PDEs with Distributed Control and Delay in the Loop: Transfer Function Poles, Exponential Modes and Robustness of Stability *

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Abstract: In this paper we consider a large class of partial differential equations (PDEs) with distributed control and with a time-delay in the feedback loop. We analyze the relationship between the poles of the closed-loop transfer function and the exponential modes of the underlying retarded PDE in order to derive internal stability properties from external ones. Our approach is based on a combination of input-output methods and modal analysis. We give a number of sufficient conditions for robustness/nonrobustness of closed-loop modal stability with respect to delays. The theory is illustrated by two examples.

Keywords: Partial differential equations; distributed control; small time delays; stability; robustness; transfer functions.

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1. Introduction

The literature on robustness and lack of robustness of distributed parameter systems with respect to small delays in the feedback loop can be roughly divided into two types. Those papers dealing with individual partial differential equations (PDEs) focus on the existence, or lack of existence, of exponentially growing modes, see for instance Datko [3], Datko *et al.* [4] and Desch and Wheeler [5]. Those papers considering input-output systems in the frequency domain deal with input-output stability, see for instance Barman *et al.* [1] and Logemann *et al.* [11].

In Logemann and Rebarber [10] we examined the relationship between the external concept of *spectral stability* and the internal concept of *modal stability* for a large class of boundary control systems in one space dimension (allowing for in-span control). Spectral stability simply means that the transfer function of the system has no poles in the closed right-half plane, whilst modal stability means that any exponential solutions (modes) of the form $e^{st}\phi(x)$ must satisfy $\operatorname{Re} s < 0$. It was shown in [10] that the frequency-domain results in [10, 11] on robustness/nonrobustness of spectral stability translate to results on robustness/nonrobustness of modal stability. This paper addresses similar questions, but instead of boundary control systems we consider here systems with distributed control action, where it is more complicated to unravel the relationship between modes and poles. Distributed control is of interest since in certain problems the control action is more accurately modelled by a function with "small" support than by a δ-function formulation. The underlying class of PDEs for the control systems under consideration consists of linear PDEs of spatial dimension 1, where on different parts of the space interval different partial differential equations are satisfied. The coefficients may depend on the spatial variable. The boundary conditions are general enough to allow all natural coupling conditions. It is not assumed that the controlled and observed systems has any useful (A, B, C, D) state-space representation, and in fact many systems in our class do not. Moreover, we do not even assume that the free dynamics are described by a strongly continuous semigroup.

In order to relate modal stability of the PDE to the spectral stability of the transfer function, a relationship between the modes of a PDE and the poles of the associated transfer function is needed. As in the case of the boundary control systems considered in [10], we show in this paper that a pole s_0 of the transfer function leads to a mode with exponent s_0 of the corresponding free dynamics. However, this does not follow in a straightforward way from [10], but requires a different analysis. Of course, as in finite dimensions, it is possible that a mode will not appear as a pole of the associated transfer function, since any possible effect of the mode on the output might be anihilated by the observation and control operators. However, it will be shown that for any system in our class the exponents of the unstable modes coincide with the unstable poles of the transfer function, provided the closed-loop system (without delay) is modally stable. This result will in turn be applied to prove that for a large class of systems robustness/nonrobustness of spectral stability implies robustness/nonrobustness of modal stability.

The paper is organized as follows. In Section 2 we introduce the class of systems under consideration, and we show for the open-loop as well as for the closed-loop system with delay how the transfer function poles are related to the exponents of the system modes. In Section 3 we briefly describe the input-output robustness/nonrobustness results that we need from [11], and in Corollaries 3.3 and 3.4 we give the related results for robustness/nonrobustbess of modal stability for the class of systems described in Section 2. Finally, in Section 4 we present two examples illustrating the results in the previous sections.

Notation and terminology

For any $\alpha \in \mathbb{R}$, let $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$. We define

$$H_- := \{f : \Omega_f \to \mathbb{C} \,|\, \Omega_f \text{ open}, \Omega_f \supset \mathbb{C}_0^{cl} \text{ and } f \text{ holomorphic}\}.$$

The field of all meromorphic functions on $\mathbb{C}_\alpha$ is denoted by M_α, while H_α^∞ denotes the algebra of all bounded holomorphic functions defined on $\mathbb{C}_\alpha$. We write H^∞ for H_0^∞. If $f \in M_\alpha$ and $g \in M_\beta$, where $\alpha < \beta$, and if $f(s) = g(s)$ for all $s \in \mathbb{C}_\beta$, then we shall identify f and g.

Let $\Omega \subset \mathbb{C}$. A function $\mathbf{H} : \Omega \to \mathbb{C}$ is called a *transfer function* if $\mathbb{C}_\alpha \subset \Omega$ for some $\alpha \in \mathbb{R}$ and $\mathbf{H}|_{\mathbb{C}_\alpha} \in M_\alpha$. A transfer function $\mathbf{H}$ is called *well-posed* if $\mathbf{H} \in H_\alpha^\infty$ for some $\alpha \in \mathbb{R}$. Moreover, a transfer function $\mathbf{H}$ is called *regular* if it is well-posed and if the limit $\lim_{\xi \to +\infty} \mathbf{H}(\xi) = D$ exists (where $\xi \in \mathbb{R}$). The number D is called the *feedthrough* of $\mathbf{H}$. If $\mathbf{H}$ is not well-posed, we say that it is *ill-posed.*

2. PDEs with distributed control and delay in the loop: The relationship between transfer functions poles and exponential modes

In the following we introduce a class of controlled and observed partial differential equations which are linear in time and of spatial dimension 1, with coefficients which may depend upon the spatial variable. On different parts of the space interval different PDEs are satisfied. They are coupled via the boundary conditions at the in-span points. The control action is assumed to be distributed.

We suppose that the space variable x belongs to some closed interval $[a, b]$. Without loss of generality we assume that $[a, b] = [0, 1]$. Let $\lambda \in \mathbb{N}$, and $\{x_l\}_{l=1}^{\lambda} \subset (0, 1)$, where $x_1 < x_2 < \ldots < x_\lambda$. These numbers determine a decomposition of $(0, 1)$ into $\lambda + 1$ open intervals $\{I_k\}_{k=0}^{\lambda}$. Let $\iota \in \mathbb{N}$. For $j = 0, \ldots, \iota$ and $k = 0, \ldots, \lambda$, let p_j^k be polynomials, let

a_j^k be continuous functions on I_k^{cl}, and let $b_j^k \in L^2(0,1;\mathbb{C})$. Let D_x denote differentiation with respect to x, and consider the controlled PDE

$$\sum_{j=0}^{\iota} a_j^k(x) p_j^k(D_x) \frac{\partial^j w}{\partial t^j}(x,t) = \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j u}{\partial t^j}(t)\,, \qquad x \in I_k, \quad t > 0\,. \tag{2.1}$$

Set $n_k := \max_{0 \le j \le \iota} \deg p_j^k$ and let $p_k(x,s)$ be the coefficient of $D_x^{n_k}$ in the expression

$$\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(D_x)\,,$$

where s is a complex variable. We introduce the following assumption:

(**A 1**) There exists an open set $\Omega \supset \mathbb{C}_0^{cl}$ such that for any $k = 0, \ldots, \lambda$

$$p_k(x,s) \neq 0 \qquad \text{for all } x \in I_k^{cl}\,,\ s \in \Omega\,. \qquad \diamond$$

This condition guarantees that when the Laplace transform of the PDE in I_k is taken, the resulting ordinary differential equation is not degenerate.

