Integral Control of Infinite-Dimensional Linear Systems Subject to Input Saturation *

H. LOGEMANN and E.P. RYAN
School of Mathematical Sciences,
University of Bath,
Claverton Down, Bath BA2 7AY, United Kingdom
Email: `hl@maths.bath.ac.uk` , `epr@maths.bath.ac.uk`

S. TOWNLEY †
Department of Mathematics,
University of Exeter,
North Park Roand, Exeter EX4 4QE, United Kingdom
Email: `townley@maths.ex.ac.uk`

December 1996

Abstract: Closing the loop around an exponentially stable single-input single-output regular linear system, subject to a globally Lipschitz and non-decreasing actuator nonlinearity and compensated by an integral controller, is shown to ensure asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive integrator gain is sufficiently small and (c) the reference value is feasible in a very natural sense. The class of actuator nonlinearities under consideration contains standard nonlinearities important in control engineering such as saturation and deadzone.

Keywords: Regular infinite-dimensional systems; integral control; actuator nonlinearities; input saturation; robust tracking; operator Riccati equations.

AMS subject classifications: 93C10, 93C20, 93C25, 93D09, 93D10, 93D21.

---

* This work was supported by the Human Capital and Mobility programme (Project number CHRX-CT93-0402) and NATO (Grant CRG 950179).

† Also with the Centre for Systems and Control Engineering, School of Engineering, University of Exeter, UK.

1. Introduction

The synthesis of low-gain integral (I) and proportional-plus-integral (PI) controllers for uncertain stable plants has received considerable attention in the last 20 years. The following principle is well known (see Davison [5], Lunze [19] and Morari [23]): closing the loop around a stable, finite-dimensional, continuous-time, single-input, single-output plant, with transfer function $\mathbf{G}(s)$, compensated by a pure integral controller k/s (see Fig. 1), will result in a stable closed-loop system which achieves asymptotic tracking of arbitrary constant reference signals, provided that $|k|$ is sufficiently small and $k\mathbf{G}(0) > 0$. Therefore, if a plant is known to be stable and if the sign of $\mathbf{G}(0)$ is known (this information can be obtained from plant step response data), then the problem of tracking by low-gain integral control reduces to that of tuning the gain parameter k. Such a controller design approach ("tuning regulator theory"[5]) has been successfully applied in process control, see, for example, Coppus et al. [3] and Lunze [18].

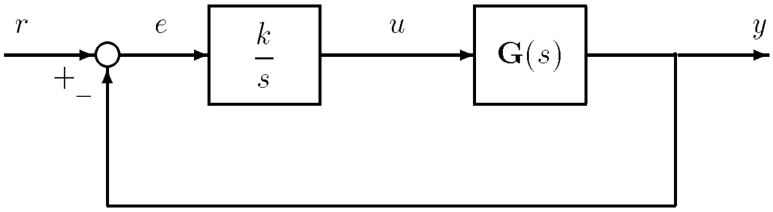

Fig. 1: Low-gain control system

An analogous result holds for finite-dimensional multivariable systems, under suitable assumptions on $\mathbf{G}(0)$, see [5], [19] and [23]. Moreover, the result has been extended by Logemann et al. [13], Logemann and Owens [14], Logemann and Townley [17], Pohjolainen [26, 27] and Pohjolainen and Lätti [28] to various classes of (abstract) infinite-dimensional systems, and by Jussila and Koivo [9] and Koivo and Pohjolainen [11] to differential delay systems. Furthermore, the problem of tuning the integrator gain adaptively has been addressed recently in a number of papers, see Cook [2] and Miller and Davison [21, 22] for the finite-dimensional case and Logemann and Townley [16, 17] for the infinite-dimensional case.

In this paper we present results which show that the above principle remains true if the plant to be controlled is a single-input, single-output, infinite-dimensional, linear system subject to an input nonlinearity (see Fig. 2). More precisely, we prove that, for an exponentially stable system with $\mathbf{G}(0) > 0$, there exists a number $K > 0$ such that, for all non-decreasing globally Lipschitz nonlinearities ϕ with Lipschitz constant λ and all $k \in (0, K/\lambda)$, the output $y(t)$ of the closed-loop system shown in Fig. 2 converges to r as $t \to \infty$, provided that $[\mathbf{G}(0)]^{-1}r \in \mathrm{clos}\,(\mathrm{im}\,\phi)$. The number K is the supremum of the set of all numbers $\kappa^* > 0$ such that the function

$$1 + \kappa \mathrm{Re}\, \frac{\mathbf{G}(s)}{s}$$

is positive real for all $\kappa \in (0, \kappa^*)$. The essence of our approach is to invoke a particular coordinate transformation and perform a Liapunov-type analysis on the tranformed system. A parametrized operator Riccati equation plays a central role in the latter analysis which further develops an idea presented in Townley and Kamstra [33].

The linear, infinite-dimensional part of the plant in Fig. 2 is assumed to be regular. The class of regular linear infinite-dimensional systems, introduced by Weiss [34]–[38], is rather general. It includes most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. Although there exist well-posed abstract infinite-dimensional systems which are not regular, the authors are of the opinion that any physically-motivated, *well-posed*, linear, time-invariant control system is regular. We emphasize that our assumptions on the actuator nonlinearity allow for standard nonlinearities occurring in control engineering such as saturation and deadzone.

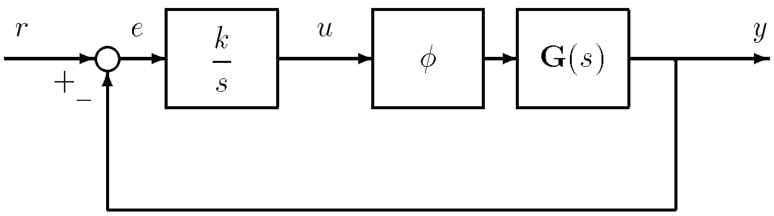

Fig. 2: Low-gain control with input nonlinearity

To our knowledge some of the results in this paper are new even for the finite-dimensional case. Whilst Desoer and Lin [6] consider the low-gain tracking problem for a class of nonlinear finite-dimensional systems, their framework does not include input saturation.

The paper is organized as follows. Definitions and fundamental facts pertaining to regular systems are assembled in Section 2. Section 3 contains the main result of the paper as outlined above. Examples and simulations illustrating our results are given in Section 4. The proofs of three technical lemmas are given in the Appendix.

Notation:

- For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$.
- For $\alpha \in \mathbb{R}$ and H a Hilbert space, we define the exponentially weighted L^2-space $L^2_\alpha(\mathbb{R}_+, H) := \{f \in L^2_{loc}(\mathbb{R}_+, H) \,|\, f(\cdot)\exp(-\alpha\,\cdot) \in L^2(\mathbb{R}_+, H)\}$.

- If A is a linear operator, then the domain, spectrum and resolvent set of A are denoted by $\mathrm{dom}\,(A)$, $\sigma(A)$ and $\varrho(A)$, respectively.
- The set of all linear bounded operators from H_1 to H_2 (where H_1,H_2 are Hilbert spaces) is denoted by $\mathcal{B}(H_1, H_2)$. We write $\mathcal{B}(H)$ for $\mathcal{B}(H, H)$.
- The Laplace transform is denoted by $\mathfrak{L}$.

2. Preliminaries on regular systems

In this section we give some background on well-posed linear systems: the reader is referred to Weiss [34]–[38] for full details.

First, we introduce some further notation. For any Hilbert space H and any $\tau \geq 0$, $\mathbf{R}_\tau$ denotes the right-shift by τ on $L^2_{loc}(\mathbb{R}_+, H)$. The truncation operator $\mathbf{P}_\tau : L^2_{loc}(\mathbb{R}_+, H) \to L^2(\mathbb{R}_+, H)$ is given by

$$(\mathbf{P}_\tau u)(t) = \begin{cases} u(t) & \text{if} \quad t \in [0, \tau] \\ 0 & \text{if} \quad t > \tau \end{cases} \quad .$$

For $u, v \in L^2_{loc}(\mathbb{R}_+, H)$ and $\tau \geq 0$, the *τ-concatenation* $u \overset{\tau}{\diamondsuit} v$ is defined by

$$u \overset{\tau}{\diamondsuit} v = \mathbf{P}_\tau u + \mathbf{R}_\tau v\,.$$

The fundamental concept of a well-posed linear system was introduced by Weiss [38]: an equivalent definition can be found in Salamon [32].

Definition 2.1 Let U, X and Y be real Hilbert spaces. A *well-posed* linear system with *state-space* X, *input-space* U and *output-space* Y is a quadruple $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$, where

(1) $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X,

(2) $\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to X such that

$$\mathbf{\Phi}_{\tau+t}(u \overset{\tau}{\diamondsuit} v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v\,,$$

for all $u, v \in L^2(\mathbb{R}_+, U)$ and all $\tau, t \geq 0$,

(3) $\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, Y)$ such that

$$\mathbf{\Psi}_{\tau+t} x_0 = \mathbf{\Psi}_\tau x_0 \overset{\tau}{\diamondsuit} \mathbf{\Psi}_t \mathbf{T}_\tau x_0\,,$$

for all $x_0 \in X$ and all $\tau, t \geq 0$, and $\mathbf{\Psi}_0 = 0$,

(4) $\mathbf{F} = (\mathbf{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ such that

$$\mathbf{F}_{\tau+t}(u \overset{\tau}{\diamondsuit} v) = \mathbf{F}_\tau u \overset{\tau}{\diamondsuit} (\mathbf{\Psi}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v)\,,$$

$u, v \in L^2(\mathbb{R}_+, U)$ and all $\tau, t \geq 0$, and $\mathbf{F}_0 = 0$. $\diamond$

Let an input $u \in L^2_{loc}(\mathbb{R}_+, U)$ and an initial state $x_0 \in X$ be given. The state $x(t) = x(t; x_0, u)$ of $\mathbf{\Sigma}$ at time $t \geq 0$ and the output $y(\cdot) = y(\cdot\,; x_0, u)$ of $\mathbf{\Sigma}$ are defined by

$$x(t) = \mathbf{T}_t x_0 + \mathbf{\Phi}_t \mathbf{P}_t u\,, \tag{2.1a}$$
$$\mathbf{P}_t y = \mathbf{\Psi}_t x_0 + \mathbf{F}_t \mathbf{P}_t u\,. \tag{2.1b}$$

The state trajectory $x(\cdot)$ is a continuous function from $\mathbb{R}_+$ to X, and the output $y(\cdot)$ is in $L^2_{loc}(\mathbb{R}_+, Y)$.

