Title: Estimates on the heat kernel of parabolic equations with advection

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Abbreviated title: Estimates on the heat kernel.

Abstract: The paper considers heat kernels of second–order parabolic equations in $\mathbb{R}^N$, with constant uniform diffusion and advective coefficients bounded in the maximum norm. Two critical cases, corresponding to upper and lower solutions, are identified, and explicit solutions are constructed for them in terms of the error function. They are shown to bound above and below all other heat kernels satisfying the same constraints on their advective coefficients, using a method of proof which relates two heat kernels together in a way which resembles the classical parametrix construction. Sharp bounds on the corresponding parabolic solution operators in $L_1(\mathbb{R}^N)$ are obtained as a consequence.

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1 Introduction

Upper and lower pointwise bounds on the heat kernels of second–order parabolic operators were first derived, for the general case of variable diffusion and advection coefficents in L_p spaces, by D. G. Aronson [1]. These initial qualitative estimates have more recently been sharpened, in the selfadjoint case, by E. B. Davies [2] and E. Fabes & D. W. Stroock [3]. For the non–selfadjoint case of equations with advection, sharper estimates have been obtained by J. R. Norris & Stroock [6].

Here, we consider the heat kernel $\Gamma(x,\, t;\, y,\, s)$ of the equation

$$Lu(x,t) \equiv \frac{\partial u}{\partial t} - \Delta u + a(x,\, t).\nabla u = 0, \quad (x,\, t) \in \mathbb{R}^N \times (0,\, \infty), \tag{1.1}$$

with constant uniform diffusion, and aim to obtain sharper estimates in this physically significant special case. Our methods are classically based, and thus we initially take a to obey assumption (A):

(A) $a(x,\, t)$ is a continuous function, Hölder continuous in x, (exponent α), uniformly with respect to $(x,\, t)$ in subsets of the form $\mathbb{R}^N \times [0,\, T)$, for $T > 0$. The a_i also satisfy

$$\text{ess. sup}_{(x,\, t) \in \mathbb{R}^N \times [0,\, \infty)} |a_i(x,\, t)| \leq M_i, \qquad \text{for } M_i \geq 0, \quad i = 1,\, \ldots,\, N. \tag{1.2}$$

We recall that the classical heat kernel $\Gamma(x,\, t;\, y,\, s)$, $x,\, y \in \mathbb{R}^N$, $t > s \geq 0$, may be characterised as a solution of (1.1), twice continuously differentiable in space and once in time, such that

$$\lim_{t \searrow s} \int_{\mathbb{R}^N} \Gamma(x,\, t;\, y,\, s) f(y)\, dy = f(x), \tag{1.3}$$

for all continuous $f \in L_1(\mathbb{R}^N)$.

To derive a pointwise upper bound on $\Gamma(x,\, t;\, y, s)$, for fixed y and s, we explicitly construct a function $G_M(x - y,\, t - s)$, which is the solution of

$$\frac{\partial u}{\partial t} - \Delta u + \sum_{i=1}^{N} M_i \text{sign}[x_i - y_i] \frac{\partial u}{\partial x_i} = 0 \qquad (x,\, t) \in \mathbb{R}^N \times (0,\, \infty), \tag{1.4}$$

such that

$$\lim_{t \searrow s} \int_{\mathbb{R}^N} G_M(x - y,\, t - s) f(y)\, dy = f(x), \tag{1.5}$$

for all continuous $f \in L_1(\mathbb{R}^N)$. Here, we use the convention $\text{sign}[0] = 0$, and $M = [M_1, M_2, \ldots, M_N]^T$, where the M_i are as in (1.2).

The constructed function $G_M(x-y, t-s)$ also satisfies

$$\text{sign}[x_i - y_i]\frac{\partial G_M}{\partial x_i} \leq 0, \qquad i = 1, \ldots, N, \tag{1.6}$$

and is therefore one of the solutions of the nonlinear equation

$$\frac{\partial u}{\partial t} - \Delta u - \sum_{i=1}^{N} M_i \left|\frac{\partial u}{\partial x_i}\right| = 0, \qquad (x, t) \in \mathbb{R}^N \times (0, \infty). \tag{1.7}$$

We remark that the maximality of non–singular solutions of (1.7) has previously been considered by C. Pucci [7].

