# THE GEOMETRY OF SINGULAR QUATERNIONIC KÄHLER QUOTIENTS

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Abstract. Two descriptions of quaternionic Kähler quotients by proper group actions are given: the first as a union of smooth manifolds, some of which come equipped with quaternionic Kähler or locally Kähler structures; the second as a union of quaternionic Kähler orbifolds. In particular the quotient always has an open set which is a smooth quaternionic Kähler manifold. When the original manifold and the group are compact, we describe a length space structure on the quotient.

Similar descriptions of singular hyperKähler and 3-Sasakian quotients are given.

## 1. Introduction

If $M$ is a symplectic manifold admitting a proper symplectic action by a Lie group $G$, then in certain circumstances there is a $G$-equivariant moment map $\mu\colon M \to \mathfrak{g}^*$. It is well known that if $\mu^{-1}(0)/G$ is a manifold, then it admits a symplectic structure. In general, Sjamaar & Lerman [17] have shown that decomposing $M$ by orbit type induces a canonical decomposition of the symplectic quotient into a union of symplectic manifolds. Furthermore, they show that this procedure provides a stratification of $\mu^{-1}(0)/G$. Notice that as $\mu^{-1}(0)$ may be singular, even the statement that decomposing by orbit type expresses the quotient as a union of manifolds is a non-trivial result.

We shall investigate this decomposition for some other quotient constructions arising in differential geometry. Our primary interest will be in quaternionic Kähler quotients, but we shall also discuss the hyperKähler and 3-Sasakian quotients. Some of the results of this paper were announced in [6].

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A hyperKähler structure on a $4n$-dimensional manifold $M$ is a Riemannian metric $g$ together with a triple $I, J, K$ of covariant-constant complex structures, Hermitian with respect to $g$, which satisfy the quaternionic multiplication relations. Equivalently, the holonomy of $g$ is contained in $Sp(n)$. Each complex structure, together with the metric, defines a Kähler (and hence symplectic) form. If a group $G$ acts preserving the hyperKähler structure then in particular it will preserve the symplectic forms. When moment maps for these symplectic structures exist, we can combine them into a hyperKähler moment map $\mu\colon M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$ and call $\mu^{-1}(0)/G$ the hyperKähler quotient of $M$ by $G$. If this quotient is a manifold, then Hitchin et al. [11] showed that it is hyperKähler. Our first result is that in general, the decomposition of $M$ into $G$-orbit types provides a description of the hyperKähler quotient $\mu^{-1}(0)/G$ as a union of hyperKähler manifolds.

A quaternionic Kähler structure on a $4n$-dimensional manifold $M$ consists of a metric $g$ and real rank-three subbundle $\mathcal{G}$ of $\operatorname{End} TM$ preserved by the Levi-Civita connection and locally generated by almost-Hermitian structures $I, J, K$ behaving under composition like the multiplicative pure imaginary quaternions. Although $I, J, K$ need not exist globally, there is a globally defined $\mathcal{G}$-valued two-form $\Theta$, which we shall call the structure tensor, given locally by

$$\Theta = \omega_I \otimes I + \omega_J \otimes J + \omega_K \otimes K,$$

where $\omega_I$ is the two-form associated to $I$ via $g$, etc. An equivalent definition of quaternionic Kähler is that the holonomy of $g$ reduces to $Sp(n)\,Sp(1)$. Any quaternionic Kähler manifold is Einstein.

Of course, a hyperKähler manifold may be regarded as a special type of quaternionic Kähler manifold. In this case, $I, J, K$ exist globally and define a covariant constant trivialisation of $\mathcal{G}$. We shall exclude this case, even locally, when we talk about quaternionic Kähler manifolds in this paper. Equivalently, we require our quaternionic Kähler manifolds to have non-zero scalar curvature.

There is a quotient construction for quaternionic Kähler manifolds, due to Galicki & Lawson [8]. If $G$ acts preserving the quaternionic

Kähler structure of $M$, then for each $\xi \in \mathfrak{g}$, we may contract $\Theta$ with the vector field $X_\xi$ generated by $\xi$ to obtain an element of $\Omega^1(\mathcal{G})$. Assuming that $M$ is not locally hyperKähler, it can be shown that there exists a unique $\mu_\xi \in \Omega^0(\mathcal{G})$ such that $\nabla \mu_\xi = X_\xi \lrcorner \Theta$. The equation $\langle \mu(x), \xi \rangle = \mu_\xi(x)$ then defines a $G$-equivariant section $\mu$ of $\mathfrak{g}^* \otimes \mathcal{G}$. The quaternionic Kähler quotient is defined to be the set $\mu^{-1}(0)/G$, and is quaternionic Kähler when $G$ acts freely.

We shall show that a slight refinement of the Sjamaar-Lerman decomposition of $M$ induces a description of $\mu^{-1}(0)/G$ as a union of manifolds. An example of Galicki & Lawson shows that these manifolds are not always quaternionic Kähler. However we shall give a sufficient condition for the manifolds to be quaternionic Kähler and show, in particular, that for an orbit type with connected stabiliser, the associated manifold in the decomposition of $\mu^{-1}(0)/G$ is always quaternionic Kähler. We shall also give a sufficient condition for a piece of the quotient $\mu^{-1}(0)/G$ to have a Kähler covering space. For circle actions this will be seen to imply that each manifold in the decomposition is either quaternionic Kähler or has a Kähler cover. This conclusion also holds for torus actions on the classical Wolf spaces.

We shall also describe another, somewhat coarser, decomposition which expresses the quotient as a union of quaternionic Kähler orbifolds. So at the price of allowing orbifold singularities we can obtain a decomposition of $\mu^{-1}(0)/G$ into quaternionic Kähler pieces. This enables us to show that $\mu^{-1}(0)/G$ always has an open set which is a smooth quaternionic manifold. We conclude by discussing a way of giving each component of the quotient the structure of a length space.

In this paper we shall assume that the action of $G$ is proper. As a result of Yau [20] states that the isometry group of a Riemannian manifold acts properly, this assumption will be satisfied whenever $G$ is a closed subgroup of the isometry group.

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## 2. HyperKähler Quotients

We first recall some notation associated to group actions. Let a group $G$ act on a set $W$, and let $H$ be a subgroup of $G$. We write $W^H$ for the set of points fixed by $H$, and let $W_H$ denote the set of points whose stabiliser is precisely $H$. We write $W_{(H)}$ for the set of points with stabiliser conjugate to $H$ in $G$. Usually we shall assume that $G$ acts effectively.