To define boundary operators for the PDE (2.1), we note that, by (**A 1**), the PDE has spatial order n_k in I_k, so we need

$$n := \sum_{k=0}^{\lambda} n_k$$

boundary conditions. While the boundary for $\bigcup_{k=0}^{\lambda} I_k$ is $\{x_l\}_{l=1}^{\lambda} \cup \{0,1\}$, for the purpose of defining boundary conditions each x_j should be represented by x_j^+ and x_j^-. This allows coupling conditions (for example $D_x w(0.5^-,t) = D_x w(0.5^+,t)$) and in-span control. Therefore, we consider the boundary set to be $\{x_l^-\}_{l=1}^{\lambda} \cup \{x_l^+\}_{l=1}^{\lambda} \cup \{0,1\}$, which we rename as $\{z_l\}_{l=1}^{\mu}$, where $\mu = 2(\lambda+1)$. For any piecewise continuous function $f : [0,1] \to \mathbb{C}$ and for $l = 1, \ldots, \lambda$ we define

$$f(x_l^-) := \lim_{x \nearrow x_l} f(x)\,, \quad f(x_l^+) := \lim_{x \searrow x_l} f(x)\,,$$

so that $f(z_l)$ is a well-defined complex number for all $l = 1, \ldots, \mu$.

Let $q_{l,j}^i$ be polynomials for $i = 1, \ldots, n$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$. We define boundary operators B_i on solutions $w(x,t)$ of (2.1) by

$$(B_i w)(t) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} q_{l,j}^i(D_x) \frac{\partial^j}{\partial t^j} w(z_l,t)\,. \tag{2.2}$$

We need to impose bounds on the order of the spatial derivatives in (2.2). In particular, we do not wish to take spatial derivatives at the boundary of I_k which are of order larger than $n_k - 1$. To this end it is useful to introduce the function $\kappa : \{1, \ldots, \mu\} \to \{0, \ldots, \lambda\}$ given by

$$\kappa(l) := \begin{cases} 0 & \text{if } z_l = 0 \\ \lambda & \text{if } z_l = 1 \\ l_0 & \text{if } z_l = x_{l_0}^+ \\ l_0 - 1 & \text{if } z_l = x_{l_0}^- \,. \end{cases}$$

We assume that for any $i = 1, \ldots, n$, $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$

$$\deg q_{l,j}^i \le n_{\kappa(l)} - 1 \,.$$

The boundary operators (2.2) are sufficiently general to allow higher order differential equations at the boundary, as in the "hybrid systems" in Littman and Markus [9]. We consider the following boundary conditions for the PDE (2.1):

$$(B_i w)(t) = 0 \,, \qquad i = 1, \ldots, n \,. \tag{2.3}$$

The observation operator for the system will be quite general, since we will allow for distributed and point observation. For $j = 0, \ldots, \iota$, $l = 1, \ldots, \mu$, let $r_{l,j}$ be polynomials, let $f_j \in L^2(0, 1; \mathbb{C})$, and define the observation operator by

$$(Cw)(t) = \sum_{l=1}^{\mu} \sum_{j=0}^{\iota} r_{l,j}(D_x) \frac{\partial^j}{\partial t^j} w(z_l, t) + \sum_{j=0}^{\iota} \int_0^1 f_j(x) \frac{\partial^j}{\partial t^j} w(x, t)\, dx \,,$$

where we assume that

$$\deg r_{l,j} \le n_{\kappa(l)} - 1 \,.$$

The observation $y(t)$ is then given by

$$y(t) = (Cw)(t) \,. \tag{2.4}$$

We refer to the observed distributed control system given by (2.1), (2.3) and (2.4) as the *open-loop system*. In the following it will be denoted by (**OLS**).

Application of output feedback of the form $u(t) = v(t) - y(t - \varepsilon)$ leads to

$$\sum_{j=0}^{\iota} a_j^k(x) p_j^k(D_x) \frac{\partial^j w}{\partial t^j}(x, t) = \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j v}{\partial t^j}(t) - \sum_{j=0}^{\iota} b_j^k(x) \frac{\partial^j}{\partial t^j}(Cw)(t - \varepsilon) \,, \quad x \in I_k, \ t > 0 \,, \tag{2.5}$$

where $\varepsilon \ge 0$ is a time-delay and $v(t)$ denotes the input of the feedback system. We refer to the system given by (2.5), (2.3) and (2.4) as the *closed-loop system with delay* ε. In the following it will be denoted by ($\mathbf{CLS}_\varepsilon$). If $\varepsilon = 0$, then we will call ($\mathbf{CLS}_0$) the *undelayed closed-loop system*. If $u(t) \equiv 0$ (resp. $v(t) \equiv 0$), then we refer to (**OLS**) (resp. ($\mathbf{CLS}_\varepsilon$)) as the *uncontrolled* open-loop system (resp. *uncontrolled* closed-loop system with delay ε).

We first find conditions under which exponential solutions exist, and then find the transfer functions of (**OLS**) and ($\mathbf{CLS}_\varepsilon$).

Existence of exponential solutions

We will be looking for exponential solutions of the form

$$w(x,t) = e^{st}\phi(x)\,, \quad \text{where } s \in \mathbb{C},\ \phi \in L^2(0,1;\mathbb{C}),\ \phi \not\equiv 0\,, \tag{2.6}$$

for the uncontrolled open-loop and closed-loop systems. Such a solution will be called a *mode* of (**OLS**) (resp. ($\mathbf{CLS}_\varepsilon$)). The complex number s is called the *exponent* of the mode.

Suppose that (2.6) is a solution of the uncontrolled system (**OLS**). Since ϕ will in general depend on s, we write $\phi(x) = \phi(x,s)$. Clearly, $\phi(x,s)$ satisfies

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(D_x)\right)\phi(x,s) = 0\,, \quad x \in I_k\,. \tag{2.7}$$

Using assumption (**A 1**), we see that for every $s \in \Omega \supset \mathbb{C}_0^{cl}$ and every $k = 0,\dots,\lambda$, (2.7) is an ordinary differential equation of order n_k on I_k^{cl}. Let $\{e_j^k\}_{j=1}^{n_k}$ be a basis of $\mathbb{C}^{n_k}$ and let $\{\phi_j^k(\cdot,s)\}_{j=1}^{n_k}$ be solutions of (2.7) on I_k^{cl} satisfying

$$\left(\phi_j^k(x_k,s), D_x\phi_j^k(x_k,s),\dots,D_x^{n_k-1}\phi_j^k(x_k,s)\right) = e_j^k\,, \qquad j = 1,\dots,,n_k;\ s \in \Omega\,,$$

where $x_0 := 0$. Clearly, for any $k = 0,\dots,\lambda$ and any $s \in \Omega$, the functions $\phi_j^k(\cdot,s)$ are linearly independent, and hence span the solution space of (2.7) on I_k^{cl}. In particular, for every $s \in \Omega$, $\phi_j^k(\cdot,s) \in L^2(I_k^{cl},\mathbb{C})$, and every solution of (2.7) in I_k can be written in the form

$$\sum_{j=1}^{n_k} A_j^k(s)\phi_j^k(x,s)\,, \qquad x \in I_k\,,$$

for some coefficients $A_j^k(s)$. For $k = 0,\dots,\lambda$ and $j = 1,\dots,n_k$, let us define

$$\Phi_j^k(x,s) := \begin{cases} \phi_j^k(x,s) & \text{for } x \in I_k^{cl} \\ 0 & \text{for } x \in [0,1] \setminus I_k^{cl}\,. \end{cases}$$

It will be convenient to rename these n vectors as $\{\Phi_p(x,s)\}_{p=1}^n$. It is clear that any solution of (2.7) can be written as

$$\phi(x,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s)\,, \quad x \in \bigcup_{k=0}^{\lambda} I_k\,,$$
$$\phi(z_l,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(z_l,s)\,, \quad l = 1,\dots\,\mu\,,$$

for some coefficients $A_p(s)$.

