, tend to 0. To estimate (9), we observe that $S(a_i x/a_j) \to 0$ in $C^1(2\bar{B} \backslash (1/2)\sqrt{k}\bar{B})$ when $a_i/a_j \to \infty$, thanks to the last statement of Lemma 1. Hence (9) goes to 0 in $C^1(2\bar{B})$ uniformly in $v_1, \ldots, v_n$ if a_j/a_i, $j > i$, tend to 0. As (7) and (9) are small with respect to (8) in $C^1(2\bar{B})$, a homotopy gives a solution of (5) in $(M_\epsilon)^n$. More specifically, we compose two homotopies. The first one is at fixed k and suppresses the terms (7) and (9). The second homotopy decreases the value of k to $k = 0$. Taking k, (7) and (9) small enough, this can be achieved without creating any fixed point on the boundary of $(M_\epsilon)^n$. The conclusion follows from the equality $\deg((I - S_0)^n, (M_\epsilon)^n, 0) = (-1)^n$. ∎

From now on, we choose a sequence $(a_i : i \in \mathbb{N})$ converging quickly to 0 in such a way that Theorem 3 holds for $(a_1, \dots, a_n)$ and all $n \in \mathbb{N}$.

Theorem 4 *For each $n \geq 1$, equation (2) admits some solution w_n consisting of n nested vortex rings. Moreover the total kinetic energy corresponding to w_n is uniformly bounded in n if the sequence $(a_i, i \in \mathbb{N})$ converges quickly to 0.*

Proof. Let us fix $n \geq 1$. To obtain a solution of (2) from the $v_{n,i}$ given by Theorem 3, we set

$$\tilde{v}_{n,i}(y) = W_i v_{n,i}(y/a_i) + G_i(0, v_{n,1}, \dots, v_{n,i-1}) = W_i v_{n,i}(y/a_i) + 1 - W_i,$$

for $y \in 2a_i\bar{B} \backslash (2/3)a_i\sqrt{k}\bar{B}$ and $1 \leq i \leq n$. We get successively

$$\begin{aligned}
\tilde{v}_{n,i}(y) &= \sum_{j=1}^{n} W_j(v_{n,1}, \dots, v_{n,j-1}) S(y/a_j, v_{n,j}, k) \\
&= \sum_{j=1}^{n} W_j \int_{2\bar{B}} \omega^{-1} |y/a_j - \tilde{x}|^{-3} H(v_{n,j}(\tilde{x}) - 1 - k\tilde{r}^{-2})) d\tilde{x} \\
&= \sum_{j=1}^{n} W_j \int_{2\bar{B}} \omega^{-1} |y/a_j - \tilde{x}|^{-3} H\left(\operatorname{sgn}(W_j)(\tilde{v}_{n,j}(a_j\tilde{x}) - 1 - W_j k\tilde{r}^{-2})\right) d\tilde{x} \\
&= \sum_{j=1}^{n} W_j a_j^{-2} \int_{2a_j\bar{B}} \omega^{-1} |y - \tilde{x}|^{-3} H\left(\operatorname{sgn}(W_j)(\tilde{v}_{n,j}(\tilde{x}) - 1 - a_j^2 W_j k\tilde{r}^{-2})\right) d\tilde{x}.
\end{aligned}$$

Hence $w_n \in C^1_{sym}(\mathbb{R}^5)$ defined by

$$\begin{aligned}
w_n(x) &:= \sum_{j=1}^{n} W_j a_j^{-2} \int_{2a_j\bar{B}} \omega^{-1} |x - \tilde{x}|^{-3} H\left(\operatorname{sgn}(W_j)(\tilde{v}_{n,j}(\tilde{x}) - 1 - a_j^2 W_j k\tilde{r}^{-2})\right) d\tilde{x} \\
&= \sum_{j=1}^{n} W_j a_j^{-2} \int_{2a_j\bar{B}} \omega^{-1} |x - \tilde{x}|^{-3} H\left(\operatorname{sgn}(W_j)(w_n(\tilde{x}) - 1 - a_j^2 W_j k\tilde{r}^{-2})\right) d\tilde{x}
\end{aligned}$$

is a solution of (2). Its core has exactly one component in each $3a_i/2\bar{B}\backslash 2a_i\sqrt{k}/3\bar{B}$, $i=1,\ldots,n$, and the vorticity has opposite sign on consecutive components. By Lemma 1, Part 2, each component is homeomorphic to a solid torus.

