Travelling-wave solutions for reaction-diffusion-convection systems

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Abstract

We prove the existence of travelling-wave solutions of the parabolic system

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + G(u, \frac{\partial u}{\partial x})\frac{\partial u}{\partial x} + f(u),$$

where the reaction term f is of *'bistable'* type and the convection term G is a nonlinear matrix-valued function satisfying certain growth conditions. A monotone travelling wave connecting two equilibria is shown to exist using degree theory and a homotopy to a convectionless system previously studied by Vol'pert, Vol'pert and Vol'pert. Their theory of reaction-diffusion systems is thereby extended to include nonlinear convection terms.

Résumé

Nous prouvons l'existence d'une onde progressive du système parabolique

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + G(u, \frac{\partial u}{\partial x})\frac{\partial u}{\partial x} + f(u).$$

Ici le terme de réaction f est *'bistable'*, et le terme de convection G est une fonction matricielle non linéaire qui satisfait une condition de croissance. L'existence d'une onde progressive monotone entre deux equilibres est demontrée par la theorie du degré. On homotope à un système sans convection qui était étudié par Vol'pert, Vol'pert et Vol'pert. On étend leur théorie en permettant des termes de convection non linéaires.

1 Introduction

Systems of nonlinear parabolic partial differential equations arise in a vast number of physical and biological models. In [15], Vol'pert and Vol'pert discuss the existence of travelling-wave solutions of the *reaction-diffusion* system

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(u), \qquad u \in \mathbb{R}^N, x \in \mathbb{R}, t \in [0,\infty), \tag{1}$$

where A is a real, positive-definite, $N \times N$ matrix and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuously differentiable nonlinear function. The vector u may represent, for example, the concentrations of chemicals or the population densities of interacting species, the interactions between components of u being modelled by the *reaction* term $f(u)$ and their *diffusion* by $A\dfrac{\partial^2 u}{\partial x^2}$. Travelling waves are solutions u of (1) in the form

$$u(x,t) = w(x - ct). \tag{2}$$

where $w : \mathbb{R} \to \mathbb{R}^N$ is the profile of the wave which propagates through the one-dimensional spatially homogeneous domain at the (*a priori* unknown) constant velocity c.

This paper is concerned with an extension of the Vol'pert theory of travelling waves to systems with nonlinear dependence on the gradient with respect to x of the dependent variable u. Such gradient dependences arise in applications in which there is an underlying drift; for example, chromatography, convection in chemical reactions or wind effects in biology - see Murray [10], pages 292,322 and 420 for details. (Some work has been done on scalar-valued gradient-dependent problems, references to which are contained in [16], page 111, but here we are concerned with systems.) We prove an existence theorem, which reduces to that in [15] when $G \equiv 0$, for the system

$$Aw''(s) + cw'(s) + \varsigma(s)G(w(s), w'(s))w'(s) + f(w(s)) = 0. \tag{3}$$

Here, as in [15], A is a positive-definite diagonal matrix and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuously differentiable function satisfying

(f1) f is locally monotone;

(f2) $f(S) = f(T) = 0$ where $S < T$;

(f3) S and T are both stable equilibria of f;

(f4) there is a non-zero finite number m of equilibria u^k of f with $S < u^k < T$ ($k = 1, \ldots, m$); furthermore, each such u^k is unstable.

(Terms such as *locally monotone* and *stable* and the partial ordering of vectors, which are standard, are recalled at the end of this Introduction.) Also, $\varsigma : \mathbb{R} \to [-1,1]$ is uniformly continuous on $\mathbb{R}$ and $G : \mathbb{R}^N \times \mathbb{R}^N \to M^{N\times N}$ and

(G1) G is a continuously differentiable, diagonal-matrix-valued function;

(G2) there exist continuous functions $\beta, \gamma : \mathbb{R}^N \to \mathbb{R}_+$ such that for each $u, v \in \mathbb{R}^N$,

$$\|G(u,v)v\| \leq \beta(u) + \gamma(u)\|v\|,$$

where $\|\cdot\|$ denotes a norm on $\mathbb{R}^N$;

(G3) $G(u,0) = 0$ for each $u \in \mathbb{R}^N$ with $f(u) = 0$.

When $\varsigma \equiv 1$, a special case of our result gives the existence of a monotone travelling-wave solution of the system of *reaction-diffusion-convection* equations

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + G(u, \frac{\partial u}{\partial x})\frac{\partial u}{\partial x} + f(u). \tag{4}$$

The Vol'pert existence theory for travelling wave solutions of (1) is degree theoretic. They reformulate (3) with $G \equiv 0$ as a problem involving $(S)_+$ operators in a weighted Sobolev space. Roughly speaking, $(S)_+$ operators are of the form *compact* + *monotone* and topological degree theory for them was introduced in the 1970s, independently by Browder [4] and Skrypnik [13]. Our idea for treating the general case of (3) is to homotopy from the gradient-dependent system to the system with $G \equiv 0$, and thereby to deduce the required existence result for (3) from the known existence of travelling-wave solutions of (1) with non-zero degree, as established in [15]. Although the operators associated with (3) are not in the domain of the degree function for $(S)_+$ operators except when $G \equiv 0$, that difficulty can be overcome by approximation, followed by a limiting argument. We first consider the modified system

$$Av''(s) + cv'(s) + \sigma_R(s)\varsigma(s)G(v(s), v'(s))v'(s) + f(v(s)) = 0, \ s \in \mathbb{R} \tag{5}$$

where for $R > 1$, $\sigma_R \in C(\mathbb{R}, [0,1])$ is supported in $[-R, R]$, and $\sigma_R(s) \to 1$ as $R \to \infty$ for each $s \in \mathbb{R}$. The corresponding operators are shown to be $(S)_+$. The fact that the system is bistable yields the monotone term. The remainder is compact because of the use of weighted spaces in which the weights grow at infinity and of the fact that σ_R has compact support. For each $R > 1$, the existence of a monotone solution to (5) asymptotic to S and T is proved via a Leray-Schauder principle which be can be established using $(S)_+$ degree theory; solutions to (3) are then obtained by taking the limit of these solutions as $R \to \infty$.

The paper is organized as follows. In Section 2 problem (5) is formulated in terms of $(S)_+$ operators in weighted Sobolev spaces and the appropriate degree theory is reviewed. Krasnosel'skii's idea of "functionalizing the parameter" is recalled from [15] and the unknown velocity c is replaced by a functional of u without sacrifice of generality. The rest of the section is concerned with showing that for each $R > 1$, an operator associated with (5) in a weighted function space is $(S)_+$. The compact support of σ_R and **(G2)** are vital for the compactness of the contribution of the term $\sigma_R \varsigma G(v, v')v'$.

Section 3 deals with uniform *a priori* estimates for monotone solutions of the approximate system (5). The aim is to prove that, *independent* of R, the difference between a monotone solution of (5) and a fixed C^∞ function ψ lies in a bounded set in a weighted Sobolev space. In fact we establish some estimates for problems more general than (5), in which σ_R is replaced by σ, a uniformly continuous function, bounded between -1 and 1 but not necessarily of compact support. The bounds obtained are *independent* of the choice of σ. This independence leads to the required Sobolev-space bound; here essential use is made of conditions **(G1)**, **(G2)** and **(G3)**.

The existence of travelling-wave solutions for our class of reaction-diffusion-convection systems is obtained in Section 4. The uniform bounds from Section 3 give the existence of a weak limit u of a sequence $\{u_n\}$, where $u_n + \psi$ is a monotone solution of (5) corresponding to $R = n \in \mathbb{N}$. We show that $v = u + \psi$ is a monotone solution of (3) and ensure that the asymptotic boundary conditions are satisfied - that is, $v(s) \to S, T$ as $s \to -, +\infty$.

1.1 Preliminaries

We recall some standard definitions and notation. The notation ':=' means 'is defined to equal'. Denote the usual inner products and corresponding norms on $\mathbb{R}^N$ and $\mathbb{C}^N$ by $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$, $\|\cdot\|_{\mathbb{C}^N}$ respectively.

For vectors $p, q \in \mathbb{R}^N$, we say that $p < q$ $(p \le q)$ if $p_k < q_k$ $(p_k \le q_k)$ for each k, $1 \le k \le N$ and p_k denotes the k^{th} coordinate of p. When $q < p$, (q, p) is the set $\{x \in \mathbb{R}^N : q < x < p\}$ and $[q, p]$ is the set $\{x \in \mathbb{R}^N : q \le x \le p\}$. A vector $x \in \mathbb{R}^N$ is *positive* (*non-negative*) if all its components are positive (non-negative). We write $x > 0$ $(x \ge 0)$. We denote the positive cone in $\mathbb{R}^N$ of non-negative vectors by $\mathbb{R}^N_+$; when $N = 1$, $\mathbb{R}^1_+$ is abbreviated to $\mathbb{R}_+$ (that is, $\mathbb{R}_+ := [0, \infty)$). The set of all real $N \times N$ matrices with strictly positive off-diagonal elements is denoted by $P^{N\times N}$. The set of all real $N \times N$ matrices is denoted as usual by $M^{N \times N}$. The entry in the $i-th$ row and $j-th$ column of a matrix M is denoted by M_{ij}. If $M \in M^{N \times N}$ is diagonal, then we use the shorthand M_i for the $i-th$ element on the diagonal of M. Throughout, A denotes a real, positive-definite diagonal matrix.

For a Fréchet differentiable function $f : \mathbb{R}^N \to \mathbb{R}^N$, we denote the Fréchet derivative of f at $w \in \mathbb{R}^N$ by $df[w]$. Recall that $df[w]$ can be identified with its Jacobian matrix, the matrix of partial derivatives of f at w with respect to the standard basis of $\mathbb{R}^N$. Without further comment, we will use the notation $df[w]$ to denote the derivative or its Jacobian matrix, as appropriate.

Let Ω be an open subset of a real finite-dimensional space (for example, $\mathbb{R}, \mathbb{R}^N$, $M^{N \times N}$, $\mathbb{R}^N \times \mathbb{R}^N$). Denote the closure of Ω by $\overline{\Omega}$, the boundary by $\partial\Omega$ and the interior by Ω^o. Let Y be a subset of a (possibly different) real finite-dimensional space. Then for $k \in \mathbb{N}$, we write $C^k(\Omega, Y)$ for the linear space of functions $f : \Omega \to Y$ which, together with all the partial derivatives of each component function of f of order at most k are continuous on Ω. Let $C^\infty(\Omega, Y) = \cap_{k=0}^{\infty} C^k(\Omega, Y)$, and denote by $C_0^\infty(\Omega, Y)$ the subspace of $C^\infty(\Omega, Y)$ consisting of those functions with compact support in Ω.

Let $C^k(\overline{\Omega}, Y)$ be comprised of those $f \in C^k(\Omega, Y)$ for which all the partial derivatives of each component function of f of order at most k are bounded and uniformly continuous on Ω. Then $C^k(\overline{\Omega}, Y)$ is a Banach space with norm

$$\|f\|_{C^k(\overline{\Omega}, Y)} = \Sigma_{i,|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha f_i| \tag{6}$$

where $D^\alpha f_i$ denotes the partial derivative of the i^{th} component of f given by the multi-index α. Note that we may suppress explicit mention of the set Y if it is clear from the context.

Now let Ω be an open subset of $\mathbb{R}$. Let $L_\infty(\Omega, \mathbb{R}^N)$ comprise of equivalence classes of measurable functions $u : \Omega \to \mathbb{R}^N$ that are essentially bounded; this space is endowed with the norm

$$\|u\|_{L_\infty(\Omega, \mathbb{R}^N)} = \text{ ess sup}_{x \in \Omega} \|u(x)\|, \tag{7}$$

Denote by $L_2(\Omega, \mathbb{R}^N)$ the linear space of equivalence classes of measurable functions $u : \Omega \to \mathbb{R}^N$ such that

$$\|u\|_{L_2(\Omega, \mathbb{R}^N)} := \langle u, u \rangle_{L_2(\Omega, \mathbb{R}^N)} < \infty, \tag{8}$$

where for u, v measurable, the inner product is given by

$$\langle u, v\rangle_{L_2(\Omega,\mathbb{R}^N)} := \int_\Omega \langle u(s), v(s)\rangle ds. \tag{9}$$

We write $W_2^1(\Omega,\mathbb{R}^N)$ for the subset of $L_2(\Omega,\mathbb{R}^N)$ consisting of functions $u \in L_2(\Omega,\mathbb{R}^N)$ for which $\mathcal{D}u \in L_2(\Omega,\mathbb{R}^N)$, where $\mathcal{D}u$ is the vector of weak derivatives of the components of u. Recall from Adams [1] that the Sobolev space $W_2^1(\Omega,\mathbb{R}^N)$ is a separable Hilbert space, with inner product

$$\langle u, v\rangle_{W_2^1(\Omega,\mathbb{R}^N)} := \langle u, v\rangle_{L_2(\Omega,\mathbb{R}^N)} + \langle \mathcal{D}u, \mathcal{D}v\rangle_{L_2(\Omega,\mathbb{R}^N)} \quad (u, v \in W_2^1(\Omega,\mathbb{R}^N)) \tag{10}$$

and norm $\|\cdot\|_{W_2^1(\Omega,\mathbb{R}^N)}$, and $C_0^\infty(\mathbb{R},\mathbb{R}^N)$ is dense in $W_2^1(\mathbb{R},\mathbb{R}^N)$. If Ω is clear from the context, then we will suppress explicit mention of it in the subscripts on $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ and write $\langle\cdot,\cdot\rangle_{L_2}$, etcetera.

The spaces of complex-valued functions $\tilde{L}_2(\mathbb{R},\mathbb{C}^N)$, $\tilde{W}_2^1(\mathbb{R},\mathbb{C}^N)$, $\tilde{C}_0^\infty(\mathbb{R},\mathbb{C}^N)$ will occasionally be required. A function $u : \mathbb{R} \to \mathbb{C}^N$ belongs to $\tilde{L}_2(\mathbb{R},\mathbb{C}^N)$ if

$$\int_\mathbb{R} \langle u(s), u(s)\rangle_{\mathbb{C}^N} ds < \infty. \tag{11}$$

$u : \mathbb{R} \to \mathbb{C}^N$ belongs to $\tilde{W}_2^1(\mathbb{R},\mathbb{C}^N)$ if $u, \mathcal{D}u \in \tilde{L}_2(\mathbb{R},\mathbb{C}^N)$. $\tilde{C}_0^\infty(\mathbb{R},\mathbb{C}^N)$ is comprised of functions $u : \mathbb{R} \to \mathbb{C}^N$ whose real and imaginary parts belong to $\tilde{C}_0^\infty(\mathbb{R},\mathbb{R}^N)$.

We now turn to definitions required to describe the system (3). An *equilibrium* of $f \in C^1(\mathbb{R}^N,\mathbb{R}^N)$ is a point $w_0 \in \mathbb{R}^N$ where $f = 0$. An equilibrium is said to be *stable* if all the eigenvalues of $df[w_0]$ are in the open left-half complex plane, and *unstable* if there is an eigenvalue of $df[w_0]$ in the open right-half plane.

Now let $S, T \in \mathbb{R}^N$ be equilibria of f such that $S < T$. $E \in \mathbb{R}^N$ is said to be an intermediate equilibrium of f if $f(E) = 0$ and $S_k \le E_k \le T_k$ for each $k \in \{1,\dots,N\}$. (The existence of such intermediate equilibria is discussed in Crooks [6].) The system (1) is said to be *bistable* if both S and T are stable equilibria of f. In [15] it is shown that for some $c \in \mathbb{R}$, there exists a heteroclinic orbit $\{w(s) : s \in \mathbb{R}\}$ of (3) with $G \equiv 0$, the components of which v_k are monotone and asymptotic to S_k and T_k as $s \to -\infty$ and $s \to \infty$ respectively. Such solutions are said to be *monotone*, They are sometimes called *travelling fronts* or *monotone connections*, and correspond to smooth transitions between steady states of the systems being modelled by (1).