Setting

$$(\hat{B}_i\Phi_p)(s) = \sum_{l=1}^{\mu}\sum_{j=0}^{\iota} s^j q_{l,j}^i(D_x)\Phi_p(z_l,s)\,, \tag{2.8}$$

we introduce the $n \times n$ matrix

$$\mathbf{\Delta}(s) := \Big((\hat{B}_i\Phi_p)(s)\Big)\,, \quad i,p = 1,\ldots,n\,. \tag{2.9}$$

By lemma 4.1 in [10], $\mathbf{\Delta}(s)$ is holomorphic on Ω. Thus the following result can be obtained in the same way as proposition 4.2 in [10].

Proposition 2.1 *Suppose that* (**A 1**) *is satisfied and let* $s_0 \in \Omega$. *Then* (**OLS**) *has a mode with exponent* s_0 *if and only if* $\det \mathbf{\Delta}(s_0) = 0$.

In order to prove a similar result for the closed-loop system ($\mathbf{CLS}_\varepsilon$), suppose that

$$w(x,t) = e^{st}\phi(x)\,, \quad \text{where } \phi \in L^2(0,1;\mathbb{C}),\ \phi \not\equiv 0\,,$$

is a solution of the uncontrolled system ($\mathbf{CLS}_\varepsilon$). Again we will indicate the dependence of ϕ on s explicitly by writing $\phi(x) = \phi(x,s)$. We then obtain from (2.5) with $v(t) \equiv 0$

$$\left(\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(D_x)\right)\phi(x,s) = -e^{-\varepsilon s}b^k(x,s)(\hat{C}\phi)(s)\,, \qquad x \in I_k\,, \tag{2.10}$$

where

$$b^k(x,s) := \sum_{j=0}^{\iota} b_j^k(x)s^j\,,$$

and $(\hat{C}\phi)(s)$ is defined by

$$(\hat{C}\phi)(s) := \sum_{l=1}^{\mu}\sum_{j=0}^{\iota} s^j r_{l,j}(D_x)\phi(z_l,s) + \sum_{j=0}^{\iota} s^j \int_0^1 f_j(x)\phi(x,s)\,dx\,. \tag{2.11}$$

Let η_k be the left endpoint of I_k, and let $\Psi(x,s)$ be the solution of the initial value problem

$$\left(\sum_{j=0}^{\iota} a_j^k(x)s^j p_j^k(D_x)\right)\phi(x,s) = b^k(x,s)\,, \qquad x \in I_k\,, \tag{2.12}$$
$$D_x^j\phi(\eta_k,s) = 0\,, \qquad j = 0,\ldots,n_k - 1\,.$$

Clearly, the solution $\phi(x,s)$ of (2.10) can be written in the form

$$\phi(x,s) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s) + a(s)\Psi(x,s)\,, \tag{2.13}$$

where $a(s), A_p(s) \in \mathbb{C}$ are suitable coefficients. The boundary conditions (2.3) become

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) + a(s)(\hat{B}_i\Psi)(s) = 0\,, \qquad i = 1,\dots,n\,, \tag{2.14}$$

where $(\hat{B}_i\Psi)(s)$ is defined as in (2.8) with Φ_p replaced by Ψ. Plugging (2.13) into (2.10) and noting that Φ_p satisfies (2.7) and Ψ satisfies (2.12), we obtain

$$b^k(x,s)\left(\sum_{p=1}^{n} A_p(s)(\hat{C}\Phi_p)(s) + a(s)\left((\hat{C}\Psi)(s) + e^{\varepsilon s}\right)\right) = 0\,, \qquad x \in I_k\,,\ s \in \Omega\,, \tag{2.15}$$

where $(\hat{C}\Phi_p)(s)$ and $(\hat{C}\Psi)(s)$ are defined as in (2.11) with ϕ replaced by Φ_p and Ψ, respectively. Let

$$\mathbf{c} := \left(\hat{C}\Phi_1,\dots,\hat{C}\Phi_n\right)^T\,, \qquad \mathbf{b} := -\left(\hat{B}_1\Psi,\dots,\hat{B}_n\Psi\right)^T\,,$$

and

$$\boldsymbol{\Delta}_\varepsilon(s) := \begin{pmatrix} \boldsymbol{\Delta}(s) & -\mathbf{b}(s) \\ \mathbf{c}^T(s) & (\hat{C}\Psi)(s) + e^{\varepsilon s} \end{pmatrix}\,, \tag{2.16}$$

where $\boldsymbol{\Delta}$ is given by (2.9). We are now in the position to prove the following result.

Proposition 2.2 *Suppose that* (**A 1**) *is satisfied and let* $s_0 \in \Omega$. *Then* ($\mathbf{CLS}_\varepsilon$) *has a mode with exponent* s_0 *if and only if* $\det \boldsymbol{\Delta}_\varepsilon(s_0) = 0$.

Proof: Suppose that ($\mathbf{CLS}_\varepsilon$) has a mode with exponent s_0. Then using the above notation, $e^{s_0 t}\phi(x,s_0)$ is a solution of ($\mathbf{CLS}_\varepsilon$), where $\phi(\cdot,s_0) \not\equiv 0$.

Case 1: Assume that there exists $k_0 \in \{0,\dots,\lambda\}$ and $\xi_0 \in I_{k_0}$ such that $b^{k_0}(\xi_0,s_0) \neq 0$. Then, by (2.15)

$$\sum_{p=1}^{n} A_p(s_0)(\hat{C}\Phi_p)(s_0) + a(s_0)\left((\hat{C}\Psi)(s_0) + e^{\varepsilon s_0}\right) = 0\,. \tag{2.17}$$

Setting $A(s) := (A_1(s),\dots,A_n(s))^T$ and $\tilde{A}(s) := (A^T(s), a(s))^T$, then a combination of (2.14) and (2.17) yields that $\boldsymbol{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$. Since $\phi(\cdot,s_0) \not\equiv 0$, it follows from (2.13) that $\tilde{A}(s_0) \neq 0$ and hence that $\det \boldsymbol{\Delta}_\varepsilon(s_0) = 0$.

Case 2: Now assume that $b^k(x,s_0) = 0$ for all $x \in I_k$ and for all $k \in \{0,\dots,\lambda\}$. Then $\Psi(x,s_0) = 0$ for all $x \in [0,1]$ and hence $\mathbf{b}(s_0) = 0$. Moreover, by (2.14)

$$\sum_{p=1}^{n} A_p(s_0)(\hat{B}_i\Phi_p)(s_0) = 0\,, \qquad i = 1,\dots,n\,. \tag{2.18}$$

Since $\Psi(\cdot, s_0) \equiv 0$ and $\phi(\cdot, s_0) \not\equiv 0$, (2.13) yields that $A(s_0) = (A_1(s_0), \ldots, A_n(s_0))^T \neq 0$. So, combining (2.18) with (2.9), we see that $\det \mathbf{\Delta}(s_0) = 0$. Now

$$\mathbf{\Delta}_\varepsilon(s_0) = \begin{pmatrix} \mathbf{\Delta}(s_0) & 0 \\ \mathbf{c}^T(s_0) & e^{\varepsilon s_0} \end{pmatrix},$$

and thus $\det \mathbf{\Delta}_\varepsilon(s_0) = e^{\varepsilon s_0} \det \mathbf{\Delta}(s_0) = 0$.

