A multiplicity result for solitary gravity-capillary waves in deep water via critical-point theory

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1 Introduction

1.1 The main result

This paper considers the existence of solitary-wave solutions to the classical water-wave problem, which concerns the two-dimensional, irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The hydrodynamic problem involves the physical quantities g, σ, h and c, which denote respectively the acceleration due to gravity, the coefficient of surface tension, the asymptotic depth of the water and the velocity of the solitary wave. Introducing dimensionless variables, one finds that the equations of motion depend upon the parameters $\alpha = gh/c^2$ and $\beta = \sigma/hc^2$ (see Section 2). The past ten years have seen the development of mathematically rigorous existence theories for small-amplitude solitary-wave solutions of the water-wave problem in certain regions of (β, α)-parameter space, namely near the curves C_1, C_2, C_3, C_4 shown in Figure 1. The number of purely imaginary eigenvalues of the linearised problem changes as one crosses C_2, C_3, C_4, and the curve C_1 depicts the locus of points where four non-zero, real, simple eigenvalues become complex without passing through zero (Kirchgässner 1988).

Amick & Kirchgässner (1989) looked for solitary waves near the curve $C_4 \setminus \{(1/3, 1)\}$. They found a solitary wave of depression just above this curve and proved it to be unique. The parameter region to the left of the curve C_1 and near the point $(1/3, 1)$ was studied by Buffoni, Groves & Toland (1996), who proved the existence of infinitely many distinct solitary-wave solutions. These solutions are waves of depression which have an arbitrary but finite number of troughs, between which there are distributed smaller local maxima and minima, and which have an exponentially decaying oscillatory tail at infinity. The existence of *generalised solitary waves* which have an oscillatory tail of exponentially small amplitude at infinity has been established just below the curve $C_3 \setminus \{(1/3, 1)\}$ (Beale 1991; Sun 1991; Iooss & Kirchgässner 1992; Lombardi 1995).

The present article is concerned with the parameter region just to the right of $C_2 \setminus \{(1/3, 1)\}$ (the shaded region in Figure 1). This region was first discussed by Iooss & Kirchgässner (1990), who showed that solitary waves correspond to homoclinic solutions of a fourth-order, reversible dynamical system. They applied a normal-form transformation and demonstrated that at each order the normal form admits two homoclinic solutions provided certain coefficients have the correct signs. Dias & Iooss (1993) calculated the coefficients in question and thereby verified the condition on their signs, and Iooss & Pérouème (1993) completed the existence proof by showing that the homoclinic solutions of the normal form persist for the full dynamical system. The corresponding solutions of the water-wave problem are modulated solitary waves whose envelopes are symmetric and decay exponentially to zero at infinity. The dominant feature of one wave is its central crest, while the dominant feature of the other is its central trough. Notice that by moving sufficiently high up along the curve C_2 one can treat water of arbitrarily large depth. The limiting case $h = \infty$ has recently been considered by Kirrmann & Iooss (1995), who obtained a similar result with the qualitative difference that the waves decay algebraically rather than exponentially to zero at infinity.

This article extends the work of the above authors; the main result may be stated as follows. For each $(\beta_s, \alpha_s) \in C_2 \setminus \{(1/3, 1)\}$ there exists a real number $\epsilon > 0$ such that for each (β, α)

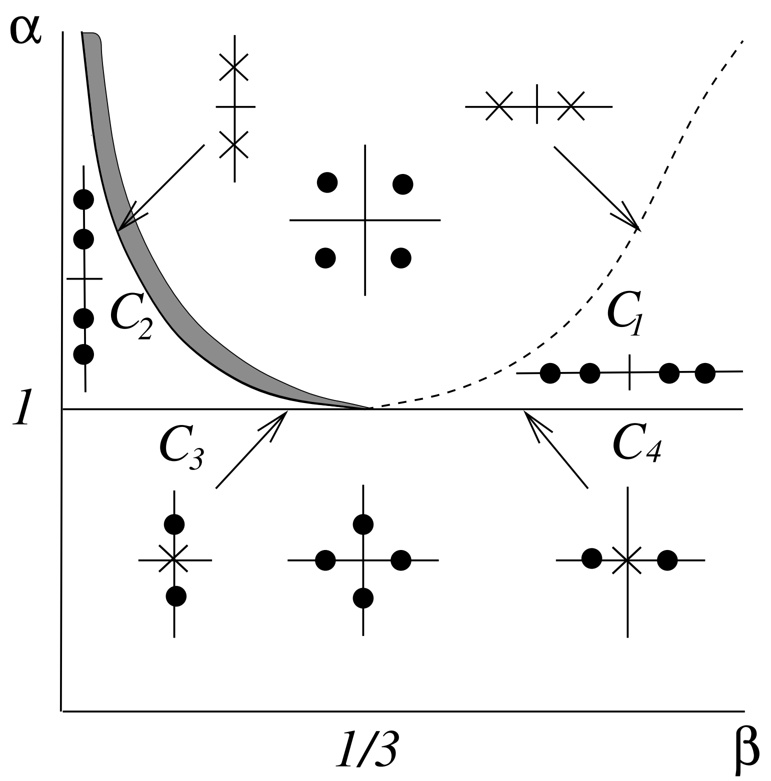

Figure 1: The curves C_2, C_3, C_4 consist of points in (β, α)-parameter space at which the number of purely imaginary eigenvalues of the linearised problem changes; at C_1 four non-zero, real eigenvalues become complex without passing through zero. Dots and crosses denote respectively simple and double eigenvalues. The shaded region is the region to be studied in this paper.

with

$$\beta = \beta_s, \qquad \alpha = \alpha_s + \epsilon^2$$

there are infinitely many geometrically distinct modulated solitary-wave solutions of the water-wave problem. The solitary waves are associated with critical points of a functional, and one of them, the *primary* wave, corresponds to a critical point of mountain-pass type. If this critical point is isolated in a certain function space, the other waves are *multibump* solutions whose envelopes have an arbitrary but finite number of crests and exponentially decaying tails.

1.2 The method

The method used here, as in the references cited above, is a reduction of the mathematical problem for solitary waves to an equivalent system of ordinary differential equations using a centre-manifold reduction technique due to Mielke (1988). The first step in this method is to formulate the problem as an infinite-dimensional, quasilinear dynamical system; Mielke (1991) has examined the case when such a system is Hamiltonian and has shown that the resulting reduced system is also Hamiltonian. In Sections 2 and 3 of the present paper Mielke's theory is applied to a Hamiltonian formulation of the mathematical problem for solitary waves. The result is a fourth-order Hamiltonian dynamical system analogous to that studied by Iooss & Kirchgässner (1990), who did not use any Hamiltonian formalism. In Section 4 a Hamiltonian counterpart of the normal-form transformation used by Iooss & Kirchgässner is applied, and after a Legendre transformation and a scaling of the dependent and independent variables, one arrives at the Lagrangian system with Lagrangian

$$\tilde{L}(v,\dot{v},\tau,\epsilon)=\frac{1}{2}|\dot{v}|^2+\frac{1}{2}C|v|^2-\frac{1}{4}D|v|^4+\epsilon G(|v|^2,(s\mathcal{I}v,\dot{v}),\epsilon)+\tilde{S}(v,\dot{v},\tau,\epsilon).$$

Here G is a polynomial satisfying $G(a,b,\epsilon)=O(|a|+|b|)$ as $(a,b)\to 0$, C, D, s are positive constants, $v\in H^1(\mathbb{R},\mathbb{R}^2)$, $|\cdot|$, $(\cdot,\cdot)$ are the usual norm and inner product on $\mathbb{R}^2$, and

$$\mathcal{I}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix};$$

the function $\tilde{S}(\cdot,\cdot,\cdot,\epsilon)$ possesses derivatives with respect to $(v,\dot{v},\tau)$ which are all $O(\epsilon)$ as $\epsilon\to 0$.

When $\epsilon=0$, Lagrange's equations for the Lagrangian $\tilde{L}(v,\dot{v},\epsilon)$ give a completely integrable system and one may write down explicitly its family of homoclinic solutions (see Section 5.1). The critical issue here is the persistence of some of the solutions for small $\epsilon>0$. The function $\tilde{L}(v,\dot{v},\cdot,\epsilon)$ is periodic and its period tends to zero as $\epsilon\to 0$, so that the distance between the stable and unstable manifolds of the zero equilibrium cannot be assessed by a direct application of Melnikov theory. In the present paper the results of Ambrosetti & Coti Zelati (1992), Bessi (1995) and Buffoni & Séré (1996) are adapted to prove the existence of a primary homoclinic solution for $\epsilon>0$ and to deduce the existence of infinitely many non-reversible homoclinic solutions. Ambrosetti & Coti Zelati (1992) consider large perturbations of a completely integrable Lagrangian system which has a family of homoclinic solutions and show that two persist as critical points of the associated action functional; Section 5 and Appendix C explain how a variant of their method can be applied in the present setting. Bessi (1995) implements a Morse-reduction procedure to obtain homoclinic chaos for time-periodic dynamical systems. As a by-product, his technique is most helpful in dealing with situations where there is no obvious function space in which to study the action functional. Buffoni & Séré (1996) extend Bessi's method to autonomous Lagrangian systems whose linearisation has a saddle-focus equilibrium at the origin. The results of Bessi and Buffoni & Séré are in the spirit of those of Séré (1992), where a variational characterisation of a 'primary' homoclinic solution leads to the existence of infinitely many other homoclinic solutions. It should also be noted that a variational approach to the above type of perturbed Lagrangian system has also been given by Angenent (1993).

In both the present work and that of Buffoni, Groves & Toland (1996), the existence of infinitely many solitary waves is proved by using the centre-manifold reduction technique to obtain a four-dimensional Hamiltonian system whose linearisation has a saddle-focus equilibrium at the origin. The reduced systems and the techniques used to analyse them are, however, quite different. The system obtained by Buffoni, Groves & Toland (1996) has a homoclinic orbit which is the transverse intersection of the stable and unstable manifold in the zero energy surface. One may therefore apply the result of Devaney (1976) to conclude that there is a Smale horseshoe in the dynamics within the zero energy surface. More specifically, there is a Poincaré map that has an invariant Cantor set upon which it is topologically conjugate to a full shift on a countably infinite set of symbols; a full and precise description of the consequent infinite set of homoclinic orbits can therefore be given (see Buffoni, Champneys & Toland (1994)). The same result would of course also be true for the present system if one could establish the existence of a transverse homoclinic orbit; for the reasons given above this task would appear to be difficult. The nondegeneracy condition stipulated here is weaker than transversality, and has the added advantage that its failure automatically implies the existence of an infinite family of homoclinic solutions. The main drawback of the present approach is that the description of the solution set is less precise. It is not possible to give a complete description of the primary homoclinic orbit, for example to determine whether it is reversible or a wave of elevation or depression, and similar comments hold for the other homoclinic orbits if the nondegeneracy condition fails. However if the nondegeneracy condition is satisfied one can at least say that the other homoclinic orbits resemble multiple copies of the primary orbit. For their fourth-order system Buffoni & Séré (1996) show that one is able to give a more precise result if the system is analytic, namely that the following alternative holds: either the stable and unstable manifolds coincide or multibump solutions exist. Unfortunately, the centre-manifold technique leads to a system which is not necessarily analytic, but of class C^k for an arbitrarily large but fixed value of k. However, Buffoni & Séré state as a remark that analyticity is in fact not necessary for their alternative to hold. For the present problem, coincidence of the stable and unstable manifolds would imply the existence of a 'circle' of homoclinic solutions, that is a continuous family of homoclinic solutions parameterised by $\theta \in [0, 2\pi)$.

2 Hamiltonian formulation of the hydrodynamic problem

The classical water-wave problem concerns the two-dimensional, irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid occupies the domain $D_\eta = \{(x, y) : x \in \mathbb{R},\ y \in (0, \eta(x, t))\}$, where $\eta > 0$ is a function of the spatial coordinate x and time t, and its motion is specified in terms of an Eulerian velocity potential $\phi(x, y, t)$. In the steady water-wave problem one considers motions which are stationary with respect to a uniformly translating frame of reference, so that $\eta(x, t) = \eta(x - ct)$, $\phi(x, y, t) = \phi(x - ct, y)$. In dimensionless variables, a *solitary wave* is a nontrivial solution of Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{for} \quad 0 < y < \eta(x) \tag{1}$$

with boundary conditions

$$\phi_y = 0 \qquad \text{on} \quad y = 0, \tag{2}$$

$$\phi_y = \eta_x \phi_x - \eta_x \qquad \text{on} \quad y = \eta(x), \tag{3}$$

$$\left(-\phi_x + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2\right) + \alpha(\eta - 1) - \beta \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2}}\right]_x = 0 \qquad \text{on} \quad y = \eta(x) \tag{4}$$

and asymptotic conditions

$$\eta \to 1, \qquad (\phi_x, \phi_y) \to 0 \qquad \text{as} \quad x \to \pm\infty. \tag{5}$$

Here x is now a shorthand for $x - ct$, 'trivial' means $\eta(x) = 1$ for all x and $(\phi_x, \phi_y) = 0$ for all x and y, and $\alpha = 1/F^2$, $\beta = \tau/F^2$, where $F = \sqrt{c^2/gh}$, $\tau = \sigma/gh^2$ are the *Froude* and *Bond* numbers.