Let $G$ be a Lie group acting smoothly and properly on a manifold $M$. If $H$ is a subgroup of $G$ such that $M_H$ is non-empty, then $H$ is the stabiliser of a point for a proper action, and hence is compact. It follows that $M^H$ is a manifold with tangent space $(T_xM)^H$. Moreover, it is a corollary of the slice theorem for proper actions [15] that $M_H$ is an open subset of $M^H$, and hence is also smooth, with tangent space at $x$ given by $(T_xM)^H$.

If $N(H)$ denotes the normaliser of $H$ in $G$, then $L := N(H)/H$ acts freely and properly on $M_H$ and so $M_{(H)}/G = M_H/L$ is a manifold. Therefore $M/G$ is a union of smooth manifolds $M_{(H)}/G$, where $(H)$ runs over conjugacy classes of stabilisers.

For a symplectic manifold $(M,\omega)$ and a symplectic action of $G$, we have that $M_H$ is symplectic, since, for $x \in M_H$, $H$ acts on $T_xM$ as a subgroup of the group of symplectic transformations and hence the fixed-point set $(T_xM)^H$ is a symplectic subspace. Sjamaar & Lerman [17] prove that the restriction $\mu'$ of the moment map $\mu\colon M \to \mathfrak{g}^*$ to the space $M'_H$ consisting of those components of $M_H$ meeting $\mu^{-1}(0)$, is a moment map for the action of $L$ on $M'_H$. To see this, first note that the equivariance of $\mu$ implies that $\mu'(M'_H) \subset (\mathfrak{g}^*)^H$. As $H$ acts trivially on $M'_H$, we see that $\langle \mu, h\rangle$ is locally constant on $M'_H$ for all $h \in \mathfrak{h}$, but each component of $M'_H$ meets $\mu^{-1}(0)$, so $\langle \mu, h\rangle$ is zero on $M'_H$ and $\mu'(M'_H)$ is contained in $\operatorname{Ann}_{\mathfrak{g}^*}\mathfrak{h}$, the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Identifying

$\mathfrak{l}^*$ with $\operatorname{Ann}_{\mathfrak{n}^*}\mathfrak{h}$, where $\mathfrak{n}$ is the Lie algebra of $N(H)$, now enables us to regard $\mu'$ as taking values in $\mathfrak{l}^*$.

Since $L$ acts freely on $M'_H$, we have that $0$ is a regular value of $\mu'$ and $(M_{(H)} \cap \mu^{-1}(0))/G = (\mu')^{-1}(0)/L$ is a manifold and carries a symplectic structure. The symplectic quotient of $M$ by $G$ is therefore a union of symplectic manifolds. Note that these strata need not be of pure dimension and can indeed be empty. This remark will also apply to the partitions we give of hyperKähler and quaternionic Kähler quotients.

An analogous result for hyperKähler quotients follows almost immediately. In certain cases, this seems to be a "folk" result; for example it is used in the discussion by Nakajima of quiver varieties, which arise as hyperKähler quotients of flat space [13]. However we have not seen a proof in the literature, or a statement of the result in generality.

**Theorem 2.1.** *Suppose $(M, g, I, J, K)$ is a hyperKähler manifold with a smooth, proper action of a Lie group $G$ preserving the hyperKähler structure. If there is a hyperKähler moment map $\mu\colon M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$, then the hyperKähler quotient $\mu^{-1}(0)/G$ is a union of the hyperKähler manifolds $(M_{(H)} \cap \mu^{-1}(0))/G$, where $(H)$ runs over conjugacy classes of stabilisers of points of $M$.*

*Proof.* The main observation is that the manifold $M_H$ is now hyperKähler, since it is symplectic with respect to each of the symplectic structures on $M$. Writing $\mu = \mu_I i + \mu_J j + \mu_K k$ and applying Sjamaar & Lerman's proof to the symplectic moment maps $\mu_I$, $\mu_J$ and $\mu_K$, we see that $\mu' = \mu|_{M'_H}$ is a hyperKähler moment map for the free action of $L$ on $M'_H$, and the result follows. $\square$

Note that we have not shown that the hyperKähler quotient is a decomposed space in the sense of Sjamaar & Lerman; the frontier condition is missing. In the symplectic case, this is proved using a local model constructed from the symplectic normal bundle to an orbit $G/H$ in $\mu^{-1}(0)$. Although the corresponding manifold in the hyperKähler case admits a hyperKähler metric [5], it does not provide such a universal model.

Boyer, Galicki and Mann [3, 4] have developed a quotient construction for 3-Sasakian manifolds. Given a 3-Sasakian manifold $M$, one may construct a hyperKähler cone metric on $P = \mathbb{R}_{>0} \times M$, and any action on $M$ preserving the 3-Sasakian structure extends to a hyperKähler action on $P$. From above, we see that $P_H$ is hyperKähler, and we can deduce that $M_H$ is 3-Sasakian. The restriction of the 3-Sasakian moment map to $M_H$ is now the 3-Sasakian moment map for the action of $L$ on $M_H$. We obtain the following result.

**Theorem 2.2.** *Let $M$ be a 3-Sasakian manifold with a smooth proper action of $G$ preserving the 3-Sasakian structure. If the moment map $\mu$ exists, then the quotient $\mu^{-1}(0)/G$ is the union of the smooth, 3-Sasakian manifolds $(M_{(H)} \cap \mu^{-1}(0))/G$.*

## 3. A Quaternionic Kähler Counterexample

The following example, due to Galicki & Lawson [8], shows that the quaternionic Kähler quotient of a quaternionic Kähler manifold need not be a union of quaternionic Kähler manifolds.

Let $U(1)$ act on $M = \mathbb{HP}(n)$ via $e^{i\theta} \cdot [u, v] = [e^{pi\theta}u, e^{qi\theta}v]$, for $u \in \mathbb{H}^n$, $v \in \mathbb{H}$, where $q < p$ are coprime, positive integers. In the simplest case with $p + q$ odd and $p \neq q + 1$, one has the following stabilisers:

$U(1)$ on $\mathbb{CP}(n-1) = \{\, [a, 0] : a \in \mathbb{C}^n \setminus 0 \,\}$ and $\{[0, 1]\}$,
$\mathbb{Z}/2p$ on $\{\, [u, 0] : u \in \mathbb{H}^n \,\} \setminus \mathbb{CP}(n-1)$,
$\mathbb{Z}/p+q$ on $\{\, [ja, 1] : a \in \mathbb{C}^n \setminus 0 \,\}$,
$\mathbb{Z}/p-q$ on $\{\, [a, 1] : a \in \mathbb{C}^n \setminus 0 \,\}$,
$\{1\}$ elsewhere.

