An Invariant Moving Mesh Scheme for the Nonlinear Diffusion Equation

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Abstract

We consider the Cauchy problem for the nonlinear diffusion equation, $u_t = (u^m u_x)_x$ which is posed on an infinite domain. The PDE and a conservation law are invariant to a Lie group of stretchings which is used to construct the invariant quantities, $xt^{-1/(2+m)}$ and $ut^{1/(2+m)}$. Using these invariants as similarity variables the problem is reduced to a second order ODE and then integrated to give the well known similarity solutions.

The problem is semi-discretised using the method of lines. The mesh movement is governed by the conservation of mass law, so that the computational domain expands as the solution diffuses. The resulting semi-discretisation is a system of ODE's which is invariant to the same Lie group as the PDE and so the mesh $X_i(t)$ behaves like the discretised invariant $Y_i t^{1/(2+m)}$ and the solution $U_i(t)$ like $V_i t^{-1/(2+m)}$. Furthermore it is shown that, using these invariants, the same reduction and integration is possible in the semi-discrete case as in the continuous case.

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1 Introduction

The nonlinear diffusion equation is an example of a partial differential equation (PDE) that is invariant to a Lie group of stretching transformations. We can use these Lie groups to rescale the PDE's and reduce them to ordinary differential equations (ODE's), which can in turn, be solved exactly to give explicit self-similar solutions - see Barenblatt [1], Dresner [10], Bluman and Cole [2], Bluman and Kumei [3] and Olver [18]. This type of solution often describes the long time behaviour of the PDE with general initial data.

When solving such Cauchy problems numerically we can use the information given by the Lie group to discretise the problem and the mesh it is solved on. The aim of this approach is to derive discrete solutions which have the same asymptotic behaviour as solutions of the continuous problem and this is done by ensuring that the scheme has the same conserved quantities and the same invariants as the continuous problem.

In this paper we present such a scheme to solve the nonlinear diffusion equation. The scheme is shown to be both convergent and effective in reproducing the dynamics of the PDE. Its performance is significantly better than that of fixed or non-invariant moving mesh schemes.

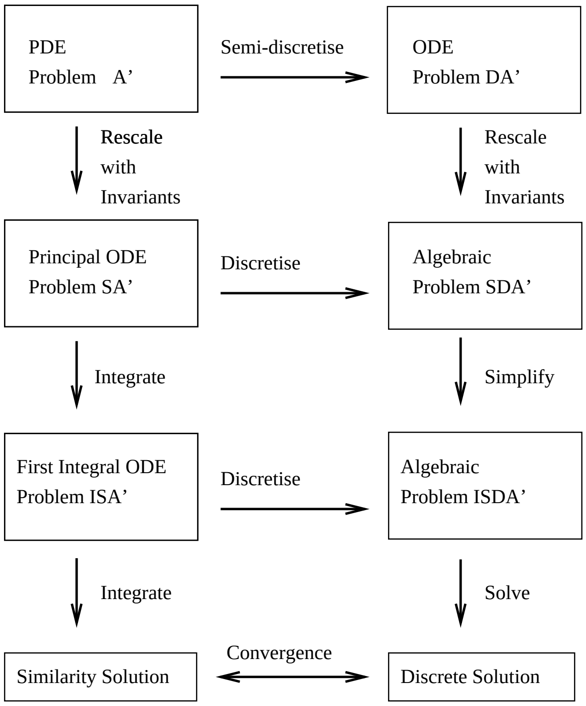

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Figure 1: Commuting relation.

In particular, rescaling and integrating the PDE problem has an analogue in the discrete case as suggested in figure 1. The left hand side describes the standard approach of reducing the PDE to a similarity solution, which we go through in section 2. In section 3 we derive an adaptive method for the PDE based on the method of lines procedure described in [13] and [14]. Such a procedure admits a similarity reduction for the discrete solution as outlined by the right hand side and described in sections 4 and 5. An overview of numerical and analytic convergence results is given in section 6.

The method of invariant moving meshes has been applied with success to blow up problems in Budd, Huang and Russell [5] and to diffusion problems with similarity solutions that have a compactly supported moving boundary in Budd [6]. Fronteau and Combis in [12] study invariant moving mesh methods for Fokker-Planck equations with conserved probability. Extensive work on invariant meshes for finite difference equations has been done by Dorodnitsyn in [7], [8] and [9].

2 The Cauchy Problem for Nonlinear Diffusion

2.1 General Theory

The problem considered is the Cauchy problem for nonlinear diffusion,

Problem A

$$u_t = (u^m u_x)_x \tag{1}$$

$$u(0,x) = u_0(x) \tag{2}$$

$$u^m u_x \to 0 \text{ as } x \to \pm\infty. \tag{3}$$

Where $-2 < m \in \Re$ and $u_0(x) \in L_1(\Re)$. For simplicity it is assumed that the centre of mass of $u_0(x)$, $x_0 \equiv \int_{-\infty}^{\infty} xudx$ is at 0. This restriction implies no loss of generality because problem A is invariant to translations in the x direction.

The nonlinear diffusion equation has been used to model many diffusive processes. The linear case $m = 0$ models a variety of heat transfer processes in isotropic media. The case $m > 0$ is called the porous medium equation, see Vazquez [20]. Diffusion in Silicon has been modelled by the equation with $-1 < m < 0$, see King [16], [17].