Let us next formulate (1), (2), (3), (4) as an infinite-dimensional Hamiltonian system. The Hamiltonian system (M, Ω, H) in question is defined on the manifold

$$M = \{(\eta, \omega, \Phi, \Gamma) \in (0, \infty) \times \mathbb{R} \times H^1(0,1) \times H^1_0(0,1) : \int_0^1 \Phi \, dz = 0, |W| < \beta\}, \tag{6}$$

where

$$W = \omega + \frac{1}{\eta} \int_0^1 z\Phi_z(\Gamma_z - 1)\, dz = \omega - \frac{1}{\eta}\Phi(1) + \frac{1}{\eta}\int_0^1 z\Phi_z\Gamma_z \, dz,$$

which is an open subset of the Hilbert space

$$\mathcal{H} = \{(\eta, \omega, \Phi, \Gamma) \in \mathbb{R} \times \mathbb{R} \times H^1(0,1) \times H^1_0(0,1) : \int_0^1 \Phi \, dz = 0\}.$$

The symplectic 2-form Ω on M is defined (independently of $m \in M$) by

$$\Omega|_m((\eta_1, \omega_1, \Phi_1, \bar{\Gamma}_1), (\eta_2, \omega_2, \Phi_2, \bar{\Gamma}_2)) = \omega_2\eta_1 - \eta_2\omega_1 + \int_0^1 (\Gamma_{2z}\Phi_1 - \Gamma_{1z}\Phi_2)\, dz \tag{7}$$

and the Hamiltonian function is

$$H(\eta, \omega, \Phi, \Gamma, \alpha, \beta) = \frac{1}{2\eta}\int_0^1 (\Gamma_z^2 - \Phi_z^2)\, dz - \frac{1}{2}\alpha(\eta-1)^2 + \frac{1}{2}\left(\frac{1}{\eta} - 1\right) + \frac{1}{2}(\eta - 1) + \beta - \sqrt{\beta^2 - W^2}. \tag{8}$$

The domain $\mathcal{D}(v_H)$ of the Hamiltonian vector field v_H is the set of all $(\rho, \omega, \Phi, \Gamma) \in M$ such that $(\eta, \omega, \Phi, \Gamma)$ belongs to $(0, \infty) \times \mathbb{R} \times H^2(0,1) \times H^2(0,1)$ and satisfies

$$\Phi_z = 0 \qquad \text{on} \quad z = 0, \tag{9}$$

$$\Phi_z - \frac{W(\Gamma_z - 1)}{\sqrt{\beta^2 - W^2}} = 0 \qquad \text{on} \quad z = 1, \tag{10}$$

and the smooth vector field v_H is defined on $\mathcal{D}(v_H)$ by

$$\begin{pmatrix} \eta \\ \omega \\ \Phi \\ \Gamma \end{pmatrix} \mapsto \begin{pmatrix} \frac{W}{\sqrt{\beta^2 - W^2}} \\ \frac{1}{2\eta^2}\int_0^1 (\Gamma_z^2 - \Phi_z^2)\, dz + \frac{W}{\eta^2\sqrt{\beta^2 - W^2}}\int_0^1 z\Phi_z(\Gamma_z - 1)\, dz + \frac{1}{2\eta^2} + \alpha(\eta - 1) - \frac{1}{2} \\ \frac{\Gamma_z}{\eta} + \frac{W}{\eta\sqrt{\beta^2 - W^2}}(z\Phi_z - \Phi(1)) \\ -\frac{\Phi_z}{\eta} + \frac{zW(\Gamma_z - 1)}{\eta\sqrt{\beta^2 - W^2}} \end{pmatrix}.$$

A continuously-differentiable path $\gamma : (a, b) \to \mathcal{D}(v_H) \subset M$ given by

$$\gamma(x) = (\eta(x), \omega(x), \Phi(x)(\cdot), \Gamma(x)(\cdot)).$$

is therefore a trajectory of v_H if it satisfies Hamilton's equations

$$\dot{\gamma}(x) = v_H(\gamma(x)), \qquad x \in (a, b). \tag{11}$$

A straightforward argument of the kind given by Buffoni, Groves & Toland (1996, §2(b)) shows that a trajectory of Hamilton's equations for (M,Ω,H) results in functions $\eta\in C^2(x_1,x_2)$ and $\phi\in C^2[\{(x,y):x\in(x_1,x_2),y\in[0,\eta(x)]\}]$, $a<x_1<x_2<b$, which solve the steady water-wave problem. (Note that Buffoni, Groves & Toland (1996, p. 581) wrongly state that $\int_0^\eta\phi\,dy=0$. An examination of the definition of ϕ in their calculations shows that this condition is not necessarily satisfied; it was intended to state that $\int_0^1\Phi\,dz=0$.)

The Hamiltonian system (M,Ω,H) introduced above poses problems for analysis because of the nonlinear boundary condition (10). The next step is therefore to find a change of variable which leads to an equivalent problem in a linear space. The present paper is ultimately concerned with solutions of the steady water-wave problem with η close to unity, ω, Φ, Ψ close to zero and (β,α) close to and to the right of the curve $C_2\setminus\{(1/3,1)\}=\{(\beta_s,\alpha_s):s\in(0,\infty)\}$, where

$$(\beta_s,\alpha_s)=\left(\frac{-1}{2\sinh^2(s)}+\frac{1}{2s\tanh(s)},\frac{s^2}{2\sinh^2(s)}+\frac{s}{2\tanh(s)}\right).$$

Let us therefore set

$$\eta=1+\rho,\qquad \alpha=\alpha_s+\mu,\qquad \beta=\beta_s, \tag{12}$$

for some fixed $s\in(0,\infty)$ and consider the manifold

$$N=\{(\rho,\omega,\Phi,\Gamma)\in\mathcal{H}:\max\{|\rho|,|\omega|,\|\Phi\|_{H^1(0,1)},\|\Gamma\|_{H_0^1(0,1)}\}<\epsilon\}, \tag{13}$$

in which ϵ is a positive, real constant chosen so that $\epsilon<1$ and

$$\epsilon+\frac{\epsilon(1+\epsilon)}{1-\epsilon}<\beta_s.$$

The manifold N is independent of α and β and on N one has that

$$|W|<|\omega|+\frac{1}{1-|\rho|}\|\Phi\|_{H^1(0,1)}\left(1+\|\Gamma\|_{H_0^1(0,1)}\right)<\beta_s=|\beta|,$$

so that solutions on N of the system

$$\dot\rho = \frac{W}{\sqrt{\beta_s^2-W^2}}, \tag{14}$$

$$\dot\omega = \frac{1}{2(\rho+1)^2}\int_0^1(\Gamma_z^2-\Phi_z^2)\,dz+\frac{W}{(\rho+1)^2\sqrt{\beta_s^2-W^2}}\int_0^1 z\Phi_z(\Gamma_z-1)\,dz + \frac{1}{2(\rho+1)^2}+(\alpha_s+\mu)\rho-\frac{1}{2}, \tag{15}$$

$$\dot\Phi = \frac{\Gamma_z}{\rho+1}+\frac{W}{(\rho+1)\sqrt{\beta_s^2-W^2}}\left(z\Phi_z-\Phi(1)\right), \tag{16}$$

$$\dot\Gamma = -\frac{\Phi_z}{\rho+1}+\frac{zW(\Gamma_z-1)}{(\rho+1)\sqrt{\beta_s^2-W^2}} \tag{17}$$

with boundary conditions

$$\Phi_z = 0 \qquad \text{on}\quad z=0, \tag{18}$$

$$\Phi_z-\frac{W(\Gamma_z-1)}{\sqrt{\beta_s^2-W^2}} = 0 \qquad \text{on}\quad z=1 \tag{19}$$

correspond to solutions on M of (11) with boundary conditions (9), (10) via the formulae (12). Let us write (14), (15), (16), (17) as

$$\dot{\gamma}_1(x) = g(\gamma_1(x), \mu),$$

where $\gamma_1(x) = (\rho(x), \omega(x), \Phi(x), \Gamma(x))$ and the domain of g is the set of $(\rho, \omega, \Phi, \Gamma, \mu)$ in $N \times \mathbb{R}^2$ such that $\Phi, \Gamma \in H^2(0,1)$ and satisfy (18), (19).

Now define $G : N \times \mathbb{R} \to \mathcal{H} \times \mathbb{R}$ by

$$G(\rho, \omega, \Phi, \Gamma, \mu) = (\rho, v, \Psi, \Gamma, \mu),$$

where $v = \Phi(1)$ and

$$\Psi(z) = \Phi(z) - \frac{W}{\sqrt{\beta_s^2 - W^2}} \left(\int_0^z s(\Gamma_z - 1)\, ds - \int_0^1 \int_0^z s(\Gamma_z - 1)\, ds\, dz \right).$$

Observe that G is a smooth map from $N \times \mathbb{R}$ into $\mathcal{H} \times \mathbb{R}$ with $G(0) = 0$ and its derivative at 0 is

$$dG[0](\rho, \omega, \Phi, \Gamma, \mu) = \left(\rho, \Phi(1), \Phi + \frac{1}{2\beta_s} \left(z^2 - \frac{1}{3} \right) (\omega - \Phi(1)), \Gamma, \mu \right).$$

Notice also that $dG[0]$ is a bijection with inverse

$$(dG[0])^{-1}(\rho, v, \Psi, \Gamma, \mu) = \left(\rho, 3\beta_s \Psi(1) + (1 - 3\beta_s) v, \Psi + \frac{3}{2} \left(z^2 - \frac{1}{3} \right) (v - \Psi(1)), \Gamma, \mu \right).$$

It follows from the inverse function theorem that G is a smooth diffeomorphism from a neighbourhood $V_1 \times \Lambda_1$ of 0 onto a neighbourhood $V_2 \times \Lambda_2$ of 0 in $\mathcal{H} \times \mathbb{R}$. The inverse of G may in fact be found in closed form; one has that $G^{-1}(\rho, v, \Psi, \Gamma, \mu) = (\rho, \omega, \Phi, \Gamma, \mu)$, where

$$\begin{aligned} \omega &= \frac{\beta_s R}{\sqrt{1 + R^2}} - \frac{R}{1 + \rho} \int_0^1 z^2 (\Gamma_z - 1)^2\, dz - \frac{1}{1 + \rho} \int_0^1 z(\Gamma_z - 1) \Psi_z\, dz, \\ \Phi &= \Psi + R A(\Gamma), \end{aligned}$$

and

$$\begin{aligned} A(\Gamma) &= \int_0^z s(\Gamma_z - 1)\, ds - \int_0^1 \int_0^z s(\Gamma_z - 1)\, ds\, dz, \\ R &= \frac{v - \Psi(1)}{A(\Gamma)(1)}. \end{aligned}$$

(The procedure for calculating G^{-1} is given by Buffoni, Groves & Toland (1996, p. 594).)

Observe that G is a diffeomorphism in a neighbourhood of zero in $(\mathcal{H} \times \mathbb{R}) \cap (\mathbb{R}^2 \times H^2(0,1) \times H^2(0,1) \times \mathbb{R})$ with respect to the usual norm on $\mathbb{R}^2 \times H^2(0,1) \times H^2(0,1) \times \mathbb{R}$. For each μ, it follows that $\{\gamma_1(x) : x \in (a,b)\} \subset V_1$ is a solution of (14), (15), (16), (17) if and only if $\{\gamma_2(x) : x \in (a,b)\} \subset V_2$ is a solution of

$$\dot{\gamma}_2(x) = \hat{g}(\gamma_2(x), \mu), \tag{20}$$

where

$$\hat{g}(\gamma_2(x),\mu) = d_1 G[G^{-1}(\gamma_2(x),\mu)](g(G^{-1}(\gamma_2(x),\mu)))$$

and the domain $\mathcal{D}(\hat{g})$ of $\hat{g}$ is the linear space of $(\eta, \upsilon, \Psi, \Gamma, \mu) \in V_2 \times \mathbb{R}$ such that $\Psi, \Gamma \in H^2(0,1)$ with

$$\Psi_z(0) = \Psi_z(1) = 0.$$

A straightforward calculation similar to that given by Buffoni, Groves & Toland (1996, Appendix A) shows that equations (20) are given explicitly by

$$\begin{aligned}
\dot{\rho} &= R, \\
\dot{\upsilon} &= = \frac{\Gamma_z(1)}{\rho+1} + \frac{R}{\rho+1}[R(\Gamma_z(1)-1)-\upsilon], \\
\dot{\Psi} &= -\frac{A(\Gamma)(1+R^2)^{\frac{3}{2}}}{\beta_s}\left[\frac{1}{2(\rho+1)^2} + (\alpha+\mu)\rho - \frac{1}{2} + \frac{R^2(\Gamma_z(1)-1)^2}{2(\rho+1)^2} + \frac{\Gamma_z(1)(\Gamma_z(1)-2)}{2(\rho+1)^2}\right] \\
&\quad + \frac{\Gamma_z}{\rho+1} + \frac{R}{\rho+1}[2z\Psi_z + Rz^2(\Gamma_z - 1) - \upsilon - \Psi - \Psi(1)], \\
\dot{\Gamma} &= -\frac{\Psi_z}{\rho+1}.
\end{aligned}$$

Let us now return to the Hamiltonian system (N, Ω, H) defined by (7), (8), (13). In keeping with the above theory, one sets $\beta = \beta_s$, $\alpha = \alpha_s + \mu$, $\mu \in \Lambda_1$, where Λ_1 is a neighbourhood of 0 in $\mathbb{R}$, and attention is restricted to a neighbourhood V_1 of 0 in N. The Hamiltonian system of interest here is therefore (V_1, Ω, H_1), where

$$\begin{aligned}
H_1(\rho,\omega,\Phi,\Gamma,\mu) &= \frac{1}{2(1+\rho)}\int_0^1 (\Gamma_z^2 - \Phi_z^2)\,dz - \frac{1}{2}(\alpha_s+\mu)\rho^2 + \frac{1}{2}\left(\frac{1}{1+\rho} - 1\right) \\
&\quad + \frac{1}{2}\rho + \beta_s - \sqrt{\beta_s^2 - W^2},
\end{aligned}$$

in which

$$W = w - \frac{\Phi(1)}{1+\rho} + \frac{1}{1+\rho}\int_0^1 z\Phi_z\Gamma_z\,dz$$

and $\mu \in \Lambda_1$. The diffeomorphism G transforms the Hamiltonian system (V_1, Ω, H_1) into the Hamiltonian system (V_2, Ω_2, H_2), where

$$\begin{aligned}
H_2(\rho,\upsilon,\Psi,\Gamma,\mu) &= H_1(G^{-1}(\rho,\upsilon,\Psi,\Gamma,\mu)) \\
&= \frac{1}{2(1+\rho)}\int_0^1 [\Gamma_z^2 - \Psi_z^2 - 2zR\Psi_z(\Gamma_z-1) - z^2R^2(\Gamma_z-1)^2]\,dz \\
&\quad - \frac{1}{2}(\alpha_s+\mu)\rho^2 + \frac{1}{2}\left(\frac{1}{1+\rho}-1\right) + \frac{1}{2}\rho + \beta_s\left(1 - \frac{1}{\sqrt{1+R^2}}\right)
\end{aligned}$$

for $\mu \in \Lambda_2$, and Ω_2 is given in coordinates by

$$\begin{aligned}
\Omega_2|_m(w^1,w^2) &= \Omega|_{G_1^{-1}(m,\mu)}(d_1G_1^{-1}[m,\mu](w^1), d_1G_1^{-1}[m,\mu](w^2)), \\
&= \Omega_0(w^1,w^2) + O(\| m \| \| w^1 \| \| w^2 \|)
\end{aligned} \tag{21}$$

as $m \to 0$, where $G^{-1}(m,\mu) = (G_1^{-1}(m,\mu),\mu)$. The weakly nondegenerate 2-form Ω_0 is independent of m and is defined in coordinates by

$$\begin{aligned}\Omega_0(w^1,w^2) &= 3\beta_s(\Psi^2(1)\rho^1 - \Psi^1(1)\rho^2) + (1-3\beta_s)(\upsilon^2\rho^1 - \upsilon^1\rho^2) + \int_0^1 (\Psi^1\Gamma_z^2 - \Psi^2\Gamma_z^1)\,dz \\ &= +\frac{3}{2}(\upsilon^1 - \Psi^1(1))B_1(\Gamma^2) - \frac{3}{2}(\upsilon^2 - \Psi^2(1))B_1(\Gamma^1),\end{aligned}$$

in which $w^1 = (\rho^1, \upsilon^1, \Psi^1, \Gamma^1)$, $w^2 = (\rho^2, \upsilon^2, \Psi^2, \Gamma^2)$ and

$$B_1(\Gamma) = \int_0^1 z^2 \Gamma_z(z)\,dz. \tag{22}$$

3 Centre-manifold reduction

This section is concerned with the application of centre-manifold reduction theory to the steady water-wave problem. The key theorem (Theorem 1 below) is due to Mielke (1991, Theorem 2.1). A complete discussion of the theorem, its proof and generalisations is given by Mielke (1986; 1988; 1990).