Attention in this paper, and in [15], is restricted to functions f which are *locally monotone*: whenever $u \in \mathbb{R}^N$ is such that $f_i(u) = 0$ for some i, $1 \le i \le N$

$$\frac{\partial f_i(u)}{\partial u_j} > 0 \text{ for } j \neq i, \;\; j = 1, \dots, N, \tag{12}$$

where f_i denotes the i^{th} component function of f. This is a natural hypothesis for mutualistic interactions in biology, and for certain kinds of chemical kinetics. Such a monotonicity property allows exploitation of Perron-Frobenius theory. If $w_0 \in \mathbb{R}^N$ is such that $\dfrac{\partial f_i}{\partial u_j}(w_0) > 0$ for $i \neq j$, $i, j = 1, \dots, N$, then the off-diagonal elements of $df[w_0]$ are positive - that is, $df[w_0]$ is a Perron-Frobenius matrix. Hence if f satisfies the local monotonicity property (12), then $df[u]$ is a Perron-Frobenius matrix for each $u \in \mathbb{R}^N$ with $f(u) = 0$. The following well-known version of the Perron-Frobenius theorem (see, for example, Seneta [11]) will be useful.

Theorem 1.1 *(Perron-Frobenius). Let $M \in P^{N\times N}$. Then M has a real, simple eigenvalue $\mu_{PF}(M)$ such that an associated eigenvector is positive , and every other eigenvalue of M has real part less than $\mu_{PF}(M)$. Moreover, any positive eigenvector must be a multiple of the eigenvector corresponding to $\mu_{PF}(M)$. We call $\mu_{PF}(M)$ the Perron-Frobenius eigenvalue of M, and the associated positive eigenvector x with $\|x\| = 1$ the Perron-Frobenius eigenvector of M.*

Such eigenvalues satisfy the following property - see, for example, Varga [14].

Theorem 1.2 *Increasing (decreasing) elements of a matrix $M \in P^{N\times N}$ increases (decreases) $\mu_{PF}(M)$.*

We close the Introduction by noting the following useful consequence of condition **(G3)**.

Lemma 1.3 *Let Ω be a bounded subset of $\mathbb{R}^N$ and let $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ satisfy* **(G2)**. *Then there exists a positive constant α^*, depending on Ω, such that for each $u \in \Omega$,*

$$\|G(u, v)v\| \le \alpha^* \|v\|$$

for every $v \in \mathbb{R}^N$.

Proof. As G is continuous, there exists $\alpha_0 > 0$ such that

$$\|G(u,v)\| \leq \alpha_0 \tag{13}$$

for each $u \in \Omega$ and $v \in \mathbb{R}^N$ with $\|v\| \leq 1$. Also, if $v \in \mathbb{R}^N$ has $\|v\| \geq 1$, then by **(G2)**,

$$\|G(u,v)v\| \leq \beta(u) + \gamma(u)\|v\| \leq [\beta(u) + \gamma(u)]\|v\| \tag{14}$$

for each $u \in \mathbb{R}^N$. Thus (13) and (14) yield that

$$\|G(u,v)v\| \leq \max\{\alpha_0, \beta(u) + \gamma(u)\}\|v\| \tag{15}$$

for each $u \in \Omega, v \in \mathbb{R}^N$. The result follows.

□

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2 Degree theory for the analysis of travelling waves

We begin the present section with some technical machinery on weighted function spaces and degree theory. We then turn to a precise formulation of the approximate travelling-wave problem (5) in this language. The rest of the section is concerned with the non-trivial task of showing that operators associated with (5) lie in the domain of definition of the degree.

2.1 Technical machinery

2.1.1 Weighted function spaces

Let $\mu(s) := 1 + s^2$ $(s \in \mathbb{R})$. Let Ω be an open subset of $\mathbb{R}$. Then we define the *weighted* space $L_{2,\mu}(\Omega, \mathbb{R}^N)$ to be the linear space of equivalence classes of measurable functions $u : \mathbb{R} \to \mathbb{R}^N$ such that

$$\|u\|_{L_{2,\mu}(\Omega,\mathbb{R}^N)} := \langle u, u\rangle^{\frac{1}{2}}_{L_{2,\mu}(\Omega,\mathbb{R}^N)} < \infty, \tag{16}$$

with inner product

$$\langle u, v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)} := \int_\Omega \langle u(s), v(s)\rangle \mu(s) ds,\ (u, v \in L_{2,\mu}(\Omega, \mathbb{R}^N)). \tag{17}$$

The weighted Sobolev space $W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ is then defined to be the subspace of $L_{2,\mu}(\Omega, \mathbb{R}^N)$ comprising of functions $u \in L_{2,\mu}(\Omega, \mathbb{R}^N)$ for which $\mathcal{D}u \in L_{2,\mu}(\Omega, \mathbb{R}^N)$. of the components of u.) This space is endowed with the inner product

$$\langle u, v\rangle_{W^1_{2,\mu}(\Omega,\mathbb{R}^N)} := \langle u, v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)} + \langle \mathcal{D}u, \mathcal{D}v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)},\ (u, v \in W^1_{2,\mu}(\Omega, \mathbb{R}^N)) \tag{18}$$

and norm $\|\cdot\|_{W^1_{2,\mu}(\Omega,\mathbb{R}^N)}$. For notational convenience, we make the following definitions;

$$\omega(s) := \sqrt{\mu(s)} \text{ and } \nu(s) := \mu^{-1}(s)\mu'(s),\ \ s \in \mathbb{R}. \tag{19}$$

Lemma 2.1 *The linear multiplication operators* $M : W^1_2(\Omega, \mathbb{R}^N) \to W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ *and* $N : W^1_{2,\mu}(\Omega, \mathbb{R}^N) \to W^1_2(\Omega, \mathbb{R}^N)$, *defined by*

$$Mu(s) = \omega^{-1}(s)u(s) \quad (u \in W^1_2(\Omega, \mathbb{R}^N)) \text{ and} \tag{20}$$

$$Nu(s) = \omega(s)u(s) \qquad (u \in W^1_{2,\mu}(\Omega, \mathbb{R}^N)) \tag{21}$$

respectively, are bounded.

Proof. Let $u \in W^1_{2,\mu}(\Omega, \mathbb{R}^N)$. Then

$$\begin{aligned}\|Mu\|^2_{W^1_{2,\mu}(\Omega,\mathbb{R}^N)} &= \int_\Omega \left\{\|\omega^{-1}(s)u(s)\|^2 + \|\omega^{-1}(s)\mathcal{D}u(s) - \frac{1}{2}\omega^{-1}(s)\nu(s)u(s)\|^2\right\}\mu(s)\, ds \\ &\leq \int_\Omega \left\{\|u(s)\|^2 + 2\|\mathcal{D}u(s)\|^2 + \frac{1}{2}\|\nu(s)u(s)\|^2\right\} ds \\ &\leq 2\|u\|^2_{W^1_2(\Omega,\mathbb{R}^N)},\end{aligned}$$

using the fact that $|\nu(s)| = \dfrac{2|s|}{1+s^2} \leq 1$ for each $s \in \mathbb{R}$. The result follows for M; the argument for N is similar.

□

Using Lemma 2.1, we deduce from the corresponding properties of $W^1_2(\Omega, \mathbb{R}^N)$ (see section 2.1) that $W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ is a separable Hilbert space. It also follows that $C^\infty_0(\mathbb{R}, \mathbb{R}^N)$ is dense in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. This enables proof of the following vital estimate.

Lemma 2.2 *For $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,*

$$\|u(s)\| \leq \frac{\|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}}{\sqrt{\mu(s)}} \tag{22}$$

for each $s \in \mathbb{R}$.

Proof. Let $u \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$. Then for $s \leq 0$,

$$\begin{aligned}
\|u(s)\|^2 &= 2\int_{-\infty}^{s} \langle u(t), u'(t)\rangle \, dt \\
&= 2\int_{-\infty}^{s} \langle u(t), u'(t)\rangle \mu(t) \frac{1}{\mu(t)} \, dt \\
&\leq \frac{2}{\mu(s)} \int_{-\infty}^{s} \|u(t)\| \|u'(t)\| \mu(t) \, dt \\
&\leq \frac{1}{\mu(s)} \int_{-\infty}^{s} \left\{ \|u(t)\|^2 + \|u'(t)\|^2 \right\} \mu(t) \, dt \\
&\leq \frac{\|u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}}{\mu(s)}.
\end{aligned}$$

A similar argument, integrating from s to ∞, shows that (22) holds for $s > 0$. Now $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and if a sequence converges in $L_2(\mathbb{R}, \mathbb{R}^N)$, then a subsequence converges pointwise almost everywhere. Hence (22) holds for almost every $s \in \mathbb{R}$. Since

$$W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \subset W^1_2(\mathbb{R}, \mathbb{R}^N) \subset C(\mathbb{R}, \mathbb{R}^N),$$

the result follows.

$\square$

2.2 Degree theory for $(S)_+$ operators

A degree theory applicable here is that for operators of class $(S)_+$, as developed in [13, 4, 2], and discussed in [15].

Let X denote a real, reflexive, separable Banach space, and let $W : X \to X^*$. Then W is of class $(S)_+$, or $W \in (S)_+$, if for $\{u_n\} \subset X$,

$$\left.\begin{array}{r} u_n \rightharpoonup u_o \text{ in } X \\ \text{and } \limsup\limits_{n\to\infty} (W(u_n))(u_n - u_o) \leq 0 \end{array}\right\} \Rightarrow u_n \to u_o \text{ in } X. \tag{23}$$

Suppose that W is also bounded and demicontinuous (continuous from the strong topology of X to the weak topology of X^*). Let Ω be a bounded open subset of X; denote the closure and boundary of Ω by $\overline{\Omega}$ and $\partial\Omega$ respectively. If

$$0 \notin W(\partial\Omega),$$

then an integer-valued degree, $deg_{(S)_+}(W, \Omega, 0)$, is defined via Galerkin approximations.

The usual properties of a degree function hold. If $deg_{(S)_+}(W, \Omega, 0) \neq 0$, then there exists $u \in \Omega$ with $W(u) = 0$. Also, the degrees of functions which are $(S)_+$-homotopic relative to Ω are equal; if $W_1, W_2 : X \to X^*$ are bounded demicontinuous $(S)_+$ mappings, W_1 and W_2 are said to be $(S)_+$- *homotopic relative to* Ω if there is a bounded demicontinuous operator $W : \overline{\Omega} \times [0,1] \to X^*$ such that

$$(i) \qquad 0 \notin W(\partial\Omega \times [0,1]) \tag{24}$$

$$(ii) \qquad W(u,0) = W_1(u), W(u,1) = W_2(u) \text{ for all } u \in \overline{\Omega}, \text{ and} \tag{25}$$

$$(iii) \qquad \text{For } \{u_n, t_n\} \subset \overline{\Omega} \times [0,1],$$

$$\left.\begin{array}{c} u_n \rightharpoonup u_o \text{ in } X, \\ t_n \to t_o, \text{ and} \\ \limsup\limits_{n\to\infty} (W(u_n, t_n))(u_n - u_o) \leq 0 \end{array}\right\} \Rightarrow u_n \to u_o \text{ in } X. \tag{26}$$

2.3 Preliminaries for the approximate system

2.3.1 Formulation of the problem

Let $R > 1$ and $\sigma_R \in C(\mathbb{R}, [-1,1])$ be given by

$$\sigma_R(s) := \begin{cases} 0, & s \leq -R \\ \frac{1}{R}s + 1, & -R < s \leq 0 \\ -\frac{1}{R}s + 1, & 0 < s \leq R \\ 0, & s > R. \end{cases} \tag{27}$$

It is straightforward to see that $\{\sigma_R\}$ is uniformly bounded and equicontinuous as a family parametrised by R. We seek a constant c and a function $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that

$$Aw''(s) + cw'(s) + \sigma_R(s)\varsigma(s)G(w(s), w'(s))w'(s) + f(w(s)) = 0, \; s \in \mathbb{R}, \tag{28}$$

$$w'(s) > 0, \; s \in \mathbb{R}, \tag{29}$$

and

$$w(s) \to S \text{ as } s \to -\infty, \quad w(s) \to T \text{ as } s \to \infty. \tag{30}$$

Let

$$\psi(s) = T\alpha(s) + S(1 - \alpha(s)), \tag{31}$$

and seek w in the form

$$w(s) = u(s) + \psi(s), \tag{32}$$

where $\alpha \in C^\infty(\mathbb{R}, [0,1])$ is an arbitrary but fixed monotone function with $\alpha(s) = 0$ when $s \leq -1$ and $\alpha(s) = 1$ when $s \geq 1$. Then w satisfies (28), (29) and (30) if and only if u satisfies

$$A(u''(s) + \psi''(s)) + c[u'(s) + \psi'(s)] + \sigma_R(s)\varsigma(s)G(u(s) + \psi(s), u'(s) + \psi'(s))(u'(s) + \psi'(s))$$

$$+f(u(s) + \psi(s)) = 0, \tag{33}$$

$$u'(s) + \psi'(s) > 0, \quad s \in \mathbb{R} \tag{34}$$

and

$$u(s) \to 0 \text{ as } s \to \pm\infty. \tag{35}$$

Following [15], we seek a solution u to (33), (34) and (35) in the weighted Sobolev space $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ defined in section 5.1.1. Note that estimate (22) implies that (35) is satisfied for *every* $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. The fact that the constant c is not known *a priori* is overcome by functionalising the parameter. This idea, due to Krasnosel'skii, is developed in [15]. We review some key points. Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(s) := \begin{cases} e^s & \text{, if } s < 0 \\ 1 & \text{, if } s \geq 0. \end{cases} \tag{36}$$

For $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, set

$$\rho(u) := \left(\int_{\mathbb{R}} \|u(s) + \psi(s) - T\|^2 \phi(s)\, ds \right)^{\frac{1}{2}} \tag{37}$$

and

$$c(u) := \log\, \rho(u), \tag{38}$$

where ψ is defined in (31). Define

$$u_h(s) := u(s+h) + \psi(s+h) - \psi(s),\ s \in \mathbb{R}, \tag{39}$$

for each $h \in \mathbb{R}$. Then $c(\cdot)$ has the following two properties, which are proved in [15].

Lemma 2.3 *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Then $c(u_h)$ is monotone in h, and*

$$c(u_h) \to \pm\infty \text{ as } h \to \mp\infty.$$

Lemma 2.4 *The functional c is Lipschitz continuous on bounded subsets of $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.*

Now consider the equation

$$A(u''(s)+\psi''(s))+c(u)[u'(s)+\psi'(s)]+\sigma_R(s)\varsigma(s)G(u(s)+\psi(s),u'(s)+\psi'(s))(u'(s)+\psi'(s))$$

$$+f(u(s)+\psi(s)) = 0. \tag{40}$$

It is clear that if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ satisfies (40), then u satisfies (33) with $c = c(u)$. Conversely, if a function u satisfies (33) with velocity c, then there is some $h \in \mathbb{R}$ with $c(u_h) = c$. Thus u_h satisfies (40) with $\sigma_R(\cdot)\varsigma(\cdot)$ replaced by $\sigma_R(\cdot + h)\varsigma(\cdot + h)$. Whence there is an equivalence between (33) and (40). Henceforth, a solution of (40) and (34) will be sought in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ (recall that (35) is satisfied for all functions in this space).

Our approach is to consider the following continuous deformation of (40):

$$A(u''+\psi'')+c(u)[u'+\psi']+\tau\sigma_R\varsigma G(u+\psi,u'+\psi')(u'+\psi')+f(u+\psi) = 0,\ \tau \in [0,1]. \tag{41}$$

When $\tau = 1$, (41) becomes (40), whilst when $\tau = 0$, (41) corresponds to the system treated in [15]. We will use the known existence of a solution with non-zero degree when $\tau = 0$ to deduce existence when $\tau = 1$ using the degree theory for $(S)_+$ operators introduced in section 5.1.2.

2.4 Construction of $(S)_+$ operators

To invoke the degree theory for $(S)_+$ operators, an $(S)_+$ operator associated with (40) is required, together with a suitable $(S)_+$ homotopy associated with (41). First, define $P_R : [0,1] \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$ by

$$\begin{aligned}(P_R(\tau, u))(v) = \int_{\mathbb{R}} \langle A\mathcal{D}u(s), \mathcal{D}(v\mu)(s)\rangle \, ds \; - \\ \int_{\mathbb{R}} \langle \{A\psi''(s) + c(u)[\mathcal{D}u(s) + \psi'(s)] + \tau\sigma_R\varsigma G(u(s) + \psi(s), \mathcal{D}u(s) + \psi'(s))[\mathcal{D}u + \psi'(s)] \\ + f(u(s) + \psi(s))\}, v(s)\rangle \mu(s) \, ds \end{aligned} \tag{42}$$

for $\tau \in [0,1]$, $u, v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.