We say that $\mathbf{\Sigma}$ is *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable, i.e.

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \log \|\mathbf{T}_t\| < 0\,.$$

If $\mathbf{\Sigma}$ is exponentially stable, then the operators $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are uniformly bounded. It is clear that there exist unique operators $\mathbf{\Psi}_\infty : X \to L^2_{loc}(\mathbb{R}_+, Y)$ and $\mathbf{F}_\infty : L^2_{loc}(\mathbb{R}_+, U) \to L^2_{loc}(\mathbb{R}_+, Y)$ such that for all $\tau \geq 0$

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty\,, \quad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty\,.$$

It follows easily that $\mathbf{P}_\tau \mathbf{F}_\infty = \mathbf{P}_\tau \mathbf{F}_\infty \mathbf{P}_\tau$ for all $\tau \geq 0$, i.e. $\mathbf{F}_\infty$ is a *causal* operator. Moreover, if $\mathbf{\Sigma}$ is exponentially stable, then $\mathbf{\Psi}_\infty$ is a bounded operator from X into $L^2(\mathbb{R}_+, Y)$ and $\mathbf{F}_\infty$ maps $L^2(\mathbb{R}_+, U)$ boundedly into $L^2(\mathbb{R}_+, Y)$.

The generator of $\mathbf{T}$ is denoted by A. Let X_1 be the space $\operatorname{dom}(A)$ endowed with the graph norm. The norm on X is denoted by $\|\cdot\|$, whilst $\|\cdot\|_1$ denotes the graph norm. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(sI - A)^{-1}x\|$, where $s \in \varrho(A)$ is fixed. We have $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0-semigroup on X_1 and extended to a C_0-semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, its generator) and the associated restrictions and extensions. With this convention, we may write $A \in \mathcal{B}(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X.

By a representation theorem due to Salamon [32] (see also Weiss [36, 37]) there exist unique operators $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$ (the *control operator* and the *observation operator* of $\mathbf{\Sigma}$, respectively) such that, for all $t \geq 0$, $u \in L^2_{loc}(\mathbb{R}_+, U)$ and $x_0 \in X_1$,

$$\mathbf{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \quad \text{and} \quad (\mathbf{\Psi}_\infty x_0)(t) = C \mathbf{T}_t x_0\ .$$

B is called *bounded* if $B \in \mathcal{B}(U, X)$ (and *unbounded* otherwise), whereas C is called *bounded* if it can be extended continuously to X (and *unbounded* otherwise). If $\mathbf{T}$ is exponentially stable, then there exist constants $\alpha, \beta > 0$ such that for all $t \geq 0$, $u \in$

$L^2(\mathbb{R}_+, U)$ and $x_0 \in X_1$

$$\|\mathbf{\Phi}_t \mathbf{P}_t u\| = \|\int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau\| \le \alpha \|u\|_{L^2(0,t;U)}\,, \tag{2.2}$$

$$\|(\mathbf{\Psi}_\infty x_0)(\cdot)\|_{L^2(0,t;Y)} = \left(\int_0^t \|C\mathbf{T}_\tau x_0\|^2\, d\tau\right)^{1/2} \le \beta \|x_0\|\,. \tag{2.3}$$

As in [37], the *Lebesgue extension* of C is defined by

$$C_L x_0 = \lim_{t\to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\tau x_0\, d\tau\,,$$

where $\operatorname{dom}(C_L)$ is the set of all those $x_0 \in X$ for which the above limit exists. Clearly $X_1 \subset \operatorname{dom}(C_L) \subset X$ and, for any $x_0 \in X$, we have $\mathbf{T}_t x_0 \in \operatorname{dom}(C_L)$ for almost every (a.e.) $t \ge 0$. Furthermore,

$$(\mathbf{\Psi}_\infty x_0)(t) = C_L \mathbf{T}_t x_0 \quad \text{for a.e. } t \ge 0\,.$$

It can be shown (see Weiss [35]) that, if $\alpha > \omega(\mathbf{T})$ and $u \in L^2_\alpha(\mathbb{R}_+, U)$, then $\mathbf{F}_\infty u \in L^2_\alpha(\mathbb{R}_+, Y)$ and there exists a unique holomorphic $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathcal{B}(U, Y)$ such that, for all $s \in \mathbb{C}_\alpha$,

$$\mathbf{G}(s)(\mathfrak{L}u)(s) = [\mathfrak{L}(\mathbf{F}_\infty u)](s)\,.$$

In particular, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ for all $\alpha > \omega(\mathbf{T})$. The function $\mathbf{G}$ is called the *transfer function* of $\mathbf{\Sigma}$.

$\mathbf{\Sigma}$ and its transfer function $\mathbf{G}$ are said to be *regular* if, for any $u \in U$, the limit

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s)u = Du$$

exists. It follows, from the principle of uniform boundedness, that $D \in \mathcal{B}(U, Y)$. The operator D is called the *feedthrough operator* of $\mathbf{\Sigma}$. If $\mathbf{\Sigma}$ is regular, then for any $x_0 \in X$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$, the functions $x(\cdot)$ and $y(\cdot)$, defined by (2.1), satisfy the equations

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = x_0\,, \tag{2.4a}$$

$$y(t) = C_L x(t) + Du(t) \tag{2.4b}$$

for a.e. $t \ge 0$ (in particular $x(t) \in \operatorname{dom}(C_L)$ for a.e. $t \ge 0$). The derivative on the left-hand side of (2.4a) has to be understood in X_{-1}. In other words, if we consider the initial value problem (2.4a) in the space X_{-1}, then for any $x_0 \in X$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$, the classical solution of (2.4a) is given by the variation of parameters formula

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau\,.$$

It has been demonstrated in [35] that, if $\mathbf{\Sigma}$ is regular, then $(sI - A)^{-1}BU \subset \operatorname{dom}(C_L)$ for all $s \in \varrho(A)$ and the transfer function $\mathbf{G}$ can be expressed in the following way

$$\mathbf{G}(s) = C_L(sI - A)^{-1}B + D\,, \qquad \text{for all } s \in \mathbb{C}_{\omega(\mathbf{T})}\,,$$

which is familiar from finite-dimensional systems theory. The operators A, B, C and D are called the *generating operators* of $\mathbf{\Sigma}$.

The following lemma will be needed in Section 3. Certainly, it should be well known. However, since we could not find it in the literature, we include the proof.

Lemma 2.2 *Suppose that $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ is exponentially stable. Then the following statements hold:*

(1) There exist $\alpha, \beta > 0$ such that, for any $x_0 \in X$ and any $u \in L^2(\mathbb{R}_+, U)$, the solution $x(\cdot)$ of the initial-value problem (2.4a) satisfies

$$\|x\|_{L^2(\mathbb{R}_+,X)} \leq \alpha \|u\|_{L^2(\mathbb{R}_+,U)} + \beta \|x_0\|$$

(2) If $u \in L^\infty(\mathbb{R}_+, U)$ and $lim_{t\to\infty} u(t) = u_\infty$ exists, then for any $x_0 \in X$, $x(\cdot)$ defined by (2.4a) satisfies

$$\lim_{t\to\infty} \|x(t) + A^{-1}Bu_\infty\| = 0\,.$$

Proof: By the exponential stability we may assume, without loss of generality, that $x_0 = 0$. Consequently, we have $x(t) = \int_0^t \mathbf{T}_{t-\tau} Bu(\tau)\, d\tau$ for all $t \geq 0$. Let $H^2(\mathbb{C}_0, X)$ denote the usual Hardy space of holomorphic functions defined on $\mathbb{C}_0$ with values in X. Appealing to the Paley-Wiener theorem, statement (1) will follow if we can show that there exists $\alpha > 0$ such that for all $u \in L^2(\mathbb{R}_+, U)$

$$\|\mathfrak{L}x\|_{H^2(\mathbb{C}_0,X)} \leq \alpha \|\mathfrak{L}u\|_{H^2(\mathbb{C}_0,U)}\,. \tag{2.5}$$

To this end, set $\omega_0 := \omega(\mathbf{T})$ and recall from [34] that for any $\omega > \omega_0$ there exists $M_\omega > 0$ such that, for all $s \in \mathbb{C}_\omega$,

$$\|(sI - A)^{-1}B\|_{\mathcal{B}(U,X)} \leq \frac{M_\omega}{\sqrt{\operatorname{Re} s - \omega}}\,. \tag{2.6}$$

It is clear that $s \mapsto (sI - A)^{-1}B$ is a holomorphic $\mathcal{B}(U, X_{-1})$-valued function: using the resolvent identity, it follows that it is also holomorphic as an $\mathcal{B}(U, X)$-valued function. The Laplace transform $\mathfrak{L}x$ of x satisfies

$$(\mathfrak{L}x)(s) = (sI - A)^{-1}B(\mathfrak{L}u)(s) \quad \forall\, s \in \mathbb{C}_{\omega_0}\,. \tag{2.7}$$

By hypothesis, $\omega_0 < 0$ and $\mathfrak{L}u \in H^2(\mathbb{C}_0, X)$. Therefore, combining (2.6) and (2.7) we see that (2.5) holds with $\alpha = M_{\omega_0}/\sqrt{|\omega_0|}$.