Theorem 1 *Consider the differential equation*

$$\dot{x} = Kx + f(x,\lambda), \tag{23}$$

where x belongs to a Hilbert space X, $\lambda \in \mathbb{R}^n$ is a parameter, $K : \mathcal{D}(K) \subset X \to X$ is a closed linear operator and 0 is an equilibrium point when $\lambda = \lambda_0 \in \mathbb{R}^n$. Regarding $\mathcal{D}(K)$ as a Hilbert space equipped with the graph norm, suppose that

(H1) X admits a direct-sum decomposition $X = X_1 \oplus X_2$, where X_1, X_2 are closed, K-invariant subspaces, so that (23) can be written as the linearly-decoupled system

$$\dot{x}_1 = K_1 x_1 + f_1(x_1 + x_2, \lambda), \tag{24}$$
$$\dot{x}_2 = K_2 x_2 + f_2(x_1 + x_2, \lambda), \tag{25}$$

in which $K_i = K|_{\mathcal{D}(K)\cap X_i} : \mathcal{D}(K) \cap X_i \subset X_i \to X_i$, $i = 1, 2$;

(H2) X_1 is finite dimensional and the spectrum of K_1 lies on the imaginary axis;

(H3) the imaginary axis lies in the resolvent set of K_2 and

$$\| (K_2 - iaI)^{-1} \| \leq \frac{C}{1+|a|}, \qquad a \in \mathbb{R},$$

for some constant C that is independent of a;

(H4) there exists a natural number k and neighbourhoods $\Lambda \subset \mathbb{R}^n$ of λ_0 and $U \subset \mathcal{D}(K)$ of 0 such that f is $(k+1)$ times continuously differentiable on $U \times \Lambda$, its derivatives are bounded and uniformly continuous on $U \times \Lambda$ and

$$f(0,\lambda_0) = 0, \qquad d_1 f[0,\lambda_0] = 0.$$

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ of λ_0 and $\tilde{U}_1 \subset U \cap X_1$, $\tilde{U}_2 \subset U \cap X_2$ of zero and a reduction function $h : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2 \subset X_2$ with the following properties. The reduction function h is k times continuously differentiable on $\tilde{U}_1 \times \tilde{\Lambda}$, its derivatives are bounded and uniformly continuous on $\tilde{U}_1 \times \tilde{\Lambda}$ and

$$h(0, \lambda_0) = 0, \qquad d_1 h[0, \lambda_0] = 0. \tag{26}$$

The graph

$$M_C^\lambda = \{x_1 + h(x_1, \lambda) \in \tilde{U}_1 \times \tilde{U}_2 : x_1 \in \tilde{U}_1\}$$

is a centre manifold for (23), so that

(1) M_C^λ is a locally invariant manifold of (23): through every point in M_C^λ there passes a unique solution of (23) that remains on M_C^λ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$;

(2) Every small bounded solution $x(t)$, $t \in \mathbb{R}$ of (23) that satisfies $x_1(t) \in \tilde{U}_1$, $x_2(t) \in \tilde{U}_2$ lies completely in M_C^λ;

(3) Every solution $\tilde{x} : (a, b) \to \tilde{U}_1$ of the reduced equation

$$\dot{x}_1 = K_1 x_1 + f_1(x_1 + h(x_1, \lambda), \lambda) \tag{27}$$

generates a solution

$$x(t) = x_1(t) + h(x_1(t), \lambda)$$

of the full equation (23).

The differential equation (20) is in a form suitable for the application of Theorem 1. One may write (20) as

$$\dot{y} = Ky + f(y, \mu),$$

where $X = \mathcal{H}$, $y = (\rho, v, \Psi, \Gamma) \in \mathcal{H}$ and K is the closed linear operator defined by

$$K \begin{pmatrix} \rho \\ v \\ \Psi \\ \Gamma \end{pmatrix} = \begin{pmatrix} 3(\Psi(1) - v) \\ \Gamma_z(1) \\ \Gamma_z + \frac{1}{2\beta_s}\left(z^2 - \frac{1}{3}\right)\left((\alpha_s - 1)\rho - \Gamma_z(1)\right) \\ -\Psi_z \end{pmatrix} \tag{28}$$

with

$$\mathcal{D}(K) = \{(\rho, v, \Psi, \Gamma) \in \mathcal{H} : \Psi, \Gamma \in H^2(0,1), \Psi_z(0) = \Psi_z(1) = 0\}.$$

The task in hand is therefore to verify hypotheses (H1), (H2), (H3), (H4) of Theorem 1. Notice immediately that (H4) is satisfied with $\lambda = \mu \in \mathbb{R}$, $\lambda_0 = 0$, $\Lambda = \Lambda_2$ and $U = V_2 \cap \mathcal{D}(K)$ for any natural number k; hypothesis (H3) follows from the following result, whose proof is given in Appendix A.

Proposition 2 *A complex number ν is an eigenvalue of K if and only if*

$$\nu \cos \nu = \left(\alpha_s - \beta_s \nu^2\right) \sin \nu. \tag{29}$$

Each ia with $a \in \mathbb{R}$, $|a| > 2s$ lies in the resolvent set of K and

$$\| (K - iaI)^{-1} \| \leq \frac{C}{1 + |a|}$$

for some constant C that is independent of a.

To verify the remaining hypotheses, take ν_0 in the resolvent set of K. It follows from the Rellich embedding theorem that $(K-\nu_0 I)^{-1}$ is compact. The spectrum of K therefore consists of isolated eigenvalues of finite multiplicity (Kato 1976, Theorem III.6.29), and the only purely imaginary eigenvalues of K are $\pm is$. Now define the spectral projection P corresponding to the subset $\{-is, is\}$ of $\sigma(K)$ by

$$P(x) = \frac{1}{2\pi i}\int_C (K-zI)^{-1}x\,dz,$$

where C is a closed curve in the resolvent set of K enclosing $-is, is$ and no other point of $\sigma(K)$. Defining $X_1 = P(\mathcal{H})$ and $X_2 = (I-P)(\mathcal{H})$, one finds that the hypotheses (H1), (H2) of Theorem 1 are satisfied (see Kato (1976, Theorem III.6.17)).

The generalised eigenspace X_1 is four-dimensional and has a basis consisting of the generalised eigenvectors

$$e_1 = \begin{pmatrix} \sinh(s) \\ -i\cosh(s) + \frac{i}{s}\sinh(s) \\ \left(\frac{isz^2}{2} + \frac{i}{s} - \frac{is}{6}\right)\sinh(s) - i\cosh(sz) \\ \sinh(sz) - z\sinh(s) \end{pmatrix},$$

$$e_2 = \begin{pmatrix} -i\cosh(s) \\ \frac{1}{s}\cosh(s) - \left(1+\frac{1}{s^2}\right)\sinh(s) \\ \left(\frac{sz^2}{2} + \frac{1}{s} - \frac{s}{6}\right)\cosh(s) + \left(\frac{z^2}{2} - \frac{1}{6} - \frac{1}{s^2}\right)\sinh(s) - z\sinh(sz) \\ iz\cosh(s) - iz\cosh(sz) \end{pmatrix}$$

and $e_3 = \overline{e_1}$, $e_4 = \overline{e_2}$, where $Ke_1 = ise_1$, $Ke_2 - ise_2 = e_1$, $Ke_3 = -ise_3$, $Ke_4 + ise_4 = e_3$. (Note that e_2 does not lie in the range of $K - isI$, since equation (54) is not satisfied when $\lambda = is$, $(\rho^\star, \nu^\star, \Psi^\star, \Gamma^\star) = e_2$. One finds similarly that e_4 does not lie in the range of $K + isI$.) Direct calculations show that $\Omega_0(e_1, e_4) = -\Omega_0(e_2, e_3) = \gamma_1$, $\Omega_0(e_2, e_4) = i\gamma_2$, where

$$\begin{aligned} \gamma_1 &= -\frac{1}{2} + s\coth(s) - \frac{\sinh(2s)}{4s}, \\ \gamma_2 &= \frac{1}{24s\sinh^2(s)}(-3 - 8s^2 - 16s^2\cosh(2s) + 3\cosh(4s)) \end{aligned}$$

and all other combinations $\Omega_0(e_i, e_j)$, $i \geq j$ are zero. The generalised eigenvectors

$$f_1 = \frac{1}{\sqrt{\gamma_1}}e_1, \qquad f_2 = \frac{1}{\sqrt{\gamma_1}}\left(e_2 - \frac{i\gamma_2}{2\gamma_1}e_1\right)$$

and $f_3 = \overline{f_1}$, $f_4 = \overline{f_2}$ therefore form a basis for X_1 with $\Omega_0(f_1, f_4) = -\Omega_0(f_2, f_3) = 1$ and all other combinations $\Omega_0(f_i, f_j)$, $i \geq j$ are zero.

4 Calculation of the flow on the centre manifold

4.1 Theoretical results

The following theorem is a parameterised version of a result due to Mielke (1991, Theorem 4.1) (see Buffoni, Groves & Toland (1996, Theorem 4.1)). It is stated here in the notation of Theorem 1 applied to Hamilton's equations for the Hamiltonian system (V_2, Ω_2, H_2).

Theorem 3 *The centre manifold M_C^μ is a symplectic submanifold of V_2 for each $\mu \in \tilde{\Lambda}$. Moreover, the flow determined by the Hamiltonian system $(M_C^\mu, \tilde{\Omega}, \tilde{H})$, where the tilde denotes restriction to M_C^μ, coincides with the flow on M_C^μ determined by (V_2, Ω_2, H_2).*

The centre manifold

$$M_C^\mu = \{u_1 + u_2 \in \tilde{U}_1 \times \tilde{U}_2 : u_2 = h(u_1, \mu)\}$$

is modelled on the single coordinate chart $\tilde{U}_1$, which is an open subset of the finite-dimensional space X_1. The coordinate map $\chi : M_C^\mu \to \tilde{U}_1$ is defined by

$$\chi^{-1}(u_1) = u_1 + h(u_1, \mu).$$

Working in these coordinates, one finds that the reduced equations (27) are Hamilton's equations for $(M_C^\mu, \tilde{\Omega}, \tilde{H})$ and the 2-form $\tilde{\Omega}$ is given by

$$\tilde{\Omega}|_{(u_1,\mu)}(w^1, w^2) = \Omega_2|_{u_1+h(u_1,\mu)}(w^1 + d_1h[u_1, \mu](w^1), w^2 + d_1h[u_1, \mu](w^2))$$

It follows from this equation and equation (21) that

$$\tilde{\Omega}|_{(u_1,\mu)}(w^1, w^2) = \Omega_0(w^1, w^2) + \Omega_\mu(w^1, w^2) + O(\| u_1 \|\| w^1 \|\| w^2 \|)$$

as $(u_1, \mu) \to 0$, where Ω_μ is a constant 2-form which satisfies the estimate

$$\Omega_\mu(w^1, w^2) = O(|\mu| \, \| w^1 \|\| w^2 \|)$$

as $\mu \to 0$. Writing $u_1 = z_1 f_1 + z_2 f_2 + \overline{z_1}\overline{f_1} + \overline{z_2}\overline{f_2}$, one therefore has that

$$\tilde{\Omega}|_{(z_1,z_2,\bar{z_1},\bar{z_2},\mu)}(w^1, w^2) = \Upsilon(w^1, w^2) + \Omega_\mu(w^1, w^2) + O(|(z_1, z_2, \overline{z_1}, \overline{z_2})| \, \| w^1 \|\| w^2 \|)$$

as $(z_1, z_2, \overline{z_1}, \overline{z_2}) \to 0$, where Υ is the canonical symplectic 2-form defined by

$$\Upsilon((z_1^1, z_2^1, \overline{z_1^1}, \overline{z_2^1}), (z_1^2, z_2^2, \overline{z_1^2}, \overline{z_2^2})) = z_1^1\overline{z_2^2} - z_1^2\overline{z_2^1} + \overline{z_1^1}z_2^2 - \overline{z_1^2}z_2^1.$$

Theorem 4 *Consider the Hamiltonian system $(M_C^\mu, \tilde{\Omega}, \tilde{H})$, in which M_C^μ is equipped with the coordinate system described above. There is a near identity change of coordinates with the property that in the new coordinates $\tilde{\Omega}$ is the canonical 2-form Υ.*

Proof Because M_C^μ is finite-dimensional, the canonical symplectic 2-form Υ is strongly nondegenerate. By replacing $\tilde{\Lambda}$ by a smaller neighbourhood of the origin in $\mathbb{R}$ if necessary, one may therefore assume that $\Upsilon + \Omega_\mu$ is strongly nondegenerate for all $\mu \in \tilde{\Lambda}$. It follows from elementary linear algebra that there is a linear coordinate change of the form $z' = z + O(|\mu||z|)$ as $\mu \to 0$ which transforms $\Upsilon + \Omega_\mu$ to Υ. Dropping the primes, one has in the new coordinates that

$$\tilde{\Omega}|_{(z_1,z_2,\overline{z_1},\overline{z_2},\mu)}(w^1, w^2) = \Upsilon(w^1, w^2) + O(|(z_1, z_2, \overline{z_1}, \overline{z_2})| \, \| w^1 \|\| w^2 \|)$$

as $(z_1, z_2, \overline{z_1}, \overline{z_2}) \to 0$.

The next step is to apply Darboux' theorem as formulated by Mielke (1991, Theorem 5.15) (see Abraham, Marsden & Ratiu (1988, Theorem 8.1.2) for a general description of the theorem).