It is shown that the first integral of the solution, herein referred to as mass, is conserved for all time,

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} udx = \int_{-\infty}^{\infty} (u^m u_x)_x dx = [u^m u_x]_{-\infty}^{\infty} = 0.$$

The PDE (1) and this conservation law are invariant to a Lie group of stretchings,

$$u^* = \lambda^{-1/(2+m)}u \quad x^* = \lambda^{1/(2+m)}x \quad t^* = \lambda t. \tag{4}$$

Using this group, and the fact that the problem is invariant to translations in t, give three invariant quantities which can be considered as the natural scalings of the problem,

$$y = x(t+t_0)^{-1/(2+m)} \tag{5}$$

$$w = u(t+t_0)^{1/(2+m)} \tag{6}$$

$$s = ux. \tag{7}$$

2.2 Nonlinear Diffusion on an Expanding Interval

Problem A, posed on an infinite domain, cannot be solved by a finite numerical scheme. To overcome this we introduce a computational region $x \in [x_l(t), x_r(t)]$ which adapts the domain to the computed solution as it diffuses. The limits $x_l(t)$ and $x_r(t)$ are chosen so that the mass of the solution in the interval is constant, giving,

$$0 \equiv \quad \frac{\partial}{\partial t}\int_{x_l(t)}^{x_r(t)} u(x,t)dx = \int_{x_l(t)}^{x_r(t)} u_t(x,t)dx + u(x_r,t)\frac{dx_r}{dt} - u(x_l,t)\frac{dx_l}{dt},$$

$$\text{thus} \quad u(x_r,t)\frac{dx_r}{dt} - u(x_l,t)\frac{dx_l}{dt} = -\int_{x_l(t)}^{x_r(t)} (u^m u_x)_x dx = -u_r^m u_x(x_r,t) + u_l^m u_x(x_l,t).$$

This is satisfied if,

$$u_r\frac{dx_r}{dt} = -u_r^m u_x(x_r,t) \text{ and } u_l\frac{dx_l}{dt} = -u_l^m u_x(x_l,t). \tag{8}$$

It can be shown that two extra conditions are needed to ensure that the truncated problem has the correct degree of determinacy. To obtain these, we set the invariant ux (7) to be equal to a constant at the ends of the domain. This condition is important later when the conservation of mass in the discretised scheme is considered.

Problem A is truncated to the related but numerically tractable problem A′,

Problem A′

$$u_t = (u^m u_x)_x \quad \text{for } x_l < x < x_r \tag{9}$$

$$\int_{x_r(t)}^{x_l(t)} udx = \int_{x_r(0)}^{x_l(0)} u_0(x)dx = a \tag{10}$$

$$\Longrightarrow \begin{cases} u_r\frac{dx_r}{dt} = -u_r^m u_x \\ u_l\frac{dx_l}{dt} = -u_l^m u_x \end{cases} \tag{11}$$

$$x_r(t)u_r(t) = x_r(0)u_r(0) = r \tag{12}$$

$$x_l(t)u_l(t) = x_l(0)u_l(0) = l \tag{13}$$

$$u(0,x) = u_0(x) \text{ for } x \in [x_l(0), x_r(0)] = \Omega \subset (-\infty,\infty). \tag{14}$$

$$u_0 \in L_1(\Omega), \int_{-\infty}^{\infty} xu_0(x)dx = 0.$$

2.3 Derivation of the Self-Similar Solution

The group analytic techniques described in Dresner [10] and Barrenblatt [1] are used to reduce the problem A′ to its principal ODE, which, when integrated twice, gives self-similar solutions.

Rescaling Problem A$'$ in terms of the invariants (5) and (6) gives the principal ODE problem,

Problem SA$'$

$$\frac{-1}{2+m}\left(\frac{dw}{dy}y+w\right)=\frac{d}{dy}\left(w^m\frac{dw}{dy}\right) \text{ on } y\in(y_l,y_r) \tag{15}$$

$$\int_{y_l}^{y_r} wdy = a \tag{16}$$

$$\frac{1}{m+2}wy=-w^m\frac{dw}{dy} \text{ on } y=y_l\ ,\ y_r \tag{17}$$

$$y_r w_r = r \tag{18}$$

$$y_l w_l = l. \tag{19}$$

The ODE (15) can be integrated to,

$$\frac{-1}{2+m}wy=w^m\frac{dw}{dy}+K,$$

and condition (17) implies that the constant K is 0. This gives the first integral of Problem SA$'$,

Problem ISA$'$

$$\frac{-1}{2+m}wy=w^m\frac{dw}{dy} \tag{20}$$

$$\int_{y_l}^{y_r} wdy = a \tag{21}$$

$$y_r w_r = r \tag{22}$$

$$y_l w_l = l. \tag{23}$$

Integrating 20 gives the self-similar solutions,

$$w(y)=\begin{cases}(\gamma(a)-\frac{m}{2(2+m)}y^2)^{1/m} \text{ for } m\neq 0\\ \gamma(a)\exp(-\frac{1}{4}y^2) \text{ for m =0.}\end{cases} \tag{24}$$

Where $\gamma(a)$ is determined by the mass of the initial distribution so that $\int_{y_l}^{y_r} wdy=\int_{x_l}^{x_r} u_0dx=a$. The similarity solution for the case $m>0$ is 0 outside the moving support $x=Yt^{1/(m+2)}$. For $-2<m\leq 0$ the solution is positive everywhere.

It has been shown that these solutions are attractors of the problem for suitable initial conditions. The linear case is well documented, see for example Friedman [11]. For the porous medium equation case $m>0$, the convergence proof is in Kamenomostskaya [15]. The case $-2<m<0$ is dealt with in King [16]. Using these results we establish the following result in [4].