To prove statement (2), choose $t^* > 0$ such that $\|\mathbf{T}_t\| \leq 1/2$ for all $t \geq t^*$. Moreover, let (t_n) be a sequence of real numbers satisfying

$$t^* \leq t_{n+1} - t_n \leq 2t^*\,.$$

Set $\beta = \sup\{\|\mathbf{T}_t\| \,|\, 0 \le t \le 2t^*\}$. For $t \ge t_n$ we have

$$x(t) = \mathbf{T}_{t-t_n} x(t_n) + \int_{t_n}^{t} \mathbf{T}_{t-\tau} B u(\tau)\, d\tau\,.$$

Hence, by exponential stability and (2.2), there exists $\alpha > 0$ such that for all $n \in \mathbb{N}$

$$\|x(t)\| \le \beta \|x(t_n)\| + \alpha\sqrt{2t^*}\|u\|_{L^\infty(t_n,t_{n+1})}\,, \qquad \text{if } t \in [t_n, t_{n+1}]\,. \tag{2.8}$$

Moreover,

$$\|x(t_{n+1})\| \le \frac{1}{2}\|x(t_n)\| + \alpha\sqrt{2t^*}\|u\|_{L^\infty(t_n,t_{n+1})}\,. \tag{2.9}$$

We first consider the case when $u_\infty = 0$. Then

$$\lim_{n\to\infty} \|u\|_{L^\infty(t_n,t_{n+1})} = 0\,, \tag{2.10}$$

and (2.9) implies that

$$\lim_{n\to\infty} \|x(t_n)\| = 0\,. \tag{2.11}$$

Combining (2.8), (2.10) and (2.11) shows that $\lim_{t\to\infty}\|x(t)\| = 0$. Finally, if $u_\infty \ne 0$, then, by writing $u(t) = (u(t) - u_\infty) + u_\infty$, it is clear that it suffices to show that

$$\lim_{t\to\infty} \Big\| \int_0^t \mathbf{T}_\tau B u_\infty \, d\tau + A^{-1} B u_\infty \Big\| = 0\,. \tag{2.12}$$

Setting $z(t) = \int_0^t \mathbf{T}_\tau B u_\infty \, d\tau$ we have that

$$\lim_{t\to\infty} \|\dot z(t)\|_{-1} = \lim_{t\to\infty} \|\mathbf{T}_t B u_\infty\|_{-1} = 0\,. \tag{2.13}$$

The function $z(\cdot)$ is the classical solution of the initial-value problem $\dot z(t) = Az(t) + Bu_\infty$, $z(0) = 0$, considered in X_{-1}, and so we may write

$$z(\cdot) + A^{-1} B u_\infty = A^{-1}\dot z(\cdot)\,. \tag{2.14}$$

Since $A^{-1} \in \mathcal{B}(X_{-1}, X)$, (2.12) follows from (2.13) and (2.14). $\square$

3. Integral control in the presence of nonlinearities

In the following, let (A, B, C, D) be the generating operators of a linear,single-input, single-output regular system with state space X and transfer function $\mathbf{G}$. Suppose that the system is subject to an input nonlinearity ϕ, where $\phi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Denoting the constant reference signal by r, an application of the integrator

$$u(t) = u_0 + k\int_0^t [r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau\,,$$

where k is a real parameter (see Fig. 2), leads to the following nonlinear system of differential equations

$$\dot{x} = Ax + B\phi(u)\,, \quad x(0) = x_0 \in X \tag{3.1a}$$

$$\dot{u} = k[r - C_L x - D\phi(u)]\,, \quad u(0) = u_0 \in \mathbb{R}\,. \tag{3.1b}$$

For $a \in (0, \infty]$, a continuous function

$$[0, a) \to X \times \mathbb{R}\,, \quad t \mapsto (x(t), u(t))$$

is called a *solution* of (3.1) if $(x(\cdot), u(\cdot))$ is absolutely continuous as a $(X_{-1} \times \mathbb{R})$-valued function, $x(t) \in \mathrm{dom}\,(C_L)$ for a.e. $t \in [0, a)$, $(x(0), u(0)) = (x_0, u_0)$ and the differential equations in (3.1) are satisfied a.e. on $[0, a)$. Of course, the derivative on the right-hand side on (3.1a) has to be understood in X_{-1}.†

An application of a well-known result on abstract Cauchy problems (see Pazy [25], theorem 2.4, p. 107) shows that a continuous $(X \times \mathbb{R})$-valued function $(x(\cdot), u(\cdot))$ is a solution of (3.1) if, and only if, it satisfies the following integrated version of (3.1)

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B\phi(u(\tau))\, d\tau\,, \tag{3.2a}$$

$$u(t) = u_0 + k \int_0^t [r - C_L x(\tau) - D\phi(u(\tau))]\, d\tau\,. \tag{3.2b}$$

The next result shows that (3.1) has a unique solution.

Proposition 3.1 *For any pair $(x_0, u_0) \in X \times \mathbb{R}$ of initial conditions, there exists a unique solution $(x(\cdot), u(\cdot))$ of (3.1) defined on a maximal interval $[0, a_{max})$. If $a_{max} < \infty$, then*

$$\limsup_{t \to a_{max}} \|(x(t), u(t))\| = \infty\,. \tag{3.3}$$

If ϕ is globally Lipschitz, then $a_{max} = \infty$.

For the proof of the above result it will be useful to consider the following initial-value problem for u

$$\dot{u} = k[r - \mathbf{\Psi}_\infty x_0 - \mathbf{F}_\infty \phi(u)]\,, \quad u(0) = u_0\,. \tag{3.4}$$

Clearly, (3.4)‡ is obtained from (3.1b) on noting that $C_L x(t) + D\phi(u(t)) = (\mathbf{\Psi}_\infty x_0)(t) + (\mathbf{F}_\infty \phi(u))(t)$. An absolutely continuous function $u : [0, a) \to \mathbb{R}$ is a *solution* of (3.4) if $u(0) = u_0$ and the differential equation in (3.4) is satisfied a.e. on $[0, a)$.

---

† Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive. Hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration, see [1], theorem 3.1, p. 10.

‡ Strictly speaking, to make sense of (3.4), we have to give a meaning to $\mathbf{F}_\infty v$, when v is a continuous function defined on a *finite* interval $[0, a)$ (recall that $\mathbf{F}_\infty$ operates on the space of locally square-integrable functions defined on the *infinite* interval $[0, \infty)$). This can be easily done using the causality of $\mathbf{F}_\infty$. Moreover, by slight abuse of notation, the expression $\phi(u)$ on the r.h.s. of (3.4) denotes the function $t \mapsto \phi(u(t))$.

Lemma 3.2 *Let $x_0 \in X$. For any initial condition $u_0 \in \mathbb{R}$ there exists a unique solution $u(\cdot)$ of (3.4) defined on a maximal interval $[0, a_{max})$. If $a_{max} < \infty$, then*

$$\limsup_{t \to a_{max}} |u(t)| = \infty \,. \tag{3.5}$$

If ϕ is globally Lipschitz, then $a_{max} = \infty$.

The proof of this lemma is relegated to the Appendix.

Proof of Proposition 3.1: Let $u : [0, a_{max}) \to \mathbb{R}$ be the unique maximal solution of (3.4) (whose existence is guaranteed by Lemma 3.2) and *define* $x(\cdot)$ to be the unique solution of

$$\dot{x} = Ax + B\phi(u)\,, \quad x(0) = x_0\,.$$

Then $(x(\cdot), u(\cdot))$ is the unique solution of (3.1) which satisfies (3.3) if $a_{max} < \infty$. Moreover, it follows trivially from Lemma 3.2 that $a_{max} = \infty$ if ϕ is globally Lipschitz. $\square$

Henceforth, let $\mathcal{M}$ denote the set of all bounded measures on $[0, \infty)$. A measure $\mu \in \mathcal{M}$ can be written in the form

$$\mu(dt) = a(t)dt + \sum_{i=0}^{\infty} a_i \delta_{t_i}(dt) + \mu_s(dt)\,,$$

where $a(\cdot) \in L^1(0, \infty)$, $\sum_{i=0}^{\infty} a_i \delta_{t_i}$ and μ_s, respectively, represent the absolutely continuous, the discrete and the singular parts of μ. In particular, δ_{t_i} denotes the unit point mass at $t_i \geq 0$ and the a_i are real numbers such that $\sum_{i=0}^{\infty} |a_i| < \infty$.

Furthermore, for $\lambda > 0$, let $\mathcal{N}(\lambda)$ denote the set of all non-decreasing globally Lipschitz nonlinearities $\phi : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant λ. Finally, if $\mathbf{G}(0) > 0$, it is easy to show that

$$1 + \kappa \mathrm{Re}\, \frac{\mathbf{G}(s)}{s} \geq 0\,, \qquad \text{for all } s \in \mathbb{C}_0\,, \tag{3.6}$$

for all sufficiently small $\kappa > 0$, see lemma 3.10 in [17]. We define K to be the supremum of the set of all numbers $\kappa^* > 0$ so that (3.6) holds for all $\kappa \in (0, \kappa^*)$.

The main result of this section is the following theorem.

Theorem 3.3 *Let $\lambda > 0$ and $\phi \in \mathcal{N}(\lambda)$. Assume that $\mathbf{T}_t$ is exponentially stable, $\mathbf{G}(0) > 0$, $k \in (0, K/\lambda)$ and $r \in \mathbb{R}$ is such that*

$$\phi_r := [\mathbf{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im} \phi)\,. \tag{3.7}$$

If C is bounded, then for all $(x_0, u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot), u(\cdot))$ of (3.1) exists on $[0, \infty)$ and satisfies

(1) $\lim_{t \to \infty} \phi(u(t)) = \phi_r$,

(2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\phi_r\| = 0$,

(3) $\lim_{t\to\infty}(r - y(t)) = 0$, *where* $y(t) = Cx(t) + D\phi(u(t))$.