This result states that there is a near identity coordinate change that transforms $\tilde{\Omega}|_{(z_1,z_2,\overline{z}_1,\overline{z}_2,\mu)}$ to $\tilde{\Omega}|_{(0,\mu)} = \Upsilon$. The required coordinate change is the composition of those above. $\square$

Hamilton's equations for the Hamiltonian system $(M_C^\mu, \tilde{\Omega}, \tilde{H})$ are given in the new coordinates by the equations

$$\begin{aligned}\dot{z}_1 &= isz_1 + z_2 + f_1(z_1, z_2, \overline{z_1}, \overline{z_2}, \mu),\\ \dot{z}_2 &= isz_2 + f_2(z_1, z_2, \overline{z_1}, \overline{z_2}, \mu)\end{aligned}$$

together with their complex conjugates. Here

$$f_1(z_1, z_2, \overline{z_1}, \overline{z_2}, \mu), f_2(z_1, z_2, \overline{z_1}, \overline{z_2}, \mu) = O(|\mu||(z_1, z_2, \overline{z_1}, \overline{z_2})|) + O(|(z_1, z_2, \overline{z_1}, \overline{z_2})|^2)$$

as $(z_1, z_2, \overline{z_1}, \overline{z_2}, \mu) \to 0$. The final step is the use of normal-form theory developed by Elphick *et al.* (1987) and extended to Hamiltonian systems by Meyer & Hall (1992, Chapter VII). In the present setting, the theory states that there is a near identity, symplectic change of variable

$$z = Z + \Theta(Z, \mu),$$

where Θ is a polynomial of degree $n \geq 2$ in the variable (Z, μ) with the property that the transformed Hamiltonian takes the form

$$\tilde{H} = is(\bar{B}A - \bar{A}B) + B\bar{B} + \sum_{j=3}^{n+1} \tilde{H}_j(Z, \mu) + O(|Z|^2|(\mu, Z)|^n)$$

as $(Z, \mu) \to 0$. Here A, B, $\bar{A}$, $\bar{B}$ are the components of Z and the functions $\tilde{H}_j$, $j = 3, \ldots, n+1$ are polynomials which are homogeneous of degree j in (Z, μ) and have the property that

$$\tilde{H}_j(e^{L^* t}Z, \mu) = \tilde{H}_j(Z, \mu) \tag{30}$$

for all $\mu \in \tilde{\Lambda}$ and $t \in \mathbb{R}$; the operator L is defined by

$$L = \begin{pmatrix} z_1 \\ z_2 \\ \overline{z_1} \\ \overline{z_2} \end{pmatrix} = \begin{pmatrix} is & 1 & 0 & 0 \\ 0 & is & 0 & 0 \\ 0 & 0 & -is & 1 \\ 0 & 0 & 0 & -is \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \overline{z_1} \\ \overline{z_2} \end{pmatrix}$$

and L^* is its adjoint. Equation (30) states that $\tilde{H}_j$ is an invariant of the linear system

$$\dot{Z} = L^* Z.$$

Because this system has three independent invariants $A\bar{A}$, $A\bar{B} - \bar{A}B$, and $-isA/B + \log(B)$, it follows that $\tilde{H}_j$ is a function of these quantities. The fact that $\tilde{H}_j$ is a polynomial of degree j in (Z, μ) implies that it is a polynomial function of $A\bar{A}$ and $A\bar{B} - \bar{A}B$ with coefficients that depend upon μ. Finally, the reality condition (that Hamilton's equations for $(M_C^\mu, \tilde{\Omega}, \tilde{H})$ take the form of two complex-conjugate pairs) shows that $\tilde{H}_j$ is a real polynomial function of $A\bar{A}$ and $i(A\bar{B} - \bar{A}B)$ with coefficients that depend upon μ. Moreover $\tilde{H}_j(0, \mu) = 0$.

4.2 Calculations

Let us henceforth work with the coordinates $Z = (A, B, \bar{A}, \bar{B})$ obtained above. Taking $n \geq 3$, one finds that

$$\begin{aligned}\tilde{H}_3 + \tilde{H}_4 &= \mu c_1 A\bar{A} + \mu i c_2 A\bar{B} - \mu i c_2 \bar{A}B \\ &\quad + c_3 A^2\bar{A}^2 + ic_4 A^2\bar{A}\bar{B} - ic_4 A\bar{A}^2 B - c_5 A^2\bar{B}^2 + 2c_5 A\bar{A}B\bar{B} - c_5\bar{A}^2B^2 \\ &\quad + \mu^2 c_6 A\bar{A} + \mu^2 ic_7 A\bar{B} - \mu^2 ic_7\bar{A}B\end{aligned}$$

for some real constants $c_1, \ldots, c_7$, and Hamilton's equations for the Hamiltonian system $(M_C^\mu, \tilde{\Omega}, \tilde{H})$ are

$$\begin{aligned}\dot{A} &= isA + B + c_2\mu iA + c_4 iA^2\bar{A} - 2c_5 A^2\bar{B} + 2c_5 A\bar{A}B + c_7\mu^2 iA + O(|Z||(\mu, Z)|^3), \\ \dot{B} &= isB - c_1\mu A + c_2\mu iB - 2c_3A^2\bar{A} - c_4 iA^2\bar{B} + 2c_4 iA\bar{A}B - 2c_5 AB\bar{B} + 2c_5\bar{A}B^2 \\ &\quad - c_6\mu^2 A + c_7\mu^2 iB + O(|Z||(\mu, Z)|^3),\end{aligned} \tag{31}$$

together with their complex conjugates, as $(\mu, Z) \to 0$. It is crucial to the theory below that $c_1 < 0$ and $c_3 > 0$; the task in hand is therefore to compute these coefficients.

Let us write the Hamiltonian vector field for $(M_C^\mu, \tilde{\Omega}, \tilde{H})$ as $Z \to KZ + P(Z, \mu)$, where

$$P(Z, \mu) = \sum_{r+i+j+k+\ell=2}^{3} P_{r,ijk\ell}\mu^r A^i B^j \bar{A}^k \bar{B}^\ell + O(|(A, B, \bar{A}, \bar{B})||(\mu, A, B, \bar{A}, \bar{B})|^3)$$

as $(Z, \mu) \to 0$ for some complex numbers $P_{r,ijk\ell}$. In similar notation, the reduction function $h : \tilde{U}_1 \times \tilde{\Lambda} \to X_2$ defining the centre manifold takes the form

$$h(Z, \mu) = \sum_{r+i+j+k+\ell=2}^{3} h_{r,ijk\ell}\mu^r A^i B^j \bar{A}^k \bar{B}^\ell + O(|(A, B, \bar{A}, \bar{B})||(\mu, A, B, \bar{A}, \bar{B})|^3)$$

as $(Z, \mu) \to 0$, where each $h_{r,ijk\ell}$ is an element of $\tilde{U}_2 \subset X_2$. Writing (20) in the form (23) and substituting $u = Z + h(Z, \mu)$, $\dot{Z} = KZ + P(Z, \mu)$ into this equation, one obtains the relations

$$Kh_{0,2}(Z) - dh_{0,2}[Z](KZ) = -f_{0,2}(Z, Z) + P_{0,2}(Z), \tag{32}$$

$$\begin{aligned}Kh_{0,3}(Z) - dh_{0,3}[Z](KZ) &= dh_{0,2}[Z](P_{0,2}(Z)) - 2f_{0,2}(Z, h_{0,2}(Z)) \\ &\quad - f_{0,3}(Z, Z, Z) + P_{0,3}(Z),\end{aligned} \tag{33}$$

in which

$$P_{m,n} = \sum_{i+j+k+\ell=n} P_{m,ijk\ell}\mu^m A^i B^j \bar{A}^k \bar{B}^\ell, \qquad h_{m,n} = \sum_{i+j+k+\ell=n} h_{m,ijk\ell}\mu^m A^i B^j \bar{A}^k \bar{B}^\ell$$

and $f_{0,2}(\cdot, \cdot) = \frac{1}{2}d_1^2[0, 0](\cdot, \cdot)$, $f_{0,3}(\cdot, \cdot, \cdot) = \frac{1}{6}d_1^3 f[0, 0, 0](\cdot, \cdot, \cdot)$.

Observe that the first part of the proof of Proposition 2 remains true when α_s is replaced by $\alpha_s + \mu$: the eigenvalues ν of the operator obtained by replacing α_s by $\alpha_s + \mu$ in the definition (28) of K satisfy the equation

$$\nu \cos \nu = (\alpha_s + \mu - \beta_s \nu^2) \sin \nu. \tag{34}$$

On the other hand it follows from the reduced equations (31) that these eigenvalues satisfy

$$\nu = i(s + c_2\mu) \pm \sqrt{-c_1}\sqrt{\mu} + O(\mu^{\frac{3}{2}}) \tag{35}$$

as $\mu \to 0$. The value of c_1 may therefore be determined by substituting the expression (35) for ν into (34). One finds that $c_1 < 0$; it is given explicitly in Appendix B.

To calculate c_3, first note that $P_{0,2} = 0$ (see equations (31)). Equation (32) consists of ten equations which determine $h_{0,2}$; two of them of particular interest here are

$$\begin{aligned} Kh_{0,1010} &= -2f_{0,2}(f_1, \overline{f_1}), \\ Kh_{0,2000} - 2ish_{0,2000} &= -f_{0,2}(f_1, f_1). \end{aligned}$$

The unique solutions to these equations may be calculated explicitly using the method explained in the proof of Proposition 2 and are given in Appendix B.

Next note that $P_{0,2010} = c_4 i f_1 - 2c_3 f_2$ (see equations (31)), and the equation in (33) which contains this term is

$$Kh_{0,2000} - ish_{0,2000} = -2f_{0,2}(f_1, h_{0,1010}) - 2f_{0,2}(\overline{f_1}, h_{0,2000}) - 3f_{0,3}(f_1, f_1, \overline{f_1}) + c_4 i f_1 - 2c_3 f_2. \tag{36}$$

Because $(K - is)h_{0,2010} \in \mathcal{R}(K - is) \cap \tilde{U}_1$ and $f_2 \notin \mathcal{R}(K - is)$, the vector $(K - is)h_{0,2010}$ has no component in the f_2-direction. It follows from equation (36) that

$$c_3 = \Omega_0(f_{0,2}(f_1, h_{0,1010}), \overline{f_1}) + \Omega_0(f_{0,2}(\overline{f_1}, h_{0,2000}), \overline{f_1}) + \frac{3}{2}\Omega_0(f_{0,3}(f_1, f_1, \overline{f_1}), \overline{f_1}).$$

The coefficient c_3 may be calculated explicitly from the above formula; it is found to be positive and is given explicitly in Appendix B.

4.3 *Lagrangian formulation*

Let us proceed by studying the Hamiltonian system $(M_C^\mu, \tilde{\Omega}, \tilde{H})$ in the real coordinates q_1, q_2, p_1, p_2 defined by

$$A = \frac{1}{\sqrt{2}}(q_1 + iq_2), \qquad B = \frac{1}{\sqrt{2}}(p_1 + ip_2).$$

In these coordinates, one finds that the 2-form $\tilde{\Omega}$ is the canonical symplectic 2-form Υ, where

$$\Upsilon((q_1^1, q_2^1, p_1^1, p_2^1), (q_1^2, q_2^2, p_1^2, p_2^2)) = q_1^1 p_1^2 + q_2^1 p_2^2 - q_1^2 p_1^1 - q_2^2 p_2^1,$$

and the Hamiltonian is given by

$$\tilde{H} = \frac{1}{2}|p|^2 + s(p, \mathcal{I}q) + P(|q|^2, (p, \mathcal{I}q), \mu) + Q(q, p, \mu).$$

Here $p = (p_1, p_2)$, $q = (q_1, q_2)$, the symbols $(\cdot, \cdot)$ and $|\cdot|$ denote the usual inner product and norm on $\mathbb{R}^2$ and

$$\mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The third term in the Hamiltonian represents the normal form, so that $P(|q|^2, (p, \mathcal{I}q), \mu)$ is a polynomial of degree $n + 1$ in (q, p, μ) such that

$$P(|q|^2, (p, \mathcal{I}q), \mu) = O(|(q, p)|^2 |(\mu, q, p)|)$$

as $(q, p, \mu) \to 0$. Finally, the remainder term Q belongs to $C^k(\tilde{U}, \mathbb{R})$ and satisfies the estimate

$$Q(q, p, \mu) = O(|(q, p)|^2 |(\mu, q, p)|^n)$$

as $(q, p, \mu) \to 0$.

Lemma 5 *The Hamiltonian system $(M_C^\mu, \tilde{\Omega}, \tilde{H})$ admits a Lagrangian formulation in which the Lagrangian L is given in the coordinates $(q, \dot{q})$ by the formula*

$$L(q, \dot{q}, \mu) = \frac{1}{2}|\dot{q} - s\mathcal{I}q|^2 + \tilde{P}(|q|^2, (s\mathcal{I}q, \dot{q} - s\mathcal{I}q), \mu) + \tilde{Q}(q, \dot{q} - s\mathcal{I}q, \mu).$$

Here $\tilde{P}(|q|^2, (s\mathcal{I}q, \dot{q} - s\mathcal{I}q), \mu)$ is a polynomial of degree $n + 1$ in $(q, \dot{q}, \mu)$ such that

$$\tilde{P}(|q|^2, (s\mathcal{I}q, \dot{q} - s\mathcal{I}q), \mu) = O(|(q, \dot{q} - s\mathcal{I}q)|^2 |(\mu, q, \dot{q} - s\mathcal{I}q)|)$$

as $(q, \dot{q}, \mu) \to 0$. The remainder term $\tilde{Q}$ is a C^{k-1}-function of its arguments and satisfies the estimate

$$\tilde{Q}(q, \dot{q} - s\mathcal{I}q, \mu) = O(|(q, \dot{q} - s\mathcal{I}q)|^2 |(\mu, q, \dot{q} - s\mathcal{I}q)|^n)$$

as $(q, \dot{q}, \mu) \to 0$.

Proof The equation $\dot{q} = \partial\tilde{H}/\partial p$ implies that

$$\dot{q} = p + s\mathcal{I}q + \partial_2 P(|q|^2, (\mathcal{I}q, p), \mu)\mathcal{I}q + \partial_2 Q(q, p, \mu). \tag{37}$$

An application of the implicit-function theorem shows that the equation

$$z = \partial_2 P(|q|^2, -z|q|^2 + (\mathcal{I}q, \dot{q} - s\mathcal{I}q), \mu)$$

has a C^∞-solution $z = \Gamma(|q|^2, (\mathcal{I}q, \dot{q} - s\mathcal{I}q), \mu)$ for small $|(|q|^2, (\mathcal{I}q, \dot{q} - s\mathcal{I}q), \mu)|$ with the property that $\Gamma(0, 0, 0) = 0$. Writing $\dot{q} = p + s\mathcal{I}q + \Gamma\mathcal{I}q + v$ in (37), one finds that

$$\begin{aligned} v &= \partial_2 P(|q|^2, (\mathcal{I}q, \dot{q} - s\mathcal{I}q) - \Gamma|q|^2 - (\mathcal{I}q, v), \mu)\mathcal{I}q - \partial_2 P(|q|^2, (\mathcal{I}q, \dot{q} - s\mathcal{I}q) - \Gamma|q|^2, \mu)\mathcal{I}q \\ &\quad + \partial_2 Q(q, \dot{q} - s\mathcal{I}q - \Gamma\mathcal{I}q - v, \mu), \end{aligned}$$

and the implicit-function theorem shows that for small $|(q, \dot{q} - s\mathcal{I}q, \mu)|$ this equation has a solution v which is a C^{k-1}-function of q, $\dot{q} - s\mathcal{I}q$ and μ of high order (similar to that of $\partial_2 Q$). The conclusion follows from the formula $L = (p, \dot{q}) - \tilde{H}$. □

Let us now work in the parameter regime $\mu = \epsilon^2$, $\epsilon \geq 0$ and introduce new independent and dependent variables in Lagrange's equations defined by

$$q(t) = \epsilon\mathcal{R}(st)v(\tau), \qquad \tau = \epsilon t,$$

where

$$\mathcal{R}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

The corresponding scaled Lagrangian is defined by

$$\tilde{L}(v, \dot{v}, \tau, \epsilon) = \epsilon^{-4} L(q, \dot{q}, \mu),$$

and a straightforward calculation shows that

$$\tilde{L}(v, \dot{v}, \tau, \epsilon) = \frac{1}{2}|\dot{v}|^2 + \frac{1}{2}C|v|^2 - \frac{1}{4}D|v|^4 + \epsilon G(|v|^2, (s\mathcal{I}v, \dot{v}), \epsilon) + \tilde{S}(v, \dot{v}, \tau, \epsilon),$$

in which $C = -c_1$, $D = 2c_3$ are positive constants and

$$\tilde{S}(v, \dot{v}, \tau, \epsilon) = \epsilon^{-4} \tilde{Q}(\epsilon \mathcal{R}(s\epsilon^{-1}\tau)v, \epsilon^2 \mathcal{R}(s\epsilon^{-1}\tau)\dot{v}, \epsilon^2).$$

Here G is a polynomial such that $G(a, b, \epsilon^2) = O(|a| + |b|)$ as $(a, b) \to 0$. By choosing n and k sufficiently large one finds that $\tilde{S}(\cdot, \cdot, \cdot, \epsilon)$ possesses derivatives with respect to $(v, \dot{v}, \tau)$ which are all $O(\epsilon)$ as $\epsilon \to 0$.

