Title: Dissipativity of Runge-Kutta methods in Hilbert spaces

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Abstract: This paper concerns the discretization by Runge-Kutta methods of the initial value problem $u_t = f(u)$, under the dissipative structural condition that there exist $\alpha \geq 0$, $\beta > 0$, such that

$$f : W \longrightarrow H, \quad \Re e\langle f(w), w\rangle_H \leq \alpha - \beta |w|_H^2, \quad \forall\, w \in W,$$

for complex Hilbert spaces $W \subseteq H$. It is shown that strong A-stability is necessary to ensure the dissipativity of the method, whilst algebraic stability plus $|R(\infty)| < 1$ is a sufficient condition in the case of DJ-irreducible methods.

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1 Introduction

The Navier-Stokes equation and several other equations in the mathematical sciences satisfy two characteristic criteria. Firstly, they may be written as an ordinary differential equation on a Hilbert space; secondly, the solution obeys a dissipative energy inequality, see e.g. Temam [11]. In this paper, we consider Runge-Kutta discretizations of dissipative differential equations on a Hilbert space, and identify those methods which inherit dissipativity.

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and let W be a dense continuously imbedded Hilbert subspace of H. Let $f : W \longrightarrow H$ be a locally Lipschitz continuous function such that, for some $\alpha \geq 0$, $\beta > 0$,

$$\Re e\langle f(w), w\rangle \leq \alpha - \beta |w|^2, \qquad \forall\, w \in W. \tag{1.1}$$

(The inner product and norm here are those of H, not W.) For such an f, we consider the differential equation

$$u_t = f(u), \quad t > 0, \quad u(0) = u_0 \in H. \tag{1.2}$$

Assuming that (1.2) possesses a solution $u \in C[[0, \infty); H] \cap C[(0, \infty) : W]$, one may take the inner product of (1.2) with $u(t)$, apply (1.1) and integrate over time with integrating factor $e^{2\beta t}$ to obtain

$$|u(t)|^2 \leq e^{-2\beta t}|u_0|^2 + \frac{\alpha}{\beta}(1 - e^{-2\beta t}), \qquad \forall\, t \geq 0. \tag{1.3}$$

If a solution is defined for all $u_0 \in H$, then (1.3) implies that all trajectories enter the ball $B(0, \sqrt{\alpha/\beta} + 1)$ after a time bounded in terms of $|u_0|$. This property of the equation is known as dissipativity, and is central to the dynamics of many systems, see Temam [11].

We shall consider the inheritance of dissipativity by the Runge-Kutta method

$$\begin{aligned} y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i f(g_i), \quad n \geq 0 && (1.4)\\ g_i &= y_n + h \sum_{j=1}^{s} a_{ij} f(g_j), \quad i = 1, \ldots, s, && (1.5) \end{aligned}$$

for $y_0 = u_0$. The case of $H = W = \mathbb{C}^N$ has already been studied by Humphries & Stuart [10], who extended earlier work by Butcher [3] and Burrage & Butcher [1] on monotone systems to show that algebraic stability is a sufficient condition for the inheritance of dissipativity, when the method is DJ-irreducible. Classically, it is known that A-stability is necessary for such inheritance in general. Furthermore, difficulties arising for A-stable methods, when the stability function R satisfies $|R(\infty)| = 1$, have been identified by Ehle [6] and others, see also Hairer & Wanner [8, Section IV.3].

For the case of linear multistep methods approximating (1.1, 1.2), we showed in [9], using the ideas of Dahlquist's [4] G-stability theory, that strong A-stability is both necessary and sufficient for the inheritance of dissipativity in general. Here, we show that, for Runge-Kutta methods, strong A-stability is necessary and that, for DJ-irreducible methods, algebraic stability plus $|R(\infty)| < 1$ is sufficient. Our work leans heavily on that of Burrage & Butcher [2].

2 Results

We follow Hairer & Wanner [8, Chapter IV] in our terminology for Runge-Kutta methods. In particular $A = (a_{ij})$, $b = [b_1, \ldots, b_s]^T$ and $\mathbb{1} = [1, \ldots, 1]^T$. We recall the following definitions.

Definition 2.1. The stability function is defined by

$$R(z) = 1 + zb^T(I - zA)^{-1}\mathbb{1}. \tag{2.1}$$

Since R is rational, $R(z)$ is analytic except at poles and $R(\infty) \equiv \lim_{|z|\to\infty} R(z)$ exists when $\limsup_{|z|\to\infty} |R(z)|$ exists.