(4) *if* $\phi_r \in \operatorname{im}\phi$, *then*

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \phi^{-1}(\phi_r)) = 0\,, \tag{3.8}$$

(5) *if* $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$, *then* $u(\cdot)$ *is bounded.*

If C *is unbounded, then the statements (1)–(5) remain true provided that* $x_0 \in \operatorname{dom}(A)$ *and* $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$.

In particular, statement (4) says that $u(t)$ converges as $t \to \infty$ if the set $\phi^{-1}(\phi_r)$ is a singleton, which, in turn, is true if ϕ_r is not a critical value of ϕ.

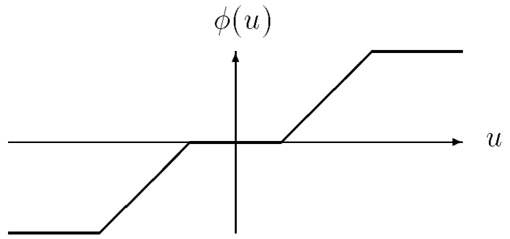

Fig. 3: Nonlinearity with saturation and deadzone

The conditions imposed in Theorem 3.3 on ϕ are satisfied by saturation and deadzone nonlinearities and combinations of the two, as shown in Fig. 3. The assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$ is not very restrictive and seems to be satisfied in all practical examples of systems with H^∞-transfer functions (in applications one usually has $\mu_s = 0$). In principle, the constant K can be obtained from frequency-response experiments performed on the linear part of the plant, see [15] for details.

For the proof of Theorem 3.3 two lemmas are required, the proofs of which can be found in the Appendix.

Lemma 3.4 *Suppose that* $\mathbf{T}_t$ *is exponentially stable and* $\mathbf{G}(0) > 0$. *Set* $A_\kappa := A - \kappa A^{-1}BC_L$ *with* $\operatorname{dom}(A_\kappa) = \operatorname{dom}(A)$ *and define*

$$\mathbf{H}(s) = \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0))\,.$$

If $0 < 2\kappa < K$, *then*

$$\|\mathbf{H}(1 + \kappa\mathbf{H})^{-1}\|_\infty < \frac{1}{\kappa}\,, \tag{3.9}$$

and there exists $P \in \mathcal{B}(X)$, *with* $P = P^* \geq 0$, *and such that the Riccati equation*

$$\langle A_\kappa x_1, Px_2\rangle + \langle Px_1, A_\kappa x_2\rangle + \kappa^2\langle C_L x_1, C_L x_2\rangle + \langle (A^{-1}B)^*Px_1, (A^{-1}B)^*Px_2\rangle = 0 \tag{3.10}$$

is satisfied for all $x_1, x_2 \in \operatorname{dom}(A_\kappa)$.

Lemma 3.5 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz, let $u : [0,\infty) \to \mathbb{R}$ be absolutely continuous, and let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty} \varepsilon_n = 0$. For $\xi \in \mathbb{R}$ define*

$$\delta\phi(\xi) = \limsup_{n\to\infty} \frac{\phi(\xi+\varepsilon_n) - \phi(\xi)}{\varepsilon_n}.$$

Then $\delta\phi \in L^\infty_{loc}(-\infty,\infty)$ ($\delta\phi \in L^\infty(-\infty,\infty)$ if ϕ is globally Lipschitz) and $\delta\phi \circ u$ is measurable. Moreover, $\phi \circ u$ is differentiable a.e. and

$$\frac{d}{dt}(\phi\circ u)(t) = \delta\phi(u(t))\dot{u}(t), \quad \text{for a.e. } t \in [0,\infty).$$

Proof of Theorem 3.3: By Proposition 3.1, there exists a unique solution of (3.1) on $[0,\infty)$. We denote this solution by $(x(\cdot), u(\cdot))$ and introduce new variables by defining

$$z(t) := x(t) + A^{-1}B\phi(u(t)), \quad v(t) := \phi(u(t)) - \phi_r \quad \text{for all } t \geq 0.$$

By regularity it follows that $z(t) \in \operatorname{dom}(C_L)$ for a.e. $t \in [0,\infty)$. Moreover, by Lemma 3.5, $\dot{v}(t) = \delta\phi(u(t))\dot{u}(t)$ for a.e. $t \in [0,\infty)$. Therefore an easy calculation yields

$$\dot{z} = Az - k\delta\phi(u)A^{-1}B(C_L z + \mathbf{G}(0)v), \quad z(0) = z_0 := x_0 + A^{-1}B\phi(u_0) \tag{3.11a}$$
$$\dot{v} = -k\delta\phi(u)(C_L z + \mathbf{G}(0)v), \quad v(0) = v_0 := \phi(u_0) - \phi_r. \tag{3.11b}$$

The derivative on the left-hand side of (3.11a) has to be understood in X_{-1}. Notice that, since ϕ is non-decreasing, $\delta\phi(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. We observe that, whilst in these new variables we still have an unbounded operator $A^{-1}BC_L$, the operator $A^{-1}B$ is in $\mathcal{B}(\mathbb{R}, X)$. We will investigate the stability properties of (3.11) using a Liapunov approach.

Since $0 < k\lambda < K$, it follows that there exists $\mu > \lambda/2$ such that $0 < 2\mu k < K$, and therefore, by Lemma 3.4

$$\|\mathbf{H}(1 + \mu k\mathbf{H})^{-1}\|_\infty < \frac{1}{\mu k}.$$

By the same lemma, the Riccati equation (3.10) with $\kappa = \mu k$ has a solution $P \in \mathcal{B}(X)$ satisfying $P = P^* \geq 0$. Set

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & \mu k\mathbf{G}(0) \end{pmatrix},$$

and define

$$\tilde{A}_k = \begin{pmatrix} A - \mu kA^{-1}BC_L & -\mu kA^{-1}B\mathbf{G}(0) \\ -\mu kC_L & -\mu k\mathbf{G}(0) \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} A^{-1}B \\ 1 \end{pmatrix}, \quad \tilde{C} = (C_L \ \ \mathbf{G}(0)).$$

where $\operatorname{dom}(\tilde{A}_k) = \operatorname{dom}(A) \times \mathbb{R}$. The operator $\tilde{A}_k$ generates a C_0-semigroup. Using (3.10), it is easy to show that

$$\langle \tilde{A}_k\tilde{x}_1, \tilde{P}\tilde{x}_2\rangle + \langle \tilde{P}\tilde{x}_1, \tilde{A}_k\tilde{x}_2\rangle + \mu^2k^2\langle \tilde{C}\tilde{x}_1, \tilde{C}\tilde{x}_2\rangle + \langle \tilde{B}^*\tilde{P}\tilde{x}_1, \tilde{B}^*\tilde{P}\tilde{x}_2\rangle = 0 \tag{3.12}$$

is satisfied for all $\tilde{x}_1, \tilde{x}_2 \in \mathrm{dom}(\tilde{A}_k)$.

Setting $\tilde{z}(\cdot) = (z(\cdot), v(\cdot))$, (3.11) can be reformulated as

$$\dot{\tilde{z}} = \tilde{A}_k \tilde{z} + k(\mu - \delta\phi(u))\tilde{B}\tilde{C}\tilde{z}\,, \quad \tilde{z}(0) = \tilde{z}_0 := \begin{pmatrix} z_0 \\ v_0 \end{pmatrix}, \tag{3.13}$$

where the derivative on the left-hand side has to be understood in $X_{-1} \times \mathbb{R}$. For an intermediate step in the Liapunov analysis we need differentiability in $X \times \mathbb{R}$, and therefore we will use an approximation argument. To this end let $T > 0$ be fixed, but arbitrary, and choose $(w_n) \subset W^{1,2}(0,T;\mathbb{R})$ and $(\tilde{z}_0^n) \subset \mathrm{dom}(\tilde{A}_k)$ such that

$$\lim_{n\to\infty} \|k(\mu - \delta\phi(u))\tilde{C}\tilde{z} - w_n\|_{L^2(0,T)} = 0\,, \quad \lim_{n\to\infty} \|\tilde{z}_0 - \tilde{z}_0^n\|_{X\times\mathbb{R}} = 0\,. \tag{3.14}$$

Consider the system

$$\dot{\eta}(t) = \tilde{A}_k \eta(t) + \tilde{B} w_n(t)\,, \qquad \eta(0) = \tilde{z}_0^n\,. \tag{3.15a}$$

$$\xi(t) = \tilde{C}\eta(t)\,. \tag{3.15b}$$

The abstract initial-value problem (3.15a) has a strong solution $\tilde{z}_n$ on $[0,T]$ in the sense that $\tilde{z}_n(0) = \tilde{z}_0^n$ and (3.15a) is satisfied for a.e. $t \in [0,T]$ (see Pazy [25], corollary 2.10, p. 109). Using (3.14) we obtain

$$\lim_{n\to\infty} \|\tilde{z} - \tilde{z}_n\|_{L^2(0,T)} = 0\,; \qquad \lim_{n\to\infty} \|\tilde{z}(t) - \tilde{z}_n(t)\|_{X\times\mathbb{R}} = 0\,, \quad \text{for all } t \in [0,T]\,. \tag{3.16}$$

Setting $\xi_n(t) = \tilde{C}\tilde{z}_n(t)$, it follows from the regularity of (3.15) that

$$\lim_{n\to\infty} \|\tilde{C}\tilde{z} - \xi_n\|_{L^2(0,T)} = 0\,. \tag{3.17}$$

Differentiating the function

$$\tau \mapsto V_n(\tau) = \langle \tilde{z}_n(\tau), \tilde{P}\tilde{z}_n(\tau)\rangle$$

shows that for a.e. $\tau \in [0,T]$

$$\dot{V}_n(\tau) = \langle \tilde{z}_n(\tau), \tilde{P}\tilde{A}_k\tilde{z}_n(\tau)\rangle + \langle \tilde{A}_k\tilde{z}_n(\tau), \tilde{P}\tilde{z}_n(\tau)\rangle + 2\langle \tilde{B}w_n(\tau), \tilde{P}\tilde{z}_n(\tau)\rangle\,. \tag{3.18}$$