Conversely, suppose that $\det \mathbf{\Delta}_\varepsilon(s_0) = 0$ for some $s_0 \in \Omega$. Then there exists $\tilde{A}(s_0) = (A^T(s_0), a(s_0))^T \in \mathbb{C}^{n+1}$, $\tilde{A}(s_0) \neq 0$, such that $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$. Writing $A(s_0) = (A_1(s_0), \ldots, A_n(s_0))^T$ and defining $\phi(x, s_0)$ by (2.13), we obtain

$$-\, e^{-\varepsilon s_0}(\hat{C}\phi)(s_0) = -e^{-\varepsilon s_0}[\mathbf{c}^T(s_0)A(s_0) + a(s_0)(\hat{C}\Psi)(s_0)] = -e^{-\varepsilon s_0}[-e^{\varepsilon s_0}a(s_0)] = a(s_0)\,, \tag{2.19}$$

where we used the fact that $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$ implies that

$$\mathbf{c}^T(s_0)A(s_0) + a(s_0)\left((\hat{C}\Psi)(s_0) + e^{\varepsilon s_0}\right) = 0\,.$$

Now

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s_0^j\, p_j^k(D_x)\right) \phi(x, s_0) = a(s_0) b^k(x, s_0)\,, \qquad x \in I_k\,,$$

and so, using (2.19) it follows that $\phi(x, s_0)$ satisfies (2.10) for $s = s_0$. Finally, since $\mathbf{\Delta}_\varepsilon(s_0)\tilde{A}(s_0) = 0$, it is clear that (2.14) is satisfied for $s = s_0$. Thus the function $w(x,t) := e^{s_0 t}\phi(x, s_0)$ is a solution of $(\mathbf{CLS}_\varepsilon)$, and hence $(\mathbf{CLS}_\varepsilon)$ has a mode with exponent s_0. $\square$

Transfer functions

We shall need the following assumption:

(A 2) $$\det \mathbf{\Delta}(s) \not\equiv 0\,. \qquad \diamond$$

For a given $s_0 \in \Omega$, let $\mathcal{E}_{s_0}$ denote the vector space of all functions $\psi : \left(\bigcup_{k=0}^{\lambda} I_k\right) \times [0, \infty) \to \mathbb{C}$ of the form $\psi(x,t) = e^{s_0 t}\tilde{\psi}(x)$, where $\tilde{\psi} : \bigcup_{k=0}^{\lambda} I_k \to \mathbb{C}$ solves the ordinary differential equation (2.7) for $s = s_0$. It is easy to see that assumption **(A 2)** is equivalent to

(A 2′) There exists $s_0 \in \Omega$ such that the restricted boundary operators $B_i|_{\mathcal{E}_{s_0}}$ are linearly independent, i.e. if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are such that $\sum_{i=1}^n \alpha_i B_i \psi = 0$ for all $\psi \in \mathcal{E}_{s_0}$, then $\alpha_i = 0$ for all $i = 1, \ldots, n$. $\diamond$

If **(A 2)** is satisfied, then the function

$$\mathbf{H}(s) := \mathbf{c}^T(s)\mathbf{\Delta}^{-1}(s)\mathbf{b}(s) + (\hat{C}\Psi)(s) \tag{2.20}$$

is well-defined. While the functions $\mathbf{c}(s)$ and $\mathbf{\Delta}(s)$ depend on the choice of the basis $\{\phi_j^k\}_{j=1}^{n_k}$ of the solution space of (2.7) on I_k^{cl}, it is easy to prove that the product

$\mathbf{c}^T(s)\mathbf{\Delta}^{-1}(s)$ does not. If (**A 1**) is satisfied, then it follows as in the proof of lemma 4.1 in [10] that $\hat{C}\Psi \in H_-$ and that $\mathbf{c}$ and $\mathbf{b}$ have all their entries in H_-. Moreover, we know that the entries of $\mathbf{\Delta}$ are in H_-. As a consequence, $\mathbf{H}$ is a transfer function in the sense of Section 1 (see the subsection "Notation and terminology"), and we say that $\mathbf{H}$ is the transfer function of (**OLS**). Setting

$$\mathbf{D} := \begin{pmatrix} \mathbf{\Delta} & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{pmatrix}, \qquad \tilde{\mathbf{N}} := \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix}, \qquad \mathbf{N} := (\mathbf{c}^T, \hat{C}\Psi), \tag{2.21}$$

we can write

$$\mathbf{H}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\tilde{\mathbf{N}}(s). \tag{2.22}$$

In order to show that $\mathbf{H}$ admits the usual dynamical interpretation, let $u(\cdot)$ be a sufficiently smooth Laplace transformable input function with zero initial conditions, i.e. $(d^j u/dt^j)(0) = 0$ for all $j = 0, \ldots, \iota - 1$. Denoting the corresponding solution of (**OLS**) with zero initial conditions by $w(x,t;u)$, we see that $\hat{w}(x,s;u)$ satisfies the ordinary differential equation

$$\left(\sum_{j=0}^{\iota} a_j^k(x) s^j p_j^k(D_x)\right) \phi(x,s) = b^k(x,s)\hat{u}(s), \qquad x \in I_k,$$

and hence $\hat{w}(x,s;u)$ can be written in the form

$$\hat{w}(x,s;u) = \sum_{p=1}^{n} A_p(s)\Phi_p(x,s) + \hat{u}(s)\Psi(x,s). \tag{2.23}$$

The boundary conditions (2.3) then lead to

$$\sum_{p=1}^{n} A_p(s)(\hat{B}_i\Phi_p)(s) + \hat{u}(s)(\hat{B}_i\Psi)(s) = 0, \qquad i = 1, \ldots, n.$$

This is equivalent to

$$\mathbf{\Delta}(s)A(s) = \mathbf{b}(s)\hat{u}(s), \tag{2.24}$$

where $A(s) = (A_1(s), \ldots, A_n(s))^T$. Using (2.23), the Laplace transform of the observation (2.4) can be expressed as follows

$$\hat{y}(s) = \mathbf{c}^T(s)A(s) + (\hat{C}\Psi)(s)\hat{u}(s).$$

Therefore, using (2.24) and the fact that $\det \mathbf{\Delta}(s) \not\equiv 0$ by (**A 2**), we obtain

$$\hat{y}(s) = \left(\mathbf{c}^T\mathbf{\Delta}^{-1}(s)\mathbf{b}(s) + (\hat{C}\Psi)(s)\right)\hat{u}(s) = \mathbf{H}(s)\hat{u}(s).$$

Using (2.22) and lemma 2.4 in [10], the transfer function $\mathbf{G}_\varepsilon(s)$ of ($\mathbf{CLS}_\varepsilon$) is given by

$$\mathbf{G}_\varepsilon(s) = \mathbf{H}(s)\left(1 + e^{-\varepsilon s}\mathbf{H}(s)\right)^{-1} = \mathbf{N}(s)\left(\mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)\right)^{-1}\tilde{\mathbf{N}}(s). \tag{2.25}$$

In contrast to the results for boundary control systems given in [10], it is not true that $\mathbf{\Delta}_\varepsilon(s) = \mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)$. However, we can show the following.

Theorem 2.3 *Suppose that* $(\mathbf{A\,1})$ *holds. Then the zeros of the functions* $\det \mathbf{\Delta}_\varepsilon(s)$ *and* $\det\left(\mathbf{D}(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)\right)$ *in* Ω *coincide (counting multiplicities).*

Proof: Using (2.21) we obtain that

$$\mathbf{D}(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)=\begin{pmatrix}\mathbf{\Delta}(s)+e^{-\varepsilon s}\mathbf{b}(s)\mathbf{c}^T(s) & e^{-\varepsilon s}(\hat{C}\Psi)(s)\mathbf{b}(s)\\ e^{-\varepsilon s}\mathbf{c}^T(s) & 1+e^{-\varepsilon s}(\hat{C}\Psi)(s)\end{pmatrix}. \tag{2.26}$$

For any matrices M_1, M_2, M_3, M_4, where M_1 is square and M_4 is square and invertible,

$$\det\begin{pmatrix}M_1 & M_2\\ M_3 & M_4\end{pmatrix}=\det M_4\det(M_1-M_2M_4^{-1}M_3)\,, \tag{2.27}$$

see e.g. Gantmacher [6], p. 46. Setting $h_\varepsilon(s):=1+e^{-\varepsilon s}(\hat{C}\Psi)(s)$ we distinguish between two cases.