Theorem 2.1 *For $u(x,0)\in L_1$ and $a>2r$ there exists a unique solution $u(x,t)$ to Problem A$'$, such that as $t\to\infty$,*

$$\begin{aligned} u(x,t) &= (t+t_0)^{-1/(2+m)}(\gamma-\frac{m}{2(2+m)}x^2(t+t_0)^{-2/(2+m)})^{1/m} && (25)\\ \textit{or } u(x,t) &= (t+t_0)^{-1/2}\gamma\exp(-\frac{1}{4}x^2(t+t_0)^{-1}) \textit{ for } m=0 && (26)\\ x_r &= (t+t_0)^{1/(2+m)}Y_r \\ x_l &= (t+t_0)^{1/(2+m)}Y_l \end{aligned}$$

where γ, Y_r and Y_l are uniquely determined by the initial conditions (10), (12) and (13). $\square$

3 A Moving Mesh Scheme

3.1 Discretisation away from the Boundary

This section describes how the solution $u(x,t)$ to Problem A′ is approximated on n moving nodes $X_i(t)$ as $u(X_i(t),t) \approx U_i(t)$ for $i = 1, n$. The approximate solution is computed on a finite number of nodes and so holds on a finite domain, $x \in [X_1(t), X_n(t)]$ which expands as the solution diffuses. Following the work of Huang, Ren and Russell in [13] and [14] we see that the Lagrangian derivative is,

$$\frac{dU_i}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dX_i}{dt}.$$

Approximating u_x with central differences gives,

$$\dot{U}_i(t) - \frac{U_{i+1}(t) - U_{i-1}(t)}{X_{i+1}(t) - X_{i-1}(t)}\dot{X}_i(t) \approx u_t.$$

The PDE (1) is then discretised in space using central differences,

$$\dot{U}_i - \frac{U_{i+1} - U_{i-1}}{X_{i+1} - X_{i-1}}\dot{X}_i = \frac{\frac{(U_{i+1}^m + U_i^m)(U_{i+1} - U_i)}{X_{i+1} - X_i} - \frac{(U_i^m + U_{i-1}^m)(U_i - U_{i-1})}{X_i - X_{i-1}}}{X_{i+1} - X_{i-1}} \quad \text{for } i = 2 \ldots n-1 \qquad (27)$$

Where the explicit dependence of X_i and U_i on t is dropped for notational simplicity.

The discretisation will not hold on the end nodes, U_1 and U_n because there is no node to the left or right respectively with which to form a central difference approximation. As suggested in the analysis of the continuous problem, two extra conditions on the end nodes are needed and these conditions must be invariant to the given group. These conditions become clear once the conservation properties of the mesh equations are considered.

3.2 Discrete Conservation of Mass

The conservation of mass law has been used to form two conditions on the end nodes in Problem A′ (8). To obtain the correct semi-discretisation of these two conditions and to derive the end node conditions for U_1 and U_n it is prescribed that a discrete approximation of the mass given by the trapezium rule, is kept constant so,

$$\sum_{i=2}^{n-1} U_i(X_{i+1} - X_{i-1}) + U_1(X_2 - X_1) + U_n(X_n - X_{n-1}) = \text{ const.}$$

Differentiating gives,

$$\sum_{i=2}^{n-1} \dot{U}_i(X_{i+1} - X_{i-1}) + \sum_{i=2}^{n-1} U_i(\dot{X}_{i+1} - \dot{X}_{i-1})$$
$$+\dot{U}_1(X_2 - X_1) + \dot{U}_n(X_n - X_{n-1}) + U_1(\dot{X}_2 - \dot{X}_1) + U_n(\dot{X}_n - \dot{X}_{n-1}) = 0.$$

With the discretisation (27) this becomes,

$$\sum_{i=2}^{n-1}[\frac{(U_{i+1}^m+U_i^m)(U_{i+1}-U_i)}{X_{i+1}-X_i}-\frac{(U_i^m+U_{i-1}^m)(U_i-U_{i-1})}{X_i-X_{i-1}}]+\sum_{i=2}^{n-1}(U_{i+1}-U_{i-1})\dot{X}_i$$
$$+\sum_{i=2}^{n-1}U_i(\dot{X}_{i+1}-\dot{X}_{i-1})+\dot{U}_1(X_2-X_1)+\dot{U}_n(X_n-X_{n-1})+U_1(\dot{X}_2-\dot{X}_1)+U_n(\dot{X}_n-\dot{X}_{n-1})=0.$$

Rearranging the summation indices gives,

$$\sum_{i=2}^{n-1}\frac{(U_{i+1}^m+U_i^m)(U_{i+1}-U_i)}{X_{i+1}-X_i}-\sum_{i=1}^{n-2}\frac{(U_{i+1}^m+U_i^m)(U_{i+1}-U_i)}{X_{i+1}-X_i}+\sum_{i=3}^{n}U_i\dot{X}_{i-1}-\sum_{i=1}^{n-2}U_i\dot{X}_{i+1}$$
$$+\sum_{i=2}^{n-1}U_i(\dot{X}_{i+1}-\dot{X}_{i-1})+\dot{U}_1(X_2-X_1)+\dot{U}_n(X_n-X_{n-1})+U_1(\dot{X}_2-\dot{X}_1)+U_n(\dot{X}_n-\dot{X}_{n-1})=0.$$

which cancels to,

$$\frac{(U_n^m+U_{n-1}^m)(U_n-U_{n-1})}{X_n-X_{n-1}}-\frac{(U_2^m+U_1^m)(U_2-U_1)}{X_2-X_1}-\sum_{i=3}^{n-2}U_i(\dot{X}_{i+1}-\dot{X}_{i-1})+\sum_{i=2}^{n-1}U_i(\dot{X}_{i+1}-\dot{X}_{i-1})$$
$$+U_{n-1}\dot{X}_{n-2}-U_2\dot{X}_3+\dot{U}_1X_2-\dot{U}_nX_{n-1}-\dot{U}_1X_1-\dot{X}_1U_1+\dot{U}_nX_n+\dot{X}_nU_n=0.$$

and finally,

$$\frac{(U_n^m+U_{n-1}^m)(U_n-U_{n-1})}{X_n-X_{n-1}}-\frac{(U_2^m+U_1^m)(U_2-U_1)}{X_2-X_1}$$
$$+\dot{X}_nU_{n-1}-\dot{X}_iU_2+\dot{U}_1X_2-\dot{U}_nX_{n-1}-\dot{U}_1X_1-\dot{X}_1U_1+\dot{U}_nX_n+\dot{X}_nU_n=0.$$