In order to minimise clutter, independent variables will be omitted when no confusion results.

Lemma 2.5 *P_R is well-defined and maps each bounded subset of $[0,1]\times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ into a bounded subset of $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$.*

Proof. Let $\tau \in [0,1], \Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ be bounded, and $u \in \Omega$. The linearity of $P_R(\tau, u)$ is elementary. Let $v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Use of the Cauchy-Schwarz inequality in $L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and the fact that $\mu^{-1}\mu'$ is uniformly bounded yields that

$$\begin{aligned}\left|\int_{\mathbb{R}} \langle A\mathcal{D}u, \mathcal{D}(v\mu)\rangle \, ds\right| &= \left|\int_{\mathbb{R}} \langle A\mathcal{D}u, \mu^{-1}\mu' v + \mathcal{D}v\rangle \mu \, ds\right| \\ &\le \alpha_1 \|\mathcal{D}u\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \|v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \alpha_2 \|\mathcal{D}u\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \|\mathcal{D}v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \\ &\le \alpha_3 \|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \end{aligned} \tag{43}$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ independent of u and v.

Lemma 2.4, together with the fact that ψ' and ψ'' have compact support, give the existence of $\alpha_4(\Omega) \in \mathbb{R}$ (dependent only on Ω), such that

$$\left|\int_{\mathbb{R}} \langle A\psi'' + c(u)[\mathcal{D}u + \psi'], v\rangle \mu \, ds\right| \le \alpha_4(\Omega)(1 + \|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}) \|v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}. \tag{44}$$

Now using condition **(G2)**,

$$\begin{aligned}&\int_{\mathbb{R}} \|\tau\sigma_R\varsigma G(u + \psi, \mathcal{D}u + \psi')[\mathcal{D}u + \psi']\|^2 \mu \, ds \\ &\le \int_{-R}^{R} [\beta(u + \psi) + \gamma(u + \psi)\|\mathcal{D}u + \psi'\|]^2 \mu \, ds. \end{aligned} \tag{45}$$

$\mathcal{D}u \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, β and γ are continuous, and by estimate (22), $\|u(s)\|$ is bounded by $\|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}$ for each $s \in \mathbb{R}$. Thus the right-hand side of (45) is bounded independently of $u \in \Omega$, $\tau \in [0, 1]$.

Also, since $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, for $s \in \mathbb{R}$,

$$f(u(s) + \psi(s)) = f(\psi(s)) + b(s)u(s), \tag{46}$$

where

$$b(s) = \int_0^1 df[tu(s) + \psi(s)]\, dt. \tag{47}$$

It follows once again from estimate (22) and the fact that $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ that $b(s)$ is bounded independently of $s \in \mathbb{R}$ and $u \in \Omega$. Using estimate (22) together with the facts that $f(\psi(s)) = 0$ when $|s| \geq 1$ and $u \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, it follows that $f(u+\psi) \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $\|f(u + \psi)\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)}$ is bounded independently of $u \in \Omega$.

Thus

$$\left| \int_{\mathbb{R}} \langle \tau\sigma_R\varsigma G(u + \psi, \mathcal{D}u + \psi')(\mathcal{D}u + \psi') + f(u + \psi), v\rangle \mu \, ds \right| \leq \alpha(\Omega)\|v\|_{L_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \tag{48}$$

for some $\alpha(\Omega) \in \mathbb{R}$, dependent only on Ω. The result follows from (43), (44) and (48).

□

The proof of the following result is omitted; it adapts an idea of J.M. Lasry, as discussed in Ekeland and Temam [8], and stems from an original result due to Krasnosel'skii.

Lemma 2.6 *P_R is jointly continuous in τ and u (from the strong topology of $[0, 1] \times W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ to the strong topology of $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$).*

It is standard to prove the following relationship between the operator P_R and (41).

Lemma 2.7 *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $\tau \in [0, 1]$. Then u is a solution of $P_R(\tau, u) = 0$ if and only if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ and u satisfies (41).*

Following [15], we will construct an operator $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ that satisfies the following.

Theorem 2.8 *There exists a bounded linear positive-definite self-adjoint operator* $S_\mu : W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ *and a function* $\theta : W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \times [0,1]$, *such that for* $\tau \in [0,1]$ *and* $u, u_0 \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$,

$$(P_R(\tau,u))(S_\mu(u-u_0)) \geq \|u-u_0\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta(u,u_0,\tau), \tag{49}$$

and $\theta(u_n,u_0,\tau) \to 0$ *uniformly for* $\tau \in [0,1]$ *as* $u_n \rightharpoonup u_0$ *in* $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$.

We will return to the proof of this result in a moment. First note that if H is a Hilbert space and $S : H \to H$ is a bounded linear self-adjoint positive-definite operator, then $S^* : H^* \to H^*$ is bounded linear and injective. (Recall that S^* is the conjugate of S, defined by $(S^*f)(u) = f(Su)$ for each $u \in H$, $f \in H^*$.) Together with Lemma 2.7,this yields that for $\tau \in [0,1]$ and $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$,

$$S^*_\mu P_R(\tau,u) = 0 \Leftrightarrow u \in C^2(\mathbb{R},\mathbb{R}^N) \text{ and } u \text{ satisfies (41)}. \tag{50}$$

Recall the discussion of degree theory for $(S)_+$ operators in section 2.2. Lemmas 2.5 and 2.6 show that $S^*_\mu P_R$ is jointly continuous in τ and u, and maps each bounded subset of $[0,1] \times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ into a bounded subset of $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$. Moreover, (49) can be rewritten as

$$(S^*_\mu P_R(\tau,u))(u-u_0) \geq \|u-u_0\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta(u,u_0,\tau). \tag{51}$$

So for each $\tau \in [0,1]$, $S^*_\mu P_R(\tau,\cdot)$ satisfies condition (23), and hence is of class $(S)_+$. Also, since $\theta(u,u_0,\tau) \to 0$ *uniformly for* $\tau \in [0,1]$ as $u \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, the operator $S^*_\mu P_R : [0,1] \times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ satisfies condition (26) required for an admissible $(S)_+$ homotopy.

So if $\Omega \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ is open, bounded and such that

$$0 \notin S^*_\mu P_R([0,1] \times \partial\Omega), \tag{52}$$

then $S^*_\mu P_R(0,\cdot)$ and $S^*_\mu P_R(1,\cdot)$ are $(S)_+$ homotopic relative to Ω. Hence

$$deg_{(S)_+}(S^*_\mu P_R(1,\cdot),\Omega,0) = deg_{(S)_+}(S^*_\mu P_R(0,\cdot),\Omega,0). \tag{53}$$

Further, if $0 \notin S^*_\mu P_R(\tau,\cdot)(\partial\Omega)$ and $deg_{(S)_+}(S^*_\mu P_R(\tau,\cdot),\Omega,0) \neq 0$, then there exists $u \in \Omega$ such that $S^*_\mu P_R(\tau,u) = 0$, so by (50), $u \in C^2(\mathbb{R},\mathbb{R}^N)$ satisfies (41).

Section 3 is devoted to proving *a priori* bounds for solutions of $S_\mu^* P_R(\tau, u) = 0$ where $u + \psi$ is monotone (as defined in the Introduction); by virtue of (50), this is equivalent to bounding monotone solutions of (41). The rest of the current section is concerned with proving Theorem 2.8. Our approach here extends that of [15].

The following lemma is a concatenation of Theorems 1-3 of section 3 of [15]; the reader is referred to [15] for the appropriate proofs. We correct a slight oversight in their proof of the existence of an operator T in Theorem 1, page 70. It is necessary to replace their constructed operator T by the operator $T^* : W_2^1(\mathbb{R}, \mathbb{R}^N) \to W_2^1(\mathbb{R}, \mathbb{R}^N)$ defined by $T^* u = \mathrm{Re}\,(Tu),\ u \in W_2^1(\mathbb{R}, \mathbb{R}^N)$, which then satisfies the requirements of their lemma.

Lemma 2.9 *Let $A \in M^{N\times N}$ be positive-definite and diagonal, and for $i = 1, 2$ let $B_i \in P^{N\times N}$ with $\mu_{PF}(B_i) < 0$. Let the matrix B be given by*

$$B(s) = \phi_1(s) B_1 + \phi_2(s) B_2, \quad s \in \mathbb{R}, \tag{54}$$

where $\phi_1, \phi_2 \in C^\infty(\mathbb{R}, [0, 1])$ are such that $\phi_1(s) + \phi_2(s) = 1$ for each $s \in \mathbb{R}$, and $\phi_1(s) = 0$ when $s > 1$, $\phi_2(s) = 0$ when $s < -1$. Then there exists a bounded linear self-adjoint positive-definite operator $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, such that for $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$,

$$(L_\mu u)(S_\mu u) \geq \|u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta_\mu(u), \tag{55}$$

where $\theta_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to \mathbb{R}$ is such that $\theta_\mu(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and for $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, $L_\mu u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)^*$ is defined by

$$(L_\mu u)(v) = \int_{\mathbb{R}} [\langle A(\mathcal{D}u)(s), \mathcal{D}v(s)\rangle - \langle B(s)u(s), v(s)\rangle]\mu(s)\, ds \tag{56}$$

for $v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.

Conditions **(f1)** and **(f2)** imply that $df[S],\ df[T] \in P^{N\times N}$. With condition **(f3)**, this yields that $df[S]$ and $df[T]$ fulfil the conditions on the matrices B_1 and B_2 in Lemma 2.9. Let $B_1 = df[S]$, $B_2 = df[T]$, and define the matrix B by (54). Then let $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ be the bounded linear self-adjoint positive-definite operator whose existence is guaranteed by Lemma 2.9 corresponding to this matrix B and the positive-definite diagonal matrix A as in the definition of the operator P_R given in (42). It will be shown that the operator S_μ satisfies the requirements of Theorem 2.8.

Proof of Theorem 2.8. Let the sequence $\{u_n\}_{n=1}^{\infty} \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ be such that $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ as $n \to \infty$, and set $v_n = u_n - u_0$. Then for each $\tau \in [0,1]$, (42) gives that

$$\begin{aligned}(P_R(\tau,u_n))(S_\mu v_n) \quad = \quad & \int_{\mathbb{R}} [\langle A(\mathcal{D}u_n), \mathcal{D}(S_\mu v_n)\rangle + \langle A(\mathcal{D}u_n), \nu(S_\mu v_n)\rangle \\ & -\langle A\psi'' + c(u_n)[\mathcal{D}u_n + \psi'] + \tau\sigma_R\varsigma G(u_n+\psi, \mathcal{D}u_n + \psi')[\mathcal{D}u_n + \psi'] \\ & +f(u_n+\psi), S_\mu v_n\rangle]\mu\, ds \end{aligned} \tag{57}$$

where ν is as defined in (19). Consider the last part of the right-hand side of (57). We will prove that

$$\int_{\mathbb{R}} \langle f(u_n(s)+\psi(s), (S_\mu v_n)(s)\rangle \mu(s)\, ds = \int_{\mathbb{R}} \langle B(s)v_n(s), (S_\mu v_n)(s)\rangle\mu(s)\, ds + \theta(v_n), \tag{58}$$

where $\theta(v_n) \to 0$ as $n \to \infty$. With this aim, note that

$$f(u_n(s)+\psi(s)) = f(u_0(s)+\psi(s)) + B_n(s)v_n(s), \tag{59}$$

where

$$B_n(s) := \int_0^1 df[tv_n(s) + u_0(s) + \psi(s)]\, dt, \tag{60}$$

so

$$\begin{aligned} & \int_{\mathbb{R}} \langle f(u_n+\psi), S_\mu v_n\rangle \mu\, ds - \int_{\mathbb{R}} \langle Bv_n, S_\mu v_n\rangle \mu\, ds \\ & = \quad \int_{\mathbb{R}} \langle f(u_0+\psi), S_\mu v_n\rangle\mu\, ds + \int_{\mathbb{R}} \langle (B_n - df[u_0+\psi])v_n, S_\mu v_n\rangle \mu\, ds \\ & \qquad \int_{\mathbb{R}} \langle (df[u_0+\psi] - B)v_n, S_\mu v_n\rangle\mu\, ds. \end{aligned} \tag{61}$$

Since $v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, $\|v_n\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}$ is bounded independently of n, and by the Sobolev Embedding Theorem, $v_n(s) \to 0$ uniformly for s in each compact subset of $\mathbb{R}$. In fact, since $\mu(s) \to \infty$ as $|s| \to \infty$ and estimate (22) holds, $v_n(s) \to 0$ uniformly on $\mathbb{R}$. Hence

$$B_n(s) = \int_0^1 df[tv_n(s) + u_0(s) + \psi(s)]\, dt \to \int_0^1 df[u_0(s)+\psi(s)]\, dt = df[u_0(s)+\psi(s)]$$

uniformly on $\mathbb{R}$. So since $\{\|v_n\|_{W^1_2(\mathbb{R},\mathbb{R}^N)}\}$ is bounded, the second term on the right of (61) tends to zero as $n \to \infty$.

Also, estimate (22) implies that $u_0(s)$ is uniformly bounded on $\mathbb{R}$ and $u_0(s) \to 0$ as $|s| \to \infty$, so because $f \in C^1(\mathbb{R}^N,\mathbb{R}^N)$, $df[u_0(s)+\psi(s)] \to df[S]$ as $s \to -\infty$ and

$df[u_0(s)+\psi(s)] \to df[T]$ as $s \to \infty$. Hence $df[u_0(s)+\psi(s)] - B(s) \to 0$ as $|s| \to \infty$. Let $\epsilon > 0$. Then since $\{\|v_n\|_{L_2(\mathbb{R},\mathbb{R}^N)}\}$ is bounded, there exists some $\kappa > 0$ such that

$$\left| \int_{|s|>\kappa} \langle (df[u_0+\psi] - B)v_n, S_\mu v_n \rangle \mu \, ds \right| < \frac{\epsilon}{2}$$

for every $n \in \mathbb{N}$. Moreover, $v_n \to 0$ uniformly on $[-\kappa, \kappa]$ and $df[u_0(s)+\psi(s)] - B(s)$ is uniformly bounded, so there exists $N_0 \in \mathbb{N}$ so that $n \geq N_0$ implies that

$$\left| \int_{|s|\leq\kappa} \langle (df[u_0+\psi] - B)v_n, S_\mu v_n \rangle \mu \, ds \right| < \frac{\epsilon}{2}.$$

Hence the third term on the right-hand side of (61) also tends to zero.

To treat the first term on the right-hand side of (61), recall from the proof of Lemma 2.5 that $f(u_0+\psi) \in L_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ whenever $u_0 \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$. Hence the mapping $v \mapsto \int_{\mathbb{R}} \langle f(u_0+\psi), S_\mu v \rangle \mu \, ds$ is a member of $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$, and so the first term on the right of (61) tends to zero as $n \to \infty$. Thus (59) holds.

Return to (57). The first term on the right-hand side is

$$\int_{\mathbb{R}} \langle A(\mathcal{D}u_n), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds = \int_{\mathbb{R}} \langle A(\mathcal{D}v_n), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds + \int_{\mathbb{R}} \langle A(\mathcal{D}u_0), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds. \quad (62)$$

Since the mapping

$$v \mapsto \int_{\mathbb{R}} \langle A(\mathcal{D}u_0), \mathcal{D}(S_\mu v_n) \rangle \mu \, ds$$

is in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$, the second integral on the right of (62) tends to zero as $v_n \rightharpoonup 0$. The Cauchy-Schwarz inequality now gives that the second term on the right-hand side of (57) satisfies

$$\left| \int_{\mathbb{R}} \langle A(\mathcal{D}u_n), \nu S_\mu v_n \rangle \mu \, ds \right| \leq \left(\int_{\mathbb{R}} \|A\mathcal{D}u_n\|^2 \mu \, ds \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|y_n\|^2 \nu^2 \, ds \right)^{\frac{1}{2}} \quad (63)$$

where $y_n = \omega S_\mu v_n$ (where ω is as defined in (19)).