Case 1: If $h_\varepsilon(s)\not\equiv 0$, then an application of (2.27) to (2.26) yields, after simplication,

$$\det(\mathbf{D}(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s))=\frac{1}{h_\varepsilon^{n-1}(s)}\det\left(h_\varepsilon(s)\mathbf{\Delta}(s)+e^{-\varepsilon s}\mathbf{b}(s)\mathbf{c}^T(s)\right). \tag{2.28}$$

Combining (2.16) and (2.27) we obtain

$$\begin{aligned}\det\mathbf{\Delta}_\varepsilon(s)&=((\hat{C}\Psi)(s)+e^{\varepsilon s})\det\left(\mathbf{\Delta}(s)+\frac{1}{(\hat{C}\Psi)(s)+e^{\varepsilon s}}\mathbf{b}(s)\mathbf{c}^T(s)\right)\\ &=e^{\varepsilon s}h_\varepsilon(s)\det\left(\mathbf{\Delta}(s)+\frac{e^{-\varepsilon s}}{h_\varepsilon(s)}\mathbf{b}(s)\mathbf{c}^T(s)\right).\end{aligned} \tag{2.29}$$

Comparing (2.28) and (2.29) shows that

$$\det\mathbf{\Delta}_\varepsilon(s)=e^{\varepsilon s}\det\left(\mathbf{D}(s)+e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)\right). \tag{2.30}$$

Case 2: If $h_\varepsilon(s)\equiv 0$, then it follows that

$$(\hat{C}\Psi)(s)=-e^{\varepsilon s}\,. \tag{2.31}$$

Let $\varepsilon_n>0$ be such that $\lim_{n\to\infty}\varepsilon_n=\varepsilon$ and $\varepsilon_n\neq\varepsilon$ for all $n\in\mathbb{N}$. Clearly, by (2.31), $h_{\varepsilon_n}(s)\not\equiv 0$ for all $n\in\mathbb{N}$, and thus, by Case 1

$$\det\mathbf{\Delta}_{\varepsilon_n}(s)=e^{\varepsilon_n s}\det\left(\mathbf{D}(s)+e^{-\varepsilon_n s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)\right).$$

Letting $n\to\infty$ we obtain by continuity that (2.30) holds true. □

3. Modal stability and small delays: Robustness issues

Let $\mathbf{H}$ be a transfer function and consider the feedback system shown in Figure 3.1, where u is the input function, y is the output function and the block with transfer function $e^{-\varepsilon s}$ represents a delay of length $\varepsilon \geq 0$.

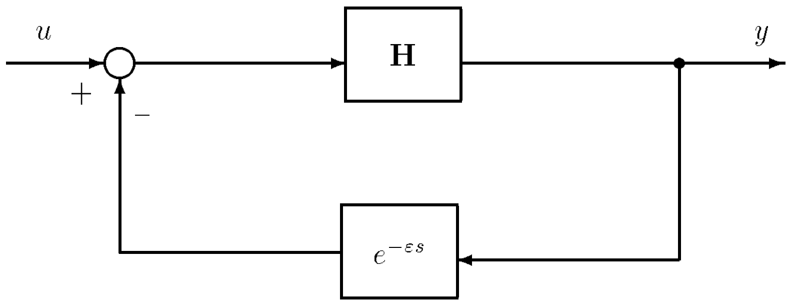

Figure 3.1: Feedback system with delay

If $1 + e^{-\varepsilon s}\mathbf{H}(s) \not\equiv 0$, then the function $\mathbf{G}_\varepsilon$ defined by

$$\mathbf{G}_\varepsilon(s) = \frac{\mathbf{H}(s)}{1 + e^{-\varepsilon s}\mathbf{H}(s)}$$

is a transfer function, the so-called closed-loop transfer function of the feedback system shown in Figure 3.1. We say that $\mathbf{G}_\varepsilon$ is *L^2-stable* if $\mathbf{G}_\varepsilon \in H^\infty$. If $\mathbf{G}_\varepsilon \in H_-$, then $\mathbf{G}_\varepsilon$ is called *spectrally stable.*

For a transfer function $\mathbf{H}$, let $\mathfrak{P}_\mathbf{H}$ denote the set of its poles. We define

$$\gamma(\mathbf{H}) := \limsup_{|s|\to\infty,\, s\in\mathbb{C}_0 \setminus \mathfrak{P}_\mathbf{H}} |\mathbf{H}(s)|\,.$$

The following destabilization result for regular transfer functions was proved by Logemann *et al.* [11].

Theorem 3.1 *Let $\mathbf{H}$ be a transfer function and suppose that $\mathbf{H}$ is regular with feedthrough D. Then the following statements hold true:*

(i) If $\mathbf{G}_0$ is L^2-stable and if $\gamma(\mathbf{H}) > 1$, then there exists sequences (ε_n) and (p_n) with

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad p_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} |\mathrm{Im}\, p_n| = \infty\,,$$

and such that for any $n \in \mathbb{N}$, p_n is a pole of $\mathbf{G}_{\varepsilon_n}$.

(ii) If $|D| > 1$, then there exists sequences (ε_n) and (p_n) with

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad p_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} \operatorname{Re} p_n = \infty\,, \quad \lim_{n\to\infty} |\operatorname{Im} p_n| = \infty\,,$$

and such that for any $n \in \mathbb{N}$, p_n is a pole of $\mathbf{G}_{\varepsilon_n}$.

There are many partial differential equation models of physical and technical systems which have transfer functions which are not well-posed, and hence in particular, are not regular. Usually this occurs in boundary control systems. In [10] a result similar to Theorem 3.1 has been proved for a large class of ill-posed transfer functions, and applications to boundary control systems were given in the same paper. Roughly speaking, since distributed control is bounded, as long as the observation is not too unbounded, systems with distributed control have regular transfer functions. For that reason, and since the two examples given in Section 4 have regular transfer functions, we do not state this result from [10] here.

The following robustness result complements Theorem 3.1.

Theorem 3.2 *Let $\mathbf{H}$ be a transfer function and suppose that $\mathbf{G}_0$ is L^2-stable (spectrally stable). If $\gamma(\mathbf{H}) < 1$, then there exists $\varepsilon^* > 0$ such that $\mathbf{G}_\varepsilon$ is L^2-stable (spectrally stable) for all $\varepsilon \in (0, \varepsilon^*)$.*

The part of the theorem which relates to L^2-stability is proved in [11], theorem 6.1. Robustness of spectral stability can be shown in a similar way and is therefore left to the reader.

In the following we shall apply the above results to the system $(\mathbf{CLS}_\varepsilon)$. We call a mode $w(x,t) = e^{st}\phi(x)$ of $(\mathbf{CLS}_\varepsilon)$ *stable* if $\operatorname{Re} s < 0$, otherwise the mode is called *unstable*. We say that the system $(\mathbf{CLS}_\varepsilon)$ is *modally stable* if all the modes of $(\mathbf{CLS}_\varepsilon)$ are stable. This is the kind of internal stability which is considered in most of the literature on robust stabilization of PDE's, see [3]–[5] and [7].

Combining Proposition 2.2, Theorem 2.3, Theorem 3.1 and equation (2.25), we obtain the following corollary.