This is satisfied exactly if the solution is symmetric, $U_n=U_1$ and $X_n=-X_1$. In general though, the discrete conservation law is conserved exactly if the conditions on the first and last nodes are,

$$\dot{X}_1U_2=-\frac{(U_2^m+U_1^m)(U_2-U_1)}{X_2-X_1}+\dot{U}_1X_2 \quad (28)$$
$$\dot{X}_nU_{n-1}=-\frac{(U_n^m+U_{n-1}^m)(U_n-U_{n-1})}{X_n-X_{n-1}}+\dot{U}_nX_{n-1}. \quad (29)$$

and the conditions on U_1 and U_n are,

$$\dot{U}_1X_1+\dot{X}_1U_1=0 \quad (30)$$
$$\dot{U}_nX_n+\dot{X}_nU_n=0. \quad (31)$$

Conditions (30) and (31) are a discrete form of the relations (12) and (13) once they have been differentiated. Substituting for $\dot{U}_1$ and $\dot{U}_n$ in (28) and (29) gives a consistent discretisation of relations (8). Thus setting the invariant ux to be constant leads to conservation of mass in the discrete scheme.

The self-similar solution (24) is a smooth function of x which gets smoother as it evolves. Therefore there is no need for complicated equidistribution strategies, which resolve sharp spatial gradients. For theoretical ease we may prescribe the movement of the internal mesh nodes $\{X_i, i = 2 \dots n-1\}$ to be uniform in space in the interval, $[X_1(t), X_n(t)]$. So that,

$$X_i = X_1 + \frac{i-1}{n-1}(X_n - X_1). \tag{32}$$

The numerical scheme described above is comprised of $2n$ ODE's for n unknown mesh variables $X_i(t)$ and n unknown solution variables U_i.

The PDE (1) can be written in a general form as,

$$\begin{aligned} u_t &= \frac{1}{\beta}(u^{m+1-\beta}(u^\beta)_x)_x \qquad\qquad \beta \in \Re \neq 0 \\ \text{or } u_t &= (u^{m+1}(\log u)_x)_x. \end{aligned}$$

There is, then, a family of discretisations parameterised by β, which we give below,

Problem DA$'$

$$\begin{aligned}
\dot{U}_1 &= -\frac{U_1 \dot{X}_1}{X_1} \\
\dot{U}_i &= \frac{1}{\beta}\frac{\frac{(U_{i+1}^{m+1-\beta}+U_i^{m+1-\beta})(U_{i+1}^\beta-U_i^\beta)}{X_{i+1}-X_i} - \frac{(U_i^{m+1-\beta}+U_{i-1}^{m+1-\beta})(U_i^\beta-U_{i-1}^\beta)}{X_i-X_{i-1}}}{(X_{i+1}-X_{i-1})} + \frac{U_{i+1}-U_{i-1}}{X_{i+1}-X_{i-1}}\dot{X}_i \\
\text{or} &= \frac{\frac{(U_{i+1}^{m+1}+U_i^{m+1})(\log U_{i+1}-\log U_i)}{X_{i+1}-X_i} - \frac{(U_i^{m+1}+U_{i-1}^{m+1})(\log U_i-\log U_{i-1})}{X_i-X_{i-1}}}{(X_{i+1}-X_{i-1})} + \frac{U_{i+1}-U_{i-1}}{X_{i+1}-X_{i-1}}\dot{X}_i \\
\text{for} \quad & i = 2 \dots n-1 \\
\dot{U}_n &= -\frac{U_n \dot{X}_n}{X_n} \\
\dot{X}_1 U_2 &= -\frac{1}{\beta}\left[\frac{(U_2^{m+1-\beta}+U_1^{m+1-\beta})(U_2^\beta-U_1^\beta)}{X_2-X_1}\right] + \dot{U}_1 X_2 \\
\dot{X}_i &= \dot{X}_1 + \frac{i-1}{n-1}(\dot{X}_n - \dot{X}_1) \text{ for } i = 2 \dots n-1 \\
\dot{X}_n U_{n-1} &= -\frac{1}{\beta}\left[\frac{(U_n^{m+1-\beta}+U_{n-1}^{m+1-\beta})(U_n^\beta-U_{n-1}^\beta)}{X_n-X_{n-1}}\right] + \dot{U}_n X_{n-1}
\end{aligned}$$

with initial conditions,

$$U_i(0) = U_i^0 \text{ is continuous and decreasing as } x \to \pm\infty \tag{33}$$

$$X_i(0) = X_i^0 \text{ for } i = 2 \dots n-1. \tag{34}$$

Where U_i^0 is given by the initial data so that its centre of mass $X_0 = \sum_{i=1}^{n-1}(X_{i+1}U_{i+1}^0 + X_i U_i^0)(X_{i+1} - X_i) = 0$. The starting mesh is generally uniform. It will be shown later in this section that it is possible to minimise truncation error using the parameter β but for now the case $\beta = m+1$ is used for theoretical simplicity.

A direct calculation shows that these ODE's are invariant to transformations by the group (4) so that problem DA$'$ can be rescaled with the discrete form of the invariants (5) and (6),

$$Y_i = X_i t^{-1/m+2}, \tag{35}$$

$$V(Y_i) \equiv V_i = U_i t^{1/m+2}, \tag{36}$$

where V_i and Y_i are not dependent on time, to give a time independent system of algebraic equations.

These discrete invariant relations directly give two remarkable results for Problem DA$'$. Firstly, using these invariants in a similar way to the continuous invariants (5) and (6), we can perform a similarity reduction of problem DA$'$ and this is done in section 5. Secondly, it is shown in section 6, that the scaled error in the scheme is independent of time.

4 Results

The system of ODE's is solved numerically using the stiff ODE solver DDASSL [19]. We consider three different types of solution that depend on the initial area a and the value of the r and l in the invariants (12) and (13).