Note that the manifolds $M_H$ are not necessarily quaternionic.

The moment map is $p\overline{u}^t iu + q\overline{v}iv$. (Of course, this is not a function on $\mathbb{HP}(n)$, but a section of the bundle $\mathcal{G}$.) The quaternionic Kähler quotient consists of three pieces: one generic, open and quaternionic Kähler coming from $M_{\{1\}}$; one quaternionic Kähler, isomorphic to $\mathrm{Gr}_2(\mathbb{C}^n)$, from $M_{\mathbb{Z}/2p}$; and one Kähler-Einstein, isomorphic to $\mathbb{CP}(n-1)$, from $M_{\mathbb{Z}/p+q}$. The other manifolds $M_H$ do not meet the zero set of the moment map. Thus if $n$ is even we have a piece whose dimension is not divisible by four and so cannot be quaternionic Kähler. We may regard this as being related to the fact that $M_{\mathbb{Z}/p+q}$ is not quaternionic.

The second named author [18] has shown that to each quaternionic Kähler manifold $M$ one may associate a manifold $\mathcal{U}(M)$ with either a hyperKähler or a pseudo-hyperKähler structure, depending on whether the scalar curvature of $M$ is positive or negative. An action on $M$ preserving the quaternionic Kähler structure lifts to an action on $\mathcal{U}(M)$ preserving the (pseudo-)hyperKähler structure. Moreover, if the action on $M$ is free, then the lifted action is also free, and the hyperKähler quotient of $\mathcal{U}(M)$ is just $\mathcal{U}(N)$, where $N$ is the quaternionic Kähler quotient of $M$.

If $M$ is $\mathbb{HP}(n)$ then $\mathcal{U}(M)$ is $\mathbb{H}^{n+1} \setminus 0$. On $\mathbb{H}^{n+1} \setminus 0$, the stabilisers of the lifted $U(1)$-action are:

$\mathbb{Z}/p$ on $\{\,(u,0) : u \in \mathbb{H}^n \setminus 0\,\}$,
$\mathbb{Z}/q$ on $\{\,(0,v) : v \in \mathbb{H} \setminus 0\,\}$,
$\{1\}$ elsewhere.

The hyperKähler quotient of the first of these gives $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))$ covering the quotient of $M_{\mathbb{Z}/2p}$. The second does not meet the zero set of the hyperKähler moment map and the hyperKähler quotient of the third covers the remaining two pieces from $M_{\{1\}} \cup M_{\mathbb{Z}/p+q}$ in the quaternionic Kähler quotient.

This discussion shows that one cannot study the stratification of quaternionic Kähler quotients just by passing to $\mathcal{U}(M)$ and applying the hyperKähler theory.

## 4. Quaternionic Kähler Quotients

In our approach to this problem, we shall first examine the structure of $M_H$, where $M$ is quaternionic Kähler and $H$ is a subgroup of a Lie group $G$ acting properly on $M$ and preserving the quaternionic Kähler structure. As $M_H$ is an open subset of the fixed point set $M^H$, we see that $M_H$ is totally geodesic in $M$.

If $x$ is a point of $M_H$ (or, more generally, if $x$ lies in $M^H$) then we have a linear action of $H$ on $T_xM$. As the action of $H$ preserves the quaternionic Kähler structure, this is actually a representation $H \to Sp(n)\,Sp(1)$. This induces a representation $\phi\colon H \to SO(3) \cong Sp(1)/\{\pm 1\}$. The group $H$ acts on a fibre $\mathcal{G}_x \cong \mathbb{R}^3$ via the composition of $\phi$ with the irreducible three-dimensional representation of $SO(3)$.

**Example 4.1.** The simplest examples of quaternionic Kähler manifolds are the Wolf spaces [19]. These are the symmetric spaces $F/K$ where $K$ contains a normal subgroup isomorphic to $Sp(1)$. In fact we may write $K$ as a local direct product $K_1\,Sp(1)$, where $K_1 \cap Sp(1)$ is either $\{1\}$ or $\{\pm 1\}$. Let $eK$ be the basepoint of such a Wolf space and let $H$ be the stabiliser of $eK$ with respect to a quaternionic Kähler action of a group $G$ on $F/K$. If $K_1 \cap Sp(1) = \{\pm 1\}$ then $\phi\colon H \to SO(3)$ is just the composition of the inclusion of $H$ into $K$ with the quotient map $K \to K/K_1 \cong Sp(1)/\{\pm 1\} \cong SO(3)$. If $K_1 \cap Sp(1) = \{1\}$ then this composition gives a map into $Sp(1)$, and to obtain $\phi$ we must follow it with the double cover $Sp(1) \to SO(3)$.

If $x$ and $y$ are two points of the same path-component of $M_H$, then parallel transport along a path in $M_H$ joining $x$ and $y$ defines an $H$-equivariant isomorphism between the tangent spaces to $M$ at $x$ and $y$. This implies that the representation $\phi\colon H \to SO(3)$ is equivalent at all points on a path component of $M_H$. (Actually this argument also works if we replace $M_H$ by $M^H$; the crucial point is that $H$ should fix each point of the path). Since the action of $L := N(H)/H$ intertwines the action of $H$, we also see that the image of $\phi$ is the same up to isomorphism on $L$-orbits.

There are now four possibilities for the isomorphism class of $\phi(H)$:

1. $\phi(H)$ is trivial,
2. $\phi(H)$ is a non-trivial cyclic group,
3. $\phi(H)$ is finite but not cyclic,
4. $\phi(H)$ has positive dimension.

We denote by $M_{H,1}$, $M_{H,\mathrm{C}}$, $M_{H,\mathrm{NC}}$ and $M_{H,\mathrm{POS}}$ respectively the set of points of $M_H$ with $\phi(H)$ as in cases (1)–(4) and use similar notation with regard to $M_{(H)}$. The above remarks show that $M_{H,1}$, $M_{H,\mathrm{C}}$, $M_{H,\mathrm{NC}}$ and $M_{H,\mathrm{POS}}$ are unions of $L$-orbits of components of $M_H$, and are therefore manifolds (possibly with components of different dimension).

We first consider case (4).