Corollary 3.3 *Assume that $(\mathbf{A\,1})$ and $(\mathbf{A\,2})$ hold and suppose that $\mathbf{H}$ given by (2.20) (or equivalently, by (2.22)) is regular. Then the following statements hold true:*

(i) If $\mathbf{G}_0$ is L^2-stable and if $\gamma(\mathbf{H}) > 1$, then there exists sequences (ε_n) and (s_n) with

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad s_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} |\operatorname{Im} s_n| = \infty\,,$$

and such that for any $n \in \mathbb{N}$, $\det \mathbf{\Delta}_{\varepsilon_n}(s_n) = 0$, i.e. for any $n \in \mathbb{N}$ the closed-loop system $(\mathbf{CLS}_{\varepsilon_n})$ has a mode with exponent s_n.

(ii) If $|D| > 1$, then there exists sequences (ε_n) and (s_n) with

$$\varepsilon_n > 0\,, \quad \lim_{n\to\infty} \varepsilon_n = 0\,, \quad s_n \in \mathbb{C}_0\,, \quad \lim_{n\to\infty} \operatorname{Re} s_n = \infty\,, \quad \lim_{n\to\infty} |\operatorname{Im} s_n| = \infty\,,$$

and such that for any $n \in \mathbb{N}$, $\det \mathbf{\Delta}_{\varepsilon_n}(s_n) = 0$, i.e. for any $n \in \mathbb{N}$ the closed-loop system $(\mathbf{CLS}_{\varepsilon_n})$ has a mode with exponent s_n.

On the other hand, if there exists $\varepsilon^* > 0$ such that $\mathbf{G}_\varepsilon$ is spectrally stable for all $\varepsilon \in [0, \varepsilon^*)$, we cannot immediately conclude modal stability of $(\mathbf{CLS}_\varepsilon)$ for all such ε. However, the following result shows that modal stability can be obtained by using Theorem 3.2 and results from [10].

Corollary 3.4 *Assume that $(\mathbf{A\,1})$ and $(\mathbf{A\,2})$ hold and suppose that $(\mathbf{CLS}_0)$ is modally stable. If $\mathbf{H}(s)$ given by (2.20) (or equivalently, by (2.22)) satisfies $\gamma(\mathbf{H}) < 1$, then there exists $\varepsilon^* > 0$ such that $(\mathbf{CLS}_\varepsilon)$ is modally stable for all $\varepsilon \in (0, \varepsilon^*)$.*

Proof: Since $(\mathbf{CLS}_0)$ is modally stable, it follows from Theorem 2.3 that

$$\det(\mathbf{D}(s) + \tilde{\mathbf{N}}(s)\mathbf{N}(s)) \neq 0 \qquad \text{for all } s \in \mathbb{C}_0^{cl}\,. \tag{3.1}$$

Thus $\mathbf{G}_0$ is spectrally stable by (2.25). An application of Theorem 3.2 shows that there exists $\varepsilon^* > 0$ such that

$$\mathbf{G}_\varepsilon \text{ is spectrally stable for all } \varepsilon \in (0, \varepsilon^*)\,. \tag{3.2}$$

Moreover using (3.1), we obtain from lemma 2.4 in [10] that the triple $(\mathbf{N}, \mathbf{D}, \tilde{\mathbf{N}})$ is bi-coprime (in the sense of [10]). Setting $\mathbf{D}_\varepsilon(s) = \mathbf{D}(s) + e^{-\varepsilon s}\tilde{\mathbf{N}}(s)\mathbf{N}(s)$, it follows from the same result that the triple $(\mathbf{N}, \mathbf{D}_\varepsilon, \tilde{\mathbf{N}})$ is bi-coprime. Combining (3.2) with proposition 2.3 in [10] shows that $\det \mathbf{D}_\varepsilon(s) \neq 0$ for all $s \in \mathbb{C}_0^{cl}$, and hence by Theorem 2.3, $(\mathbf{CLS}_\varepsilon)$ is modally stable for all $\varepsilon \in (0, \varepsilon^*)$. □

4. Examples

In this section we illustrate our results by two simple examples.

Example 1

In this example we study the heat equation in a rod made of two different materials, with homogeneous Neumann boundary conditions, localized distributed control input, and "co-located" output. We show that an application of unity output feedback yields a modally stable closed-loop system which remains stable in the presence of small delay in the feedback loop.

Consider the system given by

$$w_t(x,t) = a_1 w_{xx}(x,t) + b(x)u(t)\,, \qquad a_1 > 0, \quad x \in (0,\xi), \quad t > 0\,, \tag{4.1}$$

$$w_t(x,t) = a_2 w_{xx}(x,t) + b(x)u(t)\,, \qquad a_2 > 0, \quad x \in (\xi,1)\,, \quad t > 0\,, \tag{4.2}$$

$$w_x(0,t) = w_x(1,t) = 0, \quad w(\xi^-,t) = w(\xi^+,t), \quad a_1 w_x(\xi^-,t) = a_2 w_x(\xi^+,t)\,, \tag{4.3}$$

where $\xi \in (0,1)$ and

$$b(x) = \chi_{[x_1,x_2]}\,, \quad 0 \le x_1 < x_2 \le 1$$

is the characteristic function for the interval $[x_1,x_2]$. The last condition in (4.3) arises because the flux coming into $x = \xi$ should be equal to the flux leaving $x = \xi$. If x_1 is close to x_2, then the corresponding distributed control is an approximation of boundary or in-span point control, and is perhaps more accurate than the more typical δ-function formulation for such control. We introduce the observation

$$y(t) = k \int_0^1 b(x) w(x,t)\, dx\,, \qquad k > 0. \tag{4.4}$$

We will show that the feedback $u(t) = -y(t)$ is exponentially stabilizing, so in particular, the corresponding closed-loop system is modally stable. Using the results in Section 3 we shall then prove that modal stability is robust with respect to small delays in the loop. To this end, it will be easiest to analyze (4.1)–(4.4) in a state-space setting. Let $X = L^2(0,1)$ and let $\|\cdot\|$ denote the usual L^2-norm. Define

$$(\mathcal{A}z)(x) = \begin{cases} a_1 z_{xx}(x) & \text{for } x \in (0,\xi) \\ a_2 z_{xx}(x) & \text{for } x \in (\xi,1) \end{cases}\,,$$

with

$$\begin{aligned} \mathrm{dom}(\mathcal{A}) = \Big\{ z \,\Big|\, & z|_{(0,\xi)} \in H^2(0,\xi),\ z|_{(\xi,1)} \in H^2(\xi,1), \\ & z_x(0) = z_x(1) = 0,\ z(\xi^-) = z(\xi^+),\ a_1 z_x(\xi^-) = a_2 z_x(\xi^+) \Big\}\,. \end{aligned}$$

It is easy to check that $\mathcal{A}$ is self-adjoint,

$$\langle \mathcal{A}z, z\rangle \le 0 \tag{4.5}$$

for all $z \in \mathrm{dom}(\mathcal{A})$, and that $\mathcal{A}$ has compact resolvent. It is also clear that 0 is an eigenvalue of $\mathcal{A}$ with normalized eigenvector $e(x) \equiv 1$.