Firstly we consider initial data that breaks the assumption in Theorem 2.1 that $a > 2r$, for example the initial function $U_i(0) = 1/2\cos^2(X_i(0)/2) + 1$ on a uniform initial mesh $X_i \in [-5, 5]$. Here the initial area a is less than the area $2r$ of the rectangle with sides U_r and $X_r - X_l$. The resulting solution 'blows up' in a finite time for any m. The mesh clusters in the centre of the domain and the problem DA$'$ becomes singular. Hence inappropriate initial data leads to a spurious solution.

Problem DA$'$ has a fixed point at U_i =constant, X_i = constant. Thus for U_i^0 = constant the solution and mesh for DA$'$ remains constant. This correspond to the initial area under the graph a being equal to the area of the rectangle with sides U_r and $X_r - X_l$ which is $2r$. This solution does not occur for the Cauchy problem A as the initial condition $\notin L_1(\Re)$. However the truncated problem A$'$ admits a steady state solution $u(x,t)$ = constant which trivially conserves mass and $\in L_1[x_l(t), x_r(t)]$.

5 Similarity Reductions in the Discrete System

Theorem 2.1 holds for the case $a > 2r$ and this is the regime that is considered from now on. In figures 2 and 3 we show results for the fast diffusion case $m = -4/3$ and the slow diffusion case $m = 1$, in both the original variables and the scaled variables (35) and (36). From these pictures we see that, following a short initial transient behaviour, both the solution and the mesh evolve such that the scaled variables are asymptotically constant in time. This scheme thus reproduces the asymptotic behaviour of the continuous problem. We now examine this in more detail.

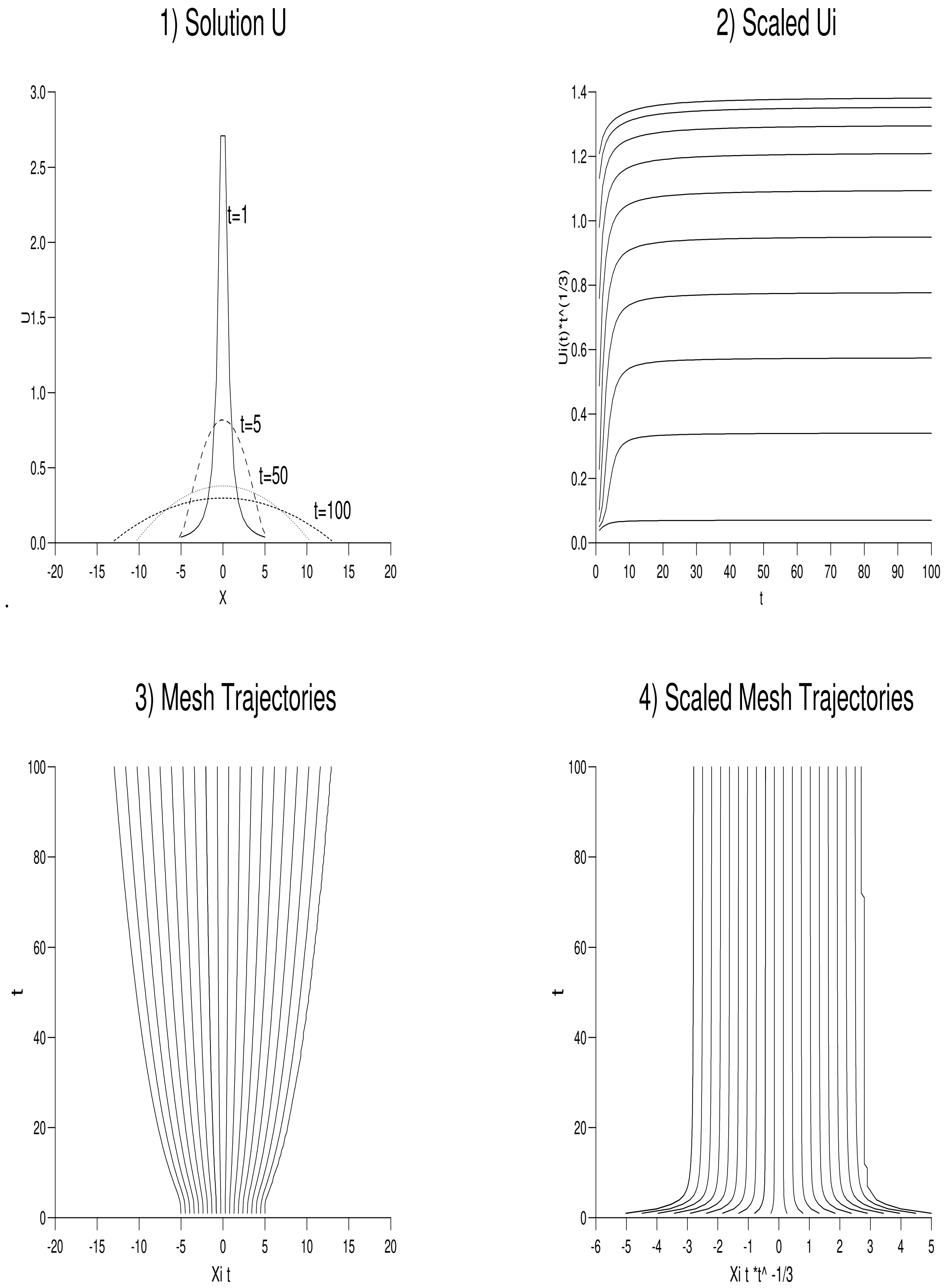

Figure 2: Slow Diffusion $u_t = (uu_x)_x$. Initial condition is $u(x,0) = (0.3 + x^2)^{-1}$ on $x_r(0) = -x_l(0) = 5$. Solved on a mesh with $n = 20$ nodes.