If one ignores the singularity at $(x, t) = (y, s)$, then intuitively $G_M(x-y, t-s)$ should bound $\Gamma(x, t; y, s)$ pointwise above, because it is an upper solution for (1.1, 1.3), in the sense of the comparison principle. In our main result we prove this relationship by representing Γ as the sum of G_M plus a non–positive integral term, in a way resembling the classical parametrix construction of the heat kernel. A lower bound is similarly obtained by considering $G_{-M}(x, t; y, s)$, the solution of (1.4, 1.5) with $-M_i$ replacing M_i.

By appealing to the machinery of Aronson [1], this classical result is extended to the case where a obeys assumption (B):

(B) $a \in L_\infty[\mathbb{R}^N \times [0, \infty)]$ and a satisfies (1.2).

The pointwise upper limit on $\Gamma(x, t; y, s)$ is actually attained within the wider class of equations obeying assumption (B). This is seen from the fact that $G_M(x-y, t-s)$ satisfies (1.4) and (1.5), and is therefore equal to $\Theta_M(z; x, t; y, s)$, the heat kernel of

$$\frac{\partial u}{\partial t} - \Delta u + \sum_{i=1}^{N} M_i \text{sign}[x_i - z_i]\frac{\partial u}{\partial x_i} = 0, \tag{1.8}$$

when $y = z$. A similar argument shows that the lower bound is also attained. Since Aronson [1] has shown that Θ_M may be pointwise approximated by the heat kernels of equations satisfying assumption (A), the upper and lower bounds given by $G_{\pm M}(x, t; y, s)$ are sharp, (though not attained), in that case.

In Section 4, we use the pointwise upper and lower bounds on $\Gamma(x, t; y, s)$ to deduce sharp upper and lower L_1 bounds on the solutions of (1.1) with $u(0) =$

$u_0 \in L_1(\mathbb{R})$. In particular, we obtain

$$\|u(t)\|_1 \leq \prod_{i=1}^{N} \left(2 + M_i^2 t\right) \|u_0\|_1.$$

2 Main result

For $M \in \mathbb{R}$, consider the function $g_M : \mathbb{R} \times (0, \infty) \to (0, \infty)$, given by

$$g_M(x, t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{(|x| - Mt)^2}{4t}\right\} + \frac{M}{4}\mathrm{erfc}\left(\frac{|x|}{2\sqrt{t}} - \frac{M\sqrt{t}}{2}\right), \tag{2.1}$$

where $\mathrm{erfc}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_x^\infty e^{-t^2}\, dt$. By direct differentiation,

$$\begin{aligned}
\frac{\partial g_M}{\partial x} &= \frac{-x}{4\sqrt{\pi t^3}} \exp\left\{-\frac{(|x| - Mt)^2}{4t}\right\}, \\
\frac{\partial^2 g_M}{\partial x^2} &= \frac{1}{4\sqrt{\pi t^3}} \left(-1 + \frac{x^2}{2t} - \frac{M|x|}{2}\right) \exp\left\{-\frac{(|x| - Mt)^2}{4t}\right\}, \\
\frac{\partial g_M}{\partial t} &= \frac{1}{4\sqrt{\pi t^3}} \left(-1 + \frac{x^2}{2t} + \frac{M|x|}{2}\right) \exp\left\{-\frac{(|x| - Mt)^2}{4t}\right\}.
\end{aligned} \tag{2.2}$$

Each of the right–hand sides is Lipschitz continuous. (Note that $\partial^3 g_M/\partial x^3$ is discontinuous at $x = 0$.) We deduce that g_M is a classical solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + M\mathrm{sign}[x]\frac{\partial u}{\partial x} = 0, \qquad (x, t) \in \mathbb{R} \times (0, \infty). \tag{2.3}$$

We define

$$G_M(x, t) = \prod_{i=1}^{N} g_{M_i}(x_i, t), \qquad \text{for } M \in \mathbb{R}^N \tag{2.4}$$

By (2.2, 2.3) and (2.4), $G_M(x - y, t)$ satisfies (1.4) and (1.6).

Theorem 2.1 *Suppose that, for some $M \in [0, \infty)^N$, $a(x, t)$ satisfies assumption (B). Then, for all $x, y \in \mathbb{R}^N$, $t > s \geq 0$,*

$$G_{-M}(x - y, t - s) \leq \Gamma(x, t; y, s) \leq G_M(x - y, t - s). \tag{2.5}$$

For each $y \in \mathbb{R}^N$, these bounds are attained for all $x \in \mathbb{R}^N$, $t > s \geq 0$ by the heat kernels $\Theta_{\pm M}(z; x, t; y, s)$ of (1.8) when $y = z$.