Definition 2.2. A method is *A-stable* if $R(z)$ is analytic for $\Re e[z] < 0$, $|R(z)| \le 1$ for $\Re e[z] \le 0$; it is *strongly A-stable* if $|R(\infty)| < 1$ as well.

Definition 2.3. A method is *algebraically stable* if $b_i \ge 0$ for $i = 1, \ldots, s$ and

$$Q = BA + A^TB - bb^T \ge 0, \qquad \text{for } B = \mathrm{diag}[b_1, \ldots, b_s], \tag{2.2}$$

where $Q \geq P$ means that $(Q - P)$ is positive semidefinite.

DEFINITION 2.4. A method is DJ-reducible if, for some non-empty index set $T \subset \{1, \ldots, s\}$,

$$b_j = 0 \quad \text{for } j \in T \qquad \text{and} \qquad a_{ij} = 0 \quad \text{for } i \notin T,\, j \in T.$$

It is otherwise DJ-irreducible.

We recall from Dahlquist & Jeltsch [5], that a DJ-irreducible, algebraically stable method satisfies

$$b_i > 0, \qquad i = 1, \ldots, s. \tag{2.3}$$

Our purpose is to find those methods which inherit dissipativity from all systems of the form (1.1, 1.2) where a solution exists for the method. Since our framework is too general for there to be sufficient information to prove existence, we shall use the following assumption for given f and h.

(A) There exist solution sequences $(y_n) \subset H$ for (1.4, 1.5, 1.1), for all $u_0 \in H$, such that $g_i \in W$, $i = 1, \ldots, s$, for all $n \geq 0$.

DEFINITION 2.5. The Runge-Kutta method (1.4, 1.5) is dissipative, under assumption (1.1), if, for all pairs H and W, all locally Lipschitz continuous $f : W \longrightarrow H$ satisfying (1.1), and all $h > 0$ for which assumption (A) is satisfied, there is an $r > 0$ such that, for all bounded subsets D of H, there is an $n_0(D)$, such that $y_n \in \overline{B}(0, r)$ for all $n \geq n_0$, whenever $u_0 \in D$.

PROPOSITION 2.1. *Dissipative Runge-Kutta methods are strongly A-stable.*

PROOF. For $H = W = \mathbb{C}$ and $f(u) \equiv \lambda u$ with $\Re e[\lambda] < 0$, (1.1) is satisfied for $\alpha = 0$ and $\beta = -\Re e[\lambda] > 0$. Dissipativity of the method and linearity of f imply that $y_n \to 0$ as $n \to \infty$. Thus, $|R(\lambda h)| < 1$ whenever $h > 0$ and $\Re e[\lambda] < 0$. By continuity of rational functions away from poles, $|R(z)| \leq 1$ for $\Re e[z] \leq 0$ and the method is therefore A-stable.

Let $H = L_2(\Omega)$ and $W = H^2(\Omega) \cap H_0^1(\Omega)$, for $\Omega = (0, \pi)$. The second derivative operator, given by

$$u(x) \longrightarrow \frac{d^2u}{dx^2}(x), \quad x \in \Omega,\ u \in C^2(\Omega) \cap C_0^1(\Omega),$$

may be extended, by taking closures in W, to a linear operator $f : W \to H$, where we write $f(w) = w_{xx}$ for $w \in W$. Since

$$\langle f(w),\, w\rangle = \langle w_{xx},\, w\rangle = -|w_x|^2 \leq -|w|^2, \qquad \forall\, w \in W,$$

(1.1) is satisfied for $\alpha = 0$, $\beta = 1$. Fix $h > 0$, and let r be as in the definition of dissipativity. For $u_0(N) = 2r\sqrt{2/\pi}\sin Nx$, $|u_0(N)| = 2r$ and $y_n(N) = R(-hN^2)^n u_0$. Since the method is A-stable and R is rational, $R(\infty)$ exists. If $|R(\infty)| = 1$, then for all $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $|R(-hN^2)^n| > 1/2$, and thus $|y_n(N)| > r$, which contradicts dissipativity for the set $D = \overline{B}(0,\, 2r)$.

□

Before our main sufficiency result, we prove two technical lemmas.