The first factor on the right-hand side of (63) is bounded independently of n, since $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$. Since $S_\mu v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, the fact that multiplication by ω is a bounded linear operator from $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ to $W^1_2(\mathbb{R},\mathbb{R}^N)$ yields that $y_n \rightharpoonup 0$ in $W^1_2(\mathbb{R},\mathbb{R}^N)$, and hence $\{y_n(s)\}$ is uniformly bounded on $\mathbb{R}$. Furthermore, $y_n(s) \to 0$ uniformly on every compact subset of $\mathbb{R}$. Thus since ν^2 is integrable, Lebesgue's Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}} \|y_n\|^2 \nu^2 \, ds \to 0$$

as $n \to \infty$, and hence the left-hand side of (63) tends to zero.

Clearly, $\int_{\mathbb{R}} \langle A\psi'', S_\mu v_n\rangle \mu\, ds \to 0$ as $v_n \rightharpoonup 0$. Since $\{u_n\}$ is bounded in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, Lemma 33 yields that $c(u_n)$ is bounded independently of n, so that

$$c(u_n) \int_{\mathbb{R}} \langle \psi', S_\mu v_n\rangle \mu\, ds \to 0$$

as $n \to \infty$.

Now hypothesis **(G2)** and the facts that $|\tau\sigma_R(s)\varsigma(s)| \le 1$ for each s and σ_R is supported in $[-R, R]$ give that

$$\begin{aligned}
&|\int_{\mathbb{R}} \tau\sigma_R\varsigma\langle G(u_n+\psi, \mathcal{D}u_n+\psi')[\mathcal{D}u_n+\psi'], S_\mu v_n\rangle \mu\, ds| && (64)\\
&\le \int_{|s|\le R} |\langle G(u_n+\psi, \mathcal{D}u_n+\psi')(\mathcal{D}u_n+\psi'), S_\mu v_n\rangle|\mu\, ds\\
&\le \int_{|s|\le R} \|G(u_n+\psi, \mathcal{D}u_n+\psi')(\mathcal{D}u_n+\psi')\|\|S_\mu v_n\|\mu\, ds\\
&\le \int_{|s|\le R} \{\beta(u_n+\psi)+\gamma(u_n+\psi)\|\mathcal{D}u_n+\psi'\|\}\|S_\mu v_n\|\mu\, ds\\
&\le \beta_0\int_{|s|\le R} \|S_\mu v_n\|\mu\, ds + \gamma_0\int_{|s|\le R} \|\mathcal{D}u_n+\psi'\|\|S_\mu v_n\|\mu\, ds\\
&\le \left\{\beta_0\sqrt{2(R)} + \gamma_0\left(\int_{|s|\le R}\|\mathcal{D}u_n+\psi'\|^2\mu\, ds\right)^{\frac{1}{2}}\right\}\left(\int_{|s|\le R}\|S_\mu v_n\|^2\mu\, ds\right)^{\frac{1}{2}}. && (65)
\end{aligned}$$

Here $\{u_n(s)\}$ is bounded independently of n and of s, so since the functions β and γ are continuous, there exist $\beta_0, \gamma_0 > 0$ such that $\beta(u_n(s)+\psi(s)) < \beta_0$ and $\gamma(u_n(s)+\psi(s)) < \gamma_0$ for all $n \in \mathbb{N}$, $s \in \mathbb{R}$.

Since $\{u_n\}$ is bounded in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $\psi \in C^\infty(\mathbb{R}, \mathbb{R}^N)$, the first factor in (65) is bounded independently of n. Also, $v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ implies that $S_\mu v_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, so $(S_\mu v_n)(s) \to 0$ uniformly for $|s| < R$. Hence the second factor in (65) tends to zero, and thus (64) tends to zero as $n \to \infty$ *uniformly* for $\tau \in [0, 1]$.

It remains to show that

$$\int_{\mathbb{R}} \langle \mathcal{D}u_n, S_\mu v_n\rangle \mu\, ds \to 0 \ \text{ as } n \to \infty. \tag{66}$$

This follows from the proof of Theorem 3.5 in [15], taking into account the correction that we noted in the paragraph before Lemma 2.9.

There thus exists a functional θ such that $\theta(u_n, u_0, \tau) \to 0$ as $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ uniformly for $\tau \in [0,1]$, and by (58) and (62),

$$\begin{aligned}
(P_R(\tau, u_n))(S_\mu v_n) &= \int_{\mathbb{R}} [\langle A(\mathcal{D}v_n)(s), \mathcal{D}(S_\mu v_n)(s)\rangle - \langle B(s)v_n(s), (S_\mu v_n(s)\rangle]\mu(s)\,ds \\
&\qquad + \theta(u_n, u_0, \tau) \\
&= (L_\mu v_n)(S_\mu v_n) + \theta_\mu(v_n) + \theta(u_n, u_0, \tau) \\
&\geq \|u_n - u_0\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta_\mu(v_n) + \theta(u_n, u_0, \tau).
\end{aligned}$$

Here L_μ, S_μ and θ_μ are as in Lemma 2.9. The result follows.

□

3 Uniform *a priori* estimates for monotone solutions of the approximate system

For $R > 0$, define

$$\mathcal{M}_R := \left\{ u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N) : \begin{array}{c} u \text{ satisfies (41) for this } R \\ \text{for some } \tau \in [0,1], \text{and} \\ w = u + \psi \text{ is monotone} \end{array} \right\}. \tag{67}$$

The goal of this section is to prove that $\mathcal{M}_R$ is bounded in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ independently of $R > 0$. In doing so, we will prove some *a priori* estimates on solutions of the general travelling-wave system (99), which are of independent interest. The fact that $\mathcal{M}_R$ is bounded *for a given* R will be used in section 4 to construct a bounded set $\Omega \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, depending on R, that satisfies (52), and is such that there are no solutions u of (41) with $w = u + \psi$ non-monotone in $\overline{\Omega}$. Existence of a monotone solution to (40) will then be established for each $R > 0$. Section 4 concludes with the proof of the existence of a monotone solution to (4) via a limiting procedure, which stems from the fact that the bound on $\mathcal{M}_R$ is independent of $R > 0$.

3.1 *A priori* estimates for a general system

We will first consider solutions $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ of

$$A(u'' + \psi'') + c[u' + \psi'] + \sigma G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0, \tag{68}$$

where f satisfies **(f1) - (f4)**, G satisfies **(G1)-(G3)**, and $\sigma \in C(\mathbb{R}, [-1,1])$. The velocity $c \in \mathbb{R}$ is for now regarded as a *parameter*, and not the functional (38). The representation of c as a functional will be reintroduced in the final stage of obtaining the uniform *a priori* bound for monotone solutions. This procedure will also rely on the fact that the set of velocities for which there is a monotone solution to (68) is uniformly bounded.

We begin by obtaining an *a priori* estimate for the speed of monotone solutions. The initial step, which will also be useful elsewhere, is to obtain estimates for the derivatives of solutions of (68) that are independent of c and the choice of $\sigma \in C(\mathbb{R}, [-1,1])$. Note that it is *not* assumed yet that the solutions are monotone. We first consider the behaviour of derivatives of any solution $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ of (68).

Lemma 3.1 *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (68) for some $c \in \mathbb{R}$ and $\sigma \in C(\mathbb{R}, [-1,1])$. Then*

$$u, u', u'' \in L_\infty(\mathbb{R}, \mathbb{R}^N),$$

and there exists $K > 0$, depending on f, G and c, such that

$$\|u'(s)\| \leq \frac{K}{\sqrt{\mu(s)}} \ \textit{for each} \ s \in \mathbb{R}, \tag{69}$$

and hence $\|u'(s)\| \to 0$ as $n \to \infty$.

Proof. That $u \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ follows from estimate (22). Since u satisfies (68),

$$u'' = -\psi'' - A^{-1}\{c[u' + \psi'] + \sigma G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi)]. \tag{70}$$

Clearly $-\psi'' - cA^{-1}[u' + \psi'] \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, because $u' \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and ψ' and ψ'' have compact support. Also, estimate (22) together with Lemma 1.3 yield that $A^{-1}\{\sigma G(u + \psi, u' + \psi')(u' + \psi')\} \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and it was shown in the proof of Lemma 2.5 that $f(u + \psi) \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Hence $u'' \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, and $u' \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. That $u' \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ and estimate (69) thus follow from (22). That $u'' \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ follows from (68), the boundedness of ψ, ψ', u and u' and the continuity of f and G.

□

Theorem 3.2 *Let $M > 0$ and $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (68) for some $c \in \mathbb{R}$ and $\sigma \in C(\mathbb{R}, [-1,1])$ and such that $\|w\|_{L_\infty(\mathbb{R}, \mathbb{R}^N)} < M$, where $w = u + \psi$. Then there exist $N_1, N_2 > 0$, dependent upon M but independent of w, c and σ, such that*

$$\|w'\|_{L_\infty(\mathbb{R}, \mathbb{R}^N)} < N_1 \ \textit{and} \ \|w''\|_{L_\infty(\mathbb{R}, \mathbb{R}^N)} < N_2.$$

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(s) := \langle Aw'(s), w'(s)\rangle, s \in \mathbb{R}$. Then h is continuously differentiable, and $h \geq 0$ for all s, since A is positive-definite. Also, Lemma 3.1 implies that $h(s) \to 0$ as $|s| \to \infty$, since $\psi'(s) = 0$ when $|s| > 1$. So h attains a maximum at a point $s_0 \in \mathbb{R}$ where $h'(s_0) = 0$. Since A is symmetric,

$$\langle Aw''(s_0), w'(s_0)\rangle = 0,$$

so taking the inner product of the right-hand side of (68) evaluated at s_0 with $w'(s_0)$ yields that

$$c\|w'(s_0)\|^2 + \sigma(s_0)\langle G(w(s_0), w'(s_0))w'(s_0), w'(s_0)\rangle + \langle f(w(s_0)), w'(s_0)\rangle = 0. \tag{71}$$

Now by hypothesis **(G2)**,

$$|\sigma(s_0)\langle G(w(s_0), w'(s_0))w'(s_0), w'(s_0)\rangle| \leq (\beta(w(s_0)) + \gamma(w(s_0))\|w'(s_0)\|)\|w'(s_0)\|,$$

so

$$\begin{aligned}|c|\|w'(s_0)\|^2 &= |\sigma(s_0)\langle G(w(s_0), w'(s_0))w'(s_0), w'(s_0)\rangle + \langle f(w(s_0)), w'(s_0)\rangle| \\ &\leq (\beta(w(s_0)) + \gamma(w(s_0))\|w'(s_0)\|)\|w'(s_0)\| + \|f(w(s_0))\|\|w'(s_0)\|. \end{aligned} \tag{72}$$

Since $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < M$ and β, γ and f are continuous, there exist $\beta_0, \gamma_0, \alpha_0 > 0$, depending *only* upon M, such that

$$\beta(w(s)) < \beta_0, \;\; \gamma(w(s)) < \gamma_0 \;\text{ and }\; \|f(w(s))\| < \alpha_0 \tag{73}$$

for all $s \in \mathbb{R}$. Hence (72) gives that if $\|w'(s_0)\| > 0$,

$$(|c| - \gamma_0)\|w'(s_0)\| \leq \beta_0 + \alpha_0. \tag{74}$$

Consider first the possibility that $|c| \geq 2\gamma_0$. Then

$$\|w'(s_0)\| \leq \frac{1}{\gamma_0}(\beta_0 + \alpha_0), \tag{75}$$

which is trivial if $w'(s_0) = 0$, and otherwise is a consequence of (74). So, as h attains a maximum at s_0,

$$h(s) \leq h(s_0) = \langle Aw'(s_0), w'(s_0)\rangle \leq a_1(\frac{1}{\gamma_0}(\beta_0 + \alpha_0))^2 \;\text{ for all }\; s \in \mathbb{R}, \tag{76}$$

where $a_1 := \max_{1\le i\le N}\{A_1,\ldots,A_N\}$. Let $a_2 := \min_{1\le i\le N}\{A_1,\ldots,A_N\}$. Then $a_2>0$ and $h(s)\ge a_2\|w'(s)\|^2$ for each $s\in\mathbb{R}$, so (76) yields

$$|c|\ge 2\gamma_0 \Rightarrow \|w'(s)\| \le \left(\frac{a_1}{a_2}\right)^{\frac{1}{2}} \frac{1}{\gamma_0}(\beta_0+\alpha_0) \text{ for all } s\in\mathbb{R}. \tag{77}$$

Suppose now that $|c|\le 2\gamma_0$. Lemma 3.1 implies that w' and w'' are uniformly bounded on $\mathbb{R}$. So Landau's inequality and the fact that $u=w-\psi$ satisfies (68) together yield that

$$\begin{aligned}\|w'\|^2_{L_\infty(\mathbb{R},\mathbb{R}^N)} &\le 4\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\|A^{-1}(cw'+\sigma G(w,w')w'+f(w))\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\\ &\le \frac{4M}{a_2}(2\gamma_0\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}+\beta_0+\gamma_0\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}+\alpha_0)\end{aligned}$$

since $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\le M, |c|\le 2\gamma_0$, and **(G2)** and (73) hold.
Hence

$$|c|\le 2\gamma_0 \Rightarrow \|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \le \max\{1, \frac{4M}{a_2}(3\gamma_0+\beta_0+\alpha_0)\}. \tag{78}$$

The existence of $N_1>0$ as in the statement of the theorem follows from (77) and (78).

To obtain the corresponding bound for w'', observe that the i^{th} equation of (68) yields that for each $s\in\mathbb{R}$,

$$|w_i''(s)| \le A_i^{-1}\left\{\sup_{s\in\mathbb{R}}|cw_i'(s)|+\sup_{s\in\mathbb{R}}|G_i(w(s),w'(s))w_i'(s)|+\sup_{s\in\mathbb{R}}|f(w(s))|\right\}. \tag{79}$$

Since $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\le M$, $\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\le N_1$, and f and G are continuous, the second and third terms on the right-hand side of (79) are bounded by some $\alpha>0$ dependent only on M.

If $c=0$, the existence of $N_2>0$ as required follows immediately. Otherwise, note that $cw_i'(s)\to 0$ as $|s|\to\infty$, so as in the first part of this proof, $|cw_i'|$ attains a maximum at some $s_0\in\mathbb{R}$ where $(cw_i')'(s_0)=cw_i''(s_0)=0$. Since $c\ne 0$, $w_i''(s_0)=0$, so from the i^{th} equation of (68),

$$cw_i'(s_0)+\sigma(s_0)G_i(w(s),w'(s))w_i'(s)+f_i(w(s_0))=0. \tag{80}$$

As argued in the last paragraph, the second and third terms of (80) are bounded by $\alpha>0$, so $cw_i'(s_0)$ is bounded by $\alpha>0$, dependent only on M. Since $|cw_i'|$ attains its maximum at s_0, the existence of $N_2>0$ as in the statement of the theorem follows from (79).

□

To obtain an *a priori* estimate for the speed of monotone solutions to (68), we require some preliminary lemmas on the signs of the components of the function f. These results are proved in [15], and are stated here. We give an alternative, possibly simpler, proof for the second lemma.