Remark: Aronson [1, Theorem 10, p. 679] shows that $\Gamma(x, t; y, s) = \tilde{\Gamma}(y, s; x, t)$ for $t > s \geq 0$, where $\tilde{\Gamma}$ is the heat kernel for the adjoint problem to (1.1),

$$\tilde{L}[u] \equiv \frac{\partial v}{\partial s} + \Delta v + \nabla.(a(y, s)v) = 0, \qquad (y, s) \in \mathbb{R}^N \times [0, t). \tag{2.6}$$

Hence, Theorem 2.1 also implies pointwise bounds for $\tilde{\Gamma}$.

3 Proof of main result

Lemma 3.1 *If (1.1) obeys assumption (A) for some $M \in [0, \infty)^N$, then*

$$\Gamma(x,\, t;\, y,\, s) = G(x-y,\, t-s) - \int_s^t \int_{\mathbb{R}^N} \Gamma(x,\, t;\, \eta,\, \sigma) LG(\eta - y,\, \sigma - s)\, d\eta d\sigma, \quad (3.1)$$

where $G(x,\, t)$ is either $G_M(x,\, t)$ or $G_{-M}(x,\, t)$.

Proof: The proof relies on the account given by Friedman [4, Chapter 1] of the parametrix method. Here, for $t_0 \in (s,\, t)$ and $D = \{z \mid |z - x| \leq 1\}$, the integral on the right–hand side of (3.1) is split up as $J_1 + J_2 + J_3$, where

$$\begin{aligned}
J_1 &= \int_s^{t_0} \int_{\mathbb{R}^N} \Gamma(x,\, t;\, \eta,\, \sigma) LG(\eta - y,\, \sigma - s)\, d\eta d\sigma, \\
J_2 &= \int_{t_0}^{t} \int_{\mathbb{R}^N \setminus D} \Gamma(x,\, t;\, \eta,\, \sigma) LG(\eta - y,\, \sigma - s)\, d\eta d\sigma, \\
J_3 &= \int_{t_0}^{t} \int_{D} \Gamma(x,\, t;\, \eta,\, \sigma) LG(\eta - y,\, \sigma - s)\, d\eta d\sigma.
\end{aligned}$$

For J_1 and J_2, $\Gamma(x,\, t;\, \eta,\, \sigma)$ has second derivatives in x, continuous with respect to $(x,\, t;\, \eta,\, \sigma)$. On the other hand, for C and c, generic positive constants,

$$\begin{aligned}
|LG(\eta - y,\, \sigma - s)| &= \left| \sum_{i=1}^N \left(a_i(\eta,\, \sigma) \mp M_i \mathrm{sign}[\eta_i - y_i] \frac{\partial G}{\partial \eta_i} \right) \right| \\
&\leq C \sum_{i=1}^N \left| \frac{\partial G}{\partial \eta_i} \right| \\
&\leq \frac{C|\eta - y|}{(\sigma - s)^{(N+2)/2}} \exp\Big(-(|\eta - y| - M(\sigma - s))^2/4(\sigma - s) \Big) \\
&\leq \frac{C}{(\sigma - s)^{3/4} |\eta - y|^{(N-1/2)}} \exp\left(-c|\eta - y|^2/(\sigma - s) \right),
\end{aligned}$$

and so is absolutely integrable. Hence, differentiation in x and t commutes with integration in J_1 and J_2. (We note here that differentiation of the upper limit t in J_2 is valid, but that this yields a term which one may show to be 0.)

For J_3, we observe that for fixed y and s

$$LG(\eta - y,\, \sigma - s) = \sum_{i=1}^N (a_i(\eta,\, \sigma) \mp M_i \mathrm{sign}[\eta_i - y_i]) \frac{\partial G}{\partial \eta_i}(\eta - y,\, \sigma - s)$$

is Hölder continuous in η, uniformly for $(\eta,\sigma) \in \mathbb{R}^N \times [t_0, t]$, since, noting (2.2) and (2.4),

$$\frac{\partial G}{\partial \eta_i}(\eta - y, \sigma - s) \qquad \text{and} \qquad \mathrm{sign}[\eta_i - y_i]\frac{\partial G}{\partial \eta_i}(\eta - y, \sigma - s)$$

are both uniformly Lipschitz continuous, and assumption (A) implies that $a(\eta,\sigma)$ is uniformly Hölder continuous.