The total kinetic energy corresponding to w_n is simply given by

$$\int_{\mathbb{R}^5} |\nabla w_n|^2\, dx/(2\pi^2).$$

By roughly estimating the gradient of the right-hand side of

$$w_n(x) = \sum_{j=1}^{n} W_j \int_{2\bar{B}} \omega^{-1}|x/a_j - \tilde{x}|^{-3} H(v_{n,j}(\tilde{x}) - 1 - k\tilde{r}^{-2}) d\tilde{x}, \tag{10}$$

we obtain, for $2 \le i \le n$ and $2a_i \le |x| < 2a_{i-1}$, the upper-bound

$$\begin{aligned}
|\nabla w_n(x)| &\le Const \left(\sum_{j=1}^{i-1} \frac{|W_j|}{a_j} + \frac{|W_i|}{a_i}(|x|/a_i)^{-4} + \sum_{j=i+1}^{n} \frac{|W_j|}{a_j}(a_i/a_j)^{-4} \right) \\
&\le Const \left(a_{i-1}^{-1} \sum_{j=1}^{i-2} |W_j| \frac{a_{i-1}}{a_j} + \frac{|W_{i-1}|}{a_{i-1}} \right. \\
&\qquad \left. + \frac{|W_i|}{|x|} + a_{i-1}^{-1} \sum_{j=i+1}^{n} |W_j| \frac{a_{i-1}}{a_i}(a_j/a_i)^3 \right) \\
&\le Const\,(|W_{i-1}|/a_{i-1} + |W_i|/|x|)
\end{aligned}$$

if the a_j/a_i, $j > i$, are small enough and where the constants may be chosen independently of n. This also holds for $i = n+1$ if we set $W_{n+1} = a_{n+1} = 0$. In particular the total kinetic energy is uniformly bounded by

$$Const \sum_{i=2}^{\infty} a_{i-1}^3 \left(W_{i-1}^2 + W_i^2 \right) + Const \left(\sum_{j=1}^{\infty} |W_j| a_j^3 \right)^2 \int_{|x|>2} |x|^{-8} dx,$$

which is finite if the sequence $(a_i, i \in \mathbb{N})$ converges quickly to 0. ■

Our result in the case $n = 1$ (unsurprisingly) gives Norbury's vortex rings. In the limit $n \to \infty$, the following theorem holds.

Theorem 5 *Taking an infinite sequence $(a_i, i \in \mathbb{N})$ tending sufficiently quickly to zero, the solutions with n rings converge (up to a subsequence) in $C^1_{loc}(\{x \in$*

$\mathbb{R}^5 : r > 0\})$ *to a solution consisting of a countable set of nested co-axial rings moving at the same speed and with finite total kinetic energy. In the moving frame, the cores accumulate to the origin, which is a point of discontinuity of the velocity.*

PROOF. We have

$$
\begin{aligned}
v_{n,i} &= W_i(v_{n,1},\ldots,v_{n,i-1})^{-1}\left(G_i(x,v_{n,1},\ldots,v_{n,i-1}) - G_i(0,\ldots)\right) \quad (11)\\
&+ S(x,v_{n,i},k) \quad (12)\\
&+ \xi(|x|)W_i(v_{n,1},\ldots,v_{n,i-1})^{-1}\sum_{j=i+1}^{n} W_j(\ldots)S(a_i x/a_j, v_{n,j}, k). \quad (13)
\end{aligned}
$$

For fixed i, the sequence $(v_{n,i} : n \in \mathbb{N})$ converges, up to a subsequence, in M_ϵ to some $v_{\infty,i}$. This is proved by induction on i, using the compactness of (12) and the fact that the sequence $\{(13) : n \in \mathbb{N}\}$ admits a Cauchy subsequence if $(a_l, l \in \mathbb{N})$ converges quickly to 0. By (10), we deduce the convergence of w_n in $C^1_{loc}(\{x \in \mathbb{R}^5 : r > 0\})$ to w_∞ satisfying

$$
w_\infty(x) = \sum_{j=1}^{\infty} W_j \int_{2\bar{B}} \omega^{-1}|x/a_j - \tilde{x}|^{-3} H(v_{\infty,j}(\tilde{x}) - 1 - k\tilde{r}^{-2})d\tilde{x}. \qquad (14)
$$

For a fixed compact set in $\{x \in \mathbb{R}^5 : r > 0\}$, the convergence of (14) is even in C^2, by Part 4 of Lemma 1, if we omit the terms (finitely many) in the series that have discontinuous second derivative. The behaviours of $|w_\infty|$, $|w'_\infty|$ and $|w''_\infty|$ for large $|x|$ are like respectively $O(|x|^{-3})$, $O(|x|^{-4})$ and $O(|x|^{-5})$. ∎

4 Conclusion

For some sequence $(a_i : i \in \mathbb{N})$ of positive numbers converging quickly to 0, we have established, for all $n \in \mathbb{N}$, the existence of w_n, which corresponds to n vortex rings moving at the same speed, as well as a solution w_∞ consisting of an infinity of vortex rings accumulating to the origin. The physical interpretation of w_∞ itself is not clear, because the corresponding velocity field has a point of discontinuity. However, w_∞ may have some interest as the limit of the w_n (up to a subsequence).

More generally, we could argue analogously for every finite subset and every subsequence of $(a_i : i \in \mathbb{N})$, obtaining in this way a family $\left\{w_\sigma : \sigma \in \{0,1\}^{\mathbb{N}}\right\}$ of solutions of Euler equation.

The main restriction lies in the fact that the consecutive vortex rings are near dilatations of Hill's vortex. As a consequence, the case with all velocities W_i of one sign is precluded. More complicated steady solutions could probably be constructed using vortices of small cross section [18, 19].

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