**Proposition 4.2.** *Let $M$ be a quaternionic Kähler manifold with a smooth, proper action of a Lie group $G$ preserving the quaternionic Kähler structure. Let $\mu \in \Omega^0(M, \mathfrak{g}^* \otimes \mathcal{G})$ be the moment map and let $H$ be a subgroup of $G$. Then $M_{H,\mathrm{POS}}$ does not meet the zero set of $\mu$. Hence $M_{(H),\mathrm{POS}}$ does not meet the zero set of $\mu$ either.*

*Proof.* Given a point $x \in M_H$, the Lie algebra $\mathfrak{h}$ of $H$ may be regarded as the space of Killing fields for the action of $G$ which vanish at $x$. Such a Killing field $X$ is determined by the value at $x$ of its covariant derivative. In turn, for each point $m$ of the ambient manifold $M$ and each Killing field $X$ (not necessarily vanishing at $m$), we have that $(\nabla X)_m$ lies in the Lie algebra $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ of the holonomy group of $M$, so we have inclusions

$$\mathfrak{h} \subseteq \{\, (\nabla X)_x : X \in \mathfrak{g} \,\} \subseteq \mathfrak{sp}(n) \oplus \mathfrak{sp}(1).$$

Moreover the composition of these inclusions is just the Lie algebra version of the isotropy representation of $H$ at $x$.

It is well-known (see [18] for example) that if $\mu^X$ is the component of the quaternionic Kähler moment map associated to the Killing field $X$ then $\mu^X(m)$ is the $\mathfrak{sp}(1)$-component of $(\nabla X)_m$ (up to multiplication by a constant depending on the scalar curvature). It follows that if $x$ is in $M_H \cap \mu^{-1}(0)$, then on the level of Lie algebras the isotropy representation of $\mathfrak{h}$ at $x$ takes values in $\mathfrak{sp}(n)$, and hence the Lie algebra map $\phi_*\colon \mathfrak{h} \to \mathfrak{so}(3) \cong \mathfrak{sp}(1)$ is identically zero. The image of the homomorphism of Lie groups $\phi\colon H \to SO(3)$ is therefore discrete. □

**Example 4.3.** Let $Sp(1)$ act on $\mathbb{HP}(n)$ by multiplication on the left. Then $Sp(1)$ has $\mathbb{RP}(n)$ as its fixed point set and $\phi\colon Sp(1) \to SO(3)$ is the covering map. However, the moment map is given by $w \mapsto (\bar{w}^t i w, \bar{w}^t j w, \bar{w}^t k w)$ which is clearly non-zero on $\mathbb{RP}(n)$.

By definition an almost quaternionic submanifold of $M$ is a submanifold whose tangent space is invariant under the action of $\mathcal{G} \subset \operatorname{End} TM$. As the tangent space to $M_H$ at a point $y$ is $(T_yM)^H$, it is readily seen that $M_H$ is almost quaternionic precisely when $H$ acts trivially on the fibre of $\mathcal{G}$ over each point of $M_H$, that is, precisely if case (1) holds.

**Proposition 4.4.** *Let $M$ be a quaternionic Kähler manifold with a smooth, proper action of a Lie group $G$ preserving the quaternionic Kähler structure. Let $\mu$ be the moment map and let $H$ be a subgroup of $G$.*

*Then $M_{H,1}$ is quaternionic Kähler and the restriction of $\mu$ to $M'_{H,1}$, the union of components of $M_{H,1}$ meeting $\mu^{-1}(0)$, is the quaternionic Kähler moment map for the free action of $L := N(H)/H$ on $M'_{H,1}$. Moreover, $(M_{(H),1} \cap \mu^{-1}(0))/G$ is the quaternionic Kähler quotient of $M'_{H,1}$ by $L$ and is therefore a quaternionic Kähler manifold.*

*Proof.* Gray [9] shows that any almost quaternionic submanifold of a quaternionic Kähler manifold is quaternionic Kähler and totally geodesic, so in particular this applies to $M_{H,1}$. Let $i\colon M_{H,1} \to M$ be the inclusion. Equivariance of $\mu$, and the fact that $H$ acts trivially on $M_{H,1}$, implies that $i^*\mu = \mu|_{M_{H,1}}$ takes values in $(\mathfrak{g}^* \otimes i^*\mathcal{G})^H$. However, since the homomorphism $\phi$ is trivial by assumption, the induced action of $H$ on $i^*\mathcal{G}$ is also trivial and we find that $(\mathfrak{g}^* \otimes i^*\mathcal{G})^H = (\mathfrak{g}^*)^H \otimes i^*\mathcal{G}$. The structural tensor of $M_{H,1}$ is $i^*\Theta$ and as $M_H$ is totally geodesic we have $i^*\nabla\langle i^*\mu, X\rangle = X \lrcorner\, i^*\Theta$ for all $X$ in the Lie algebra of $N(H)$. In particular, for $X \in \mathfrak{h}$, this gives that $\langle i^*\mu, X\rangle$ is parallel. Therefore on $M'_{H,1}$, the map $i^*\mu$ takes values in $\operatorname{Ann}_{\mathfrak{n}^*} \mathfrak{h} \otimes i^*\mathcal{G}$. The result now follows. □

Combining the preceding two propositions gives us the next result.

**Theorem 4.5.** *Suppose a Lie group $G$ acts smoothly and properly on a quaternionic Kähler manifold $M$ preserving the quaternionic Kähler structure. If all stabilisers of points are connected, then the quaternionic Kähler quotient of $M$ by $G$ is the union of the quaternionic Kähler manifolds $(M_{(H)} \cap \mu^{-1}(0))/G$, where $(H)$ runs over conjugacy classes of stabilisers of points of $M$.* □

We shall now establish the smoothness result for arbitrary stabilisers.

**Theorem 4.6.** *Let $M$ be a quaternionic Kähler manifold with a smooth, proper action of a Lie group $G$ preserving the quaternionic Kähler structure. Let $\mu$ be the moment map and let $H$ be a subgroup of $G$.*

*Then $(M_{(H)} \cap \mu^{-1}(0))/G = (M_H \cap \mu^{-1}(0))/L$ is a smooth manifold.*

*Proof.* Without loss of generality we may assume that $M_{(H)}$ is non-empty so that $H$ is the stabiliser of some point and in particular is compact. Let $H_0$ denote the identity component of $H$ and let $M^{H_0\prime}$ denote the union of all path components of $M^{H_0}$ meeting the zero level set of the moment map. At each point of $M^{H_0\prime}$, we have a map $\phi\colon H_0 \to SO(3)$ and Proposition 4.2 applied to the $H_0$-action shows that $\phi(H_0)$ is discrete and hence trivial, because $H_0$ is connected. Now the proof of Proposition 4.4 shows that $M^{H_0\prime}$ is a quaternionic Kähler manifold and that the restriction of $\mu$ to $M^{H_0\prime}$ is the moment map for the action of $\widehat{L} := N(H_0)/H_0$.