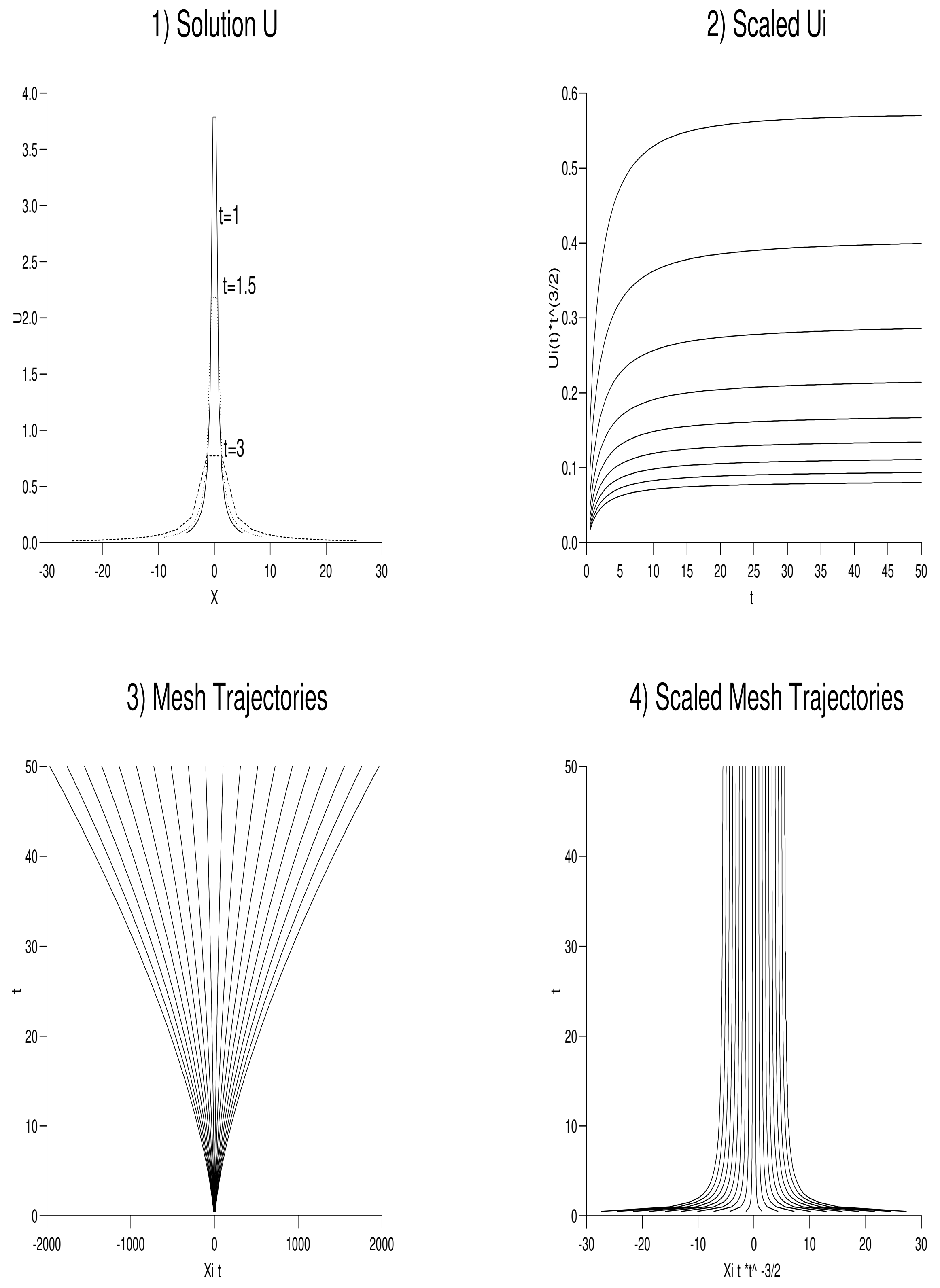

Figure 3: Fast Diffusion $u_t = (u^{-4/3}u_x)_x$. Initial condition is $u(x, 0) = (0.1 + x^2)^{-3/4}$ on $x_r(0) = -x_l(0) = 5$. Solved on a mesh with $n = 20$.

Substituting for V_i and Y_i in problem DA$'$ gives the following,

Problem SDA$'$

$$
\begin{aligned}
V_i + \frac{V_{i+1}-V_{i-1}}{Y_{i+1}-Y_{i-1}} Y_i &= -\frac{2(m+2)}{m+1} \frac{\frac{V_{i+1}^{m+1}-V_i^{m+1}}{Y_{i+1}-Y_i} - \frac{V_i^{m+1}-V_{i-1}^{m+1}}{Y_i-Y_{i-1}}}{(Y_{i+1}-Y_{i-1})} \\
&\text{for} \quad i = 2 \ldots n-1 &(37)\\
Y_1 V_2 + V_1 Y_2 &= -\frac{2(m+2)}{m+1}\left[\frac{V_2^{m+1}-V_1^{m+1}}{Y_2-Y_1}\right] &(38)\\
Y_i &= Y_1 + \frac{i-1}{n-1}(Y_n - Y_1) &(39)\\
&\text{for} \quad i = 2 \ldots n-1,\ j = 1 \ldots n \\
Y_n V_{n-1} + V_n Y_{n-1} &= -\frac{2(m+2)}{m+1}\left[\frac{V_n^{m+1}-V_{n-1}^{m+1}}{Y_n-Y_{n-1}}\right], &(40)
\end{aligned}
$$

Now we append the three conditions given by the initial data in Problem A$'$ to V_i and Y_i. That is the conditions (10), (12) and (13) which are set to be invariants of the discretisation.

$$
\begin{aligned}
\sum_{i=1}^{n-1}(V_{i+1}+V_i)(Y_{i+1}-Y_i) &= 2a &(41)\\
V_1 Y_1 &= l &(42)\\
V_n Y_n &= r. &(43)
\end{aligned}
$$