5 Variational structure of the reduced problem

5.1 Variational structure for $\epsilon = 0$

This section considers the variational structure of the scaled Lagrangian system derived in Section 4.3 when $\epsilon = 0$. The Lagrangian of interest here is therefore

$$\tilde{L}_0(v, \dot{v}) = \frac{1}{2}|\dot{v}|^2 + \frac{1}{2}C|v|^2 - \frac{1}{4}D|v|^4$$

and Lagrange's equations are

$$\ddot{v} = Cv - D|v|^2 v. \tag{38}$$

Observe that functions $v \in H^1(\mathbb{R}, \mathbb{R}^2)$ are homoclinic solutions to (38) if and only if $\mathcal{L}'(v) = 0$, where

$$\mathcal{L}(v) = \int_{-\infty}^{\infty} \tilde{L}_0(v, \dot{v})\, d\tau.$$

The dynamical system (38) is completely integrable (the functions $|\dot{v}|^2/2 - C|v|^2/2 + D|v|^4/4$ and $(\mathcal{I}v, \dot{v})$ are first integrals), and one can write down explicitly its family of non-trivial homoclinic solutions, that is the set Cr of nontrivial critical points of $\mathcal{L}$. This set is given by

$$Cr = \{\mathcal{R}(\theta)w(\cdot - \sigma) : \theta \in [0, 2\pi), \sigma \in \mathbb{R}\},$$

in which

$$w(\tau) = \begin{pmatrix} (2C/D)^{\frac{1}{2}} \mathrm{sech}(C^{\frac{1}{2}}\tau) \\ 0 \end{pmatrix},$$

and $\mathcal{L}(v) = c$ for all $v \in Cr$, where $c = 4C^{\frac{3}{2}}/3D$.

The functional $\mathcal{L} \in C^\infty(H^1(\mathbb{R}, \mathbb{R}^2), \mathbb{R})$ has a mountain-pass structure, that is 0 is a strict local minimum of $\mathcal{L}$ and there exists an element $v \in H^1(\mathbb{R}, \mathbb{R}^2)$ with $\mathcal{L}(v) < 0$. (To prove the second assertion, take any non-zero element $z \in H^1(\mathbb{R}, \mathbb{R}^2)$ and note that $\mathcal{L}(sz) < 0$ for large values of s. One can therefore set $v = sz$ for a sufficiently large value of s.) The mountain-pass lemma as stated by Brezis & Nirenberg (1991, p. 943) asserts the existence of a Palais-Smale sequence $\{u_n\} \subset H^1(\mathbb{R}, \mathbb{R}^2)$ with $\mathcal{L}(u_n) \to a$, $\mathcal{L}'(u_n) \to 0$ as $n \to \infty$, where

$$a = \inf_{h \in \Gamma} \max_{s \in [0,1]} \mathcal{L}(\gamma(s)), \qquad \Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}, \mathbb{R}^2)) : \gamma(0) = 0, \mathcal{L}(\gamma(1)) < 0\}.$$

It follows from a standard concentration-compactness result (e.g. see Buffoni & Séré (1996, Lemma 3)) that there exists $v \in Cr$ with $\mathcal{L}(v) = a$ and hence $a = c$.

Next take $z \neq 0$ with $\langle \mathcal{L}'(z), z \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $H^1(\mathbb{R}, \mathbb{R}^2)$. A direct calculation shows that

$$\begin{aligned}
\langle \mathcal{L}''(z)z, z \rangle &= \int_{-\infty}^{\infty} \left\{ |\dot{z}|^2 + C|z|^2 - 3D|z|^4 \right\} d\tau \\
&= \int_{-\infty}^{\infty} \left\{ |\dot{z}|^2 + C|z|^2 - D|z|^4 \right\} d\tau - 2D \int_{-\infty}^{\infty} |z|^4 \, d\tau \\
&= \langle \mathcal{L}'(z), z \rangle - 2D \int_{-\infty}^{\infty} |z|^4 \, d\tau \\
&= -2D \int_{-\infty}^{\infty} |z|^4 \, d\tau \\
&< 0,
\end{aligned}$$

whence $\mathcal{L}(z)$ is the unique strict maximum of $\mathcal{L}(sz)$ for $s > 0$. Since $\mathcal{L}(sz) < 0$ for sufficiently large values of s, one must have that $\mathcal{L}(z) \geq a = c$. Observe that any $v \in Cr$ satisfies $\langle \mathcal{L}'(v), v \rangle = 0$ and $\mathcal{L}(v) = c$, from which it follows that

$$c = \min\{\mathcal{L}(v) : v \in \mathcal{C}\}, \qquad \mathcal{C} := \{v \in H^1(\mathbb{R}, \mathbb{R}^2) : \langle \mathcal{L}'(v), v \rangle = 0, v \neq 0\},$$

where the minimum is achieved by the elements of Cr.

Finally, the fact that each $v \in Cr$ minimises $\mathcal{L}$ on the one-codimensional manifold $\mathcal{C}$ and satisfies $\langle \mathcal{L}''(v)v, v \rangle < 0$ implies that each $v \in Cr$ is a critical point of mountain-pass type as defined by Hofer (1984), that is the set $\mathcal{L}^{-1}(-\infty, c)$ is non-empty and disconnected in every neighbourhood of v. (The introduction of the *the natural constraint* using the manifold $\mathcal{C}$ is a standard technique in critical-point theory.)

5.2 Morse reduction

In this section the variational problem for $\epsilon > 0$ is reduced to a finite-dimensional space using a Morse-reduction technique.

Lemma 6 *There exist real numbers $\tau_1 < \ldots < \tau_N < 0 < \tau_{N+1} < \ldots < \tau_{2N}$, $\epsilon_0 > 0$ and small, closed neighbourhoods $A_1, \ldots, A_{2N}$ of respectively $w(\tau_1), \ldots, w(\tau_{2N})$, such that for every ϵ with $|\epsilon| < \epsilon_0$ and*

$$(u_1, \ldots, u_{2N}) \in \mathcal{B} := \cup_{\theta \in [0,2\pi)} \mathcal{R}(\theta) A_1 \times \ldots \times \mathcal{R}(\theta) A_{2N},$$

there exists a function $\tau \to S(\tau, u_1, \ldots, u_{2N}, \epsilon) \in \mathbb{R}^2$ with the properties that

1. $S(\tau, u_1, \ldots, u_{2N}, \epsilon)$ *is a solution of*

$$\frac{d}{d\tau}\left(\frac{\partial \tilde{L}}{\partial \dot{v}}\right) = \frac{\partial \tilde{L}}{\partial v} \tag{39}$$

on $[\tau_i, \tau_{i+1}]$, $i = 1, \ldots, 2N-1$, $(-\infty, \tau_1]$ *and* $[\tau_{2N}, \infty)$*;*

2. $S(\tau_i, u_1, \ldots, u_{2N}, \epsilon) = u_i$, $i = 1, \ldots, 2N$*;*
3. $\lim_{|\tau|\to\infty} S(\tau, u_1, \ldots, u_{2N}, \epsilon) = 0$*;*
4. S *may be supposed to be as differentiable as necessary with respect to* $(u_1, \ldots, u_{2N})$*, the derivatives being continuous with respect to all variables.*

Proof Let $W^s_{loc}(\epsilon)$ be the local stable manifold in phase space for solutions starting at time τ_{2N}. An inspection of the standard proofs of its existence and differentiability (e. g. the proofs of Theorems 4.1 and 4.2 in Chapter 13 of Coddington & Levinson (1955)) shows that $W^s_{loc}(\epsilon)$ is the graph of a function and that this function and all its derivatives (up to order $k-1$) depend continuously upon ϵ. Observe that for $\epsilon = 0$, the set $\{(w(\tau_{2N}), z) : z \in \mathbb{R}^2\}$ intersects $W^s_{loc}(0)$ transversally, and because of the rotation invariance the same is true of the set $\{(\mathcal{R}(\theta)w(\tau_{2N}), z) : z \in \mathbb{R}^2\}$ for all $\theta \in [0, 2\pi)$. This transverse intersection persists under small perturbations, that is in the previous statement one can replace $w(\tau_{2N})$ by a point $v \in \mathbb{R}^2$ sufficiently close to $w(\tau_{2N})$ and $W^s_{loc}(0)$ by $W^s_{loc}(\epsilon)$ for a sufficiently small value of ϵ. The same argument holds for the unstable manifold for solutions ending at time τ_1.

There exists $\Delta\tau > 0$ with the property that each two real numbers $\hat{\tau}, \tilde{\tau} \in [\tau_1, \tau_{2N}]$ with $0 < \tilde{\tau} - \hat{\tau} < \Delta\tau$ are not conjugate times for the solution w. An equivalent statement is that the manifolds $\{\mathcal{F}(\tilde{\tau}, \hat{\tau}, w(\hat{\tau}), z) : z \in \mathbb{R}^2\}$ and $\{(w(\tilde{\tau}), z) : z \in \mathbb{R}^2\}$ intersect transversally, where $\mathcal{F}(\cdot, \hat{\tau}, v, z)$ is a solution of Lagrange's equations for $\epsilon = 0$ which takes the value (v, z) at time $\hat{\tau}$. Because of the rotation invariance the same is true of $\{\mathcal{F}(\tilde{\tau}, \hat{\tau}, \mathcal{R}(\theta)w(\hat{\tau}), z) : z \in \mathbb{R}^2\}$ and $\{(\mathcal{R}(\theta)w(\tilde{\tau}), z) : z \in \mathbb{R}^2\}$ for all $\theta \in [0, 2\pi)$. This transverse intersection persists under the perturbations explained above.

The result follows by choosing $\tau_1, \ldots, \tau_{2N}$ such that $\max_{i\in\{1,\ldots,2N-1\}}(\tau_{i+1} - \tau_i) < \Delta\tau$. □

Let

$$M(\tau, u_1, \ldots, u_{2N}, \epsilon) := \frac{\partial \tilde{L}}{\partial \dot{v}}(S, \partial_1 S, \tau, \epsilon)$$

be the momentum corresponding to S, and observe that the continuous function $S(\cdot, u_1, \ldots, u_{2N}, \epsilon)$ is a homoclinic solution of (39) on $\mathbb{R}$ if and only if $M(\cdot, u_1, \ldots, u_{2N}, \epsilon)$ is continuous at τ_i, $i = 1, \ldots, 2N$. Now define a reduced functional $F : \mathcal{B} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ by the formula

$$F(u_1, \ldots, u_{2N}, \epsilon) = \int_{\mathbb{R}} \tilde{L}(S, \partial_1 S, \tau, \epsilon) d\tau.$$

Lemma 7 *For* $i = 1, \ldots, 2N$*, one has that*

$$\partial_i F(u_1, \ldots, u_{2N}, \epsilon) = M(\tau_i^-, u_1, \ldots, u_{2N}, \epsilon) - M(\tau_i^+, u_1, \ldots, u_{2N}, \epsilon),$$

and moreover

$$\sum_{i=1}^{2N} \partial_i F(u_1, \ldots, u_{2N}, 0)^T u_i =< \mathcal{L}'(z), z >,$$

where $z = S(., u_1, \ldots, u_{2N}, 0)$.

Proof In the whole of the present paper finite-dimensional gradients are taken to be column vectors, with the single exception of this proof, where for notational convenience they are taken to be row vectors. (Notice that M is a gradient and is thus also treated as a row vector here). Using this convention, one finds that

$$\begin{aligned}
\partial_i F &= \int_{\mathbb{R}} \left(\partial_v \tilde{L} \partial_{u_i} S + M \partial_{u_i} \partial_\tau S \right) d\tau \\
&= \int_{\mathbb{R}} (\partial_v \tilde{L} - \partial_\tau M) \partial_{u_i} S d\tau + \sum_{j=1}^{2N} M \partial_{u_i} S|_{\tau_j^+}^{\tau_j^-} \\
&= M|_{\tau_i^+}^{\tau_i^-},
\end{aligned}$$

where the fact that $M\partial_{u_i} S(\tau_j^\pm) = \delta_{ij} M(\tau_j^\pm)$ has been used. It follows that

$$\begin{aligned}
\sum_{i=1}^{2N} \partial_i F(u_1, \ldots, u_{2N}, 0) u_i &= \sum_{i=1}^{2N} M|_{\tau_i^+}^{\tau_i^-} z(\tau_i) \\
&= \int_{\mathbb{R}} (\partial_v \tilde{L} - \partial_\tau M) z d\tau + \sum_{i=1}^{2N} M z|_{\tau_i^+}^{\tau_i^-} \\
&= \int_{\mathbb{R}} \left(\partial_v \tilde{L} z + M \partial_\tau z \right) d\tau \\
&= < \mathcal{L}'(z), z > . \qquad \square
\end{aligned}$$

An immediate corollary of Lemma 7 is that $S(\cdot, u_1, \ldots, u_{2N}, \epsilon)$ is a homoclinic solution of (39) if and only if $(u_1, \ldots, u_{2N})$ is a critical point of $F(\ldots, \epsilon)$. The next result concerns the non-degeneracy of the natural constraint related to $F(\ldots, \epsilon)$.