LEMMA 2.2. *Suppose that $R(\infty)$ exists and $\mathbb{1} \in \mathrm{Im}[A]$ (the image of A). Then there is a vector ξ such that $A\xi = \mathbb{1}$ and $R(\infty) = 1 - b^T\xi$.*

PROOF. Since $R(\infty)$ exists, $\mathrm{rank}[A] = d \geq 1$. Taking the singular value decomposition of A, (see Golub & Van Loan [7]), there are orthogonal matrices $U,\, V \in \mathcal{L}(\mathbb{R}^s)$ and values $\sigma_i > 0$, $i = 1,\, \ldots,\, d$, such that

$$U^TAV = D \equiv \begin{bmatrix} \hat{D} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad \hat{D} \equiv \mathrm{diag}[\sigma_1 \ldots,\, \sigma_d].$$

Since $A = UDV^T$ and $\mathbb{1} \in \mathrm{Im}[A]$, $U^T\mathbb{1} = x$, for $x = \begin{bmatrix} \hat{x} \\ 0 \end{bmatrix}$, $\hat{x} = [x_1,\, \ldots,\, x_d]^T$ and $0 \in \mathbb{R}^{s-d}$. Now,

$$\begin{aligned} 1 - R(z) &= zb^T(I - zA)^{-1}\mathbb{1} \\ &= zb^TV(U^TV - zD)^{-1}U^T\mathbb{1} = b^TV(U^TV/z - D)^{-1}x. \end{aligned}$$

It is not difficult to show that as $|z| \to \infty$ the right-hand side tends to $b^T\xi$, where $\xi \equiv V \begin{bmatrix} \hat{D}^{-1}\hat{x} \\ 0 \end{bmatrix}$. Since $A\xi = UDV^T\xi = 1\!\!1$, the proof is complete. $\square$

LEMMA 2.3. *Suppose that* $1\!\!1 \notin \mathrm{Im}[A]$, $b^T 1\!\!1 = 1$ *and* $b_i > 0$ *for* $i = 1, \ldots, s$. *Then there is a* $\gamma \in (0, 1)$ *such that*

$$u^T bb^T u \leq \gamma u^T Bu, \qquad \forall\, u \in \mathrm{Im}[A]. \tag{2.4}$$

PROOF. By linearity, we need only consider $u \in S((\mathbb{R}^s, \|\cdot\|_2); 0, 1) \cap \mathrm{Im}[A]$, a compact sphere. Now, the Cauchy-Schwarz inequality implies

$$u^T bb^T u = \left(\sum_{i=1}^{s} u_i b_i\right)^2 \leq \sum_{i=1}^{s} b_i \sum_{j=1}^{s} b_j u_j^2 = u^T Bu,$$

with equality only if there is a constant $c \in \mathbb{R}$ such that $u_i\sqrt{b_i} = c\sqrt{b_i}$ for $i = 1, \ldots, s$. Hence, strict inequality holds for all $u \in \mathrm{Im}[A] \subseteq \mathbb{R}^s \ominus \mathrm{Span}[1\!\!1]$. Since $(u^T b)^2/u^T Bu$ is continuous on the compact sphere, (2.4) follows. $\square$

THEOREM 2.4. *Consistent, DJ-irreducible, algebraically stable methods, satisfying* $|R(\infty)| < 1$, *are dissipative.*

PROOF: To test dissipativity, we fix $h > 0$ and make assumption (A). As in Burrage & Butcher [2] and Humphries & Stuart [10], we take the inner product of (1.4) with itself, substitute (1.5) and apply (1.1) to eventually obtain the following inequality for arbitrary $k \in (0, 1)$:

$$\begin{aligned} &|y_{n+1}|^2 - k|y_n|^2 - 2h\alpha \\ &\qquad \leq -m_{11}|y_n|^2 - \sum_{i=2}^{s+1} m_{i1}\langle hf_i,\, y_n\rangle - \sum_{j=2}^{s+1} m_{1j}\langle y_n,\, hf_j\rangle - \sum_{i,j=2}^{s+1} m_{ij}\langle hf_i,\, hf_j\rangle, \end{aligned} \tag{2.5}$$

where f_i denotes $f(g_i)$, $(m_{ij}) = M(k, 2h\beta) \in \mathcal{L}(\mathbb{R}^{s+1})$ for

$$M(k, \nu) \equiv \begin{bmatrix} (k + \nu - 1) & \nu 1\!\!1^T BA \\ \nu A^T B 1\!\!1 & Q + \nu A^T BA \end{bmatrix} \tag{2.6}$$

and Q is given by (2.2).