According to Friedman [4, Theorems 1.3, 1.4 and 1.5], the first and second derivatives in x commute with integration in J_3, and

$$\frac{\partial J_3}{\partial t} = \int_{t_0}^{t}\int_D \frac{\partial \Gamma(x,t;\eta,\sigma)}{\partial t} LG(\eta - y, \sigma - s)\, d\eta d\sigma + LG(x - y, t - s).$$

The continuity of these derivatives follows from considerations similar to [4, Theorem 1.8 p. 19].

Let $J(x,t;y,s) = J_1 + J_2 + J_3$. From (2.2, 2.4), it follows that $G(x-y;t-s)$ is twice continuously differentiable in x and once in t, for $x, y \in \mathbb{R}^N$ and $t > s \geq 0$. Thus, the same property holds for $G(x-y;t-s) + J(x,t;y,s)$.

Combining the above results, we deduce that

$$\begin{aligned} &LG(x-y,t-s) - L\int_s^t\int_{\mathbb{R}^N} \Gamma(x,t;\eta,\sigma) LG(\eta - y, \sigma - s)\, d\eta d\sigma \\ &\quad = LG(x-y,t-s) - LG(x-y,t-s) \\ &\quad \int_s^t\int_{\mathbb{R}^N} L\Gamma(x,t;\eta,\sigma) LG(\eta - y, \sigma - s)\, d\eta d\sigma = 0. \end{aligned}$$

It remains to prove that

$$\lim_{t \searrow s} \int_{\mathbb{R}^N} (G(x-y,t-s) + J(x,t;y,s)) f(y)\, dy = f(x), \tag{3.2}$$

for continuous $f \in L_1(\mathbb{R}^N)$. The contribution from J may be shown to be 0 by following the proof in [4, p. 20]. On the other hand, for fixed $T > 0$, it is readily shown that

$$\begin{aligned} &\left| G(x,t;y,s) - \frac{\exp(-|x-y|^2/(4(t-s)))}{(4\pi(t-s))^{N/2}} \right| \\ &\quad \leq \frac{C}{(t-s)^{(N-1)/2}} \exp\left(\frac{-c|x-y|^2}{(t-s)}\right) \\ &\quad \leq \frac{C}{(t-s)^{1/2}|x-y|^{(N-2)}} \exp\left(\frac{-c|x-y|^2}{(t-s)}\right), \end{aligned}$$

for all $(t-s) \in (0, T]$. Hence,

$$\lim_{t \searrow s} \int_{\mathbb{R}^N} \left(G(x-y,\, t-s) - \frac{\exp(-|x-y|^2/(4(t-s)))}{(4\pi(t-s))^{N/2}} \right) f(y)\, dy = 0,$$

and so (3.2) follows from the well–known properties of the Gaussian kernel.

We deduce that $G(x-y;\, t-s) + J(x,\, t;\, y,\, s)$ satisfies the characterising equations (1.1) and (1.3), and is therefore equal to $\Gamma(x,\, t;\, y,\, s)$. □

Proof of Theorem 2.1: We first consider (3.1) under assumption (A). It is well known that $\Gamma(x,\, t;\, \xi,\, \sigma) > 0$. On the other hand, for $M \in [0,\, \infty)^N$,

$$LG_M(x-y,\, t-s) = \sum_{i=1}^{N} (a_i(x,\, t) - M_i \mathrm{sign}[x_i - y_i]) \frac{\partial G_M}{\partial x_i} \geq 0,$$

by (1.2) and (1.6). So, (3.1) implies that $\Gamma(x,\, t;\, y,\, s) \leq G_M(x-y,\, t-s)$. The lower bound follows similarly.

Under conditions which include (1.1) under assumption (B), Aronson [1] defines the weak fundamental solution $\Gamma(x,\, t;\, y,\, s)$, which coincides with the classical heat kernel when coefficients are sufficiently smooth. He considers [1, p. 681] a sequence of classical heat kernels $\Gamma^k(x,\, t;\, y,\, s)$, $k \in \mathbb{N}$, of the problem

$$u_t - \Delta u + a^k(x,\, t).\nabla u = 0, \tag{3.3}$$

for $|x| < k$, $t > 0$, with Dirichlet boundary conditions

$$u(x,\, t) = 0, \qquad |x| = k,\ t > 0,$$

and shows that $\Gamma^k(x,\, t;\, y,\, s) \to \Gamma(x,\, t;\, y,\, s)$ pointwise, as $k \to \infty$. Here, $a^k(x,\, t)$ is an integral average of $a(x,\, t)$ formed with a smooth kernel whose support lies in $|x|^2 + t^2 < k^{-2}$. $a^k(x,\, t)$ therefore satisfies assumption (A).