Lemma 8 *Without loss of generality, one may assume that*

$$\sup_{\mathcal{B} \times (-\epsilon_0, \epsilon_0)} \sum_{i,j=1}^{2N} u_i^T \partial_i \partial_j F(u_1, \ldots, u_{2N}, \epsilon) u_j < 0.$$

Proof By continuity, it is sufficient to prove the result for $\epsilon = 0$ and, because of the rotation invariance when $\epsilon = 0$, for $u_i = w(\tau_i)$, $i = 1, \ldots, 2N$. Note that $\mathcal{B}$ and ϵ_0 can be chosen as small as necessary. One has that

$$\sum_{i,j=1}^{2N} w(\tau_i)^T \partial_i \partial_j F \left(w(\tau_1), \ldots, w(\tau_{2N}), 0 \right) w(\tau_j)$$

$$
\begin{aligned}
&= \left.\frac{d^2}{dr^2} F(rw(\tau_1), \ldots, rw(\tau_{2N}), 0)\right|_{r=1} \\
&= \left.\frac{d^2}{dr^2} \mathcal{L}\left(S(\cdot, rw(\tau_1), \ldots, rw(\tau_{2N}), 0)\right)\right|_{r=1} \\
&= \left.\langle \mathcal{L}''(w)\partial_r S(\cdot, rw(\tau_1), \ldots, rw(\tau_{2N}), 0), \partial_r S(\cdot, rw(\tau_1), \ldots, rw(\tau_{2N}), 0)\rangle\right|_{r=1}.
\end{aligned}
$$

Moreover $\partial_r S(\cdot, rw(\tau_1), \ldots, rw(\tau_{2N}), 0)|_{r=1} \to w$ in $H^1(\mathbb{R}^2)$ as $N, -\tau_1, \tau_{2N}$ tend to infinity and $\max_{i\in\{1,\ldots,2N-1\}}(\tau_{i+1} - \tau_i)$ tends adequately to zero. The above expression therefore tends to the negative number $< \mathcal{L}''(w)w, w >$. □

Let us now consider the compact manifold

$$
\mathcal{C}(\epsilon) := \left\{ (u_1, \ldots, u_{2N}) \in \mathcal{B} : \sum_{i=1}^{2N} \partial F(u_1, \ldots, u_{2N}, \epsilon)^T u_i = 0,\ |u_N| = |u_{N+1}| \right\}.
$$

Note that $|w(\tau_N)| = |w(\tau_{N+1})|$, $(\dot{w}(\tau_N), w_{\tau_N}) > 0$ and $(\dot{w}(\tau_{N+1}), w(\tau_{N+1})) < 0$; in particular $\mathcal{C}(\epsilon) \neq \emptyset$. By continuity, one may suppose that

$$
(\partial_1 S(\tau_N^{\pm}, \ldots, \epsilon), S(\tau_N, \ldots, \epsilon)) > 0, \qquad (\partial_1 S(\tau_{N+1}^{\pm}, \ldots, \epsilon), S(\tau_{N+1}, \ldots, \epsilon)) < 0
$$

on $\mathcal{C}(\epsilon)$.

Theorem 9 *For small ϵ, there exists $(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon)) \in (\mathcal{C}(\epsilon))^\circ$ such that*

$$
F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon) = \min\{F(u_1, \ldots, u_{2N}, \epsilon) : (u_1, \ldots, u_{2N}) \in \mathcal{C}(\epsilon)\}.
$$

(Here $\mathcal{M}^\circ$ denotes the interior of a manifold $\mathcal{M}$.) Moreover $S(\cdot, \hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon)$ is a solution of (39) on $\mathbb{R}\backslash\{\tau_N, \tau_{N+1}\}$, and there exists $\lambda(\epsilon) \in \mathbb{R}$ such that

$$
M(\tau_N^-) - M(\tau_N^+) = \lambda(\epsilon) S(\tau_N), \qquad M(\tau_{N+1}^-) - M(\tau_{N+1}^+) = -\lambda(\epsilon) S(\tau_{N+1}).
$$

Proof For $\epsilon = 0$, Lemma 7 shows that $(u_1, \ldots, u_{2N}) \in \mathcal{C}(0)$ if and only if $S(\cdot, u_1, \ldots, u_{2N}, 0) \in \mathcal{C}$ and $|u_N| = |u_{N+1}|$. It follows that $c := \min\{F(u_1, \ldots, u_{2N}, 0) : (u_1, \ldots, u_{2N}) \in \mathcal{C}(0)\}$ is attained exactly for $(u_1, \ldots, u_{2N})$ of the form $u_i = \mathcal{R}(\theta) w(\tau_i)$, $i = 1, \ldots, 2N$, where $\theta \in [0, 2\pi)$ is independent of i, so that

$$
m := \min\{F(u_1, \ldots, u_{2N}, 0) : (u_1, \ldots, u_{2N}) \in \partial\mathcal{C}(0)\} > c. \tag{40}
$$

A straightforward continuity argument shows that

$$
\begin{aligned}
&\min\{F(u_1, \ldots, u_{2N}, \epsilon) : (u_1, \ldots, u_{2N}) \in \mathcal{C}(\epsilon)\} \\
&- \min\{F(u_1, \ldots, u_{2N}, \epsilon) : (u_1, \ldots, u_{2N}) \in \partial\mathcal{C}(\epsilon)\} \geq (m - c)/2 > 0
\end{aligned}
$$

for small ϵ, from which it follows that $\min\{F(u_1, \ldots, u_{2N}, \epsilon) : (u_1, \ldots, u_{2N}) \in \mathcal{C}(\epsilon)\}$ is attained for some $(\hat{u}_1(\epsilon), \ldots, \hat{u}_n(\epsilon)) \in (\mathcal{C}(\epsilon))^\circ$.

Because $(\hat{u}_1(\epsilon), \ldots, \hat{u}_n(\epsilon))$ minimises $F(\ldots, \epsilon)$ on $(\mathcal{C}(\epsilon))^\circ$, there exist two Lagrangian multipliers $\lambda(\epsilon), \mu(\epsilon) \in \mathbb{R}$ such that

$$\begin{aligned} \partial_i F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon) &= \mu \partial_i \left(\sum_{j=1}^{2N} \partial_j F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon)^T \hat{u}_j(\epsilon) \right) \\ &\quad + \lambda \left(\delta_{i,N} \hat{u}_N(\epsilon) - \delta_{i,N+1} \hat{u}_{N+1}(\epsilon) \right), \end{aligned} \tag{41}$$

whence

$$\begin{aligned} \sum_{i=1}^{2N} \partial_i F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon)^T \hat{u}_i(\epsilon) &= \mu \sum_{i,j=1}^{2N} \hat{u}_i(\epsilon) \partial_i \partial_j F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon)^T \hat{u}_j(\epsilon) \\ &\quad + \mu \sum_{i=1}^{2N} \partial_i F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon)^T \hat{u}_i(\epsilon) \\ &\quad + \lambda \left(|\hat{u}_N(\epsilon)|^2 - |\hat{u}_{N+1}(\epsilon)|^2 \right). \end{aligned}$$

Since the left-hand side and the last term of the right-hand side of this equation are zero and the factor multiplying μ is negative, it follows that $\mu = 0$. The second result now follows from (41) and Lemma 7. □

Corollary 10 *The point $(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon))$ is a critical point of mountain-pass type for the restriction of $F(\ldots, \epsilon)$ to*

$$\mathcal{D}(\epsilon) := \{(u_1, \ldots, u_{2N}) \in \mathcal{B} : \ |u_N| = |u_{N+1}|\}.$$

Its Morse index is 1.

Proof Theorem 9 shows that $(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon))$ minimises $F(\ldots, \epsilon)$ on $\mathcal{C}(\epsilon)$, which is a one-codimensional manifold in $\mathcal{D}(\epsilon)$, and Lemma 8 shows that

$$\sum_{i,j=1}^{2N} \hat{u}_i(\epsilon)^T \partial_i \partial_j F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon) \hat{u}_j(\epsilon) < 0. \tag{42}$$

It follows that $(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon))$ is a critical point of $F(\ldots, \epsilon)|_{\mathcal{D}(\epsilon)}$ of mountain-pass type. Its Morse index is therefore 0 or 1 (see Hofer (1984)), and (42) shows that it is 1. □

6 Existence of homoclinic solutions

6.1 *Existence of a primary homoclinic solution for* $\epsilon > 0$

Let us now suppose ϵ to be fixed, so that dependence upon ϵ will henceforth not be systematically written. Set $t_i = \tau_i/\epsilon$, $i = 1, \ldots, 2N$, let $\Gamma : \mathcal{B} \to \mathbb{R}^{4N}$ be the unitary transformation defined by

$$\Gamma(u_1, \ldots, u_{2N}) = \epsilon(\mathcal{R}(st_1)u_1, \ldots, \mathcal{R}(st_{2N})u_{2N}),$$

with $\hat{F} = F \circ \Gamma^{-1}$, $\hat{\mathcal{D}} = \Gamma(\mathcal{D}(\epsilon))$, and define

$$\begin{aligned}
\hat{S}(t, q_1, \ldots, q_{2N}) &= \epsilon \mathcal{R}(st) S\left(\epsilon t, \epsilon^{-1}\mathcal{R}(-st_1)q_1, \ldots, \epsilon^{-1}\mathcal{R}(-st_{2N})q_{2N}, \epsilon\right), \\
\hat{M}(t, q_1, \ldots, q_{2N}) &= \partial_2 L(\hat{S}, \dot{\hat{S}}, \epsilon^2), \\
\hat{q}(t) &= \epsilon \mathcal{R}(st) S(\epsilon t, \hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon), \\
\hat{p}(t) &= \partial_2 L(\hat{q}, \dot{\hat{q}}, \epsilon^2).
\end{aligned}$$

It follows that $(\hat{q}, \hat{p})$ is a solution on $\mathbb{R} \setminus \{t_N, t_{N+1}\}$ of Hamilton's equations for $(M_C^\mu, \tilde{\Omega}, \tilde{H})$; moreover

$$\Delta_1 \hat{p} := \hat{p}(t_N^-) - \hat{p}(t_N^+) = \epsilon \lambda \hat{q}(t_N)$$

and

$$\Delta_2 \hat{p} := \hat{p}(t_{N+1}^-) - \hat{p}(t_{N+1}^+) = -\epsilon \lambda \hat{q}(t_{N+1}).$$

Theorem 11 $(\hat{q}, \hat{p})$ *is a homoclinic solution of Hamilton's equations for* $(M_C^\mu, \tilde{\Omega}, \tilde{H})$.

Proof The method of proof is to use the invariance of the Hamiltonian $\tilde{H}$ to show that $\lambda = 0$. Let us introduce the notation

$$\Delta_1 \tilde{H} = \tilde{H}(\hat{q}, \hat{p}, \epsilon^2)|_{t_N^+}^{t_N^-}, \qquad \Delta_2 \tilde{H} = \tilde{H}(\hat{q}, \hat{p}, \epsilon^2)|_{t_{N+1}^+}^{t_{N+1}^-}.$$

Suppose, *ab absurdo,* that $\lambda > 0$. Because $(\hat{q}, \hat{p})$ remains in a $O(\epsilon)$-neighbourhood of the origin, in which $\tilde{H}$ is convex in p, it follows that

$$\Delta_1 \tilde{H} \geq (\dot{\hat{q}}(t_N^+), \Delta_1 \hat{p}) = \epsilon \lambda (\dot{\hat{q}}(t_N^+), \hat{q}(t_N)) > 0$$

and similarly $\Delta_2 \tilde{H} > 0$. One therefore obtains the contradiction

$$0 = \tilde{H}(\hat{q}, \hat{p}, \epsilon^2)(t_N^-) > \tilde{H}(\hat{q}, \hat{p}, \epsilon^2)(t_N^+) = \tilde{H}(\hat{q}, \hat{p}, \epsilon^2)(t_{N+1}^-) > \tilde{H}(\hat{q}, \hat{p}, \epsilon^2)(t_{N+1}^+) = 0.$$

In the same way, $\lambda < 0$ would lead to a contradiction. □

The following result is a direct consequence of the above analysis.

Theorem 12 *Suppose that the point* $(q_1, \ldots, q_{2N}) \in (\hat{\mathcal{D}})^\circ$ *is a critical point of* $\hat{F}|_{\hat{\mathcal{D}}}$. *The function* $\hat{S}(\cdot, q_1, \ldots, q_{2N})$ *is a homoclinic solution of Lagrange's equations for* L.

6.2 Multiplicity of homoclinic solutions

Let us now work under the assumption that $(\hat{q}(t_1), \ldots, \hat{q}(t_{2N}))$ *is an isolated critical point of* $\hat{F}|_{\hat{\mathcal{D}}}$. Under this hypothesis, one can now construct infinitely many multibump homoclinic solutions of Lagrange's equations for L using the method of Buffoni & Séré (1996). Observe that if the hypothesis does not hold, Theorem 12 immediately gives infinitely many homoclinic solutions.

Let $\hat{W}_{loc}^s$ and $\hat{W}_{loc}^u$ be the local stable and unstable manifolds in phase space. By the construction of S in Lemma 6, one has that $(\hat{q}(t_1), \hat{p}(t_1)) \in \hat{W}_{loc}^u$ and $(\hat{q}(t_{2N}), \hat{p}(t_{2N})) \in \hat{W}_{loc}^s$. Moreover for v near $\hat{q}(t_{2N})$, there exists a solution $\rho_s(\cdot, v)$ on $\mathbb{R}_+$ such that $\rho_s(0, v) = v$ and

$(\rho_s(0,v),\bar{\rho}_s(0,v)) \in \hat{W}^s_{loc}$, where $\bar{\rho}_s$ is the momentum corresponding to ρ_s. Similarly, for w near $\hat{q}(t_1)$, there exists a solution $\rho_u(\cdot,w)$ on $\mathbb{R}_-$ such that $\rho_u(0,w) = w$ and $(\rho_u(0,w),\bar{\rho}_u(0,w)) \in \hat{W}^u_{loc}$, where $\bar{\rho}_u$ is the momentum corresponding to ρ_u. Moreover ρ_s, ρ_u, $\bar{\rho}_s$, $\bar{\rho}_u$ are C^1-functions of their arguments. The next lemma is a variant of Lemma 6 in Buffoni & Séré (1996).