Let $\mathcal{B} : \mathbb{R} \to X$ be given by $\mathcal{B}u = b(\cdot)u$ and let $z(t) = w(\cdot,t)$. The system (4.1)–(4.4) is equivalent to the following abstract system with state space X:

$$\dot z(t) = \mathcal{A}z(t) + \mathcal{B}u(t)\,, \tag{4.6}$$

$$y(t) = k\mathcal{B}^* z(t)\,. \tag{4.7}$$

The transfer function can be written as

$$\mathbf{H}(s) = k\mathcal{B}^* R(s, \mathcal{A})\mathcal{B}\,,$$

where $R(s, \mathcal{A}) = (sI - \mathcal{A})^{-1}$. Since $b \in X$, hence $\mathcal{B}$ is a bounded operator on X, it is easy to verify that $\mathbf{H}(s)$ is analytic in $\mathbb{C}_0^{cl} \setminus \{0\}$ and regular, and that

$$\gamma(\mathbf{H}) = 0\,. \tag{4.8}$$

If $u(t) = -y(t)$, (4.6) becomes

$$\dot{z}(t) = (\mathcal{A} - k\mathcal{B}\mathcal{B}^*)z(t)\,.$$

If we can show that $\mathcal{A} - k\mathcal{B}\mathcal{B}^*$ generates an exponentially stable C_0-semigroup, then the closed-loop transfer function $\mathbf{G}_0(s) = k\mathcal{B}^* R(s, \mathcal{A} - k\mathcal{B}\mathcal{B}^*)\mathcal{B}$ is L^2-stable. To this end, note that $\mathcal{A} - k\mathcal{B}\mathcal{B}^*$ is self-adjoint and

$$\langle(\mathcal{A} - k\mathcal{B}\mathcal{B}^*)z, z\rangle = \langle\mathcal{A}z, z\rangle - k|\mathcal{B}^* z|^2\,, \qquad \text{for all } z \in \operatorname{dom}(\mathcal{A})\,. \tag{4.9}$$

Set $X_u = \{\alpha e \mid \alpha \in \mathbb{R}\}$ and $X_s = X_u^{\perp}$. If $z \in X_u$, then $z = \alpha e$ for some $\alpha \in \mathbb{R}$, and thus

$$-k|\mathcal{B}^* z|^2 = -k\alpha^2 \left(\int_0^1 b(x)\,dx\right)^2 = -k\alpha^2(x_2 - x_1)^2 = -m_1\|z\|^2, \qquad \text{for } z \in X_u\,, \tag{4.10}$$

where $m_1 = k(x_2 - x_1)^2$. Moreover, $\mathcal{A}$ is negative self-adjoint on X_s, *i.e.*

$$\langle\mathcal{A}z, z\rangle < -m_2\|z\|, \qquad \text{for } z \in X_s \cap \operatorname{dom}(\mathcal{A}), \tag{4.11}$$

for some $m_2 > 0$. Combining (4.5), (4.9), (4.10) and (4.11), we see that $\mathcal{A} - k\mathcal{B}\mathcal{B}^*$ is negative self-adjoint, hence the generator of a exponentially stable C_0-semigroup on X. Therefore, the closed-loop system is modally stable when there is no delay in the feedback loop. Combining Corollary 3.4 and (4.8) shows that there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the feedback $u(t) = y(t - \varepsilon)$ produces a modally stable closed-loop system.

Example 2

In this example we describe a system with distributed control input which can be stabilized by unity output feedback, but the modal stability of the system is not robust with respect to delays. We consider the following Euler-Bernoulli equation for a cantilever beam with distributed control.

$$w_{tt}(x,t) + w_{xxxx}(x,t) = b_1(x)u_t(t) + b_2(x)u(t)\,, \qquad x \in (0,1), \quad t \geq 0\,, \tag{4.12}$$

$$w(0,t) = w_x(0,t) = w_{xx}(1,t) = w_{xxx}(1,t) = 0\,. \tag{4.13}$$

We consider the observation

$$y(t) = w_{tx}(1,t)\,, \tag{4.14}$$

the angular velocity of w at the right end of the beam.

We first show that if b_1 and b_2 are of a certain form, and if $u(t) = -y(t)$ is an exponentially stabilizing feedback in an appropriate space, then when arbitrarily small delays are introduced into the feedback loop, the closed-loop system has unstable modes. We then construct b_1 and b_2 of the appropriate form which have the property that $u(t) = -y(t)$ is exponentially stabilizing.

It is again easiest to analyze this system with distributed control in a state-space setting. In particular, we will need at a few points in the exposition that the controlled and observed state-space system is *regular*, so we refer the reader to Weiss [16, 17] for notions about admissibility, regularity, well-posedness radii, and the relationship between the generating state-space operators and transfer function for regular systems. Let

$$H = \{(z_1,z_2)^T \mid z_1 \in H^2(0,1),\ z_2 \in L^2(0,1),\ z_1(0) = D_x z_1(0) = 0\}\,,$$

and let

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0 & I \\ -D_x^4 & 0 \end{pmatrix}\,,$$

with domain

$$\begin{aligned}\mathrm{dom}(\tilde{\mathcal{A}}) = \Big\{ &(z_1,z_2)^T \in H \cap [H^4(0,1) \oplus H^2(0,1)] \,\Big|\\ &D_x^2 z_1(1) = D_x^3 z_1(1) = z_2(0) = D_x z_2(0) = 0 \Big\}\,.\end{aligned}$$

It is well known that $\tilde{\mathcal{A}}$ has simple imaginary eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_k \neq 0$, and there exist $a_1, a_2 \in \mathbb{R}$ such that

$$\pi k + a_1 \le \sqrt{|\lambda_k|} \le \pi k + a_2\,. \tag{4.15}$$

Furthermore, the corresponding eigenvectors $\{e_k\}$ of $\tilde{\mathcal{A}}$ form a Riesz basis of H. It follows that $\tilde{\mathcal{A}}$ is a Riesz spectral operator which generates a C_0-semigroup on H (see *e.g.* Curtain and Zwart [2], pp. 41).

It is more convenient to work in the state space

$$X = \mathrm{dom}(\tilde{\mathcal{A}}^{1/2}) = \{\sum_{k\in\mathbb{Z}} \alpha_k e_k \mid \sum_{k\in\mathbb{Z}} |\lambda_k||\alpha_k|^2 < \infty\}\,.$$

$\tilde{\mathcal{A}}$ can be restricted to a generator $\mathcal{A}$ on X with the same eigenvalues, and $\mathcal{A}$ has eigenvectors

$$f_k = |\lambda_k|^{-1/2} e_k,$$

which form a Riesz basis for X. Hence $\mathcal{A}$ generates a C_0-semigroup $S(t)$ on X.

Let $\mathcal{B} : \mathbb{R} \to X$ be given by $\mathcal{B}u = (b_1(\cdot), b_2(\cdot))^T u$. By slight abuse of notation we will occasionally write $\mathcal{B} = (b_1, b_2)^T$ (for the sake of convenience). Then the unobserved system given by (4.12) and (4.13) can be written as

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t)\,.$$

We will restrict attention to those $\mathcal{B}$'s of the form

$$\mathcal{B} = \sum_{k\in\mathbb{Z}} \tilde{\beta}_k e_k = \sum_{k\in\mathbb{Z}} \beta_k f_k, \quad \{\beta_k\}_{k\in\mathbb{Z}} = \{|\lambda_k|^{1/2}\tilde{\beta}_k\}_{k\in\mathbb{Z}} \in l^\infty, \quad \beta_k \neq 0 \quad \text{for any } k \in \mathbb{Z}\,. \tag{4.16}$$

It easily follows from Ho and Russell [8] that $\mathcal{B}$ is an admissible control operator for $S(t)$. In this case $\{\tilde{\beta}_k\}_{k\in\mathbb{Z}} \in l^2$, so that $(b_1, b_2)^T \in H$; in particular, $b_1 \in H^2(0,1)$ and $b_2 \in L^2(0,1)$.

Setting

$$\mathcal{C}(z_1, z_2)^T = D_x z_2(1)\,,$$

then $\mathcal{C}$ is a bounded operator from $\mathrm{dom}(\mathcal{A})$ to $\mathbb{R}$, where the norm on $\mathrm{dom}(\mathcal{A})$ is given by the graph norm of $\mathcal{A}$. Then (4.14) can be formally written as

$$y(t) = \mathcal{C}z(t)\,.$$

Let

$$\gamma_k = \mathcal{C}f_k = |\lambda_k|^{-1/2}\mathcal{C}e_k\,.$$

In the proof of theorem 5 in Rebarber [13] it is shown that there exist numbers $m, M > 0$ such that

$$m \leq |\gamma_k| \leq M\,. \tag{4.17}$$

Results in [8] easily show that $\mathcal{C}^*$ is an admissible control operator for $S^*(t)$, hence $\mathcal{C}$ is an admissible observation operator for $S(t)$.