If $t \in [0,T]$, then integrating (3.18) from 0 to t, taking limits as $n \to \infty$, invoking (3.12), (3.14), (3.16) and (3.17) and setting

$$V(\tau) = \langle \tilde{z}(\tau), \tilde{P}\tilde{z}(\tau)\rangle$$

we obtain

$$V(t) - V(0) = -\int_0^t \mu^2 k^2 (\tilde{C}\tilde{z})^2 - \int_0^t (\tilde{B}^*\tilde{P}\tilde{z})^2 + 2\int_0^t \langle \tilde{B}k(\mu - \delta\phi(u))\tilde{C}\tilde{z}, \tilde{P}\tilde{z}\rangle\,.$$

Completing the square gives

$$V(t) - V(0) = -\int_0^t [\mu^2 k^2 - k^2(\delta\phi(u) - \mu)^2](\tilde{C}\tilde{z})^2 - \int_0^t [k(\delta\phi(u) - \mu)\tilde{C}\tilde{z} + \tilde{B}^*\tilde{P}\tilde{z}]^2 ,$$

and hence

$$V(t) - V(0) = -k^2 \int_0^t [2\mu\delta\phi(u) - (\delta\phi)^2(u)](\tilde{C}\tilde{z})^2 - \int_0^t [k(\delta\phi(u) - \mu)\tilde{C}\tilde{z} + \tilde{B}^*\tilde{P}\tilde{z}]^2 . \quad (3.19)$$

(3.19) holds for all $t \in [0, T]$. Since $T > 0$ was arbitrary, it follows that (3.19) holds for all $t \geq 0$. Therefore, using (3.19) and the definition of $\tilde{C}$,

$$k^2 \int_0^t (2\mu\delta\phi(u) - (\delta\phi)^2(u))(C_L z + \mathbf{G}(0)v)^2 \leq V(0) < \infty , \qquad \text{for all } t \geq 0 . \qquad (3.20)$$

Now recall that $2\mu > \lambda$ and $\|\delta\phi(u)\|_{L^\infty(\mathbb{R}_+)} \leq \lambda$, so that

$$2\mu(\delta\phi)(u) - (\delta\phi)^2(u) > \varepsilon(\delta\phi)^2(u)$$

for some $\varepsilon > 0$. Therefore (3.20) gives

$$\varepsilon k^2 \int_0^t (\delta\phi)^2(u)(C_L z + \mathbf{G}(0)v)^2 \leq V(0) < \infty , \qquad \text{for all } t \geq 0 .$$

It follows that

$$\delta\phi(u)(C_L z + \mathbf{G}(0)v) \in L^2(\mathbb{R}_+) . \qquad (3.21)$$

Using this in (3.11a) and appealing to the fact that A, $A^{-1}B$ and C are the generating operators of a stable regular system we may conclude that

$$C_L z \in L^2(\mathbb{R}_+) . \qquad (3.22)$$

Hence, by (3.21) and the boundedness of $\delta\phi(u)$,

$$\delta\phi(u)v \in L^2(\mathbb{R}_+) \qquad (3.23)$$

and thus

$$(C_L z)\delta\phi(u)v \in L^1(\mathbb{R}_+) . \qquad (3.24)$$

Using (3.20), (3.22)–(3.24) and the boundedness of $\delta\phi(u)$ it follows that

$$\delta\phi(u)v^2 \in L^1(\mathbb{R}_+) . \qquad (3.25)$$

Multiplying (3.11b) by $v(t)$, integrating and then using (3.24) and (3.25), shows that

$$\lim_{t\to\infty} v^2(t) = v_0^2 + 2\lim_{t\to\infty} \int_0^t v\dot{v} = \nu ,$$

for some $\nu \in [0, \infty)$. By continuity of $v(\cdot)$ it follows that

$$\lim_{t\to\infty} v(t) = \sqrt{\nu} \qquad \text{or} \qquad \lim_{t\to\infty} v(t) = -\sqrt{\nu} .$$

In the following we distinguish two cases: bounded and unbounded observation.

Let us first consider the case of bounded C. In order to prove statement (1), we have to show that $\nu = 0$. Seeking a contradiction, suppose that $\nu > 0$. Assuming that $\lim_{t\to\infty} v(t) = \sqrt{\nu}$ (the case $\lim_{t\to\infty} v(t) = -\sqrt{\nu}$ can be dealt with in an entirely analogous fashion) we obtain that

$$\phi_\infty := \lim_{t\to\infty} \phi(u(t)) > \phi_r \,. \tag{3.26}$$

By Lemma 2.2, part (2), it follows that

$$\lim_{t\to\infty} \|x(t) + A^{-1}B\phi_\infty\| = 0 \,. \tag{3.27}$$

Since C is bounded, it follows from (3.1b), (3.26) and (3.27) that

$$\lim_{t\to\infty} \dot{u}(t) = k(r + CA^{-1}B\phi_\infty - D\phi_\infty) = k\mathbf{G}(0)(\phi_r - \phi_\infty) < 0 \,. \tag{3.28}$$

Hence $\lim_{t\to\infty} u(t) = -\infty$ and so, since ϕ is non-decreasing we obtain

$$\phi_\infty = \lim_{t\to\infty} \phi(u(t)) = \inf(\operatorname{im}\phi) \le \phi_r \,,$$

contradicting (3.26). Therefore $\nu = 0$, and consequently $\lim_{t\to\infty} \phi(u(t)) = \phi_r$, which is statement (1). Statement (2) follows now from Lemma 2.2, part (2), and statement (3) is a consequence of statements (1) and (2).

To prove statement (4), let $\phi_r \in \operatorname{im}\phi$. Seeking a contradiction, suppose that the claim is not true. Then there exists a sequence of positive numbers (t_n) with $\lim_{n\to\infty} t_n = \infty$ and $\varepsilon > 0$ such that

$$\operatorname{dist}(u(t_n), \phi^{-1}(\phi_r)) \ge \varepsilon \,. \tag{3.29}$$

If the sequence $(u(t_n))$ is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity of ϕ and statement (1) we have that $\phi(u_\infty) = \phi_r$, and thus $u_\infty \in \phi^{-1}(\phi_r)$. This contradicts (3.29). So, suppose that $(u(t_n))$ is unbounded. Without loss of generality, we may then assume that $\lim_{n\to\infty} u(t_n) = \infty$. By monotonicity and statement (1) it follows that $\phi_r = \sup\phi$. Since $\phi_r \in \operatorname{im}\phi$, there exists ξ^* such that

$$\phi(\xi^*) = \phi_r = \sup\phi = \max\phi \,.$$

By monotonicity of ϕ we have

$$\phi(\xi) = \phi_r = \max\phi \,, \qquad \text{for all } \xi \ge \xi^* \,.$$

In particular, we see that $u(t_n) \in \phi^{-1}(\phi_r)$ for all sufficiently large n, contradicting (3.29).

To prove statement (5) assume that $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$. Again seeking a contradiction, suppose that the claim is not true. Then there exists a sequence of positive numbers (t_n) with $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} |u(t_n)| = \infty$. Without loss of generality, we may assume that $\lim_{n\to\infty} u(t_n) = \infty$. By monotonicity it then follows

$$\phi_r = \lim_{n\to\infty} \phi(u(t_n)) = \sup\phi \,,$$

contradicting the hypothesis $\phi_r \in \operatorname{int}(\operatorname{im}\phi)$.

In order to prove statement (1) for unbounded C, we will again be seeking a contradiction, and hence assume that $\nu > 0$. It is clear that (3.26) and (3.27) still hold. It only remains to show that (3.28) is true also in this case. To this end, write (3.1b) in the form

$$\dot{u} = k[r - C_L \mathbf{T}_t x_0 - \mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u)] . \tag{3.30}$$

Since $x_0 \in \operatorname{dom}(A)$ we have that $\lim_{t\to\infty} C_L \mathbf{T}_t x_0 = 0$. Furthermore, since $\lim_{t\to\infty} \phi(u(t)) = \phi_\infty$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$, it follows that $\lim_{t\to\infty} (\mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u))(t) = \mathbf{G}(0)\phi_\infty$ (see [8], theorem 6.1, part (ii), p. 96). Therefore, by (3.26) and (3.30)

$$\lim_{t\to\infty} \dot{u}(t) = k\mathbf{G}(0)(\phi_r - \phi_\infty) < 0 ,$$

which is (3.28). Statements (2), (4) and (5) then follow as in the bounded case. To obtain statement (3), write $y(t)$ in the form

$$y(t) = C_L \mathbf{T}_t x_0 + (\mathfrak{L}^{-1}(\mathbf{G}) \star \phi(u))(t) .$$

Since $x_0 \in \operatorname{dom}(A)$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}$, we see that

$$\lim_{t\to\infty} y(t) = \mathbf{G}(0) \lim_{t\to\infty} \phi(u(t)) .$$

Combining this with statement (1) shows that $\lim_{t\to\infty} y(t) = r$. □

One of the conditions imposed in Theorem 3.3 is that $[\mathbf{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im}\phi)$. The following proposition shows that this condition is necessary for solvability of the tracking problem.

Proposition 3.6 *Let $r \in \mathbb{R}$ and suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, $\mathbf{T}_t$ is exponentially stable and $\mathbf{G}(0) \neq 0$. If there exist an initial condition $x_0 \in X$ and a continuous function $u : [0,\infty) \to \mathbb{R}$ such that $\phi(u(\cdot))$ is bounded and*

$$\lim_{t\to\infty} [C_L x(t) + D\phi(u(t))] = r ,$$

where $x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B\phi(u(\tau))\, d\tau$, then $\phi_r = [\mathbf{G}(0)]^{-1} r \in \operatorname{clos}(\operatorname{im}\phi)$.