Assuming now that $M(k,\,2h\beta)\geq 0$ for some $k\in(0,\,1)$, (2.5) implies that

$$|y_{n+1}|^2-k|y_n|^2\leq 2h\alpha,\qquad \forall\, n\geq 0.$$

Iterating this inequality recursively, one obtains

$$|y_n|^2\leq k^n|y_0|^2+2h\alpha/(1-k). \tag{2.7}$$

Hence, the method satisfies Definition 2.5 for $r=\sqrt{2h\alpha/(1-k)}+1$. The proof is therefore completed by the following lemma.

LEMMA 2.5. *Under the assumptions of Theorem 2.4, for all $\nu>0$ there is a $k^*\in(0,\,1)$ such that $M(k^*,\,\nu)\geq 0$.*

PROOF. Case I: $\mathbb{1}\notin \mathrm{Im}[A]$. Since $Q\geq 0$,

$$M(k,\,\nu)\geq M_0(k,\,\nu)\equiv\begin{bmatrix}(k+\nu-1) & \nu b^TA\\ \nu A^Tb & \nu A^TBA\end{bmatrix}.$$

For $k\in(1-\nu,\,1)$, we consider $M_1(k,\,\nu)=U^TM_0(k,\,\nu)U$, where

$$U=\begin{bmatrix}1 & 0^T\\ -c(k,\,\nu)A^Tb & I\end{bmatrix},\ M_1(k,\,\nu)=\begin{bmatrix}(k+\nu-1) & 0^T\\ 0 & \nu A^T(B-c(k,\,\nu)bb^T)A\end{bmatrix}$$

and $c(k,\,\nu)=\nu/(k+\nu-1)>0$. Consistency implies that $b^T\mathbb{1}=1$, and (2.3) implies that $b_i>0$ for $i=1,\,\ldots,\,s$. Thus, we may apply Lemma 2.3 to conclude that there is a $\gamma\in(0,\,1)$ such that

$$A^T(B-c(k,\,\nu)bb^T)A\geq(1-\gamma c(k,\,\nu))A^TBA.$$

We observe, therefore, that $M(k^*,\,\nu)\geq 0$ for $k^*=1-(1-\gamma)\nu$.

Case II: $\mathbb{1}\in\mathrm{Im}[A]$ Following Burrage & Butcher's [2] $(k,\,l)$-algebraic stability theory, we consider $\overline{M}(k,\,\nu)=T^TM(k,\,\nu)T$, where

$$T=\begin{bmatrix}1 & 0\\ -\xi & I\end{bmatrix},\qquad \overline{M}(k,\,\nu)=\begin{bmatrix}k-\rho^2 & -\rho b+\xi^TBA\\ -\rho b+A^TB\xi & Q+\nu A^TBA\end{bmatrix},$$

ξ is the vector such that $A\xi = \mathbb{1}$, given by Lemma 2.2, and we write ρ for $R(\infty) = 1 - b^T\xi$.

Now, $\mathbb{R}^s = \mathrm{Ker}[A] \oplus V$, for some space V. $Q = 0 = A^TBA$ on $\mathrm{Ker}[A]$, whilst $Q \geq 0$ and A^TBA is strictly positive definite on V, by algebraic stability and (2.3), respectively. Hence, $A^TBA \geq \epsilon Q$ on $\mathbb{R}^s$, for some $\epsilon > 0$, and

$$\begin{aligned}
\overline{M}(k,\,\nu) \;&\geq\; \begin{bmatrix} k-\rho^2 & -\rho b + \xi^T BA \\ -\rho b + A^T B\xi & Q(1+\epsilon\nu) \end{bmatrix} \\
&\sim\; \overline{M}_0(k,\,\nu) \equiv \begin{bmatrix} (k-\rho^2)(1+\epsilon\nu) & -\rho b + \xi^T BA \\ -\rho b + A^T B\xi & Q \end{bmatrix},
\end{aligned}$$

where '$\sim$' denotes 'congruent to'. Putting $k^* = (1+\epsilon\nu\rho^2)/(1+\epsilon\nu)$, we obtain

$$\overline{M}_0(k^*,\,\nu) = \begin{bmatrix} (1-\rho^2) & -\rho b + \xi^T BA \\ -\rho b + A^T B\xi & Q \end{bmatrix} = T^T \begin{bmatrix} 0 & 0^T \\ 0 & Q \end{bmatrix} T \geq 0.$$

Hence, $M(k^*,\,\nu) \geq 0$. $\square$

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