Lemma 3.3 *Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be locally monotone. If, for some k, $1 \leq k \leq N$, $f_k(x) = 0$ and $y \geq x$, $y \neq x$ and $y_k = x_k$, then $f_k(y) > 0$. Also, $f_k(y) < 0$ if $y \leq x$, $y \neq x$ and $y_k = x_k$.*

Lemma 3.4 *Let $f \in C^1(\mathbb{R}, \mathbb{R}^N)$ satisfy* **(f1), (f2)** *and* **(f3)**. *Let $p, q > 0$ denote Perron-Frobenius eigenvectors of $df[S]$ and $df[T]$ respectively. Then there exist $t, \zeta > 0$ such that for each $i \in \{1, \ldots, N\}$,*

$$x \in \Gamma_i := \{y \in \mathbb{R}^N : S \leq y \leq S + tp, y_i = S_i + tp_i\} \Rightarrow f_i(x) < -\zeta < 0, \tag{81}$$

$$x \in \Lambda_i := \{y \in \mathbb{R}^N : T - tq \leq y \leq T, y_i = T_i - tq_i\} \Rightarrow f_i(x) > \zeta > 0. \tag{82}$$

Proof. We prove (81); the argument for (82) is similar. Suppose, for contradiction, that for some $i \in \{1, \ldots, N\}$, there is a sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}^+, t_n \downarrow 0$ as $n \to \infty$ such that $f_i(x^n) \geq 0$ for $x^n \in \mathbb{R}^N$ with $x_i^n = S_i + t_n p_i$ and $S_j \leq x_j^n \leq S_j + t_n p_j$ for $j \in \{1, \ldots, N\}, i \neq j$. Clearly $x^n \to S$ as $n \to \infty$. Since f_i is differentiable at S and $f_i(S) = 0$,

$$f_i(x^n) = B_i(x^n - S) + R_i(x^n - S),$$

where $B_i := df_i[S]$ and $R_i : \mathbb{R} \to \mathbb{R}$ is such that $\dfrac{\|R_i(x^n - S)|}{\|x^n - S\|} \to 0$ as $n \to \infty$. Hence

$$B_i \left(\frac{x^n - S}{t_n} \right) + \frac{R_i(x^n - S)}{t_n} \geq 0 \tag{83}$$

since $t_n > 0$ for each n. Now $\{y^n\} := \left\{ \dfrac{x^n - S}{t_n} \right\}$, a bounded sequence in $\mathbb{R}^N$, has a convergent subsequence, say $y^k \to y$. Also,

$$\frac{R_i(x^n - S)}{t_n} = \frac{R_i(x^n - S)}{\|x^n - S\|} \cdot \frac{\|x^n - S\|}{t_n} \to 0 \text{ as } n \to \infty,$$

since $\|x^n - S\| \leq t_n \|p\|$. So by (83),

$$B_i y \geq 0. \tag{84}$$

But for each k, $y_i^k = p_i$ and $0 \leq y_j^k \leq p_j$ for $i \neq j$. Moreover, **(f3)** yields that $B_i p < 0$, and $\dfrac{\partial f_i}{\partial x_j} > 0$ for $i \neq j$ by **(f1)**. Together, these observations imply that $B_i y < 0$, which contradicts (84). Since Γ_i is compact and $f_i > 0$ on Γ_i, there exists $\zeta > 0$ such that $f_i \geq \zeta > 0$ on Γ_i. The result follows.

□

The continuity of f leads to the following straightforward strengthening of Lemma 3.4.

Lemma 3.5 *Let $f, \Gamma_i, \Lambda_i, t, p, q$ be as in Lemma 3.4. Then there exists $\epsilon > 0$ such that for each $i \in \{1, \ldots, N\}$, $\epsilon < tp_i, \epsilon < tq_i$, and*

$$x \in \Gamma_i, y \in \mathbb{R}^N \text{ such that } \max_{1 \leq k \leq N} |y_k - x_k| \leq \epsilon \Rightarrow f_i(y) < 0, \tag{85}$$

$$x \in \Lambda_i, y \in \mathbb{R}^N \text{ such that } \max_{1 \leq k \leq N} |y_k - x_k| \leq \epsilon \Rightarrow f_i(y) > 0. \tag{86}$$

Lemmas 3.1 - 3.5 are now employed to bound the velocity of monotone solutions to (68), independently of the choice of $\sigma \in C(\mathbb{R}, [-1, 1])$.

Theorem 3.6 *There exists $\gamma > 0$, depending only on f, G and A, such that if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies (68) for some $c \in \mathbb{R}$ and some $\sigma \in C^1(\mathbb{R}, [-1, 1])$ and*

$$u'(s) + \psi'(s) > 0 \text{ for all } s \in \mathbb{R},$$

then

$$|c| < \gamma. \tag{87}$$

Proof. Let u be as in the statement of the theorem. We will use the fact that $w(s) = u(s) + \psi(s)$ converges monotonically to S as $s \to -\infty$ to obtain a lower bound for c. The existence of an upper bound can be proved similarly, using the monotonic convergence of $w(s)$ to T as $s \to \infty$.

Since $w(s) \to S$ as $s \to -\infty$ and $u'(s) + \psi'(s) > 0$ for each s, there exists $s_0 \in \mathbb{R}$ and $i \in \{1, \ldots, N\}$ such that $w(s_0) > S$ and $w(s_0) \in \Gamma_i$, where Γ_i is defined in (81). Now $w(s_0) \in \Gamma_i \Rightarrow w_i(s_0) = tp_i$, and $tp_i > \epsilon > 0$, where ϵ is as in Lemma 3.5. Choose $s_1 \in \mathbb{R}$

such that $w_i(s_0) - w_i(s_1) = \epsilon$. Since w is increasing, $s_1 < s_0$. Integrating the i^{th} equation of (68) from s_1 to s_0 gives

$$A_i(w_i'(s_0) - w_i'(s_1)) + c\epsilon + \int_{s_1}^{s_0} \sigma(s) G_i(w(s), w'(s)) w_i'(s)\, ds + \int_{s_1}^{s_0} f_i(w(s))\, ds = 0. \quad (88)$$

Since w is a *monotone* solution between S and T, there exists $M > 0$, independent of w, c and $\sigma \in C(\mathbb{R}, [-1,1])$, such that

$$\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \le M. \quad (89)$$

Thus by Theorem 3.2, there exists $N_1 > 0$, independent of w, c and $\sigma \in C(\mathbb{R}, [-1,1])$, such that

$$\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \le N_1. \quad (90)$$

So

$$|A_i(w_i'(s_0) - w_i'(s_1))| \le 2|A_i| N_1 \le 2a_1 N_1, \quad (91)$$

where $a_1 = \max\{A_1, \ldots, A_N\}$. Also, since G is continuous and (89) and (90) hold, there exists $\gamma_1 > 0$ such that for each $i \in \{1, \ldots, N\}$,

$$\|G_i\|_{L_\infty(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})} \le \gamma_1. \quad (92)$$

Hence

$$\begin{aligned} \left| \int_{s_1}^{s_0} \sigma(s) G_i(w(s), w'(s)) w_i'(s)\, ds \right| &\le \int_{s_1}^{s_0} |G_i(w(s), w'(s))||w_i'(s)|\, ds \\ &\le \gamma_1 \int_{s_1}^{s_0} |w_i'(s)|\, ds \\ &= \gamma_1 \int_{s_1}^{s_0} w_i'(s)\, ds \\ &= \gamma_1 \epsilon, \end{aligned} \quad (93)$$

since w is monotone. We now show that

$$f_i(w(s)) < 0 \ \text{ for } \ s_1 \le s \le s_0. \quad (94)$$

By Lemma 3.5, $f_i(y) < 0$ when $w_i(s_0) - \epsilon \le y_i \le w_i(s_0)$ and $y_k = w_k(s_0)$ for $k \in \{1, \ldots, N\}, k \ne i$. Thus by Lemma 3.3, $f_i(y) < 0$ when $w_i(s_0) - \epsilon \le y_i \le w_i(s_0)$ and $y_k \le w_k(s_0), k \in \{1, \ldots, N\}, k \ne i$. That (94) holds then follows from the monotonicity of w. So from (88), (91), (93) and (94), we find that

$$c \ge -\left(\gamma_1 + \frac{2a_1 N_1}{\epsilon}\right),$$

which gives a lower bound for the velocity c as required.

$\square$

The next result shows that monotone solutions of (68) satisfy uniform exponential estimates in neighbourhoods of S and T, in the following sense. Note that these estimates are *independent* of the velocity c and of the choice of $\sigma \in C(\mathbb{R}, [-1,1])$. The proof of this theorem closely follows that of Theorem 4.1 in Chapter 13 of [5], and hence we omit it. Condition **(G3)** is required to ensure the required uniformity of the estimates.

Theorem 3.7 *There exist $\kappa, \alpha, \beta > 0, 0 < \delta < \frac{1}{2}\min_{1\le i\le N}\{T_i - S_i\}$, such that if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies (68) for some $c \in \mathbb{R}$ and some $\sigma \in C(\mathbb{R}, [-1,1])$, and $u'(s) + \psi'(s) > 0$ for all $s \in \mathbb{R}$, then for $w = u + \psi$,*

$$\|w(s) - S\| \le \kappa e^{\alpha(s-s_0)}, \;\; \|w'(s)\| \le \kappa e^{\alpha(s-s_0)}, \tag{95}$$

where $s_0 \in \mathbb{R}$ is the unique point such that $\|w(s_0) - S\| = \delta$ and $s \le s_0$. Moreover,

$$\|w(s) - T\| \le \kappa e^{-\beta(s-t_0)}, \;\; \|w'(s)\| \le \kappa e^{-\beta(s-t_0)}, \tag{96}$$

where $t_0 \in \mathbb{R}$ is the unique point such that $\|w(t_0) - T\| = \delta$ and $s \ge t_0$. In addition, for $w_0 \in \mathbb{R}^N$,

$$0 < \|w_0 - S\| \le \delta \quad \Rightarrow \quad f(w_0) \neq 0, \tag{97}$$

$$0 < \|w_0 - T\| \le \delta \quad \Rightarrow \quad f(w_0) \neq 0. \tag{98}$$

The following theorem gives a necessary condition for the existence of monotone directions to which a solution tends monotonically. The idea of the proof is reasonably classical, but we have been unable to find a statement with this level of precision in $\mathbb{R}^N$, for arbitrary N, in the non-autonomous case needed here. To avoid cluttering our exposition, we give this proof in an appendix.

Theorem 3.8 *Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $w_0 \in \mathbb{R}^N$ be such that $f(w_0) = 0, df[w_0] \in P^{N\times N}$ and $\mu_{PF}(df[w_0]) \neq 0$. Let $A \in M^{N\times N}$ be a positive-definite diagonal matrix, and $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ satisfy condition* **(G3)**. *Let $\sigma \in C(\mathbb{R}, [-1,1])$. Suppose that $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies*

$$Aw''(s) + cw'(s) + \sigma(s)G(w(s), w'(s))w'(s) + f(w(s)) = 0, \; s \in \mathbb{R} \tag{99}$$

for some $c \in \mathbb{R}$. Further, suppose that

$$w(s) \not\equiv w_0 \; (s \in \mathbb{R}), \tag{100}$$

$$w(s) \to w_0 \textit{ as } s \to \infty \;\; (s \to -\infty) \tag{101}$$

and there exists $s_0 \in \mathbb{R}$ and $M > 0$ such that

$$s \geq s_0 \;\; (s \leq s_0) \Rightarrow w(s) \leq w_0 \;\; (w(s) \geq w_0) \textit{ and } \|w'(s)\| \leq M. \tag{102}$$

Then there exist $\lambda < 0$ ($\lambda > 0$) and a vector $q \in \mathbb{R}^N$ such that $q > 0$, and

$$(\lambda^2 A + \lambda c I + df[w_0])q = 0; \tag{103}$$

that is, there is a stable (unstable) monotone eigenvalue of the travelling-wave problem linearized at w_0.

The importance of the above theorem for the gradient-dependent travelling-wave problem lies in the following application, which is the motivation for hypothesis **(f4)** on the function f.

Lemma 3.9 *Let $w_0 \in (\mathbb{R}, \mathbb{R}^N)$ be such that $f(w_0) = 0$, $df[w_0] \in P^{N\times N}$ and $\mu_{PF}(df[w_0]) > 0$. Then for a fixed $c \in \mathbb{R}$, there cannot exist two functions $w_1, w_2 \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $\sigma_1, \sigma_2 \in C^1(\mathbb{R}, [-1,1])$ such that w_i satisfies (99) with $\sigma = \sigma_i$ $(i = 1,2)$, and*

$$w_1(s) \to w_0 \quad as \quad s \to -\infty, \;\; w_1'(s) \geq 0 \textit{ for sufficiently large } -s, \tag{104}$$

$$w_2(s) \to w_0 \quad as \quad s \to \infty, \;\; w_2'(s) \geq 0 \textit{ for sufficiently large } s. \tag{105}$$

Proof. Suppose that there is $c \in \mathbb{R}$ such that functions w_1, w_2 as in the statement of the lemma exist. Then it follows from Theorem 3.8 that there are real numbers $\lambda_1 < 0$, $\lambda_2 > 0$ and vectors $q_1, q_2 > 0$, such that

$$(\lambda_i^2 A + \lambda_i c I + df[w_0])q_i = 0, \;\; i = 1,2. \tag{106}$$

But it is shown in [6] that for c fixed, the Perron-Frobenius eigenvalue of $M(\lambda, c) = \lambda^2 A + c\lambda I + df[w_0]$ is a strictly convex function of λ. Thus since $\mu_{PF}(df[w_0]) = \mu_{PF}(M(0,c)) > 0$, there cannot exist both $\lambda_1 < 0$ and $\lambda_2 > 0$ such that the Perron-Frobenius eigenvalue of $\lambda_i^2 A + \lambda_i c I + df[w_0]$ is zero. This contradicts (106) because the only positive eigenvector of $M \in P^{N\times N}$ is that corresponding to $\mu_{PF}(M)$. The result follows.

□

Our final preliminary result shows that the assumption of the local monotonicity of f (condition **(f1)**) forces monotone solutions of the system (99) to be *strictly* monotone.

Lemma 3.10 *Let $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (99) for some $\sigma \in C(\mathbb{R}, [-1,1])$. Suppose that for each $s \in \mathbb{R}$,*

$$w'(s) \geq 0. \tag{107}$$

Then either there exists $w_0 \in \mathbb{R}^N$ such that

$$\begin{aligned} w(s) &\equiv w_0 \text{ for each } s \in \mathbb{R}, \text{ or} && (108) \\ w'(s) &> 0 \text{ for each } s \in \mathbb{R}. && (109) \end{aligned}$$

Proof. Suppose that there exists $s_0 \in \mathbb{R}$ for which $w'(s_0)$ has a zero component; without loss of generality, say $w_1'(s_0) = 0$. Consider the first equation of (99),

$$A_1 w_1''(s) + c w_1'(s) + \sigma(s) G_1(w(s), w'(s)) w_1'(s) + f_1(w(s)) = 0, \quad s \in \mathbb{R}. \tag{110}$$

If $f_1(w(s_0)) \neq 0$, (110) implies that $w_1''(s_0) \neq 0$, which says that w_1 has either a maximum or a minimum at s_0. Since this contradicts (107), $f_1(w(s_0)) = 0$, so $w_1''(s_0) = 0$. Now for $s \in \mathbb{R}$, the mean value theorem for real-valued functions yields the existence of $\theta(s) \in (0,1)$ such that

$$\begin{aligned} f_1(w(s)) &= f_1(w(s_0)) + df[w(s_0) + \theta(s)\{w(s) - w(s_0)\}](w(s) - w(s_0)) \\ &= \sum_{i=1}^{N} \frac{\partial f_1}{\partial w_i}(w(s_0) + \theta(s)\{w(s) - w(s_0)\})(w_i(s) - w_i(s_0)) \qquad (111) \end{aligned}$$

since $f_1(w(s_0)) = 0$. Define $v(s) = w_1(s) - w_1(s_0)$, $s \in \mathbb{R}$. Then for $s \leq s_0$, (107) gives that

$$v(s) \leq v(s_0) = 0. \tag{112}$$

Moreover, (110) and (111) yield that

$$\begin{aligned} & A_1 v''(s) + \{c + \sigma(s) G_1(w(s), w'(s))\} v'(s) + \frac{\partial f_1}{\partial w_1}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}] v(s) \\ &= -\sum_{i=2} N \frac{\partial f_1}{\partial w_i}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}](w_i(s) - w_i(s_0)). \qquad (113) \end{aligned}$$

It follows from the local monotonicity of f (condition **(f1)**) and the fact that $f_1(w(s_0)) = 0$ that $\dfrac{\partial f_1}{\partial w_i}(w(s_0)) > 0$ for $i \neq 1$. Thus since $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, there exists a neighbournood N of $w(s_0)$ such that

$$\frac{\partial f_1}{\partial w_i}(w(s_0)) > 0 \text{ for each } i \neq 1 \text{ and each } w_0 \in N. \tag{114}$$

Let $s_1 < s_0$ be such that $w(s) \in N$ for each $s \in \mathbb{R}$, $s_1 \leq s \leq s_0$. It follows from (112), (113) and (107) that

$$A_1 v''(s) + \{c + \sigma(s) G_1(w(s), w'(s))\} v'(s) + \frac{\partial f_1}{\partial w_1}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}] v(s) \geq 0 \tag{115}$$

for $s_1 \leq s \leq s_0$. Since G is continuous and $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, the coefficients of v and of v' are bounded independently of s. Hence the Hopf Boundary Lemma [12] yields that either $v'(s_0) > 0$, or $v(s) = v(s_0) = 0$ for each $s \in (s_1, s_0)$. By the choice of s_0, $v'(s_0) = 0$. So $v(s) = v(s_0) = 0$ for each $s \in (s_1, s_0)$. In fact, $v(s) = 0$ for each s, $-\infty < s \leq s_0$. For if $\beta := \inf_{s \leq s_0}\{v(t) = 0 \text{ for } s \leq t \leq s_0\}$ were finite, then $w_1'(\beta) = 0$. The above argument then yields the existence of a left-neighbourhood of β on which $v = 0$, which contradicts the definition of β. A similar argument shows that $v(s) = 0$ for each s, $s_0 \leq s \leq \infty$. Hence

$$w_1(s) = w_1(s_0) \text{ for each } s \in \mathbb{R}. \tag{116}$$

Suppose now that there exists $i \in \{2, \ldots, N\}$ such that $w_j \not\equiv w_j(s_0)$. It follows from the above argument that $w_j'(s_0) > 0$. Hence for s in a neighbourhood of s_0, $w_j(s) \neq w_j(s_0)$. Now by (116), $v(s) = 0$ for each $s \in \mathbb{R}$, where $v(s) := w_1(s) - w_1(s_0)$. Thus (113) gives that

$$\sum_{i=2}^{N} \frac{\partial f_1}{\partial w_i}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}](w_i(s) - w_i(s_0)) = 0 \tag{117}$$

for each $s \in \mathbb{R}$. Let $s \leq s_0$ be sufficiently close to s_0 that $\frac{\partial f_1}{\partial w_i}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}] > 0$ for each $i \in \{2, \ldots, N\}$ and $w - j(s) \neq w_j(s_0)$. Then (107) contradicts (117).