Let $H^k(x,\, t;\, y,\, s)$ be the heat kernel of (3.3) in the extended domain $\mathbb{R}^N \times (0,\, \infty)$. Then $\Gamma^k(x,\, t;\, y,\, s) = H^k(x,\, t;\, y,\, s) + v^k(x,\, t;\, y,\, s)$, for $|x|,\, |y| \leq k$, where $v^k(x,\, t;\, y,\, s)$ is the interior solution of (3.3) such that

$$v^k(x,\, t;\, y,\, s) = -H^k(x,\, t;\, y,\, s), \qquad |x| = k, \quad t > 0.$$

Since H^k satisfies the upper and lower bounds (2.5), it is clear from the parabolic maximum principle [4, Chapter 2], that $v^k(x,\, t;\, y,\, s) \to 0$ pointwise, as $k \to \infty$, for fixed $(x,\, t;\, y,\, s)$. Hence, $\Gamma(x,\, t;\, y,\, s)$ satisfies (2.5).

The last part of the theorem is completed by following the argument given in the Introduction. □

4 L^1 bounds

Aronson [1, Section 9] shows, under conditions which include the case of (1.1) with $a(x, t)$ satisfying assumption (B), that if $u_0 \in L_1(\mathbb{R}^N)$ then

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) u_0(y)\, dy \tag{4.1}$$

is the unique weak solution of the equation in $L_2[[\delta, T]; L_2(\mathbb{R}^N)]$ for any $T > \delta > 0$, and that $u(x, t)$ also satisfies the continuity condition

$$\lim_{t \to 0} \|u(t) - u_0\|_1 = 0.$$

Here, we use the representation formula (4.1) and the pointwise bounds (2.5) on $\Gamma(x, t; y, 0)$ to obtain upper and lower bounds on $\|u(t)\|_1 / \|u_0\|_1$.

Since u_0, and hence $u(t)$, may be split as the sum of non–negative and non–positive parts, it is sufficient to consider non–negative u_0 for the purposes of obtaining L_1 bounds. In this case, Fubini's theorem implies that

$$\begin{aligned} \|u(t)\|_1 &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) u_0(y)\, dy\, dx \\ &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \Gamma(x, t; y, 0)\, dx \right) u_0(y)\, dy, \qquad t > 0. \end{aligned}$$

From (2.5) we deduce that, provided u_0 is not null,

$$\inf_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} G_{-M}(x - y, t)\, dx \le \frac{\|u(t)\|_1}{\|u_0\|_1} \le \sup_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} G_M(x - y, t)\, dx.$$

That there is in fact no y dependence involved is apparent on a change of variable in the integrals, and one obtains

$$\int_{\mathbb{R}^N} G_{-M}(x, t)\, dx \le \frac{\|u(t)\|_1}{\|u_0\|_1} \le \int_{\mathbb{R}^N} G_M(x, t)\, dx. \tag{4.2}$$

Remark: Suppose that we take a sequence of smooth non–negative initial data $u_0^k(x)$, $k \in \mathbb{N}$, for (1.8), such that $\|u_0^k\|_1 = 1$ and the support of u_0^k is lies in the set $|x - z| < k^{-1}$. Let the corresponding solution be denoted $u^k(x, t)$. Fixing $t > 0$ and noting that the heat kernel $\Theta_M(z; x, t; y, 0)$ of (1.8) is equal to $G_M(x - z; t)$ when $y = z$, we deduce that

$$\lim_{k \to \infty} \frac{\|u^k(t)\|_1}{\|u_0^k\|_1} = \int_{\mathbb{R}^N} G_M(x, t)\, dx.$$

If one exchanges M_i for $-M_i$, a similar argument shows that the lower bound in (4.2) is also sharp.