The proof of the above proposition requires some preparation. Recall the concept of an ω-limit point (and ω-limit set $\Omega(\psi)$) of a continuous function $\psi : [0,\infty) \to \mathbb{R}$. A point ψ^* is an *ω-limit point* of ψ if there exists an increasing sequence $(t_n) \subset [0,\infty)$ such that $t_n \to \infty$ and $\psi(t_n) \to \psi^*$ as $n \to \infty$. The set $\Omega(\psi)$ of all ω-limit points is the *ω-limit set* of ψ.

The following lemma is probably standard: however, we were unable to locate it in the literature and so include a proof for completeness.

Lemma 3.7 *Let $\psi : [0,\infty) \to \mathbb{R}$ be continuous and bounded. Then*

$$\lim_{s\to 0,\, s>0} [s(\mathfrak{L}\psi)(s)] = \omega \quad \Longrightarrow \quad \omega \in \Omega(\psi)\,.$$

Proof: It suffices to prove the result in the case $\omega = 0$ (if $\omega \neq 0$, then simply replace ψ by $\psi_\omega : t \mapsto \psi(t) - \omega$). It is well-known that $\Omega(\psi)$ is compact and is approached by $\psi(t)$ as $t \to \infty$ (see for example [10], p. 113). Seeking a contradiction, suppose $0 \notin \Omega(\psi)$. Then there exists $\varepsilon > 0$ and $T > 0$ such that for all $t \geq T$, $|\psi(t)| \geq \varepsilon$. Since ψ is continuous, we may restrict attention, without loss of generality, to the case $\psi(t) \geq \varepsilon$ for all $t \geq T$. Then, for all $s \in (0,\infty)$, we have

$$(\mathfrak{L}\psi)(s) = \int_0^\infty e^{-st}\psi(t)\,dt \geq \int_0^T e^{-st}\psi(t)\,dt + \varepsilon\int_T^\infty e^{-st}\,dt = \int_0^T e^{-st}\psi(t)\,dt + \frac{\varepsilon e^{-sT}}{s}\,,$$

whence the contradiction

$$0 = \lim_{s\to 0,\, s>0} s(\mathfrak{L}\psi)(s) \geq \varepsilon > 0\,.$$

$\square$

Proof of Proposition 3.6: For $\delta \in (0, \pi/2)$ define the open sector $\mathbb{S}(\delta) \subset \mathbb{C}_0$ by

$$\mathbb{S}(\delta) := \{\rho e^{i\alpha} \mid \rho \in (0,\infty),\ \alpha \in (-\delta,\delta)\}\,.$$

Setting $\psi(t) = \phi(u(t))$ and $y(t) = C_L x(t) + D\psi(t)$ we obtain

$$(\mathfrak{L}y)(s) = \mathbf{G}(s)(\mathfrak{L}\psi)(s) + C(sI - A)^{-1}x_0\,,$$

and so by the final-value theorem (see [7], Satz 34.2, or [24], theorem 14, pp. 95),

$$r = \lim_{t\to\infty} y(t) = \lim_{s\to 0,\, s\in\mathbb{S}(\delta)} s(\mathfrak{L}y)(s) = \lim_{s\to 0,\, s\in\mathbb{S}(\delta)} s\mathbf{G}(s)(\mathfrak{L}\psi)(s)\,.$$

Since $\mathbf{G}(0) \neq 0$, it follows using Lemma 3.7 that

$$\phi_r = [\mathbf{G}(0)]^{-1}r = \lim_{s\to 0,\, s\in\mathbb{S}(\delta)} s(\mathfrak{L}\psi)(s) \in \Omega(\psi) \subset \operatorname{clos}(\operatorname{im}\phi)\,.$$

$\square$

A result similar to Proposition 3.6 was stated without proof by Miller and Davison [21] in a finite-dimensional context. However, their approach (as outlined by Miller [20]) does not extend to infinite-dimensional regular systems.

4. Example: controlled diffusion process with output delay

Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $I = [0, 1]$, with scalar nonlinear pointwise control action (applied at point $x_b \in I$, via a nonlinearity ϕ with Lipschitz constant $\lambda > 0$) and delayed (delay $h \geq 0$) pointwise scalar observation (output at point $x_c \in I$, $x_c \geq x_b$.). We formally write this single-input, single-output system as

$$\begin{aligned} &z_t(t,x) = a z_{xx}(t,x) + \delta(x - x_b)\phi(u(t)), \quad y(t) = z(t-h, x_c) \\ &z(t,0) = 0 = z(t,1)\,, \qquad \text{for all } t > 0\,. \end{aligned}$$

For simplicity, we assume zero initial conditions:

$$z(t,x) = 0\,, \qquad \text{for all } (t,x) \in [-h, 0] \times [0,1]\,.$$

With input $\phi(u(\cdot))$ and output $y(\cdot)$, this example qualifies as a regular linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sh} \sinh\left(x_b\sqrt{(s/a)}\right) \sinh\left((1-x_c)\sqrt{(s/a)}\right)}{a\sqrt{(s/a)}\sinh\sqrt{(s/a)}}\,.$$

In this case, a detailed analysis (see [15] for related investigations) yields

$$\begin{aligned} K &:= \sup\{\kappa^* > 0 |\ (3.6) \text{ holds for all } \kappa \in (0, \kappa^*)\} \\ &= \frac{1}{|\mathbf{G}'(0)|} = \frac{6a^2}{x_b(1-x_c)(6ha + 1 - x_b^2 - (1-x_c)^2)}. \end{aligned}$$

Therefore, by Theorem 3.3, for each $k \in (0, K/\lambda)$, the integral control

$$u(t) = k\int_0^t [r - y(t)]\, dt$$

guarantees asymptotic tracking of every constant reference signal r satisfying

$$\frac{r}{\mathbf{G}(0)} = \frac{ar}{x_b(1-x_c)} \in \ \text{clos}\,(\text{im}\,\phi)\,.$$

For purposes of illustration, we adopt the following values

$$a = 0.1, \quad x_b = \frac{1}{3}, \quad x_c = \frac{2}{3}, \quad h = 1, \quad r = 1.$$

We consider a nonlinearity ϕ of saturation type, defined as follows

$$u \mapsto \phi(u) := \begin{cases} 1, & u \geq 1 \\ u, & u \in (0,1) \\ 0, & u \leq 0 \end{cases}$$

in which case $\lambda = 1$ and

$$K = \frac{243}{620} \; (\approx 0.3919).$$

For $r = 1$, we have

$$\frac{r}{\mathbf{G}(0)} = \frac{a}{x_b(1 - x_c)} = 0.9 \in [0, 1] = \operatorname{clos}(\operatorname{im} \phi).$$

In each of the following three cases of admissible controller gains

$$\text{(i) } k = 0.39, \quad \text{(ii) } k = 0.26, \quad \text{(iii) } k = 0.13,$$

Fig.4 depicts the output behaviour of the system under integral control, while Fig.5 depicts the corresponding control input. These figures were generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 10, was adopted to model the diffusion process.

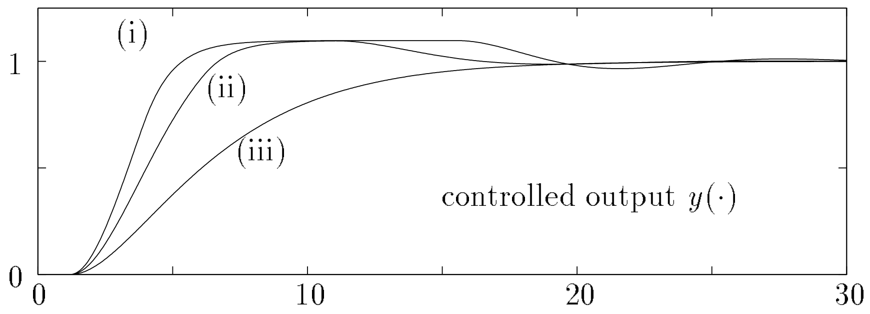

Fig.4: Controlled output

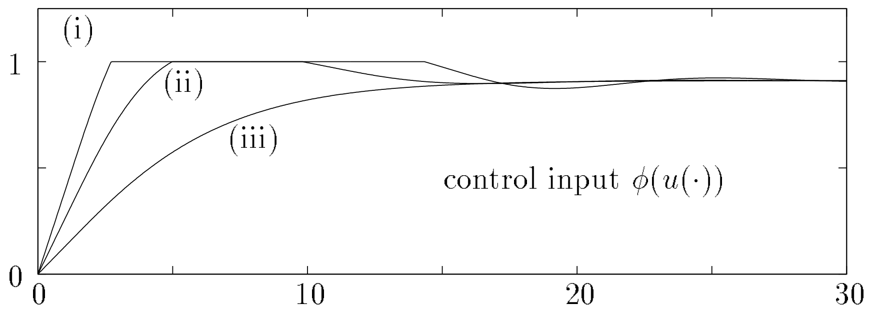

Fig.5: Control input

Appendix

Proof of Lemma 3.2

In proving Lemma 3.2, we will study an initial-value problem which is slightly more general than (3.4). Let $\alpha \geq 0$ and let $w \in C([0,\alpha],\mathbb{R})$. Consider the initial-value problem

$$\dot{u}(t) = k[r - (\boldsymbol{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)]\,, \quad t \geq \alpha\,, \tag{A.1a}$$
$$u(t) = w(t)\,, \quad t \in [0,\alpha]\,. \tag{A.1b}$$

Lemma A.1 *Let $x_0 \in X$. For any initial function $w \in C([0,\alpha],\mathbb{R})$ there exists $\varepsilon > 0$ and a unique function $u \in C([0,\alpha+\varepsilon],\mathbb{R})$ with $u(t) = w(t)$ for all $t \in [0,\alpha]$ and such that (A.1a) is satisfied for a.e. $t \in [\alpha,\alpha+\varepsilon]$.*