Lemma 13 *There exist real numbers $\delta, \nu > 0$ such that for every $v, w \in \mathbb{R}^2$ and $\alpha > 0$ with $|v - \hat{q}(t_{2N})|, |w - \hat{q}(t_1)|, 1/\alpha \leq \delta$, there is a solution $\sigma(\cdot, v, w, \alpha)$ of Lagrange's equations for L with the properties that*

1. *$\sigma(0,v,w,\alpha) = v$ and $\sigma(\alpha,v,w,\alpha) = w$;*
2. $$\bar{\sigma}(0,v,w,\alpha) - \bar{\rho}_s(0,v) \to 0$$
   and
   $$\bar{\sigma}(\alpha,v,w,\alpha) - \bar{\rho}_u(0,w) \to 0$$
   uniformly in v and w as $\alpha \to \infty$, where $\bar{\sigma}$ is the momentum corresponding to σ;
3. *there exists $\tilde{\gamma} \in [0, 2\pi s^{-1})$ such that the sequence $l_m := 2\pi m s^{-1} + \tilde{\gamma}$ satisfies*
   $$\tilde{H}(\sigma(\cdot,v,w,l_m - \nu\delta), \bar{\sigma}(\cdot,v,w,l_m - \nu\delta)) > 0,$$
   $$\tilde{H}(\sigma(\cdot,v,w,l_m + \nu\delta), \bar{\sigma}(\cdot,v,w,l_m + \nu\delta)) < 0$$
   for $l_m \geq 1/\delta$;
4. *σ, $\bar{\sigma}$ are C^1-functions of their arguments.*

For $\xi \leq \delta$, set $U_\xi = \{(q_1,\ldots,q_{2N}) \in \hat{\mathcal{D}} : |q_i - \hat{q}(t_i)| \leq \xi, 1 \leq i \leq 2N\}$. Given $k \geq 2$, $z_1,\ldots,z_k \in U_\xi$ and $\alpha_1,\ldots,\alpha_{k-1} \geq 1/\delta$, define $q = E_k(z_1,\ldots,z_k,\alpha_1,\ldots,\alpha_{k-1}) \in H^1(\mathbb{R},\mathbb{R}^2)$ by

1. $q(t) = \rho_u(t, z_{1,1})$ for $t \leq 0$;
2. $q\left((j-1)(t_{2N}-t_1) + \sum_{l=1}^{j-1}\alpha_l + t\right) = \hat{S}(t+t_1, z_j)$ for $t \in [0, t_{2N}-t_1]$ and $j = 1,\ldots,k$;
3. $q\left(j(t_{2N}-t_1) + \sum_{l=1}^{j-1}\alpha_l + t\right) = \sigma(t, z_{j,2N}, z_{j+1,1}, \alpha_j)$ for $t \in [0,\alpha_j]$ and $j = 1,\ldots,k-1$;
4. $q\left(k(t_{2N}-t_1) + \sum_{l=1}^{k-1}\alpha_l + t\right) = \rho_s(t, z_{k,2N})$ for $t \geq 0$,

so that q is a function with k bumps, the time-interval between the jth and $(j+1)$th bumps being of length α_j. Finally, for $\xi < \delta$, define $G_k \in C\left((U_\xi)^k \times (1/\delta,\infty)^{k-1}, (\mathbb{R}^{4N})^k \times \mathbb{R}^{k-1}\right)$ by

$$\begin{aligned} & \langle G_k(z_1,\ldots,z_k,\alpha_1,\ldots,\alpha_{k-1}), (u_1,\ldots,u_k,\beta_1,\ldots,\beta_{k-1})\rangle \\ = & \sum_{j=1}^{k}\langle \hat{F}'(z_j), u_j\rangle + \sum_{j=1}^{k-1}(\bar{\rho}_s(0,z_{j,2N}) - \bar{\sigma}(0,z_{j,2N},z_{j+1,1},\alpha_j), u_{j,2N}) \end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=2}^{k}\left(\bar{\rho}_u(0,z_{j,1})-\bar{\sigma}(\alpha_{j-1},z_{j-1,2N},z_{j,1},\alpha_{j-1}),u_{j,1}\right)\\
&-\sum_{j=1}^{k-1}\beta_j\tilde{H}\left(\sigma(\cdot,z_{j,2N},z_{j+1,1},\alpha_j),\bar{\sigma}(\cdot,z_{j,2N},z_{j+1,1},\alpha_j)\right)
\end{aligned}
$$

for all $u_1 \in T\mathcal{D}|_{z_1}$, ..., $u_k \in T\mathcal{D}|_{z_k}$ and $\beta_1, \ldots, \beta_{k-1} \in \mathbb{R}$.

Theorem 14 *For ξ small (and independent of k), $(z_1, \ldots, z_k, \alpha_1, \ldots, \alpha_{k-1})$ is a solution of $G_k = 0$ precisely when $q = E_k(z_1, \ldots, z_k, \alpha_1, \ldots, \alpha_{k-1})$ is a homoclinic solution of Lagrange's equations for L.*

Proof Observe that $G_k = 0$ precisely when

1. $\partial_i \hat{F}(z_j) = 0$ for $1 \leq j \leq k$, $i \in \{2, \ldots, N-1, N+2, \ldots, 2N-1\}$;
2. $\hat{M}(t_N^-, z_j) - \hat{M}(t_N^+, z_j) = \epsilon\lambda_j \hat{S}(t_N, z_j)$ and $\hat{M}(t_{N+1}^-, z_j) - \hat{M}(t_{N+1}^+, z_j) = -\epsilon\lambda_j \hat{S}(t_{N+1}, z_j)$ for $1 \leq j \leq 2N$, where the λ_j are Lagrangian multipliers;
3. $\partial_1 \hat{F}(z_j) - \bar{\rho}_u(0, z_{j,1}) + \bar{\sigma}(\alpha_{j-1}, z_{j-1,2N}, z_{j,1}, \alpha_{j-1}) = 0$ for $j = 2, \ldots, 2N$;
4. $\partial_{2N} \hat{F}(z_j) + \bar{\rho}_s(0, z_{j,2N}) - \bar{\sigma}(0, z_{j,2N}, z_{j+1,1}, \alpha_j) = 0$ for $j = 1, \ldots, 2N-1$;
5. $\tilde{H}\left(\sigma(\cdot, z_{j,2N}, z_{j+1,1}, \alpha_j), \bar{\sigma}(\cdot, z_{j,2N}, z_{j+1,1}, \alpha_j)\right) = 0$ for $j = 1, \ldots, 2N-1$.

These conditions are equivalent to

1. $\hat{M}(t_i^-, z_j) = \hat{M}(t_i^+, z_j)$ for $j = 1, \ldots, k$, $i \in \{2, \ldots, N-1, N+2, \ldots, 2N-1\}$;
2. remains the same;
3. $\hat{M}(t_1^+, z_j) = \bar{\sigma}(\alpha_{j-1}, z_{j-1,2N}, z_{j,1}, \alpha_{j-1})$ for $j = 2, \ldots, 2N$;
4. $\hat{M}(t_{2N}^-, z_j) = \bar{\sigma}(0, z_{j,2N}, z_{j+1,1}, \alpha_j)$ for $j = 1, \ldots, 2N-1$;
5. remains the same.

The momentum corresponding to q is therefore discontinuous at no more than two values of t for each bump. Moreover, $\tilde{H}$ takes the value 0 between any two consecutive bumps. As in the proof of Theorem 11, one concludes that $\lambda_j = 0$ for $j = 1, \ldots, 2N$ and that q is a homoclinic solution of Lagrange's equations for L. □

Let us now construct a homoclinic solution with $k \geq 2$ bumps. For sufficiently small ξ, the point $(\hat{q}(t_1), \ldots, \hat{q}(t_{2N}))$ is the unique critical point of $\hat{F}$ in U_ξ and $\inf \|\hat{F}'\| > 0$ on the boundary of this set. Moreover taking the α_j sufficiently large, one may take the terms in the definition of G_k involving $\bar{\rho}_s - \bar{\sigma}$ and $\bar{\rho}_u - \bar{\sigma}$ to be as small as necessary. Defining

$$
V_{k-1,\vec{n}} = \bigotimes_{j=1}^{k-1} [l_{n_j} - \nu\delta, l_{n_j} + \nu\delta],
$$

where $\vec{n}$ denotes the vector $(n_1, \ldots, n_{k-1})$, notice that there exists $\hat{n} > 0$ (independent of k) such that for $n_j > \hat{n}$, $j = 1, \ldots, k-1$, the linear functional G_k is homotopic in the sense of degree theory to $\tilde{G}_k \in C\left((U_\xi)^k \times (1/\delta, \infty)^{k-1}, (\mathbb{R}^{4N})^k \times \mathbb{R}^{k-1}\right)$ given by

$$\begin{aligned} & \langle \tilde{G}_k(z_1, \ldots, z_k, \alpha_1, \ldots, \alpha_{k-1}), (u_1, \ldots, u_k, \beta_1, \ldots, \beta_{k-1}) \rangle \\ = \; & \sum_{j=1}^{k} \langle \hat{F}'(z_j), u_j \rangle - \sum_{j=1}^{k-1} \beta_j \tilde{H}\left(\sigma(\cdot, \hat{q}(t_{2N}), \hat{q}(t_1), \alpha_j), \bar{\sigma}(\cdot, \hat{q}(t_{2N}), \hat{q}(t_1), \alpha_j)\right). \end{aligned}$$

Corollary 10 shows that $(\hat{q}(t_1), \ldots, \hat{q}(t_{2N}))$ is a critical point of $\hat{F}|_{\hat{\mathcal{D}}}$ of mountain-pass type, so that

$$\deg(\hat{F}|_{\hat{\mathcal{D}}}, (\hat{q}(t_1), \ldots, \hat{q}(t_{2N})), 0) = -1$$

(Hofer 1984). It follows that

$$\deg(G_k, U_\xi^k \times V_{k-1,\vec{n}}, 0) = \deg(\tilde{G}_k, U_\xi^k \times V_{k-1,\vec{n}}, 0) = (-1)^k$$

and hence G_k takes the value 0 at some $(z_1, \ldots, z_k, \alpha_1, \ldots, \alpha_{k-1}) \in U_\xi^k \times V_{k-1,\vec{n}}$.

In this way one obtains a countable family of homoclinic solutions of Lagrange's equations for L parameterised by k and $\vec{n}$.

Appendix A Proof of Proposition 2

Take $(\rho^\star, \nu^\star, \Psi^\star, \Gamma^\star) \in \mathcal{H}$ and consider the equations

$$\begin{aligned} 3(\Psi(1) - \nu) - \lambda\rho &= \rho^\star, & (43) \\ \Gamma_z(1) - \lambda\nu &= \nu^\star, & (44) \\ \Gamma_z + \frac{1}{2\beta_s}\left(z^2 - \frac{1}{3}\right)((\alpha_s - 1)\rho - \Gamma_z(1)) - \lambda\Psi &= \Psi^\star, & (45) \\ -\Psi_z - \lambda\Gamma &= \Gamma^\star, & (46) \end{aligned}$$

in which $(\rho, \nu, \Psi, \Gamma) \in \mathcal{D}(K)$ and $\lambda \in \mathbb{C}$. Let us examine the case $\lambda \neq 0$ first.

Differentiating (45) with respect to z and using equation (46), one finds that

$$\Gamma_{zz} + \lambda^2\Gamma = -\gamma z + \Psi_z^\star - \lambda\Gamma^\star, \tag{47}$$

where

$$\gamma = \frac{1}{\beta_s}((\alpha_s - 1)\rho - \Gamma_z(1)). \tag{48}$$

The solution of (47) with $\Gamma(0) = 0$ is

$$\Gamma(z) = A\sin(\lambda z) - \frac{\gamma z}{\lambda^2} + g(z), \tag{49}$$

in which A is a complex constant and

$$g(z) = -\frac{\sin(\lambda z)}{\lambda}\int_z^1 (\Psi_z^\star - \lambda\Gamma^\star)\cos(\lambda t)\,dt - \frac{\cos(\lambda z)}{\lambda}\int_0^z (\Psi_z^\star - \lambda\Gamma^\star)\sin(\lambda t)\,dt.$$

The next step is to evaluate (45) at $z = 1$ and use (43), (44), whereby

$$\gamma = \lambda^2(\rho - \delta), \qquad \delta := -\frac{1}{\lambda^2}(\lambda\rho^\star + 3\Psi^\star(1) - 3\nu^\star). \tag{50}$$

Equations (49) and (50) imply that

$$\Gamma(z) = A\sin(\lambda z) - \rho z + \delta z + g(z),$$

and it follows form the fact that $\Gamma(1) = 0$ that

$$\begin{aligned} \rho &= A\sin(\lambda) + \delta + g(1), && (51)\\ \Gamma &= A\sin(\lambda z) - A\sin(\lambda)z - g(1)z + g(z). && (52) \end{aligned}$$

Substituting the above expressions for ρ and Γ into (48) and equating the two formulae (48), (50) for γ, one arrives at the relationship

$$[(\alpha_s - \lambda^2\beta_s)\sin(\lambda) - \lambda\cos(\lambda)]A = \delta(1-\alpha_s) + (\lambda^2\beta_s - \alpha_s)g(1) + g_z(1).$$

This equation determines A uniquely unless

$$(\alpha_s - \lambda^2\beta_s)\sin(\lambda) = \lambda\cos(\lambda); \tag{53}$$

in this case there is no solution of (43), (44), (45), (46) unless

$$\delta(1-\alpha_s) + (\lambda^2\beta_s - \alpha_s)g(1) + g_z(1) = 0. \tag{54}$$

Equations (44), (46) now determine ν and Γ uniquely in terms of A. It follows that equations (43), (44), (45), (46) determine $(\rho, \nu, \Psi, \Gamma)$ uniquely in terms of $(\rho^\star, \nu^\star, \Psi^\star, \Gamma^\star)$ if and only if (53) does not hold. The complex number $\lambda \neq 0$ is therefore an eigenvalue of K if and only if (53) holds; in this case the range of $K - \lambda I$ is the set of $(\rho^\star, \nu^\star, \Psi^\star, \Gamma^\star) \in \mathcal{H}$ which satisfy the condition (54).

The case $\lambda = 0$ is dealt with in a similar fashion. Equations (47) and (48) arise as before, the solution of (47) with $\Gamma(0) = \Gamma(1) = 0$ is

$$\Gamma = \frac{\gamma}{6}(z - z^3) + \int_0^z \Psi^\star(s)\,ds, \tag{55}$$

and the counterpart of (50) is

$$\gamma = 3(\Psi^\star(1) - \nu^\star). \tag{56}$$

Equations (48), (55) determine ρ uniquely, equation (46) determines Ψ uniquely and (43) determines ν uniquely.

To establish that $(K - iaI)^{-1}$, $a \in \mathbb{R}$, $|a| > s$ is bounded and satisfies the desired inequality, one proceeds as follows. Let $\|\cdot\|$ denote the $L^2(0,1)$-norm and note that the norm

$$\| (\rho, \nu, \Psi, \Gamma) \|_{\mathcal{H}} = |\rho| + |\nu| + \| \Psi_z \| + \| \Gamma_z \|$$

is equivalent to the usual norm on $\mathcal{H}$ because $\Gamma(0) = \Gamma(1) = 0$ and $\int_0^1 \Psi\,dz = 0$. Equations (43), (44) may be rewritten as

$$\begin{pmatrix} \rho \\ \nu \end{pmatrix} = \frac{1}{a^2}\begin{pmatrix} ia\rho^\star - 3ia\Psi(1) - 3(\nu^\star - \Gamma_z(1)) \\ ia(\nu^\star - \Gamma_z(1)) \end{pmatrix}.$$

There is therefore a constant C_1 such that

$$
\begin{aligned}
|\rho| &\le C_1 \left\{ \frac{\rho^\star}{|a|} + \frac{\| \Psi_z \|}{|a|} + \frac{|\nu^\star|}{|a|^2} + \frac{|\Gamma_z(1)|}{|a|^2} \right\}, &(57)\\
|\nu| &\le \frac{C_1}{|a|}(\nu^\star + |\Gamma_z(1)|) &(58)
\end{aligned}
$$

for sufficiently large $|a|$. Here the inequality

$$
|\Psi(1)| = \left| \int_0^1 z\Psi_z \, dz \right| \le \| \Psi_z \|
$$

has been used to obtain (57). (Each term C_n appearing the the analysis below is a constant which is independent of a. In order to avoid unnecessary repetition, this fact will not be stated explicitly each time a new C_n appears.)

The next step is to differentiate (45), (46) with respect to z to obtain the equations

$$
\begin{aligned}
\Gamma_{zz} - ia\Psi_z &= \Psi_z^\star - \frac{\gamma z}{3\beta_s}, &(59)\\
\Psi_{zz} - ia\Gamma_z &= -\Gamma_z^\star.
\end{aligned}
$$

Taking absolute values, squaring, integrating over $(0,1)$ and adding, one finds that

$$
\begin{aligned}
\| \Gamma_{zz} \|^2 + \| \Psi_{zz} \| + a^2(\| \Gamma_z \|^2 + \| \Phi_z \|^2) &= \| \Gamma_z^\star \|^2 + \| \Psi_z^\star - \frac{\gamma z}{3\beta_s} \|^2 \\
&\le \| \Gamma_z^\star \|^2 + 2 \| \Psi_z^\star \|^2 + \frac{2}{27\beta_s^2}|\gamma|^2. &(60)
\end{aligned}
$$

Here the fact that $\Psi_z(0) = \Psi_z(1) = 0$ has also been used.