Since

$$R(s,\mathcal{A})\mathcal{B} = \sum_{k\in\mathbb{Z}} [\beta_k/(s-\lambda_k)]f_k\,,$$

and

$$\sum_{k\in\mathbb{Z}} \beta_k\gamma_k/(s-\lambda_k)$$

converges absolutely for any $s \neq \lambda_k$, an argument similar to that in Weiss [15], proposition 7.2, can be used to show that $R(s,\mathcal{A})\mathcal{B}$ is in the domain of $\mathcal{C}_L$ (the so-called Lebesgue extension of $\mathcal{C}$, described in [15] and [16]), and

$$\mathcal{C}_L R(s,\mathcal{A})\mathcal{B} = \sum_{k\in\mathbb{Z}} \beta_k\gamma_k/(s-\lambda_k)\,.$$

Using (4.15), (4.16), and (4.17), we see that theorem 2.7 in Rebarber [14] applies to this function, hence it is well-posed and regular, with feedthrough 0 and well-posedness radius equal to ∞. In particular, this shows that the system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is regular.

Let $\mathcal{K}(z_1, z_2) = D_x z_2(1)$ with domain $H^4(0,1) \oplus H^2(0,1)$. Then the transfer function for (4.12)–(4.14) is $\mathbf{H}(s) = \mathcal{K}R(s,\mathcal{A})\mathcal{B}$. Since $R(s,\mathcal{A})\mathcal{B} \in \mathrm{dom}(\tilde{\mathcal{A}}) \subset H^4(0,1) \oplus H^2(0,1)$, $\mathcal{K}R(s,\mathcal{A})\mathcal{B}$ and $\mathcal{C}_L R(s,\mathcal{A})\mathcal{B}$ agree for all $s \neq \lambda_k$, so the transfer function for (4.12)–(4.14) is $\mathbf{H}(s) = \mathcal{C}_L R(s,\mathcal{A})\mathcal{B}$. Hence $\mathbf{H}(s)$ is regular and has feedthrough 0.

Suppose now that $\mathcal{B}$ is such that the closed-loop generator $\mathcal{A} - \mathcal{B}\mathcal{C}_L$ generates an exponentially stable semigroup. Then, using the feedback $u(t) = -y(t)$ in (4.12)–(4.14) leads to a system which is modally stable, and the closed-loop transfer function $\mathbf{G}_0 = \mathbf{H}(1+\mathbf{H})$ is in H^∞. Since $\beta_k\gamma_k \neq 0$ for all $k \in \mathbb{Z}$, $\mathbf{H}$ has poles at $\{\lambda_k\}_{k\in\mathbb{Z}}$, so $\gamma(\mathbf{H}) = \infty$. Hence, Corollary 3.3 implies that the modal stability of the closed-loop system given by (4.12)–(4.14) and the feedback law $u(t) = -y(t)$ is not robust with respect to delays.

We now construct $\mathcal{B}$ satisfying (4.16) such that $\mathcal{A} - \mathcal{B}\mathcal{C}_L$ generates an exponentially stable C_0-semigroup. First note that $(\mathcal{A}, \mathcal{C})$ is detectable by $-\mathcal{B}$ if and only if $(\mathcal{A}^*, \mathcal{C}^*)$ is stabilizable by $-\mathcal{B}^*$ (see Weiss and Curtain [18]). Let the biorthogonal set for $\{f_k\}_{k\in\mathbb{Z}}$ be $\{g_k\}_{k\in\mathbb{Z}}$, which is also a Riesz basis for X. In this basis $\mathcal{A}^*$ is a diagonal operator with eigenvalues $\{\bar{\lambda}_k\}_{k\in\mathbb{Z}}$, and $\mathcal{C}^*$ can be represented by

$$\sum_{k\in\mathbb{Z}} \bar{\gamma}_k f_k \, .$$

We now use (4.15), (4.17) and the proof of theorem 2 in Rebarber [12] to construct a $\mathcal{B}^* : \mathrm{dom}(\mathcal{A}^*) \to \mathbb{R}$ of the form

$$\mathcal{B}^* x = \sum_{k\in\mathbb{Z}} \bar{\beta}_k \langle x, g_k \rangle \, , \qquad \{\bar{\beta}_k\}_{k\in\mathbb{Z}} \in l^\infty \, ,$$

such that $\mathcal{A}^* - \mathcal{C}^*\mathcal{B}_L^*$ generates an exponentially stable C_0-semigroup – a similar construction is shown in detail in lemma 2.10 of Rebarber [14]. Hence if we take

$$\mathcal{B} = \sum_{k\in\mathbb{Z}} \beta_k f_k \, ,$$

then this $\mathcal{B}$ is such that $\mathcal{A} - \mathcal{B}\mathcal{C}_L$ generates an exponentially stable C_0-semigroup, completing the argument.

References

[1] J.F. Barman, F.M. Callier and C.A. Desoer. L^2-stability and L^2-instability of linear time-invariant distributed feedback systems perturbed by a small delay in the loop, *IEEE Trans. Automatic Control* **18** (1973), 479–484.

[2] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[3] R. Datko. Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, *SIAM J. Control & Opt.* **26** (1988), 697–713.

[4] R. Datko, J. Lagnese and M.P. Polis. An example of the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control & Opt.* **24** (1986), 152–156.

[5] W. Desch and R.L. Wheeler. Destabilization due to delay in one-dimensional feedback, pp. 61–83 in *Control and Estimation of Distributed Parameter Systems* edited by F. Kappel, K. Kunisch and W. Schappacher, Birkhäuser Verlag, Boston, 1989.

[6] F.R. Gantmacher. *The Theory of Matrices, Vol. I*, Chelsea Publishing Company, New York, 1959.

[7] K.B. Hannsgen, Y. Renardy and R.L. Wheeler. Effectiveness and robustness with respect to time delays of boundary feedback stabilization in one-dimensional viscoelasticity, *SIAM J. Control & Opt.* **26** (1988), 1200–1234.

[8] L.F. Ho and D.L. Russell. Admissible input elementes for systems in Hilbert space and a Carleson measure criterion, *SIAM J. Control & Opt.* **21** (1983), 614–640.

[9] W. Littman and L. Markus. Stabilization of a hybrid system of elasticity by feedback damping, *Ann. Mat. Pura Appl.* **52** (1988), 281–330.

[10] H. Logemann and R. Rebarber. The effect of small-time delays on the closed-loop stability of boundary control systems, to appear in *Mathematics of Control, Signals and Systems.*

[11] H. Logemann, R. Rebarber and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop, *SIAM J. Control & Opt.* **34** (1996), 572–600.

[12] R. Rebarber. Spectral assignability for distributed parameter systems with unbounded scalar control, *SIAM J. Control & Opt.* **27** (1989), 148–169.

[13] R. Rebarber. Spectral determination for a cantilever beam, *IEEE Trans. Automatic Control* **34** (1989), 502–510.

[14] R. Rebarber. Exponential stability of coupled beams with dissipative joints: a frequency domain approach, *SIAM J. Control & Opt.* **33** (1995), 1–28.

[15] G. Weiss. Admissible observation operators for linear semigroups, *Israel J. Math.* **65** (1989), 17–43

[16] G. Weiss. Transfer functions of regular linear systems, part I: characterization of regularity, *Trans. Amer. Math. Soc.* **342** (1994), 827–854.

[17] G. Weiss. Regular linear systems with feedback, *Mathematics of Control, Signals, and Systems* **7** (1994), 23–57.

[18] G. Weiss and R. Curtain. Dynamic stabilization for regular systems, preprint.