Let $Q$ be the subset of $M^{H_0\prime}$ defined by

$$Q = \{\, x \in M^{H_0\prime} : \dim \operatorname{stab}_G(x) = \dim H_0 \,\}$$

Note that $H$ is a subgroup of $N(H_0)$ and that $M_H'$ is contained in $Q$. Since $G$ acts properly, it follows from the slice theorem that $Q$ is open in $M^{H_0\prime}$, so is smooth and quaternionic Kähler. Moreover it is acted on by $\widehat{L}$ and the restriction of $\mu$ to $Q$ is the moment map for this action.

Now, observe that $Q \subset M^{H_0}$ and so the $G$-stabiliser of a point $x$ in $Q$ contains the connected group $H_0$ and has the same dimension as $H_0$. It follows that $H_0$ is the identity component of $\operatorname{stab}_G(x)$, so $H_0$ is normal in $\operatorname{stab}_G(x)$. Thus, $\operatorname{stab}_G(x)$ is contained in $N(H_0)$, so $\operatorname{stab}_G(x) = \operatorname{stab}_{N(H_0)}(x)$.

Because $H_0 = \operatorname{stab}_G(x)_0$, the stabiliser of a point in $Q$ for the $\widehat{L}$-action is discrete (in fact finite, as stabilisers for proper actions are compact); that is, the action of $\widehat{L}$ on $Q$ is locally free. We deduce that $0$ is a regular value of $\mu|_Q$ and that $\mu^{-1}(0) \cap Q = (\mu|_Q)^{-1}(0)$ is smooth.

It follows from our remarks in the preceding paragraph that

$$\begin{aligned}\mu^{-1}(0) \cap M'_H &= \{\, x \in \mu^{-1}(0) \cap Q : \operatorname{stab}_{\widehat{L}}(x) = H/H_0 \,\} \\ &= (\mu^{-1}(0) \cap Q)_{H/H_0},\end{aligned}$$

which is smooth.

As $L = N(H)/H$ acts freely and properly on $\mu^{-1}(0) \cap M'_H$, the result is established. □

## 5. Non-quaternionic Pieces

Our next task is to study $(M_H \cap \mu^{-1}(0))/L$ in more detail when $M_H$ is not quaternionic. We first consider case (2), when $\phi(H)$ is a non-trivial cyclic subgroup of $SO(3)$.

Let $S$ be an $L$-orbit of path-components of $M_{H,\mathrm{c}}$. The image of $\phi$ is therefore the same up to isomorphism at different points of $S$. As $S$ is a union of components of $M_H$, it is a manifold. Let $S_{(H)}$ be the set of points swept out by applying the action of $G$ to $S$. Therefore $S_{(H)}$ is a subset of $M_{(H),\mathrm{c}}$, and $M_{(H),\mathrm{c}}$ is a union of sets $S_{(H)}$ for different choices of $S$.

At each point $x$ of $S$, the representation of $H$ on $\mathcal{G}_x$ splits as the sum of a trivial representation $\mathcal{L}_x$ and a non-trivial two-dimensional representation $\mathcal{L}_x^\perp$. Over $S$ we therefore obtain subbundles $\mathcal{L}$ and $\mathcal{L}^\perp$ of $\mathcal{G}$, of real rank one and two respectively. Since $M_H$, and hence $S$, is totally geodesic, and $H$ acts isometrically, we see that $\mathcal{L}$ and $\mathcal{L}^\perp$ are preserved by the Levi-Civita connection $\nabla^S$ of $S$.

Now $\mathcal{G}$ is oriented, and the holonomy of $\nabla^S$ on $\mathcal{G}$ is contained in $SO(3)$. The holonomy commutes with the action of $H$ on $\mathcal{G}$, so if the image of $\phi$ is a cyclic group of order greater than two, we see that the holonomy on $\mathcal{L}^\perp$ must lie in $SO(2)$. It follows that the holonomy on $\mathcal{L}$ is trivial, so we can find a parallel section of $\mathcal{L}$. This section defines a Kähler structure $I$ on $S$. The Kähler structure is invariant under the action of $L_0$, the identity component of $L = N(H)/H$.

If the cyclic group has order two, then we have either a Kähler structure or a local Kähler structure $I$, depending on whether or not $\mathcal{L}$ is orientable.

We must therefore distinguish two subcases

(a) $\mathcal{L}$ is trivial on $S$, and $S$ is Kähler,

(b) $\mathcal{L}$ is non-trivial on $S$, and $S$ is locally Kähler,

Let us assume first that we are in case (a), when $I$ defines a global Kähler structure. If $X \in \mathfrak{g}$, we write $\mu^X$ for the $X$-component of $\mu$. As usual, we identify $X$ with a Killing field on $M$. We denote by $i$ the inclusion of $S$ in $M$.

Locally we may write $i^*\mu = \mu_I \otimes I + \mu_J \otimes J + \mu_K \otimes K$. Here $\mu_I$ is globally defined on $S$, but $\mu_J$ and $\mu_K$ may not be. As $\mathcal{L}$ is flat, we have $d\mu_I^X = \nabla\mu_I^X = i^*(X \lrcorner\, \omega_I)$. The splitting $i^*\mathcal{G} = \mathcal{L} \oplus \mathcal{L}^\perp$ is $N_{SO(3)}(\phi(H))$-invariant, and hence $N(H)$-invariant, so $\mu_I$ is $N(H)$-equivariant. As in the symplectic case, we find that $\mu_I$ takes values in the Lie algebra of $L = N(H)/H$. Therefore $\mu_I$ is a Kähler moment map for the free Kähler action of the identity component $L_0$ of $L$ on $S$, and $(S \cap \mu_I^{-1}(0))/L_0$ is the Kähler quotient of $S$ by $L_0$.

Now, we have [8]

$$d\mu_J = \nabla\mu_J - \alpha_{JK}\mu_K \quad \text{and} \quad d\mu_K = \nabla\mu_K + \alpha_{JK}\mu_J,$$

where $\alpha$ is the connection form on $\mathcal{L}^\perp$. On the common zero set $\mathcal{Z}$ of $\mu_J$ and $\mu_K$, therefore, we have $d\mu_J^X = i^*(X \lrcorner\, \omega_J)$ and $d\mu_K^X = i^*(X \lrcorner\, \omega_K)$, for each $X$, and it readily follows that

$$d(\mu_J + i\mu_K)(IY) = i\, d(\mu_J + i\mu_K)(Y)$$

for any tangent vector $Y$ to $S$ at a point of $\mathcal{Z}$.