So $w(s) \equiv w(s_0)$, and hence (108) holds with $w_0 := w(s_0)$. The result follows.

□

3.2 Estimates for the approximate system that are independent of R

Consider now the system

$$A(u'' + \psi'') + c[u' + \psi'] + \tau \sigma_R \varsigma G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0, \tag{118}$$

where $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N)$, $\tau \in [0,1]$, $c \in \mathbb{R}$, $R > 0$ and σ_R is as in section 5.2.1. As usual, f satisfies **(f1)-(f4)**, G satisfies **(G1)-(G3)** and ς is as defined in section 1. This system is a specific case of (68), where here $\sigma = \tau\sigma_R\varsigma$. Hence the results of section 3.1 apply to (118). We will now prove *a priori* lower bounds on $w_i'(s)$, $\;i \in \{1,\ldots,N\}$, when $w(s)$ is outside the δ-neighbourhoods of S and T constructed in Theorem 3.7 and $u = w - \psi$ satisfies (118).

Theorem 3.11 *Let $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N)$ satisfy (118) for some $c \in \mathbb{R}$, $\tau \in [0,1]$ and $R > 0$, and suppose that $u'(s) + \psi'(s) > 0$ for all $s \in \mathbb{R}$. Then there exists $\chi > 0$, independent of u, c, τ, and R, such that for $w = u + \psi$,*

$$w_i'(s) > \chi, \quad i = 1,\ldots,N, \tag{119}$$

when $\|w(s) - S\| \geq \delta$ and $\|w(s) - T\| \geq \delta$, where δ is as in Theorem 3.7.

Proof. Suppose that the result is false. Then there exist sequences $\{u^k\} \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N)$, $\{\tau_k\} \subset [0,1]$, $\{c_k\} \subset \mathbb{R}$, $\{R_k\} \subset (0,\infty)$, $\{s_k\} \subset \mathbb{R}$ and $\{i^k\} \subset \{1,\ldots,N\}$ such that u^k satisfies (118) with $c = c_k$, $\tau = \tau_k$ and $R = R_k$,

$$\|w^k(s_k) - S\| \geq \delta \;\text{ and }\; \|w^k(s_k) - T\| \geq \delta, \tag{120}$$

and

$$0 < \left(w^k_{i_k}\right)'(s_k) < \frac{1}{k}, \tag{121}$$

where $w^k := u^k + \psi$. Since Theorem 3.6 gives that $\{c_k\}$ is bounded, there is a subsequence of $\{u^k\}$, $\{u^{k(j)}\}_{j=1}^\infty$ and constants $\tau_0 \in [0,1]$, $c_0 \in \mathbb{R}$ and $i_0 \in \{1,\ldots,N\}$ such that $c_{k(j)} \to c_0$, $\tau_{k(j)} \to \tau_0$ as $k \to \infty$ and $i_{k(j)} = i_0$ for each $j \in \mathbb{N}$. Without loss of generality, we suppose henceforth that $i_0 = 1$ and write k for $k(j)$.

Now let $\{t_k\} \subset \mathbb{R}$ be such that $\|w^k(t_k) - T\| = \delta$, and define

$$v^k(s) = w^k(s + t_k), \quad s \in \mathbb{R}, \tag{122}$$

so that for $s \in \mathbb{R}$,

$$A(v^k)''(s) + c_k(v^k)'(s) + \tau_k\sigma_{R_k}(s + t_k)G(v^k(s),(v^k)'(s))(v^k)'(s) + f(v^k(s)) \;=\; 0 \tag{123}$$

and

$$\|v^k(0) - T\| = \delta. \tag{124}$$

Also, by (122), (120) and (121),

$$\left(v_1^k\right)'(s_k - t_k) = \left(w_1^k\right)'(s_k) \to 0 \text{ as } k \to \infty, \tag{125}$$

and

$$\|v^k(s_k - t_k) - S\| \geq \delta, \ \ \|v^k(s_k - t_k) - T\| \geq \delta. \tag{126}$$

Since w^k is a monotone solution connecting S and T for each k,

$$\|w^k\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq M \text{ for each } k,$$

where $M > 0$ is as in (89), and is independent of w^k, c_k, τ_k and R_k. Hence by Theorem 3.2, there exist $N_1, N_2 > 0$, independent of w^k, c_k, τ_k and R_k, such that

$$\|(w^k)'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq N_1, \ \ \|(w^k)''\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq N_2, \tag{127}$$

and hence

$$\|(v^k)'(s)\| \leq N_1, \ \ \|(v^k)''(s)\| \leq N_2 \text{ for all } s \in \mathbb{R}. \tag{128}$$

We aim to pass to a limit in (123) as $k \to \infty$ through a subsequence. Define

$$\sigma^k(s) = \sigma_{R_k}(s + t_k)\varsigma(s + t_k), \ \ s \in \mathbb{R}, \tag{129}$$

for each $k \in \mathbb{N}$. Recall from the definition of σ_R that $\{\sigma_R\}$ is uniformly bounded and equicontinuous as a family parametrised by R. By hypothesis, ς is bounded and uniformly continuous. It follows that $\{\sigma^k\}$ is uniformly bounded and equicontinuous as a family parametrised by k. An Arzela-Ascoli argument yields the existence of $\sigma \in C(\mathbb{R}, [-1, 1])$ and a subsequence $\{\sigma^j\}$ of $\{\sigma^k\}$ such that

$$\sigma_{R_j}(s + t_j)\varsigma(s + t_j) = \sigma^j(s) \to \sigma(s) \text{ as } j \to \infty \tag{130}$$

for each $s \in \mathbb{R}$, the convergence being uniform on compact intervals.

Now G and f are continuous, so an Arzela-Ascoli argument using (128), (123) and (130) yields the existence of a subsequence of the v^k and $v \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $v^k \to v$ in $C^2([-r, r], \mathbb{R}^N)$ for each $r > 0$.

Passing to the limit in (123) as $j \to \infty$ along this subsequence yields that

$$Av''(s) + c_0 v'(s) + \tau_0\sigma(s)G(v(s), v'(s))v'(s) + f(v(s)) = 0, \ \ s \in \mathbb{R}. \tag{131}$$

By (124),

$$\|v(0) - T\| = \delta, \tag{132}$$

and since $S \leq v^k(s) \leq T$ for each $k \in \mathbb{N}$, $s \in \mathbb{R}$,

$$S \leq v(s) \leq T, \quad s \in \mathbb{R}. \tag{133}$$

Also, since $\left(v^k\right)'(s) > 0$ for each $k \in \mathbb{N}$, $s \in \mathbb{R}$,

$$v'(s) \geq 0, \quad s \in \mathbb{R}, \tag{134}$$

so it follows from (133) that there exist $p, q \in \mathbb{R}^N$, with $S \leq p \leq q \leq T$ such that

$$v(s) \to p \text{ as } s \to -\infty \text{ and } v(s) \to q \text{ as } s \to \infty. \tag{135}$$

Moreover, since $\|v''(s)\| \leq N_2$ by (128), Landau's inequality gives that $v'(s) \to 0$ as $|s| \to \infty$. Hence by (135) and (131), $v''(s) \to 0$ as $|s| \to \infty$. So using (133), we find that $f(p) = f(q) = 0$. It thus follows from (132), (133) and (98) that $q = T$.

Next suppose, for contradiction, that $p = S$. Let $x \in \mathbb{R}$ be such that $\|v(x) - S\| = \frac{\delta}{2}$. Then since $v^j(x) \to v(x)$ as $j \to \infty$, there exists $j_0 \in \mathbb{N}$ such that $j \geq j_0 \Rightarrow \|v^j(x) - S\| \leq \delta$. Hence $x < s_j - t_j < 0$ for $j \geq j_0$ (recall (126)). So $\{s_j - t_j\}_{j=j_0}^{\infty}$ is contained in a finite interval in $\mathbb{R}$, $[x, 0]$, thus has a convergent subsequence, say $s_j - t_j \to x_0 \in [x, 0]$. Since $v^j(s) \to v(s)$ uniformly for $s \in [x, 0]$ and $v_1^j(s_j - t_j) \to 0$ as $j \to \infty$, it follows that $v_1'(x_0) = 0$. But this contradicts Lemma 3.10 since v satisfies (131) and (134) holds. So $p \neq S$.

Recall from condition **(f4)** that there are only a finite number of zeros w_0 of f with $S \leq w_0 \leq T$. So we can choose $\epsilon > 0$ sufficiently small that

$$f(w_0) = 0,\ \|w_0 - p\| \leq \epsilon \Rightarrow w_0 = p. \tag{136}$$

Moreover, Lemma 3.3 implies that

$$f(w_0) = 0, w_{0i} = p_i \text{ for some } i \in \{1, \ldots, N\} \Rightarrow w_0 = p, \tag{137}$$

so ϵ can be chosen, smaller if necessary, so that

$$f(w_0) = 0,\ |w_{0_i} - p_i| \leq \epsilon \text{ for some } i \in \{1, \ldots, N\} \Rightarrow w_0 = p. \tag{138}$$

Now let $E := (\epsilon, \ldots, \epsilon)$. For $n \in \mathbb{N}$, choose $x_n \in \mathbb{R}$ such that $p \leq v(x_n) \leq p + \frac{E}{n}$. Since $v^k(x_n) \to v(x_n)$ as $k \to \infty$, we can choose k_n ($> k_{n-1}$) such that $p \leq v^k(x_n) \leq p + \frac{E}{n}$

also. Since $v^{k_n}(s) \to S$ as $s \to -\infty$ monotonically, there exists unique $\alpha_{k_n} \in \mathbb{R}$ such that $\|v^{k_n}(\alpha_{k_n}) - p\| = \epsilon$ and $\|v^{k_n}(s) - p\| \geq \epsilon$ for $s \leq \alpha_{k_n}$. For each $n \in \mathbb{N}$, define

$$\widetilde{v^{k_n}}(s) = v^{k_n}(s + \alpha_{k_n}), \quad s \in \mathbb{R}. \tag{139}$$

Then $\|\widetilde{v^{k_n}}(0) - p\| = \epsilon$ for each n, and

$$S \leq \widetilde{v^{k_n}}(0) \leq p + \frac{E}{n}. \tag{140}$$

Arguing as in the construction of the function v above, we obtain the existence of $\tilde{v} \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $\tilde{\sigma} \in C^1(\mathbb{R}, [0,1])$ such that (for a subsequence) $\widetilde{v^{k_n}} \to \tilde{v}$ uniformly on compact subsets of $\mathbb{R}$,

$$\|\tilde{v}(0) - p\| = \epsilon, \quad S \leq \tilde{v}(0) \leq p, \quad \tilde{v}'(s) \geq 0, \ s \in \mathbb{R}, \tag{141}$$

and

$$A\tilde{v}'' + c_0\tilde{v}' + \tau_0\tilde{\sigma}G(\tilde{v}, \tilde{v}')\tilde{v}' + f(\tilde{v}) = 0. \tag{142}$$

As above, there exist $p', q' \in \mathbb{R}^N$ such that

$$S \leq p' \leq q' \leq T, \quad \text{and}$$

$$\tilde{v}(s) \to p' \ \text{ as } \ s \to -\infty \ \text{ and } \ \tilde{v}(s) \to q' \ \text{ as } \ s \to \infty.$$

Now by condition **(f4)**, $\mu_{PF}(df[w_0]) > 0$ for each w_0, $S < w_0 < T$ with $f(w_0) = 0$. Theorem 2.4 of [6] implies the existence of a zero w_0 of f such that $\mu_{PF}(df[w_0]) < 0$ in the order interval between two zeros whose Fréchet derivatives have positive Perron-Frobenius eigenvalues. Hence there is no zero of f such that $S < w_0 < p$ or $p < w_0 < T$. Thus $p' = S$ since $\|\tilde{v}(0) - p\| = \epsilon$. Also, (138) yields that $q' \in \{p, T\}$. If $q' = T$, then as earlier in this proof, the points $s_{k_n} - t_{k_n} - \alpha_{k_n}$ at which $\left(v_1^{k_n}\right)'$ tends to zero are contained in a finite interval in $\mathbb{R}$. So there exists $z_0 \in \mathbb{R}$ such that $(\tilde{v}_1)'(z_0) = 0$, which contradicts Lemma 3.10 since (141) and (142) hold. Hence $q' = T$.

Thus we have $\tilde{v}$ satisfying (142) with $\tilde{v}(s) \leq p, \quad s \in \mathbb{R}$, $\tilde{v}(s) \to p$ as $s \to \infty$, and v satisfying (131) with $v(s) \geq p, \quad s \in \mathbb{R}$, $v(s) \to p$ as $s \to -\infty$. But since $\mu_{PF}(df[p]) > 0$, this contradicts Lemma 3.9. The result follows.

$\square$

Let $\mathcal{W}_r$ $(r \geq 0)$ denote the set of all monotone functions $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $w = u + \psi$ where $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies (68) for some $c \in \mathbb{R}$ and some $\sigma \in C(\mathbb{R}, [-1, 1])$ and

$$w_1(\hat{s}) = \frac{1}{2}(T_1 + S_1) \ \text{ for } \ |\hat{s}| \leq r. \tag{143}$$

(Here w_1 denotes the first component of w). The penultimate step in bounding $\mathcal{M}_R$ (67) in the space $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ independently of $R > 0$ is the following. The proof of this result follows immediately from the proof of the corresponding result in [15], due to the fact that the estimates of theorems 3.7 and 3.11 are independent of the choice of $\sigma \in C(\mathbb{R}, [-1, 1])$.

Lemma 3.12 *There exists constant $C_r > 0$ such that for any $w \in \mathcal{W}_r$,*

$$\|w - \psi\|_{W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)} \leq C_r. \tag{144}$$

Up to this point in section 3, the velocity c has been treated as a constant. We now reintroduce the representation of c as a functional to obtain the uniform Sobolev space bound on $\mathcal{M}_R$. The proof of the following result again closely follows [15], but we include it here for completeness.