The algebraic relation (37) is a discretisation of the principal ODE (15) on the mesh given by (38), (39) and (40). The following proposition is shown in [4],

Proposition 5.1 *Problem SDA$'$ is a discretisation of the continuous principal ODE Problem SA$'$ with truncation error $O(\frac{1}{n^2})$.*

The $2n-2$ equations of problem SDA$'$ with the three conditions (41), (42) and (43) appear to give an overdetermined system. However cancellations are performed on problem SDA$'$ to give a system that is reduced by one equation. This gives the following Theorem,

Theorem 5.1 *The Problem SDA$'$ is equivalent to $2n-3$ equations. These equations are a discretisation of the first integral Problem ISA$'$ with truncation error $O(\frac{1}{n^2})$ and give,*

Problem ISDA$'$

$$
\begin{aligned}
V_i Y_{i+1} + V_{i+1} Y_i &= -\frac{2(m+2)}{m+1}\frac{V_{i+1}^{m+1}-V_i^{m+1}}{Y_{i+1}-Y_i} &(44)\\
\sum_{i=1}^{n-1}(V_{i+1}+V_i)(Y_{i+1}-Y_i) &= 2a &(45)
\end{aligned}
$$

on the mesh given by (39), (42) and (43).

Proof. Adding $i-1$ equations (37) and the end node condition (38) gives,

$$Y_1V_2 + V_1Y_2 + \sum_{j=2}^{i} V_j(Y_{j+1} - Y_{j-1}) = -\frac{2(m+2)}{m+1}\left[\frac{V_2^{m+1} - V_1^{m+1}}{Y_2 - Y_1}\right]$$
$$-\frac{2(m+2)}{m+1}\sum_{j=2}^{i}\left[\frac{V_{j+1}^{m+1} - V_j^{m+1}}{Y_{j+1} - Y_j} - \frac{V_j^{m+1} - V_{j-1}^{m+1}}{Y_j - Y_{j-1}}\right] - \sum_{j=2}^{i}(V_{j+1} - V_{j-1})Y_j.$$

The summation indices are rearranged so that,

$$\sum_{j=2}^{i} V_j(Y_{j+1} - Y_{j-1}) + \sum_{j=3}^{i+1} V_jY_{j-1} - \sum_{j=1}^{i-1} V_jY_{j+1} + Y_1V_2 + V_1Y_2 =$$
$$-\frac{2(m+2)}{m+1}\left[\sum_{j=2}^{i}\frac{V_{j+1}^{m+1} - V_j^{m+1})}{Y_{j+1} - Y_j} - \sum_{j=1}^{i-1}\frac{V_{j+1}^{m+1} - V_j^{m+1}}{Y_{j+1} - Y_j} + \frac{V_2^{m+1} - V_1^{m+1}}{Y_2 - Y_1}\right]$$

which cancels to,

$$V_2(Y_3 - Y_1) + V_i(Y_{i+1} - Y_{i-1}) + V_{i+1}Y_i + V_iY_{i-1} - V_2Y_3 - V_1Y_2 + Y_1V_2 + V_1Y_2 =$$

$$-\frac{2(m+2)}{m+1}\left[\frac{V_{i+1}^{m+1} - V_i^{m+1}}{Y_{i+1} - Y_i} - \frac{V_2^{m+1} - V_1^{m+1}}{Y_2 - Y_1} + \frac{V_2^{m+1} - V_1^{m+1}}{Y_2 - Y_1}\right].$$

Terms cancel to give equation (44).

The condition on $i = n-1$ is identical to the end node condition in Problem SDA$'$, (40). Problem SDA$'$ has therefore been reduced to a system of $2n-3$ equations and the 3 conditions (41), (42) and (43) in $2n$ variables. Problem ISDA$'$ now has the right degree of determinacy. It is shown in [4] that the truncation error of problem ISDA$'$ is $K(\beta)/n^2 + O(1/n^3)$. □

Besides the theoretical beauty of the analogy between similarity reductions in the discrete and the continuous cases, the reduction of the ODE problem DA$'$ to an algebraic problem ISDA$'$ means that convergence results are more forthcoming. These results can be used to optimise the algorithm. Specifically the truncation error coefficient K is minimised over β.

Solving problem ISDA$'$ numerically gives us a discrete approximation of the similarity solution (24) with an error of $O(1/n^2)$. Furthermore numerical results show that,

$$t^{1/(2+m)}U_i - V_i = o(1) \text{ as } t \to \infty, \tag{46}$$

that is the scheme is stable. A more detailed discussion of stability is given in [4].

6 Convergence

The second direct consequence of the asymptotic convergence of the scaled solution and mesh to the self-similar solution, is that the scaled total error tends to a constant. This is an essential feature of the method.

Since the similarity solution (24) decreases as $t^{-1/(2+m)}$, it is expected that the error will also decrease with time. It is therefore sensible to define a scaled error,

$$\|se\|_\infty \equiv \max_{i=1\ldots n} |U_i(t) - u(X_i,t)| t^{1/(2+m)}$$

where $u(X_i,t)$ is the exact solution to problem A$'$ evaluated at X_i. Applying the triangle inequality we have,

$$\begin{aligned} \|se\|_\infty \leq & \max_i |U_i(t) - w(X_i t^{-1/(m+2)}) t^{-1/(m+2)}| t^{1/(2+m)} \\ + & \max_i |w(X_i t^{-1/(m+2)}) t^{-1/(m+2)} - u(X_i,t)| t^{1/(2+m)}, \end{aligned}$$

where w is the similarity solution (24). Now, it is shown in [20] that as $t \to \infty$ the right hand term is $o(1)$. Further, we have from Theorem 5.1 and the stability result (46) that, if the asymptotic relations (35) and (36) hold,

$$\begin{aligned} \max_i |U_i(t) - w(X_i t^{-1/(m+2)}) t^{-1/(m+2)}| t^{1/(2+m)} &= \max_i |V_i(t) - w(Y_i)| + o(1) \\ &= O(1/n^2) + o(1). \end{aligned}$$

So the dominant contribution to the total scaled error $\|se\|_\infty$ is the truncation error which is $O(1/n^2)$ and independent of time.