From Theorem 4.6, we know that $(S \cap \mu^{-1}(0))/L_0$, which is locally expressed as $(S \cap \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0))/L_0$, is a manifold. The discussion above shows that it is in fact Kähler.

Note that the action of $L/L_0$ on the Kähler quotient is free and proper, but not necessarily holomorphic. Therefore $(S_{(H)} \cap \mu^{-1}(0))/G = (S \cap \mu^{-1}(0))/L$ is a smooth manifold with a Kähler covering space.

Suppose, on the other hand, that case (b) occurs. Then similar arguments show that $(S \cap \mu^{-1}(0))/L_0$ is locally Kähler and the smooth manifold $(S \cap \mu^{-1}(0))/L$ has a Kähler cover.

**Proposition 5.1.** *Let $M$ be a quaternionic Kähler manifold with a smooth, proper action of a Lie group $G$ preserving the quaternionic Kähler structure. Write $\mu$ for the moment map. Choose $H$ a subgroup of $G$ and $S$ an $L$-orbit of components of $M_{H,\mathbb{C}}$. Let $S_{(H)} \subset M_{(H),\mathbb{C}}$ denote the set of points swept out by applying the action of $G$ to $S$.*

*Then $S$ is locally Kähler, and $(S_{(H)} \cap \mu^{-1}(0))/G$ is a smooth manifold with a Kähler covering space. Hence $(M_{(H),\mathbb{C}} \cap \mu^{-1}(0))/G = (M_{H,\mathbb{C}} \cap \mu^{-1}(0))/L$ is a smooth manifold each component of which is covered by a Kähler manifold.* □

*Remark 5.2.* The Kähler and locally Kähler pieces have a natural interpretation in terms of the twistor space $Z$ of $M$. As a real manifold, $Z$ is just the sphere bundle of $\mathcal{G}$, but it also admits an integrable complex structure [16, 1]. The action of $G$ on $M$ lifts to $Z$ as a holomorphic action, so the manifolds $Z_H$ are complex.

Now $Z_H$ fibres over a subset of $M^H$. Moreover, $M^H$ contains $M_H$ and $M_{H,\mathbb{C}}$ as open subsets. Thus $Z_{H,\mathbb{C}}$, the part of $Z_H$ lying over $M_{H,\mathbb{C}}$, is open in $Z_H$ and hence complex. When $\phi(H)$ is cyclic there are precisely two fixed almost complex structures $\pm I$ at a given point of $M$. Thus the projection $\pi\colon Z_{H,\mathbb{C}} \to M_{H,\mathbb{C}}$ is two-to-one and each component $S_0$ of $M_{H,\mathbb{C}}$ is either complex or locally complex, depending on whether the number of components of $\pi^{-1}S_0$ is two or one.

If $M$ has positive scalar curvature then $Z$ admits a Kähler metric [16] and in this case the above construction gives (locally) Kähler structures on $M_{H,\mathbb{C}}$ inherited from $Z$, quite possibly different from those obtained directly from the quaternionic Kähler metric on $M$.

Specialising to $S^1$-actions, Propositions 4.2, 4.4 and 5.1 together imply the following corollary.

**Corollary 5.3.** *Let $M$ be a quaternionic Kähler manifold with a smooth, proper action of $S^1$ preserving the quaternionic Kähler structure. Then all the manifolds $(M_{(H)} \cap \mu^{-1}(0))/G$ in the decomposition are quaternionic Kähler or have Kähler covers.* □

At this point it may be helpful to return to the example of the circle action on $\mathbb{HP}(n)$ discussed in §3. It can be checked that the image of $\phi$

is $U(1)$ on $M_{U(1)}$, cyclic on $M_{\mathbb{Z}/p+q}$ and $M_{\mathbb{Z}/p-q}$, and trivial on $M_{\mathbb{Z}/2p}$ and $M_1$. The associated manifolds in the quotient are respectively empty, Kähler, empty, quaternionic Kähler and quaternionic Kähler, which is consistent with our results.

*Remark 5.4.* Suppose that we have a torus action on one of the classical Wolf spaces $F/K$ preserving the quaternionic Kähler structure. In each case, after conjugation in $F$, we may assume that the torus is contained in the standard maximal torus $T$ of $F$. Using the description of $\phi$ for actions on Wolf spaces given in Example 4.1, it can be checked that the image under $\phi$ of each stabiliser for the action of $T$ is either a circle, cyclic or trivial. We deduce that this also holds for the original torus action. It follows that in this case, as in the case of a circle action, the quotient consists of quaternionic Kähler and locally Kähler manifolds. We can also show this for $G_2/\,SO(4)$ by exploiting its embedding as a submanifold of $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$.

Finally we consider case (3), when for each $x$ the image of $\phi$ is a subgroup of $SO(3)$ which is dihedral, tetrahedral, octahedral or icosahedral. In this situation no point of $\mathcal{G}_x$ is fixed by $H$; equivalently no local almost complex structure is preserved by $H$. We find that $M_{H,\mathrm{NC}}$ is totally real with respect to all the local almost complex structures.

The following example shows that $(M_{H,\mathrm{NC}} \cap \mu^{-1}(0))/L$ need not admit any complex or quaternionic structure.

**Example 5.5.** Consider the action of $\mathbb{Z}/2 \ltimes S^1$ on $\mathbb{HP}(n)$ defined by the circle action of the example in §3 together with left multiplication on $\mathbb{HP}(n)$ by $j$. Let $H$ be the subgroup $\mathbb{Z}/2 \ltimes \mathbb{Z}/p+q$. Then $M_H = \{\, [ju, 1] : u \in \mathbb{R}^n \setminus 0 \,\}$ and $M_H \cap \mu^{-1}(0)$ is the $(n-1)$-sphere. The group $L = N(H)/H$ is isomorphic to $\mathbb{Z}/2$ and changes the sign of $u$. The associated piece in the quaternionic Kähler quotient is $\mathbb{RP}(n-1)$.

The next example shows that such real pieces occur even in quotients by compact connected groups.