Proof: Without loss of generality, we may assume that $k = 1$. For $\delta > 0$ and $\eta > \|w\|_\infty$ define

$$\begin{aligned}\mathcal{C}_{\delta,\eta} = \{u \in C([0,\alpha+\delta],\mathbb{R}) \mid\ & |u(t) - w(t)| \leq \eta \quad \text{if } 0 \leq t \leq \alpha\,;\\ & |u(t) - w(\alpha)| \leq \eta \quad \text{if } \alpha \leq t \leq \alpha+\delta\}\,.\end{aligned}$$

Choosing $\eta > \|w\|_\infty$ guarantees that $\mathcal{C}_{\delta,\eta}$ contains the zero function. Using the causality of $\mathbf{F}_\infty$, the boundedness of the operators $\mathbf{P}_t\mathbf{F}_\infty$ and the Lipschitz continuity of ϕ, it is clear that, for given numbers $\delta > 0$ and $\eta > \|w\|_\infty$, there exists $\lambda > 0$ such that, for all $\varepsilon \in (0,\delta]$ and all $u, v \in \mathcal{C}_{\varepsilon,\eta}$,

$$\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\phi(u) - \mathbf{F}_\infty\phi(v)|^2 \leq \lambda^2 \int_0^{\alpha+\varepsilon} |u - v|^2\,.$$

Using Hölder's inequality we obtain the estimate

$$\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\phi(u) - \mathbf{F}_\infty\phi(v)| \leq \lambda\sqrt{\varepsilon}\left(\int_0^{\alpha+\varepsilon} |u-v|^2\right)^{1/2}, \tag{A.2}$$

which holds for all $u, v \in \mathcal{C}_{\varepsilon,\eta}$ and all $\varepsilon \in (0,\delta]$. Moreover, if $v = 0$, then we may conclude that, for all $u \in \mathcal{C}_{\varepsilon,\eta}$ and all $\varepsilon \in (0,\delta]$,

$$\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\phi(u)| \leq \int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}_\infty\phi(0))(\tau)|\,d\tau + \lambda\sqrt{\varepsilon}\left(\int_0^{\alpha+\varepsilon} |u|^2\right)^{1/2}. \tag{A.3}$$

Set $f(t) = r - (\boldsymbol{\Psi}_\infty x_0)(t)$ and choose $\rho > 0$ such that

$$\int_\alpha^{\alpha+\rho} (|f(\tau)| + |(\mathbf{F}_\infty\phi(0))(\tau)|)\,d\tau \leq \frac{\eta}{2}\,. \tag{A.4}$$

Now choose $\varepsilon > 0$ such that

$$\varepsilon \leq \delta\,, \quad \varepsilon \leq \rho\,, \quad \varepsilon < \frac{1}{\lambda}\,, \quad \varepsilon \leq \frac{1}{4(\alpha+\rho)}\left(\frac{\eta}{\lambda\max\{\|w\|_\infty, |w(\alpha)| + \eta)\}}\right)^2. \tag{A.5}$$

Define the operator Γ by

$$(\Gamma u)(t) = w(t)\,, \qquad 0 \le t \le \alpha\,,$$
$$(\Gamma u)(t) = w(\alpha) + \int_\alpha^t f(\tau)\,d\tau - \int_\alpha^t (\mathbf{F}_\infty \phi(u))(\tau)\,d\tau\,, \qquad t \ge \alpha\,,$$

and set

$$\tilde{\mathfrak{C}}_{\varepsilon,\eta} := \{u \in \mathfrak{C}_{\varepsilon,\eta} \mid u(t) = w(t) \quad \text{if } 0 \le t \le \alpha\}\,.$$

Clearly, $\tilde{\mathfrak{C}}_{\varepsilon,\eta}$ is a complete metric space, and the lemma follows if we can show that Γ is a contraction on $\tilde{\mathfrak{C}}_{\varepsilon,\eta}$.

We first show that $\Gamma(\tilde{\mathfrak{C}}_{\varepsilon,\eta}) \subset \tilde{\mathfrak{C}}_{\varepsilon,\eta}$. Using (A.3)–(A.5) we obtain, for all $u \in \mathfrak{C}_{\varepsilon,\eta}$ and all $t \in [\alpha, \alpha+\varepsilon]$,

$$\begin{aligned} |(\Gamma u)(t) - w(\alpha)| &\le \lambda\sqrt{\varepsilon}\left(\int_0^{\alpha+\varepsilon} |u(\tau)|^2\,d\tau\right)^{1/2} + \frac{\eta}{2} \\ &\le \frac{\eta}{2} + \lambda\sqrt{\varepsilon(\alpha+\rho)}\max\{\|w\|_\infty, |w(\alpha)| + \eta\} \\ &\le \eta\,, \end{aligned}$$

which shows that $\Gamma(\tilde{\mathfrak{C}}_{\varepsilon,\eta}) \subset \tilde{\mathfrak{C}}_{\varepsilon,\eta}$. It remains to show that Γ is a contraction on $\tilde{\mathfrak{C}}_{\varepsilon,\eta}$. To this end, let $u, v \in \tilde{\mathfrak{C}}_{\varepsilon,\eta}$. Using (A.2) we obtain

$$\sup_{0\le\tau\le\alpha+\varepsilon} |(\Gamma u)(\tau) - (\Gamma v)(\tau)| \le \lambda\sqrt{\varepsilon}\left(\int_\alpha^{\alpha+\varepsilon} |u - v|^2\right)^{1/2} \le \varepsilon\lambda \sup_{0\le\tau\le\alpha+\varepsilon} |u(\tau) - v(\tau)|\,.$$

By (A.5) we have that $\varepsilon\lambda < 1$, showing that Γ is a contraction on $\tilde{\mathfrak{C}}_{\varepsilon,\eta}$. □

Proof of Lemma 3.2: We proceed in several steps.

STEP 1. Existence and uniqueness on a small interval.

An application of Lemma A.1 with $\alpha = 0$ shows that there exists an $\varepsilon > 0$ such that (3.4) has a unique solution on the interval $[0, \varepsilon)$.

STEP 2. Extended uniqueness.

Let u_i be a solution of (3.4) on the interval $[0, a_i)$, $i = 1, 2$. We claim that $u_1(t) = u_2(t)$ for all $t \in [0, a)$, where $a = \min(a_1, a_2)$. Seeking a contradiction, assume that there exists $t \in (0, a)$ such that $u_1(t) \ne u_2(t)$. Defining

$$t^* = \inf\{t \in (0, a) \mid u_1(t) \ne u_2(t)\}\,.$$

it follows that $t^* > 0$ (by STEP 1), $t^* < a$ (by assumption) and $u_1(t^*) = u_2(t^*)$ (by continuity of x_1 and x_2). Clearly, the initial-value problem

$$\dot u(t) = k[r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \phi(u))(t)]\,, \quad t \ge t^*\,,$$
$$u(t) = u_1(t)\,, \quad t \in [0, t^*]\,,$$

is solved by u_1 and u_2. This implies (by Lemma A.1) that there exists an $\varepsilon > 0$ such that $u_1(t) = u_2(t)$ for all $t \in [0, t^* + \varepsilon)$, which contradicts the definition of t^*.

STEP 3. Continuation of solutions.

Let u be a solution of (3.4) on the interval $[0, a)$, $a < \infty$. In order to prove that u can be extended to a maximal solution (which satisfies (3.5) if $a_{max} < \infty$), it is sufficient to show that u can be continued to the right (beyond a) if u is bounded on $[0, a)$. Now $u(t) = (\Gamma u)(t)$ for all $t \in [0, a)$, where Γ is the operator defined in the proof of Lemma A.1. It is clear that $\lim_{t\to a-}(\Gamma u)(t) = \gamma$ exists and is finite. Consequently, $\lim_{t\to a-} u(t) = \gamma$, and hence setting $u(a) = \gamma$ makes u into a continuous function on $[0, a]$. Finally, Lemma A.1 shows that the initial value problem

$$\begin{aligned} \dot v &= k[r - \mathbf{\Psi}_\infty x_0 - \mathbf{F}_\infty \phi(v)]\,, \quad t \geq a\,, \\ v(t) &= u(t)\,, \quad t \in [0, a]\,, \end{aligned}$$

has a unique solution u^* on $[0, a + \varepsilon)$ for some $\varepsilon > 0$. By the causality of the map $\mathbf{F}_\infty\phi$, the function u^* is a solution of (3.4) on $[0, a + \varepsilon)$, i.e. u^* is a continuation of u.

STEP 4. Global existence if ϕ is globally Lipschitz.