Figure 4 confirms this and shows that the scheme is converging as $n \to \infty$. In Figure 5 the error is plotted for solutions to both problems DA$'$ and ISDA$'$. This figure confirms that the total scaled error for the time dependent problem DA$'$ is the same as the total error for the algebraic problem ISDA$'$ as $n \to \infty$ and is of order $1/n^2$.

We also see that the error can be greatly reduced by a careful choice of the discretisation parameter β. The optimal value of β may be calculated by minimising the value of the coefficient of $1/n^2$ in the expansion of the truncation error of problem ISDA$'$ that is derived in [4].

For a non-invariant moving mesh scheme or a fixed mesh scheme, the scaled errors will grow as the domain expands and the spatial step size grows. It is therefore essential that the solution and mesh behave as relations (35) and (36) as $t \to \infty$. Indeed table 1 shows that a fixed mesh scheme compares very badly with the moving mesh scheme. The fixed mesh is computed on a domain that is the same size as the moving mesh after time $t = 1000$. The end node of the fixed mesh case is determined by setting U_1 and U_n equal to the exact similarity solution (24).

7 Discussion

We have presented a moving mesh scheme that uses a conservation law to move end nodes so that that computational domain expands to follow the solution. We find that the mesh moves in an invariant way with the results that the discretisation is reducible and that error is independent of time. For these results to hold it is essential that a moving mesh is used.

n	Problem DA$'$ $\beta = m+1$	Problem DA$'$ $\beta = 2$	Fixed $\beta = m+1$
20	2.315×10^{-1}	7.245×10^{-2}	9.844
40	5.981×10^{-2}	1.670×10^{-2}	8.891
60	2.627×10^{-2}	7.270×10^{-3}	1.289

Table 1: Scaled error $\|se\|_\infty$ for $m = 0$ (linear diffusion). Error is measured after $t = 1000$.

Clearly the method is easily generalised to solve other PDE problems which conserve mass. In particular we have had success in solving Cauchy problems for Fokker-Planck equations - see [4]. We anticipate that the method can be used to solve PDE's that conserve quantities other than mass such as problems that blow up in a finite time, including the Cauchy problem for the quasilinear PDE $u_t = (u^m u_x)_x + u^p$. Here we use the property that the similarity solutions conserve $u^{-(m+1-p)/2}$. Since there is an evolving singularity an equidistribution strategy is also needed and the interplay between equidistribution and conservation is being studied.

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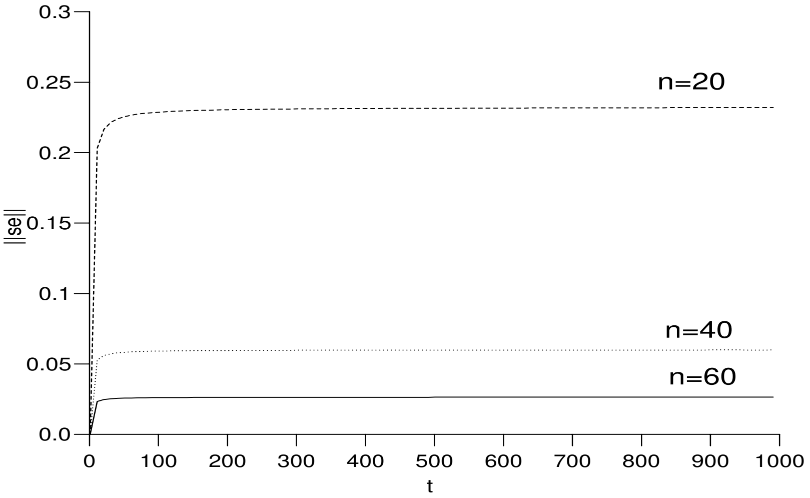

Figure 4: Computed scaled total error for m=0; Linear Diffusion. Calculations are performed on a uniform mesh with starting condition $U_0 = 10exp(-\frac{1}{4}X^2)$ and $\beta = m + 1$.

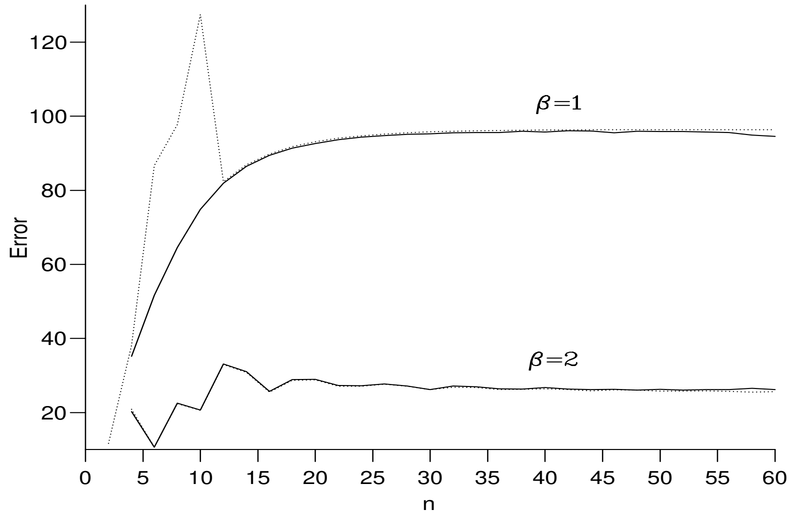

Figure 5: Computed error for Linear Diffusion. The starting condition $U_0 = 10exp(-\frac{1}{4}X^2)$ and the error is computed at $t = 1000$. The solid line is the scaled error, $||e||_\infty t^{1/2}$ for a solution to problem DA$'$. The dotted line is the total error for a solution to ISDA$'$.