**Example 5.6.** Let $SU(2)$ act irreducibly on $\mathbb{HP}(n)$ via the irreducible representation $S^{2n+1}\mathbb{C}^2 \cong \mathbb{H}^{n+1}$. Write $\{x, y\}$ for an orthonormal basis of $\mathbb{C}^2 \cong \mathbb{H}$ with $y = jx$. Then $\mathbb{H}^{n+1}$ has an orthonormal basis

$$\{\, x^{2n+1}, \sqrt{2n+1}x^{2n}y, \sqrt{\tbinom{2n+1}{2}}x^{2n-1}y^2, \dots, \sqrt{\tbinom{2n+1}{n}}x^{n+1}y^n \,\}$$

over $\mathbb{H}$, where $x^ry^s = \binom{2n+1}{r}^{-1}(x \otimes \dots \otimes x \otimes y \otimes \dots \otimes y + \cdots)$ and $\mathbb{H}$ acts on the left via

$$(c + dj)(ax^ry^s + (-1)^s bx^sy^r) = (ca - d\bar{b})x^ry^s + (-1)^s(cb + d\bar{a})x^sy^r.$$

Note that this is opposite to our conventions in the earlier examples but it is more convenient for describing the infinitesimal $SU(2)$ action, which is now $\mathbb{C}$-linear and given by

$$x^ry^s \mapsto i(r-s)x^ry^s, \tag{5.1}$$

$$x^ry^s \mapsto -sx^{r+1}y^{s-1} + rx^{r-1}y^{s+1}, \tag{5.2}$$

$$x^ry^s \mapsto isx^{r+1}y^{s-1} + irx^{r-1}y^{s+1}, \tag{5.3}$$

where terms involving $x^{-1}$ or $y^{-1}$ are to be regarded as zero.

Consider the points $q = ax^ry^s + bx^{r'}y^{s'} \in \mathbb{H}^{n+1}$ and $p = [q] \in \mathbb{HP}(n)$. We assume that $a$ and $b$ are nonzero. The point $p$ certainly has a discrete stabiliser $H$ if

$$|r - r'|, |s - s'|, |r - s'|, |s - r'| > 2. \tag{5.4}$$

On the other hand suppose $a, b \in \mathbb{R}$ and $r - s = (1-k)(r' - s')$ for some positive integer $k$. Then we claim that the stabiliser of $p$ contains $j$ (identifying $Sp(1)$ with $SU(2)$) and the cyclic group $\mathbb{Z}/k|r'-s'|$ generated by $\{\exp(2\pi i/(k|r'-s'|))\}$. First observe that the action of $j \in SU(2)$ is just that of right multiplication of $\mathbb{H}^{n+1}$ by $j$. The left- and right-actions of $j$ agree on $q$ because $a, b$ are real, so $j$ lies in the stabiliser of $p$. Also, $\exp(2\pi i/(k|r'-s'|)) \in SU(2)$ acts on $q$ as

$$e^{2\pi i/(k|r'-s'|)} \cdot q = ae^{\varepsilon 2\pi i(-1+1/k)}x^ry^s + be^{\varepsilon 2\pi i/k}x^{r'}y^{s'} = e^{\varepsilon 2\pi i/k}q,$$

where $\varepsilon$ is the sign of $r' - s'$. However, this is the action of an element $\exp(\pm 2\pi i/k)$ of $\mathbb{H}^*$ on the left and preserves the point $p \in \mathbb{HP}(n)$. Note the above conditions together with the constraint $r+s = 2n+1 = r'+s'$ force $k$ to be even.

If $X$ is a matrix in $\mathfrak{sp}(n+1) \subset M_{n+1}(\mathbb{H})$, then with respect to our conventions in this example the corresponding moment map at $[w] \in \mathbb{HP}(n)$ is $-w^t X \bar{w}$. Thus condition (5.4) implies that the moment maps corresponding to (5.2) and (5.3) vanish, whereas the moment map for (5.1) is $ia^2(r-s)\binom{2n+1}{r}^{-1} + ib^2(r'-s')\binom{2n+1}{r'}^{-1}$. Taking $r-s > 0 > r'-s'$ shows that there are points of the form $p$ lying in the zero set of the moment map. Note that the stabiliser of $q \in \mathbb{H}^{n+1}$ is now a cyclic group of order $s'-r'$.

Since $p = [q]$, the elements of $\operatorname{stab}(p)/\operatorname{stab}(q)$ act on $q \in \mathbb{H}^{n+1}$ as a subgroup $K$ of $\mathbb{H}^*$. Since $K$ is compact it lies in $Sp(1) \leqslant \mathbb{H}^*$ and we calculate $\phi(H)$ as the image $K/\{\pm 1\}$ in $SO(3)$ of $K$. The discussion above shows that $K$ contains the cyclic group $\mathbb{Z}/k = \{\exp(2\pi i/k)\}$ and the element $j$, so $\phi(H)$ contains a dihedral group of order $k$. Thus dihedral groups may occur as $\phi(H)$ even for connected group actions on $\mathbb{HP}(n)$.

Note that we have not attempted to determine necessary conditions for $\phi(H)$ to contain a dihedral group, but the conditions we give are fulfilled by the point $[x^9 + 6\sqrt{7}x^3y^6] \in \mathbb{HP}(4)$. Since $SU(2)$ has dimension three, quaternionic dimension four is the first dimension in which one expects to find non-trivial quaternionic Kähler quotients by this group.

*Remark 5.7.* If $G$ is Abelian, or more generally if $H$ is normal in $G$, we can be slightly more precise about what the quotient looks like. For in the case when $\phi(H)$ is cyclic and non-trivial, it is straightforward to show (without any hypothesis on $G$ or $H$) that $J(TM_H)$ and $K(TM_H)$ are orthogonal to $TM_H$, so the restrictions of $\omega_J$ and $\omega_K$ to $M_H$ are zero. If $H$ is normal in $G$, then all Killing fields $X$ at points of $M_H$ are tangential to $M_H$, so the restrictions to $M_H$ of $X \lrcorner \omega_J$ and $X \lrcorner \omega_K$ also vanish, and $\mu_J$, $\mu_K$ are parallel on $S$. When $(S \cap \mu^{-1}(0))/L$ is non-empty, we can therefore deduce that this set is equal to $(S \cap \mu_I^{-1}(0))/L$.

Similarly, if $\phi(H)$ is non-cyclic, we find that $A(TM_H)$ is orthogonal to $TM_H$ for all $A \in \mathcal{G}$. If $H$ is normal in $G$, we can deduce that $\mu$ is parallel on $M_H$, so $(M_H \cap \mu^{-1}(0))/L$ is either empty or is just $M_H/L$.