It follows from equations (48) and (50) that

$$
\gamma = -\frac{a^2}{\alpha_s - 1 + \beta_s a^2}\Gamma_z(1) + \frac{\alpha_s - 1}{\alpha_s - 1 + \beta_s a^2}(ia\rho^\star + 3\Psi^\star(1) - 3\nu^\star),
$$

whence

$$
|\gamma|^2 \le C_2 \left(|\Gamma_z(1)|^2 + \frac{|\rho^\star|^2}{|a|^2} + \frac{\| \Psi_z^\star \|^2}{|a|^4} + \frac{|\nu^\star|^2}{|a|^4} \right)
$$

for sufficiently large values of a. To estimate $|\Gamma_z(1)|$, let us use a trick employed by Iooss & Kirchgässner (1992, Lemma 2.1). Multiply (59) by z^k, where k is a natural number, and integrate the first term by parts to find that

$$
\Gamma_z(1) = \int_0^1 kz^{k-1}\Gamma_z \, dz + \int_0^1 z^k \Psi_z^\star \, dz + ia \int_0^1 z^k \Psi_z \, dz - \frac{\gamma}{k+2}.
$$

An application of the Cauchy-Schwarz inequality shows that

$$
\begin{aligned}
|\Gamma_z(1)|^2 &\le \frac{4k^2 \| \Gamma_z \|^2}{2k-1} + \frac{4 \| \Psi_z^\star \|^2}{2k+1} + \frac{4a^2 \| \Psi_z \|^2}{2k+1} &(61)\\
&\quad + \frac{C_3|\Gamma_z(1)|^2}{k+2} + \frac{C_3|\rho^\star|^2}{|a|^2} + \frac{C_3 \| \Psi_z^\star \|^2}{|a|^4} + \frac{C_3|\nu^\star|^2}{|a|^4}. &(62)
\end{aligned}
$$

Choosing k large enough so that

$$\frac{4C_2}{2k+1} \leq \frac{1}{2}, \qquad \frac{4}{2k+1} \leq \frac{27\beta_s^2}{8C_2},$$

one finds from (62) that

$$|\Gamma_z(1)|^2 \leq C_4 \left(\| \Gamma_z \|^2 + \| \Psi_z^\star \|^2 + \frac{|\rho^\star|^2}{|a|^2} + \frac{\| \Psi_z^* \|^2}{|a|^4} + \frac{|\nu^*|^2}{|a|^4} \right) + \frac{27\beta_s^2 a^2}{4C_2} \| \Psi_z \|^2 . \tag{63}$$

Inequalities (60), (62), (63) now show that

$$\begin{aligned} & \| \Gamma_{zz} \|^2 + \| \Psi_{zz} \| + \frac{a^2}{2}(\| \Gamma_z \|^2 + \| \Phi_z \|^2) \\ \leq \;\; & C_5 \left(\| \Gamma_z \|^2 + \| \Psi_z^\star \|^2 + \frac{|\rho^\star|^2}{|a|^2} + \frac{\| \Psi_z^* \|^2}{|a|^4} + \frac{|\nu^*|^2}{|a|^4} + \| \Gamma_z \|^2 \right), \end{aligned}$$

from which it follows that

$$\| \Gamma_{zz} \|^2 + \| \Psi_{zz} \| + \frac{a^2}{2}(\| \Gamma_z \|^2 + \| \Phi_z \|^2) \leq C_6 \| (\rho^\star, \nu^\star, \Psi^\star, \Gamma^\star) \|_{\mathcal{H}} \tag{64}$$

for sufficiently large $|a|$. Finally, (57), (58), combined with (63) and (64), now show that

$$|\rho| + |\nu| \leq C_7 \| (\rho^\star, \nu^\star, \Psi^\star, \Gamma^\star) \|_{\mathcal{H}} . \tag{65}$$

The fact that $(K - iaI)^{-1} : \mathcal{H} \to \mathcal{H}$ is bounded, together with the resolvent estimate in the statement of the theorem, now follows directly from (64), (65). $\square$

Appendix B Results of the normal-form computations

The terms $f_{0,2}$ and $f_{0,3}$ required in the computation of the normal form are defined by

$$f_{0,2}((\rho, \nu, \Psi, \Gamma), (\rho, \nu, \Psi, \Gamma)) = (\rho_2, \nu_2, \Psi_2, \Gamma_2),$$

$$f_{0,3}((\rho, \nu, \Psi, \Gamma), (\rho, \nu, \Psi, \Gamma), (\rho, \nu, \Psi, \Gamma)) = (\rho_3, \nu_3, \Psi_3, \Gamma_3),$$

in which

$$\begin{aligned} \rho_2 &= -9(\nu - \Psi(1))B_1(\Gamma), \\ \nu_2 &= -\rho\Gamma_z(1) - 9(\nu - \Psi(1))^2 + 3\nu(\nu - \Psi(1)), \\ \Psi_2 &= -\frac{B_2(\Gamma)}{\beta_s}((\alpha_s - 1)\rho - \Gamma_z(1)) + \frac{1}{2\beta_s}\left(z^2 - \frac{1}{3}\right)\left(\frac{3}{2}\rho^2 + \frac{1}{2}\Gamma_z(1)^2 + 2\rho\Gamma_z(1) + \frac{9}{2}(\nu - \Psi(1))^2\right) \\ &\quad -\rho\Gamma_z + 3(\nu + \Psi + \Psi(1) - 2z\Psi_z)(\nu - \Psi(1)) - 9z^2(\nu - \Psi(1))^2, \\ \Gamma_2 &= \rho\Gamma_z, \\ \rho_3 &= -27(\nu - \Psi(1))(B_1(\Gamma))^2, \\ \nu_3 &= \rho^2\Gamma_z(1) + 9(\nu - \Psi(1))^2(\Gamma_z(1) + \rho - 6B_1(\Gamma) + 3(3B_1(\Gamma) - \rho)\nu(\nu - \Psi(1)), \end{aligned}$$

$$\begin{aligned}
\Psi_3 &= \frac{27}{\beta_s}\left(z^2-\frac{1}{3}\right)(\nu-\Psi(1))^2((\alpha_s-1)\rho-\Gamma_z(1)) \\
&-\frac{B_2(\Gamma)}{\beta_s}\left(\frac{3}{2}\rho^2+\frac{1}{2}\Gamma_z(1)^2+2\rho\Gamma_z(1)+\frac{9}{2}(\nu-\Psi(1))^2\right) \\
&-\frac{1}{2\beta_s}\left(z^2-\frac{1}{3}\right)(2\rho^3+9(\nu-\Psi(1))^2(\Gamma_z(1)+\rho-3B_1(\Gamma))+\Gamma_z(1)^2\rho+3\rho^2\Gamma_z(1)) \\
&+\rho^2\Gamma_z+6z\Psi_z(\nu-\Psi(1))(\rho-3B_1(\Gamma))+9z^2(\nu-\Psi(1))^2(\Gamma_z-6B_1(\Gamma)+\rho) \\
&+3(\nu+\Psi+\Psi(1))(\nu-\Psi(1))(3B_1(\Gamma)-\rho), \\
\Gamma_3 &= -\rho^2\Psi_z.
\end{aligned}$$

Here $B_1(\Gamma)$ is defined by formula (22) and

$$B_2(\Gamma)=\int_0^z s\Gamma_z\,ds-\int_0^1\int_0^z s\Gamma_z\,ds\,dz.$$

The components $h_{0,1010}$ and $h_{0,2000}$ of the reduction function are given by

$$h_{0,1010}=(\rho_{0,1010},0,0,\Gamma_{0,1010}),\qquad h_{0,2000}=\frac{(\rho_{0,2000},\nu_{0,2000},\Psi_{0,2000},\Gamma_{0,2000})}{(12s\cosh(s)-9\sinh(s)-\sinh(3s))},$$

where

$$\begin{aligned}
\rho_{0,1010} &= \frac{4s\sinh^2(s)(s+\sinh(2s))}{2+2s^2-2\cosh(2s)+s\sinh(2s)}, \\
\Gamma_{0,1010} &= \frac{sz[-1+\cosh(4s)+2\cosh(sz)(\cosh(s)-\cosh(3s)+4s\sinh(s))-4s\sinh(2s)]}{-4s+2\sinh(2s)}, \\
\rho_{0,2000} &= 2s\sinh^2(s)(5\cosh(s)+\cosh(3s)), \\
\nu_{0,2000} &= \frac{i}{8s}[(26s+48s^3)\cosh(s)-27s\cosh(3s)+s\cosh(5s)] \\
&+\frac{i}{8s}[-(52+16s^2)\sinh(s)+(14-4s^2)\sinh(3s)+(2-4s^2)\sinh(5s)], \\
\Psi_{0,2000} &= -\frac{i\cosh(s)}{24s}(-282s-28s^3+84s^3z^2)-\frac{i\cosh(3s)}{24s}(279s+24s^3-72s^3z^2) \\
&-\frac{i\cosh(5s)}{24s}(3s+4s^3-12s^3z^2)-\frac{i\sinh(s)}{24s}(468-82s^2-78s^2z^2) \\
&-\frac{i\sinh(3s)}{24s}(-126-115s^2+21s^2z^2)-\frac{i\sinh(5s)}{24s}(-18-s^2+3s^2z^2) \\
&-\frac{i\cosh(sz)}{24s}(-108s+96s\cosh(2s)+12s\cosh(4s)-144s^2\sinh(2s)) \\
&-\frac{i\cosh(2sz)}{24s}(-144s^3\cosh(s)+180s^2\sinh(s)+36s^2\sinh(3s)) \\
&-\frac{i\sinh(sz)}{24s}(216s^2z-192s^2z\cosh(2s)-24s^2z\cosh(4s)+288s^3z\sinh(2s)), \\
\Gamma_{0,2000} &= \frac{sz}{2}(4\cosh(s)-3\cosh(3s)-\cosh(5s))
\end{aligned}$$

$$
\begin{aligned}
&+\frac{sz\cosh(sz)}{2}(9-8\cosh(2s)-\cosh(4s)+12s\sinh(2s))\\
&+\frac{s\sinh(2sz)}{2}(-12s\cosh(s)+15\sinh(s)+3\sinh(3s)).
\end{aligned}
$$

Finally, one finds that

$$
c_1 = \frac{2s\sinh(3s)-6s\sinh(s)}{(1+8s^2)\cosh(s)-\cosh(3s)-4s\sinh(s)}
$$

and

$$
\begin{aligned}
c_3 \;=\;& [383s-48s^3-(576s+768s^3)\cosh(2s)\\
&+(124s+48s^3)\cosh(4s)+64s\cosh(6s)+5s\cosh(8s)\\
&+(576s^4+396s^2-308)\sinh(2s)+(172-480s^2)\sinh(4s)\\
&-(4+4s^2)\sinh(6s)-6\sinh(8s)]\\
&\times(2s^5\sinh^3(s))(-\cosh(s)-8s^2\cosh(s)+\cosh(3s)+4s\sinh(s))^{-2}\\
&\times(2+2s^2-2\cosh(2s)+s\sinh(2s))^{-1}(-12s\cosh(s)+9\sinh(s)+\sinh(3s))^{-1}.
\end{aligned}
$$

Appendix C A second solution

This appendix shows how a variant of the method of Ambrosetti & Coti Zelati (1992) may be used to prove the existence of a second solution of Lagrange's equations for L. Whereas the primary solution is a *minimiser under constraint*, the second solution is found as a *critical point of mountain-pass type under constraint*. Here the notation of Sections 5.2 and 6 is used, and for simplicity let us suppose that the primary solution is isolated.

Theorem 15 *Using the notation of Theorem 9, suppose that* $(\hat{u}_1(\epsilon),\ldots,\hat{u}_{2N}(\epsilon))$ *is a* strict *local minimiser of* $F(\ldots,\epsilon)|_{C(\epsilon)}$. *Define*

$$
\tilde{c}(\epsilon) = \inf_{\gamma\in\tilde{\Gamma}}\max_{r\in[0,1]} F(\gamma(r),\epsilon),
$$

where $\tilde{\Gamma}$ *is the set of paths* $\gamma\in C([0,1],C(\epsilon))$ *with* $\gamma(0)=\gamma(1)=(\hat{u}_1(\epsilon),\ldots,\hat{u}_{2N}(\epsilon))$ *which are not homotopic in* $C(\epsilon)$ *to a point. One has that*

$$
\tilde{c}(\epsilon) > F(\hat{u}_1(\epsilon),\ldots,\hat{u}_{2N}(\epsilon),\epsilon),
$$

and there exists a critical point $(\tilde{u}_1(\epsilon),\ldots,\tilde{u}_{2N}(\epsilon))\in(C(\epsilon))^\circ$ *of* $F(\ldots,\epsilon)|_{C(\epsilon)}$ *such that* $\tilde{c}(\epsilon)=F(\tilde{u}_1(\epsilon),\ldots,\tilde{u}_{2N}(\epsilon),\epsilon)$*; in particular* $(\hat{u}_1(\epsilon),\ldots,\hat{u}_{2N}(\epsilon))\neq(\tilde{u}_1(\epsilon),\ldots,\tilde{u}_{2N}(\epsilon))$.

Proof Considering the particular path $\gamma(r)=(\mathcal{R}(2\pi r)w(\tau_1),\ldots,\mathcal{R}(2\pi r)w(\tau_{2N}))$, one finds that

$$
\tilde{c}(\epsilon)\le\max_{r\in[0,1]}F(\mathcal{R}(2\pi r)w(\tau_1),\ldots,\mathcal{R}(2\pi r)w(\tau_{2N}),\epsilon) := d(\epsilon).
$$

Because $d(\epsilon)\to c$ as $\epsilon\to 0$, equation (40) shows that

$$
d(\epsilon) < \min\{F(u_1,\ldots,u_{2N},\epsilon):(u_1,\ldots,u_{2N})\in\partial\mathcal{C}(\epsilon)\}
$$

for sufficiently small $\epsilon > 0$. It follows that all paths $\gamma \in \tilde{\Gamma}$ such that $\max_{r\in[0,1]} F(\gamma(r), \epsilon) \leq d(\epsilon)$ do not meet $\partial C(\epsilon)$. The inequality $\tilde{c}(\epsilon) > F(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon), \epsilon)$ is an immediate consequence of the fact that $(\hat{u}_1(\epsilon), \ldots, \hat{u}_{2N}(\epsilon))$ is a strict local minimiser of $F(\ldots, \epsilon)$, and the existence of $(\tilde{u}_1(\epsilon), \ldots, \tilde{u}_{2N}(\epsilon))$ now follows from the mountain-pass lemma. □

Corollary 16 *For sufficiently small $\epsilon > 0$, the function*

$$\tilde{q}(t) := \epsilon\mathcal{R}(st)S(\epsilon t, \tilde{u}_1(\epsilon), \ldots, \tilde{u}_{2N}(\epsilon), \epsilon)$$

is a homoclinic solution of Lagrange's equations for L.

Proof This result follows immediately from Theorem 12. □

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