Theorem 3.13 *There exists $C > 0$ such that if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies*

$$A(u'' + \psi'') + c(u)(u' + \psi') + \tau\sigma_R\varsigma G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0, \tag{145}$$

for some $\tau \in [0, 1]$, $R > 0$, where the functional $c(\cdot)$ is as defined in (37) and (38), and

$$u'(s) + \psi'(s) > 0 \ \text{ for each } \ s \in \mathbb{R}, \tag{146}$$

then

$$\|u\|_{W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)} \leq C. \tag{147}$$

Proof. Let u, w be as in the statement of the theorem. Denote by $\hat{s}$ the solution of the equation $w_1(\hat{s}) = \frac{1}{2}(T_1 + S_1)$. It will be proved that there exists $r > 0$, independent of u, τ, R satisfying (145), (146), such that

$$|\hat{s}| \leq r.$$

If not, then there exists a sequence $\{u^k\} \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N)$ satisfying (145), (146) for some τ_k, R_k, and a sequence $\{s_k\} \subset \mathbb{R}$ with $|s_k| \to \infty$ as $k \to \infty$, such that

$$w_1^k(s_k) = u_1^k(s_k) + \psi_1(s_k) = \frac{1}{2}(T_1 + S_1). \tag{148}$$

We will show that the set of the corresponding velocities $\{c(u^k)\}$ is unbounded, which will contradict Theorem 3.6. Define

$$v^k(s) := w^k(s + s_k) - \psi(s), \quad s \in \mathbb{R}, \tag{149}$$

so that

$$v_1^k(0) + \psi_1(0) = w_1^k(s_k) = \frac{1}{2}(T_1 + S_1) \tag{150}$$

and hence $v^k + \psi \in \mathcal{W}_0$, where $\mathcal{W}_0$ is as in Lemma 3.12. Thus there exists $C_0 > 0$, independent of k, such that

$$\|v^k\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \le C_0. \tag{151}$$

Now by (37) and (149),

$$\begin{aligned} \rho(u^k)^2 &= \int_{\mathbb{R}} \|w^k(s) - T\|^2 \phi(s)\, ds \\ &= \int_{\mathbb{R}} \|v^k(s) + \psi(s) - T\|^2 \phi(s + s_k)\, ds. \end{aligned} \tag{152}$$

Recall that $c(u^k) := \log \rho(u^k)$. Note from (151) and estimate (22) that there exists $N > 0$ independent of k such that for each $s \in \mathbb{R}$, $k \in \mathbb{N}$,

$$\|v^k(s)\| \le N(\mu(s))^{-\frac{1}{2}}. \tag{153}$$

Hence there exist $s_1, s_2 \in \mathbb{R}$, $|s_i| \ge 1, i = 1, 2$, *independent* of $k \in \mathbb{N}$, such that

$$\begin{aligned} s \le s_1 &\Rightarrow \|v^k(s)\| \le \delta, \\ s \ge s_2 &\Rightarrow \|v^k(s)\| \le \delta, \end{aligned} \tag{154}$$

where $\delta > 0$ is as in Theorem 3.6. So for each $k \in \mathbb{N}, s \in \mathbb{R}$,

$$\|v^k(s)\| \le \begin{cases} \kappa e^{-\alpha s_1} e^{\alpha s}, & s \le s_1 \\ N\mu(s)^{-\frac{1}{2}}, & s_1 < s < s_2 \\ \kappa e^{\beta s_2} e^{-\beta s}, & s \ge s_2. \end{cases} \tag{155}$$

Since the right-hand side of (155) defines a square-integrable function on $\mathbb{R}$, and $\|v^k(s) + \psi(s) - S\|$ is bounded independently of $s \in \mathbb{R}, k \in \mathbb{N}$, the form of ϕ given in (35) and

Lebesgue's Dominated Convergence Theorem together yield from (152) that $\rho(u^k) \to 0$ if $s_k \to -\infty$.

Now for $i \in \{1, \ldots, N\}$, $\psi_i(s) = T_i$ when $s \leq -1$. Choose $\tilde{s}$ such that

$$s \leq \tilde{s} \Rightarrow \|v^k(s)\| \leq \frac{1}{2}|T_i - S_i|$$

for each $i \in \{1, \ldots, N\}$, $k \in \mathbb{N}$ (which is possible by (153). So there exists $\eta > 0$ such that for each $k \in \mathbb{N}$,

$$\|v^k(s) + \psi(s) - T\|^2 \geq \eta > 0 \text{ for each } s \leq \tilde{s}. \tag{156}$$

Since $\phi(s) = 1$ for $s \geq 0$, there exists $t > 0$ such that

$$s_k \geq t \Rightarrow \phi(s + s_k) = 1 \text{ for } s \geq -t. \tag{157}$$

So if $s_k \to \infty$, For k sufficiently large, $\phi(s + s_k) = 1$ for $s \geq -s_k$. In particular, when $s_k \geq -\tilde{s}$,

$$\begin{aligned} \rho(u^k)^2 &\geq \int_{-s_k}^{-\tilde{s}} \|v^k(s) + \psi(s) - T\|^2 \phi(s + s_k)\, ds \\ &\geq \eta \int_{-s_k}^{-\tilde{s}} ds = \eta(s_k - \tilde{s}) \to \infty \text{ as } s_k \to \infty, \end{aligned} \tag{158}$$

since $\tilde{s}$ is independent of k.

It follows that $\rho(u^k) \to \infty$ as $s_k \to \infty$ and $\rho(u^k) \to 0$ as $s_k \to -\infty$. Hence since $|s_k| \to \infty$ as $k \to \infty$, $c(u^k)$ is unbounded. The result follows.

□

4 Existence of monotone solutions

The first part of this section is devoted to proving that for each $R > 0$, there exists $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u'(s) + \psi'(s) > 0$ for each $s \in \mathbb{R}$, u satisfies (40), and $u(s) + \psi(s) \to S, T$ as $s \to -\infty, +\infty$. We then tie this existence result for the approximate system together with the uniform *a priori* estimates for monotone solutions of section 3 to obtain existence of a monotone solution of the original system (3).

4.1 Existence of monotone solutions of the approximate system

Fix $R > 0$ throughout this section. With the aim of constructing a set $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ satisfying (52), the first theorem gives that non-monotone solutions $w_N = u_N + \psi$ of

(40) are bounded away from monotone solutions $w_M = u_M + \psi$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. In the following, write $u_M = w_M - \psi$ when $w'_M(s) > 0$ for each $s \in \mathbb{R}$ (that is, the subscript M means monotone), and $u_N = w_N - \psi$ when there exists $s_0 \in \mathbb{R}$ and $i \in \{1, \ldots, N\}$ such that $(w_{N_i})'(s_0) \leq 0$ (that is, the subscript N means non-monotone). Since the proof of this result is a minor modification of the proof of the corresponding result in [15], we omit it.

Theorem 4.1 *Let $R > 0$ be given. Then there exists $r > 0$ (dependent on R) such that if $u_M, u_N \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy (41) with corresponding $\tau = \tau_M, \tau_N \in [0, 1]$ for this R, and $w_M = u_M + \psi$ and $w_N = u_N + \psi$ are monotone and non-monotone respectively, then*

$$\|u_M - u_N\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \geq r. \tag{159}$$

Now define $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ by

$$\Omega := \cup_{u \in \mathcal{M}_R} \{v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) : \|u - v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} < \frac{r}{2}\} \tag{160}$$

where $\mathcal{M}_R$ is as defined in (67), and $r > 0$ is as yielded by Theorem 4.1. Clearly Ω is open, and bounded by Theorem 3.13. Moreover, with $S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ as constructed in Theorem 2.8, and P_R as defined by (42) for fixed $R > 0$,

$$0 \notin S^*_\mu P_R([0, 1] \times \partial\Omega) \tag{161}$$

by (52); if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies (41) for some $\tau \in [0, 1]$ and $w = u + \psi$ is non-monotone, then Theorem 4.1 yields that $u \notin \overline{\Omega}$; on the other hand, if $w = u + \psi$ is monotone, then $u \in \Omega^o$ (the interior of Ω). Hence (53) holds; that is,

$$deg_{(S)_+}(S^*_\mu P_R(1, \cdot), \Omega, 0) = deg_{(S)_+}(S^*_\mu P_R(0, \cdot), \Omega, 0). \tag{162}$$

Now when $\tau = 0$, (41) reduces to the system studied in [15]. We will use [15] to infer that $deg_{(S)_+}(S^*_\mu P_R(1, \cdot), \Omega, 0) \neq 0$. Observe first that the Vol'perts' argument for existence in [15] can be simplified slightly by constructing $\tilde{\Omega}$ analogous to Ω above. Then since there are no zeros of their homotopy function on $\partial\tilde{\Omega}$ for any $\tau \in [0, 1]$, it follows from their homotopy invariance and the analysis regarding the other endpoint of their homotopy that

$$deg_{(S)_+}(S^*_\mu P_R(0, \cdot), \tilde{\Omega}, 0) = 1. \tag{163}$$

(Note that the operator S_μ is constructed in this thesis in the same way as the construction in [15], correcting [15] as necessary; moreover, it is shown in [15] that the degree does not depend on the arbitrariness of the operator satisfying the conditions of Theorem 2.8.) Next, note that since all solutions u of $P_R(0,u)=0$ with $w=u+\psi$ monotone are contained in $\Omega\cap\tilde{\Omega}$, and there are no non-monotone solutions of $P_R(0,u)=0$ in $\overline{\Omega}\cup\overline{\tilde{\Omega}}$, the excision property of degree yields that

$$deg_{(S)_+}(S_\mu^* P_R(0,\cdot),\Omega,0)=deg(S_\mu^* P_R(0,\cdot),\Omega\cap\tilde{\Omega},0)=deg_{(S)_+}(S_\mu^* P_R(0,\cdot),\tilde{\Omega},0). \quad (164)$$

Thus (162) and (163) imply that

$$deg_{(S)_+}(S_\mu^* P_R(1,\cdot),\Omega,0)=1. \quad (165)$$

The following is then immediate from the existence property of degree, (52), and the fact that there are no non-monotone solutions of (41) in $\overline{\Omega}$.

Theorem 4.2 *Let $R>0$ be given. Then there exists $u\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)\cap C^2(\mathbb{R},\mathbb{R}^N)$ such that*

$$u'(s)+\psi'(s)>0 \ \text{ for each } \ s\in\mathbb{R},$$

and

$$A(u''+\psi'')+c(u)(u'+\psi')+\sigma_R G(u+\psi,u'+\psi')(u'+\psi')+f(u+\psi)=0.$$

5 Existence of monotone solutions of the autonomous system

Here we combine the uniform *a priori* estimates for monotone solutions of section 3 with the existence of monotone solutions for the approximate system proved in section 4.1. The goal of this paper is the following.

Theorem 5.1 *Let $A\in M^{N\times N}$ be a positive-definite diagonal matrix, $f\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ satisfy* **(f1)**-**(f4)**, *and $G\in C^1(\mathbb{R}^N\times\mathbb{R}^N,M^{N\times N})$ satisfy* **(G1)**-**(G3)**. *Let $\varsigma:\mathbb{R}\to[-1,1]$ be uniformly continuous. Then there exists $u\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)\cap C^2(\mathbb{R},\mathbb{R}^N)$ and $c\in\mathbb{R}$ such that*

$$u'(s)+\psi'(s)>0 \ \text{ for each } \ s\in\mathbb{R}, \quad (166)$$

and

$$A(u''+\psi'')+c[u'+\psi']+\varsigma G(u+\psi,u'+\psi')(u'+\psi')+f(u+\psi)=0. \tag{167}$$

Proof. For each $n\in\mathbb{N}$, let $u_n\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)\cap C^2(\mathbb{R},\mathbb{R}^N)$ be such that

$$u_n'(s)+\psi'(s)>0 \text{ for each } s\in\mathbb{R}, \tag{168}$$

and

$$A(u_n''+\psi'')+c(u_n)[u_n'+\psi']+\sigma_n\varsigma G(u_n+\psi,u_n'+\psi')(u_n'+\psi')+f(u_n+\psi)=0. \tag{169}$$

Such solutions exist by Theorem 4.2. By Theorem 3.6, we can take a subsequence, if necessary, so that there exists $c_0\in\mathbb{R}$ with

$$c(u_n)\to c_0 \text{ as } n\to\infty. \tag{170}$$

Also, $\{u_n\}_{n=1}^\infty$ is bounded in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ *independently* of $n\in\mathbb{N}$ by Theorem 3.13. So since closed balls in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ are weakly sequentially compact, there exists $u_0\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ such that

$$u_n\rightharpoonup u_0 \text{ in } W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \text{ as } n\to\infty. \tag{171}$$

Now let $r>0$ and let $Y_r:=[-r,r]$. Note that $u_n\rightharpoonup u_0$ in $W^1_2(\mathbb{R},\mathbb{R}^N)$ as $n\to\infty$. So the Sobolev Embedding Theorem yields that

$$u_n\to u_0 \text{ in } C(Y_r,\mathbb{R}^N) \text{ as } n\to\infty. \tag{172}$$

Since $w_n=u_n+\psi$ is monotone for each n, $\|u_n\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}$ is bounded independently of n, and thus, by Theorem 3.2, so are $\|u_n'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}$ and $\|u_n''\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}$. The Arzela-Ascoli theorem can thus be applied to yield that there is a subsequence $\{u_k\}$ of $\{u_n\}$ such that u_k converges in $C^1(Y_r,\mathbb{R}^N)$. Now recall from the construction of σ_R in section 2.3.1 that for each $s\in\mathbb{R}$, $\sigma_n\to 1$ as $n\to\infty$. It thus follows from (169), (170), the continuity of f and G and the convergence of u_k that u_k'' is uniformly convergent on Y_r. Hence there is a subsequence $\{u_l\}$ of $\{u_k\}$ and $\tilde{u}\in C^2(Y_r,\mathbb{R}^N)$, such that

$$u_l\to\tilde{u} \text{ in } C^2(Y_r,\mathbb{R}^N) \text{ as } l\to\infty. \tag{173}$$

A diagonal sequence argument yields the existence of a subsequence $\{u_j\}$ of $\{u_l\}$ that converges in $C^2(Y_r, \mathbb{R}^N)$ for each $r > 0$ to a limit $u \in C^2(\mathbb{R}, \mathbb{R}^N)$. But since $u_n \to u_0$ in $C(Y_r, \mathbb{R}^N)$ as $n \to \infty$ for each $r > 0$ (172), uniqueness of limits in $C(Y_r, \mathbb{R}^N)$ yields that

$$u = u_0. \tag{174}$$

For each $s \in \mathbb{R}$, $\sigma_n(s) \to 1$ as $n \to \infty$. Thus for each $s \in \mathbb{R}$, we can pass to the limit in (169)as $j \to \infty$ along the subsequence for which $u_j \to u_0$ to obtain that

$$A(u_0'' + \psi'') + c_0[u_0' + \psi'] + G(u_0 + \psi, u_0' + \psi')(u_0' + \psi') + f(u_0 + \psi) = 0. \tag{175}$$

Since $u_0 \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, it follows immediately from estimate (22) that $|u_0(s)| \to 0$ as $|s| \to \infty$, and hence that $w_0(s) := u_0(s) + \psi(s) \to S$ as $s \to -\infty$ and $w_0(s) \to T$ as $s \to \infty$, by (35). Also, since $u_j'(s) \to u_0'(s)$ as $j \to \infty$ for each $s \in \mathbb{R}$, and $u_j'(s) + \psi'(s) > 0$ for each j, s, we have that $w_0'(s) \geq 0$ for each s. Applying Lemma 3.10 with $\sigma \equiv 1$ thus yields that $w_0'(s) > 0$ for each $s \in \mathbb{R}$, since $w_0(s)$ is not identically constant. The result follows.

□

The following corollary follows immediately by taking $\varsigma \equiv 1$ in the above theorem.