Note from the above remarks that if $\phi(H)$ is non-trivial cyclic (respectively non-cyclic) then the dimension of $M_H$ will be at most half (respectively a quarter) of the dimension of $M$. We therefore get a bound on the dimension of the manifolds in the decomposition of the quotient which are not quaternionic Kähler.

## 6. An Orbifold Decomposition

We shall now describe an alternative, coarser decomposition of quaternionic Kähler quotients. Let $M_{([H])}$ denote the set of points whose stabiliser has identity component conjugate to $H$. By analogy with our previous notation, we define $M_{[H]}$ to be the set of points where the identity component of the stabiliser equals $H$. We can write

$$\mu^{-1}(0)/G = \bigcup(\mu^{-1}(0) \cap M_{([H])})/G, \tag{6.1}$$

where the union is taken over compact connected subgroups $H$ of $G$.

Now, each of the spaces in the union may be identified with $(\mu^{-1}(0) \cap M_{[H]})/L$, where $L = N(H)/H$ as usual. In the proof of Theorem 4.6 we showed that $M_{[H]}$ is open in the smooth quaternionic Kähler manifold $M^H$, so is itself smooth and quaternionic Kähler. The restriction of $\mu$ to $M_{[H]}$ is just the moment map for the locally free action of $L$, so we see that $(\mu^{-1}(0) \cap M_{([H])})/L$ is a quaternionic Kähler *orbifold*. We have shown the following result.

**Theorem 6.1.** *Equation* (6.1) *gives a canonical decomposition of the quaternionic Kähler quotient into a union of quaternionic Kähler orbifolds each of which is a finite quotient of a quaternionic Kähler manifold.*

Note that each of these orbifold pieces contains a relatively open quaternionic Kähler piece as follows. Suppose $G$ is a finite group acting isometrically and quaternionically on a quaternionic Kähler manifold $M$. Let $H$ be the principal stabiliser type. Then $M_{(H)}/G$ is a smooth open subset of $M/G$. We claim that $M_{(H)}/G$ is quaternionic Kähler. The quotient map $\pi\colon M_{(H)} \to M_{(H)}/G$ is étale (i.e. at each point, $\pi_*$ is invertible), so is a local isometry. But $G$ acts quaternionically and in particular preserves the bundle $\mathcal{G}$, so $\mathcal{G}$ pushes forward to a rank three bundle on $M_{(H)}/G$ giving a quaternionic Kähler structure.

The above discussion provides a convenient way of seeing the next result.

**Proposition 6.2.** *At least one of the pieces in the decomposition of $\mu^{-1}(0)/G$ into manifolds discussed in §§4 and 5 is open and quaternionic Kähler.*

*Proof.* Order those orbit types $H$ such that $M_{(H)}$ meets $\mu^{-1}(0)$ by saying $H_1 \preccurlyeq H_2$ if $H_1$ is subconjugate to $H_2$. As all the stabilisers $H$ are compact, there exists an element $H_{\text{min}}$ of this set which is minimal in the sense that if $H \preccurlyeq H_{\text{min}}$ then $H$ is conjugate to $H_{\text{min}}$. Now the slice theorem for the action of $G$ on $M$ shows that $\mu^{-1}(0) \cap M_{(H_{\text{min}})}$ is open in $\mu^{-1}(0)$. Hence $(\mu^{-1}(0) \cap M_{(H_{\text{min}})})/G$ is open in $\mu^{-1}(0)/G$.

If $H$ is the identity component of $H_{\text{min}}$, we see that $\mu^{-1}(0) \cap M_{(H_{\text{min}})}$ is open in the manifold $\mu^{-1}(0) \cap M_{([H])}$. The former set is just the union of all $G$-orbits in $\mu^{-1}(0) \cap M_{([H])}$ of stabiliser type $H_{\text{min}}$. So, as it is open, it must be the union of principal $G$-orbits in $\mu^{-1}(0) \cap M_{([H])}$. Hence $(\mu^{-1}(0) \cap M_{(H_{\text{min}})})/G$ is the smooth open piece in the quaternionic Kähler orbifold $(\mu^{-1}(0) \cap M_{([H])})/G$ so is quaternionic Kähler. □

## 7. A Length Space Structure

We have exhibited decompositions of the singular quaternionic Kähler quotient $\mu^{-1}(0)/G$ into smooth Riemannian manifolds and into Riemannian orbifolds. We conclude by showing that each path component of the quotient can be endowed in a natural way with a length space structure. It is well-known that length spaces form a natural generalisation of Riemannian manifolds. For example, the quotient of a Riemannian manifold by a compact group, or the Gromov-Hausdorff limit of a sequence of Riemannian manifolds, is always a length space even though it may not be a smooth manifold. On the other hand, length spaces have nice properties not shared by general metric spaces [10].

Assume now that $M$ is complete and that $G$ is connected. If $C$ is a path component of $\mu^{-1}(0)$, then we shall denote by $d$ the restriction to $C$ of the distance function on $M$ defined by the Riemannian metric. The singular space $(C, d)$ is then a complete metric space, but not a length space. We now let $d_i$ denote the inner metric on $C$ associated to $d$; that is, $d_i(x, y)$ is the infimum of the lengths of $d$-continuous curves $\gamma\colon [0,1] \to C$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $(C, d_i)$ is a length space.

Let us now further assume that $M$ is compact. As $M$ is Einstein, it is real-analytic as a Riemannian manifold [7], and hence the isometry group acts real-analytically [12]. Using compactness and a theorem of Nash [14], we may embed $M$ real-analytically and isometrically into a Euclidean space of some dimension. The formula for $\mu$ discussed in the proof of Proposition 4.2 shows that $\mu$ is real-analytic. Theorem 6.10 of Bierstone and Milman's paper [2] now implies that $d_i$ is a finite metric on $C$ and endows $C$ with the same topology as that inherited from $M$. In particular $(C, d_i)$ is complete. Now $d_i$ is $G$-invariant because $C$ and $d$ are, so if $G$ is compact the quotient $C/G$ is naturally a metric space (just take distance between orbits) and it is easy to check that the quotient metric is complete and makes $C/G$ into a length space.

**Theorem 7.1.** *Let $M$ and $G$ be compact and connected. Then the above procedure makes each component of $\mu^{-1}(0)/G$ into a complete finite length space. The topology given by the length metric coincides with that obtained by viewing $\mu^{-1}(0)$ as a subset of $M$ and taking the quotient topology.*

Note that exactly the same proof, again with the assumptions of compactness and connectedness, gives a length space structure on 3-Sasakian quotients.

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