The construction (2.4) of G_M implies that

$$\int_{\mathbb{R}^N} G_{\pm M}(x,\, t)\, dx = \prod_{i=1}^{N} \int_{\mathbb{R}} g_{\pm M_i}(x_i,\, t)\, dx_i. \tag{4.3}$$

A calculation using the identity [5, Formula 7.2.5 p. 299],

$$\int_x^\infty \operatorname{erfc}(t)\, dt = -x \operatorname{erfc}(x) + \frac{1}{\sqrt{\pi}} \exp(-x^2),$$

yields

$$\int_{\mathbb{R}} g_M(x,\, t)\, dx = h\left(\frac{M\sqrt{t}}{2}\right), \qquad M \in \mathbb{R},\ t \geq 0, \tag{4.4}$$

where we define

$$h(x) = \left(1 + 2x^2\right) \operatorname{erfc}(-x) + \frac{2x}{\sqrt{\pi}} \exp\left(-x^2\right). \tag{4.5}$$

The above analysis is summarised in the following theorem:

Theorem 4.1 *Suppose that L obeys assumption (B) and that non-null initial data $u_0 \in L_1(\mathbb{R})$ is given. Then the corresponding solution $u(x,\, t)$ satisfies*

$$\prod_{i=1}^{N} h\left(\frac{-M_i\sqrt{t}}{2}\right) \leq \frac{\|u(t)\|_1}{\|u_0\|_1} \leq \prod_{i=1}^{N} h\left(\frac{M_i\sqrt{t}}{2}\right). \tag{4.6}$$

For small x, the behaviour of $h(x)$ is seen from W. Gautschi [5, Formula 7.1.5 p. 297], which implies that for $x \in \mathbb{R}$,

$$\begin{aligned} h(x) &= (1+2x^2) + \frac{2\exp(-x^2)}{\sqrt{\pi}} \left(x + (1+2x^2) \sum_{n=0}^{\infty} \frac{2^n}{1.3\ldots(2n+1)} x^{2n+1} \right) \\ &= 1 + \frac{4x}{\sqrt{\pi}} + 2x^2 + \frac{4x^3}{3\sqrt{\pi}} + \ldots \end{aligned}$$

When $M_i\sqrt{t} \ll 1$ for each i, the upper bound in (4.6) grows like

$$1 + \frac{2\sum_{i=1}^{N} M_i}{\sqrt{\pi}} t^{1/2}, \tag{4.7}$$

whilst the decay of the lower bound is found by changing the sign of the M_i's.

Using an asymptotic expansion for $\mathrm{erfc}(x)$ in negative powers of $|x|$, [5, Formula 7.1.23 p. 298], for $x < 0$ one obtains

$$\begin{aligned} h(x) &= -\frac{2|x|}{\sqrt{\pi}}\exp(-x^2) + (1+2x^2)\mathrm{erfc}(|x|) \\ &\sim \frac{\exp(-x^2)}{\sqrt{\pi}|x|}\left(-2x^2 + (1+2x^2)\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3 \ldots (2n-1)}{(2x^2)^n}\right)\right). \end{aligned}$$

(If the series is truncated, the error, $h(x)-$R.H.S., is equal to $\theta(x)$ times the first neglected term, where $\theta(x) \in (0,\, 1)$.) Hence, when $M_i\sqrt{t} \gg 1$ for each i, the lower bound in (4.6) decays like

$$\left(\frac{8}{\pi^{1/2}\overline{M}^3}\right)^N t^{-3N/2} \exp\left(-\sum_{i=1}^{N} \frac{M_i^2 t}{4}\right), \qquad \text{for} \quad \overline{M} \equiv \left(\prod_{i=1}^{N} M_i\right)^{1/N}. \tag{4.8}$$

Since $\mathrm{erfc}(x)+\ \mathrm{erfc}(-x) = 2$, (4.5) implies that $h(x) = 2 + 4x^2 - h(-x)$. Thus, when $M_i\sqrt{t} \gg 1$ for each i, the upper bound in (4.6) may be approximated by

$$\prod_{i=1}^{N}(2 + M_i^2 t) \tag{4.9}$$

with very little error. This last result may be appreciated from the fact that

$$G_M(x,\, t) \approx \prod_{i=1}^{N} \frac{M_i}{2}, \qquad \text{for } -M_i t < x_i < M_i t, \quad i = 1,\, 2,\, \ldots,\, N, \tag{4.10}$$

and rapidly vanishes outside this box.

Remark: Considering non–null end–point data $u(t) = u_0 \in L_\infty(\mathbb{R}^N)$ for the adjoint problem (2.6), a duality argument leads to the bounds

$$\prod_{i=1}^{N} h\left(\frac{-M_i\sqrt{t-s}}{2}\right) \le \frac{\|u(s)\|_\infty}{\|u_0\|_\infty} \le \prod_{i=1}^{N} h\left(\frac{M_i\sqrt{t-s}}{2}\right), \qquad s \in [0,\, t]. \tag{4.11}$$

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