Corollary 5.2 *Let $A \in M^{N\times N}$ be a positive-definite diagonal matrix, $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfy* **(f1)-(f4)**, *and $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N, M^{N\times N})$ satisfy* **(G1)-(G3)**. *Then there exists $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $c \in \mathbb{R}$ such that $w'(s) > 0$ for each $s \in \mathbb{R}$, $w(s) \to S, T$ as $s \to -, +\infty$, and $u(x,t) = w(x - ct)$ is a solution of*

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial t^2} + G\left(u, \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x} + f(u). \tag{176}$$

A Appendix

Proof of Theorem 3.8. We treat the case when $s \to \infty$; the other is similar. Without loss of generality, suppose $w_0 = 0$.

Consider first $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ that is bounded for s sufficiently large, satisfies (100) and (102), and

$$Aw''(s) + cw'(s) + Bw(s) = 0 \tag{177}$$

where $B \in P^{N\times N}$ and $\mu_{PF}(B) \neq 0$. Recall that the second order system (177) is equivalent to the first order system

$$\begin{pmatrix} w' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A^{-1}B & -cA^{-1} \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} =: K_c \begin{pmatrix} w \\ y \end{pmatrix} \tag{178}$$

where $y(s) := w'(s)$, $s \in \mathbb{R}$. Furthermore, the general solution of (178) is a linear combination of terms of the form

$$s^m e^{\beta s} p \text{ or } s^n e^{\gamma s} t(s),\ s \in \mathbb{R}, \tag{179}$$

where m, n are non-negative integers, β is a real eigenvalue of K_c with generalised eigenvector p, γ is the real part of a complex eigenvalue of K_c and $t(s)$ is a vector-valued combination of sines and cosines. It thus follows from the assumption that w is bounded for s sufficiently large that w has the form

$$w(s) = \left(\Sigma_{k=0}^m s^k q_k + \Sigma_{j=0}^n s^j \phi_j(s)\right) e^{\alpha s} + z(s). \tag{180}$$

Here $m, n \in \mathbb{N}$, $q_k \in \mathbb{R}^N$ for each $k \in \{0, \ldots, m\}$ and for each $j \in \{0, \ldots, n\}$, $\phi_j(s)$ is a vector-valued combination of $\cos \beta_j s$ and $\sin \beta_j s$, where β_j are non-zero imaginary parts of eigenvalues of K_c which have real part α; $\alpha \leq 0$ and the vector q_m (that multiplies the highest power of s) satisfies

$$(\alpha^2 A + \alpha c I + B) q_m = 0. \tag{181}$$

The function z comprises of terms with a factor $e^{\beta s}$, where $\beta < \alpha$. Without loss of generality, m and n are chosen so that $q_m \neq 0$ unless $\Sigma_{k=0}^m s^k q_k \equiv 0$ and $\phi_n(s) \not\equiv 0$ unless $\Sigma_{j=0}^n s^j \phi_j(s) \equiv 0$. Also, $\Sigma_{k=0}^m s^k q_k + \Sigma_{j=0}^n s^j \phi_j(s) \not\equiv 0$ by condition (100).

Suppose first that $\Sigma_{k=0}^m s^k q_k \equiv 0$. Then $\phi_n(s) \not\equiv 0$ since $\Sigma_{k=0}^m s^k q_k + \Sigma_{j=0}^n s^j \phi_j(s) \not\equiv 0$ and

$$s^{-n} e^{-\alpha s} w(s) = \phi_n(s) + \Sigma_{j=0}^{n-1} s^{j-n} \phi_j(s) + s^{-n} e^{-\alpha s} z(s),\ s \in \mathbb{R}, \tag{182}$$

the second term on the right-hand side being included only if $n \geq 1$.

Bohr [3] defines an *almost-periodic* function to be a function f, continuous on $\mathbb{R}$, such that for each $\epsilon > 0$, there exists $L(\epsilon) > 0$ so that each interval of $\mathbb{R}$ of length $L(\epsilon)$ contains a number τ, satisfying

$$|f(x+\tau) - f(x)| \leq \epsilon \text{ for each } x \in \mathbb{R}. \tag{183}$$

It is shown in [3] that a finite sum of almost-periodic functions is almost-periodic. Since continuous periodic functions are clearly almost-periodic, it follows that ϕ_n is almost-periodic. Moreover, the mean value of ϕ_n, given by

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi_n(s)ds, \tag{184}$$

is zero, since for $\beta \in \mathbb{R}$,

$$|\int_0^T \cos\beta s ds| \leq \frac{1}{\beta} \tag{185}$$

and so $\int_0^T \phi_n(s)ds$ is bounded independently of $T > 0$. It is proved in [3] that a non-positive almost-periodic function with mean-value zero must be identically zero. Hence there exists a component $(\phi_n)_i$ of ϕ_n and $s_0 \in \mathbb{R}$ such that $(\phi_n)_i(s_0) > 0$. It follows from the definition of almost-periodicity above that there exists $\zeta > 0$ and a sequence $s_k \to \infty$ such that $(\phi_n)_i(s_k) > \zeta$. Since the second and third terms on the right-hand side of (182) tend to zero as $s \to \infty$, $w_i(s_k) > 0$ for k sufficiently large. But this contradicts the fact that $w(s) \leq 0$ for s sufficiently large (condition (102)). So $\Sigma_{k=0}^m s^k q_k \not\equiv 0$ (and hence $q_m \neq 0$, by the choice of m). Note also that $m \geq n$, since otherwise

$$s^{-n}e^{-\alpha s}w(s) = \phi_n(s) + \Sigma_{k=0}^m s^{k-n} q_k + \Sigma_{j=0}^{n-1} s^{j-n}\phi_j(s) + s^{-n}e^{-\alpha s}z(s) \tag{186}$$

and a similar argument to that in the last paragraph leads to a contradiction.

Suppose now that q_m has a positive component, say $q_{m_1} > 0$. (180) gives that

$$s^{-m}e^{-\alpha s}w(s) = q_m + \Sigma_{k=0}^{m-1} s^{k-m} q_k + \Sigma_{j=0}^{n} s^{j-m}\phi_j(s) + s^{-m}e^{-\alpha s}z(s),\ s \in \mathbb{R}. \tag{187}$$

(The second term on the right-hand side is included only if $m \geq 1$.) Recalling the above analysis for ϕ in the case when $m = n$, it follows that $w_1(s)$ takes positive values on a sequence tending to infinity. This contradicts that $w(s) \leq 0$ for s sufficiently large, as before.

Hence $q_m \leq 0$. Since $q_m \neq 0$, it follows from (181) and the fact that $B \in P^{N\times N}$ that $q_m < 0$. If $\alpha = 0$, (181) yields that $\mu_{PF}(B) = 0$, since $-q_m > 0$, and any *positive* eigenvector corresponds to the Perron-Frobenius eigenvalue (see Theorem 1.1). This contradicts the hypotheses on B, so $\alpha < 0$. Thus the conclusion of the theorem holds for the linear problem (177), taking $q = -q_m$.

Return now to w satisfying the hypotheses of the theorem. Since $w(s) \to 0$ as $s \to \infty$, there is a sequence $s_n \to \infty$ such that for n sufficiently large,

$$\frac{1}{n} = \|w(s_n)\| = \sup_{s \geq s_n} \|w(s)\|. \tag{188}$$

Define functions $w_n : \mathbb{R} \to \mathbb{R}^N$ by $w_n(s) = w(s + s_n)$; in particular, $w_n(0) = w(s_n)$. Also, write $F(y) = By + R(y)$, $y \in \mathbb{R}^N$, where $B := df[0]$ and $R : \mathbb{R} \to \mathbb{R}^N$ is such that $\dfrac{\|R(y)\|}{\|y\|} \to 0$ as $\|y\| \to 0$. (99) becomes

$$Aw_n''(s) + cw_n'(s) + \sigma(s+s_n)G(w_n(s), w_n'(s))w_n'(s) + Bw_n(s) + R(w_n(s)) = 0, \ s \in \mathbb{R}. \tag{189}$$

Now defining $v_n(s) = \dfrac{w_n(s)}{\|w_n(0)\|}$ for $s \geq 0$ yields that

$$Av_n''(s) + cv_n'(s) + \sigma(s+s_n)G(w_n(s), w_n'(s))v_n'(s) + Bv_n(s) + \frac{R(w_n(s))}{\|w_n(0)\|} = 0, \ s \geq 0. \tag{190}$$

We require that $\dfrac{\|R(w_n(s))\|}{\|w_n(0)\|} \to 0$ uniformly on $[0, \infty)$ as $n \to \infty$. Let $\epsilon > 0$ be given. There exists $t \in \mathbb{R}$ such that

$$s \geq t \Rightarrow \|R(w(s))\| \leq \epsilon \|w(s)\|. \tag{191}$$

Also, $s_n \to \infty$ as $n \to \infty$, so there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow s + s_n \geq t \text{ for every } s \geq 0. \tag{192}$$

Hence for $n \geq n_0$,

$$\frac{\|R(w_n(s))\|}{\|w_n(0)\|} \leq \frac{\|R(w_n(s))\|}{\|w_n(s)\|} = \frac{\|R(w(s+s_n))\|}{\|w(s+s_n)\|} \leq \epsilon \text{ for every } s \geq 0, \tag{193}$$

from which the required convergence follows.

The aim now is to find a subsequential limit of $\{v_n\}$, using the Arzela-Ascoli theorem, that is bounded for s sufficiently large, and satisfies (100), (102) and the linear system (177), with $B = df[0]$. The result will then follow from the above analysis for the linear system.

For this, bounds on $v_n'(s)$ and $v_n''(s)$ are required. Note first that by (99), (101) and (102), there exists $t_0 \in \mathbb{R}$ such that $\|w(s)\|$, $\|w'(s)\|$ and $\|w''(s)\|$ are bounded independently of $s \geq t_0$. Hence for $n \geq n_1$, say, $\|w_n(s)\|$, $\|w_n'(s)\|$ and $\|w_n''(s)\|$ are bounded independently of $n \geq n_1$ and $s \geq 0$, and for each *fixed* $n \geq n_1$, $\|v_n(s)\|$, $\|v_n'(s)\|$ and $\|v_n''(s)\|$

are bounded independently of $s \geq 0$. Also, since $\|w_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}$ and $\|w_n'\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}$ are bounded independently of $n \geq n_1$ and G is continuous, there exists $M_1 > 0$ such that

$$\|G(w_n(s), w_n'(s))\| \leq M_1 \text{ for } n \geq n_1,\ s \geq 0. \tag{194}$$

Now Landau's inequality [1] together with (190) yield that

$$\begin{aligned}
\|v_n'\|^2_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} &\leq 4\|v_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}\|v_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \\
&= 4\|v_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}\left\|A^{-1}(cv_n' + \sigma G(w_n, w_n')v_n' + Bv_n + R(w_n)\|w_n(0)\|^{-1}\right\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \\
&\leq 4\|A^{-1}\|\Big\{|c|\|v_n'\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} + M_1\|v_n'\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \\
&\qquad\qquad + \|B\| + \|R(w_n)\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}\|w_n(0)\|^{-1}\Big\}
\end{aligned} \tag{196}$$

since $\|v_n(s)\| = \frac{\|w_n(s)\|}{\|w_n(0)\|} \leq 1$ for each $n,\ s \geq 0$. Since $\|R(w_n(s))\|\|w_n(0)\|^{-1} \to 0$ as $n \to \infty$ *uniformly* for $s \geq 0$, $\|R(w_n)\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}\|w_n(0)\|^{-1}$ is bounded independently of n, say by $M_2 > 0$. Hence for $n \geq n_1$,

$$\|v_n'\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq \max\left\{1, 4\|A^{-1}\|(|c| + M_1 + M_2 + \|B\|)\right\} =: N_1. \tag{197}$$

It then follows immediately from (197) and (190) that $\|v_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}$ is bounded independently of $n \geq n_1$, say by $N_2 > 0$.

Now let $r > 0$ and define $X_r := [0, r]$. Then $\|v_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq 1$ for all n, so the sequence $\{v_n\}_{n=n_1}^\infty$ is uniformly bounded on X_r. Moreover, the mean value inequality for vector-valued functions gives that for $n \geq n_1,\ s, t \in X_r$, there exists $\theta \in (0, 1)$ such that for $\xi = \theta s + (1 - \theta)t$,

$$\|v_n(s) - v_n(t)\| \leq \|v_n'(\xi)\||s - t| \leq N_1|s - t|, \tag{198}$$

where N_1 is defined in (197). Hence $\{v_n\}_{n=n_1}^\infty$ is equicontinuous. The Arzela-Ascoli Theorem thus yields that there is a subsequence v_k and $u_1 \in C(X_r, \mathbb{R}^N)$ such that $v_k \to u_1$ in

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[1] A scalar-valued version of Landau's inequality is proved in Hille [9]. The following vector-valued version is a straightforward consequence.

Landau's Inequality

Let $\Lambda \subset \mathbb{R}$ be either a semi-infinite interval or $\mathbb{R}$. Let $v \in C^2(\Lambda, \mathbb{R}^N)$ be such that v, v' and v'' are uniformly bounded on Λ. Then

$$\|v'\|^2_{L_\infty(\Lambda,\mathbb{R}^N)} \leq 4\|v\|_{L_\infty(\Lambda,\mathbb{R}^N)}\|v''\|_{L_\infty(\Lambda,\mathbb{R}^N)}. \tag{195}$$

$C(X_r,\mathbb{R}^N)$. Since $\|v_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} < N_2$ for $n \geq n_1$, there is a further subsequence $\{v_j\}$ of $\{v_k\}$, and $u_2 \in C(X_r,\mathbb{R}^N)$, such that

$v_j' \to u_2$ in $C(X_r,\mathbb{R}^N)$.

Now for $n \geq n_1$, $\|w_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} = \|w_n(0)\| = \dfrac{1}{n} \to 0$ as $n \to \infty$, and by Landau's inequality,

$$\|w_n'\|^2_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq 4\|w_n\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)}\|w_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq \frac{4\gamma}{n} \tag{199}$$

since there exists $\gamma > 0$ such that $\|w_n''\|_{L_\infty(\mathbb{R}_+,\mathbb{R}^N)} \leq \gamma$ for each $n \geq n_1$. So since $G(w_0,0) = 0$ for each $w_0 \in \mathbb{R}^N$ for which $f(w_0) = 0$ (condition **(G3)**) and G is continuous,

$$G(w_n(s), w_n'(s)) \to 0 \text{ as } n \to \infty, \tag{200}$$

uniformly for $s \in X_r$. The fact that $\|v_m'(s)\| \leq N_1$ for $s \geq 0$, $m \geq n_1$ thus yields that

$$G(w_n(s), w_n'(s))v_n'(s) \to 0 \text{ as } n \to \infty, \tag{201}$$

uniformly for $s \in X_r$.

It thus follows from (190), (201), the convergence of v_j, v_j' and the fact that $\sigma(s) \in [0,1]$ for each $s \in \mathbb{R}$ that the sequence $\{v_j''\}$ is uniformly convergent on X_r. Hence $\{v_j\}$ is Cauchy in the Banach space $C^2(X_r,\mathbb{R}^N)$ (see section 2.1 for a precise definition) and thus there exists $v \in C^2(\Omega)$ such that $v_j \to v$ in $C^2(\Omega)$ as $j \to \infty$. So for each $r > 0$, there is a subsequence of the original sequence $\{v_n\}$ that converges in $C^2(X_r,\mathbb{R}^N) = C^2(X_r,\mathbb{R}^N)$; a diagonal subsequence argument now shows that $\{v_n\}$ has a subsequence that converges in $C^2([0,r],\mathbb{R}^N)$ for each $r > 0$, to a limit $v \in C^2(\mathbb{R}_+,\mathbb{R}^N)$. Letting $n \to \infty$ in (190), taking (191) and (201) into account, yields that

$$Av''(s) + cv'(s) + Bv(s) = 0 \text{ for each } s > 0. \tag{202}$$

By construction of v, it is clear that $\|v(s)\|$ and $\|v'(s)\|$ are both uniformly bounded for $s > 0$, and that $v(s) \leq 0$ for each $s > 0$. Moreover, $\|v_n(0)\| = \dfrac{\|w_n(0)\|}{\|w_n(0)\|} = 1$ for each $n \in \mathbb{N}$, so $\|v(0)\| = 1$ since $v_j \to v$ in $C^2([0,1])$ (for example). So $v(0) \neq 0$. Thus v satisfies the hypotheses for the linear analysis above. The result follows.

$\square$

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