Assume that ϕ is globally Lipschitz. Seeking a contradiction suppose that $a_{max} < \infty$. Let u be the solution of (3.4) defined on $[0, a_{max})$. Multiplying (3.4) by u and estimating we obtain that, for all $\tau \in [0, a_{max})$,

$$u(\tau)\dot u(\tau) \leq \frac{k}{2}[r^2 + (\mathbf{\Psi}_\infty x_0)^2(\tau) + 2u^2(\tau)] + |(\mathbf{F}_\infty\phi(u))(\tau)u(\tau)|\,. \tag{A.6}$$

Integrating (A.6) from 0 to t and combining the estimate

$$\int_0^t |(\mathbf{F}_\infty\phi(u))u| \leq \int_0^t |\mathbf{F}_\infty(\phi(u) - \phi(0))|\,|u| + \frac{1}{2}\left(\int_0^t (\mathbf{F}_\infty\phi(0))^2 + \int_0^t u^2\right)\,,$$

the Cauchy-Schwarz inequality and the global Lipschitz property of ϕ, it can be readily shown that there exists positive constants α and β such that, for all $t \in [0, a_{max})$,

$$u^2(t) \leq \alpha + \beta\int_0^t u^2(\tau)\,d\tau\,.$$

An application of Gronwall's lemma then shows that $u^2(t) \leq \alpha e^{\beta t}$ for all $t \in [0, a_{max})$. Hence u is bounded on $[0, a_{max})$, which by STEP 3 is in contradiction to the maximality of a_{max}. □

Proof of Lemma 3.4

Since $0 < 2\kappa < K$, it follows that there exists $\varepsilon > 0$ such that

$$1 + 2\kappa \mathrm{Re}\,\frac{\mathbf{G}(s)}{s} \geq \varepsilon\,, \qquad \text{for all } s \in \mathbb{C}_0\,.$$

Hence

$$1 + 2\kappa \mathrm{Re}\, \frac{\mathbf{G}(i\omega)}{i\omega} \geq \varepsilon\,, \qquad \text{for all } \omega \in \mathbb{R}, \omega \neq 0\,,$$

and thus

$$1 + 2\kappa \mathrm{Re}\, \mathbf{H}(i\omega) \geq \varepsilon\,, \qquad \text{for all } \omega \in \mathbb{R}\,. \tag{A.7}$$

By considering

$$e^{-(1+2\kappa \mathrm{Re}\, \mathbf{H}(s))} = \left| e^{-(1+2\kappa \mathbf{H}(s))} \right|\,,$$

applying the maximum modulus theorem and using the fact that $\mathbf{H}(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_0$, it follows from (A.7) that

$$1 + 2\kappa \mathrm{Re}\, \mathbf{H}(s) \geq \varepsilon\,, \qquad \text{for all } s \in \mathbb{C}_0\,.$$

Therefore, for all $s \in \mathbb{C}_0$,

$$\varepsilon + \kappa^2 \mathbf{H}(s)\bar{\mathbf{H}}(s) \leq (1 + \kappa \mathbf{H}(s))(1 + \kappa \bar{\mathbf{H}}(s))\,.$$

Consequently, for all $s \in \mathbb{C}_0$,

$$\mathbf{H}(s)(1 + \kappa \mathbf{H}(s))^{-1} \bar{\mathbf{H}}(s)(1 + \kappa \bar{\mathbf{H}}(s))^{-1} < \frac{1}{\kappa^2}\,,$$

yielding (3.9).

By using the identity $s(sI - A)^{-1} = A(sI - A)^{-1} + I$, we easily obtain

$$\mathbf{H}(s) = \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0)) = C_L(sI - A)^{-1}A^{-1}B\,.$$

Consider the state-space system given by the triple $(A, A^{-1}B, C_L)$. For any $T > 0$, the input-to-state map of this system maps $L^2(0, T)$ boundedly into X_1. Consequently, the triple $(A, A^{-1}B, C_L)$ defines a Pritchard-Salamon system with respect to the spaces X_1 and X, see Curtain et al. [4] or Pritchard and Townley [30]. Now, (3.9) means in particular that the closed-loop system obtained from $\mathbf{H}$ by negative output feedback with gain κ is input-output stable. By the equivalence of input-output and exponential stability (see [4] or [31]), we may conclude that the semigroup generated by A_κ, with $0 < 2\kappa < K$, is exponentially stable. Moreover, combining theorem 2.4 in Pritchard and Townley [29] (or, alternatively, theorem 1 in Logemann [12]) and (3.9), it follows that the structured complex stability radius of A_κ with respect to the weightings $A^{-1}B$ and C_L is greater than κ. Therefore an application of proposition 1.5 in [30] shows that the Riccati equation (3.10) has a self-adjoint positive-semidefinite solution $P \in \mathcal{B}(X)$ such that (3.10) holds for all $x_1, x_2 \in \mathrm{dom}(A_\kappa)$. $\square$

Proof of Lemma 3.5

It is clear that $\delta\phi \in L^\infty_{loc}(-\infty,\infty)$ if ϕ is locally Lipschitz, and that $\delta\phi \in L^\infty(-\infty,\infty)$ if ϕ is globally Lipschitz. Moreover, as the limsup of a sequence of Borel functions, $\delta\phi$ is a Borel function. Consequently, $\delta\phi \circ u$ is Lebesgue measurable. Setting $v = \phi \circ u$, it follows from the Lipschitz continuity of ϕ and the absolute continuity of u that v is absolutely continuous. If $t \in \mathbb{R}$ is such that u is differentiable at t, then we have

$$v(t+h) - v(t) = \phi(u(t) + h\dot{u}(t)) - \phi(u(t)) + \phi(u(t+h)) - \phi(u(t) + h\dot{u}(t))\,. \qquad \text{(A.8)}$$

Moreover, by Lipschitz continuity of ϕ, there exists a constant $L > 0$ such that, for all sufficiently small $|h|$,

$$\left|\frac{1}{h}\left[\phi(u(t+h)) - \phi(u(t) + h\dot{u}(t))\right]\right| \le L\left|\frac{1}{h}[u(t+h) - u(t)] - \dot{u}(t)\right|. \qquad \text{(A.9)}$$

Let $\mathcal{D} \subset \mathbb{R}$ be the set of all points t such that both u and v are differentiable at t. Then $\mathcal{D}$ is of full measure, and combining (A.8) and (A.9) yields

$$\lim_{h\to 0}\frac{1}{h}[v(t+h) - v(t)] = \lim_{h\to 0}\frac{1}{h}[\phi(u(t) + h\dot{u}(t)) - \phi(u(t)] \quad \text{for all } t \in \mathcal{D}.$$

Therefore, for every $t \in \mathcal{D}$,

$$\dot{v}(t) = 0\,, \qquad \text{if } \dot{u}(t) = 0\,, \qquad \text{(A.10a)}$$

$$\dot{v}(t) = \lim_{h\to 0}\frac{\phi(u(t) + h\dot{u}(t)) - \phi(u(t))}{h\dot{u}(t)}\,\dot{u}(t) = \phi'(u(t))\dot{u}(t)\,, \qquad \text{if } \dot{u}(t) \ne 0\,. \qquad \text{(A.10b)}$$

In particular, if $t \in \mathcal{D}_0 := \{t \in \mathcal{D} \mid \dot{u}(t) \ne 0\}$, then ϕ is differentiable at $u(t)$. For $t \in \mathcal{D}_0$ we have, of course, $\delta\phi(u(t)) = \phi'(u(t))$ and thus it follows from (A.10) that

$$\dot{v}(t) = \delta\phi(u(t))\dot{u}(t)\,, \qquad \text{for a.e. } t \in [0,\infty)\,.$$

$\square$

References

[1] V. Barbu. *Analysis and Control of Nonlinear Infinite-Dimensional Systems*, Academic Press, Boston, 1993.

[2] P.A. Cook. Controllers with universal tracking properties, *Proc. of Int. IMA Conf. on Control: Modelling, Computation, Information*, Manchester, 1992.

[3] G.W.M. Coppus, S.L. Sha and R.K. Wood. Robust multivariable control of a binary distillation column, *IEE Proceedings, 3Pt. D* **130** (1983), 201–208.

[4] R.F. Curtain, H. Logemann, S. Townley and H. Zwart. Well-posedness, stabilizability and admissibility for Pritchard-Salamon systems, *Mathematical Systems, Estimation and Control* **4** (1994) [summary pp. 493–496; full electronic manuscript = 38pp; *Address:* trick.ntp.springer.de; *User:* anonymous (personal email address); *Access:* cd jmsec; *Retrieval code:* 85404; full hardcopy to appear].

[5] E.J. Davison. Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem, *IEEE Trans. Auto. Control* **21** (1976), 35–47

[6] C.A. Desoer and C.-A. Lin. Tracking and disturbance rejection of MIMO nonlinear systems with PI controller, *IEEE Trans. Auto. Control* **30** (1985), 861–867.

[7] G. Doetsch. *Einführung in Theorie und Anwendung der Laplace-Transformation*, 3. Auflage, Birkhäuser Verlag, Basel, 1976.

[8] G. Gripenberg, S.-O. Londen and O. Staffans. *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.

[9] T.T. Jussila and H.N. Koivo. Tuning of multivariable PI-controllers for unknown delay-differential systems, *IEEE Trans. Auto. Control* **32** (1987), 364–368.

[10] H.W. Knobloch and F. Kappel. *Gewöhnliche Differentialgleichungen*, B.G. Teubner, Stuttgart, 1974.

[11] H.N. Koivo and S. Pohjolainen. Tuning of multivariable PI-controllers for unknown systems with input delay, *Automatica* **21** (1985), 81–91.

[12] H. Logemann. Circle criteria, small-gain conditions and internal stability for infinite-dimensional systems, *Automatica*, **27** (1991), 677–690.

[13] H. Logemann, J. Bontsema and D.H. Owens. Low-gain control of distributed parameter systems with unbounded control and observation, *Control–Theory and Advanced Technology* **4** (1988), 429–446.

[14] H. Logemann and D.H. Owens. Low-gain control of unknown infinite-dimensional systems: a frequency-domain approach, *Dynamics and Stability of Systems* **4** (1989), 13–29.

[15] H. Logemann, E.P. Ryan and S. Townley. Integral control of time-delay systems with actuator nonlinearities: lower bounds for the maximal regulating gain, in preparation.

[16] H. Logemann and S. Townley. Discrete-time low-gain control of uncertain infinite-dimensional systems, *IEEE Trans. Auto. Control*, to appear in Jan. 1997.

[17] H. Logemann and S. Townley. Low-gain control of uncertain regular linear systems, *SIAM J. Control & Opt.*, to appear in Jan. 1997.

[18] J. Lunze. Experimentelle Erprobung einer Einstellregel für PI-Mehrgrößenregler bei der Herstellung von Ammoniumnitrat-Harnstoff-Lösung, *Messen Steuern Regeln* **30** (1987), 2-6.

[19] J. Lunze. *Robust Multivariable Feedback Control*, Prentice Hall, London, 1988.

[20] D.E. Miller. Private communication, 1996.

[21] D.E. Miller and E.J. Davison. An adaptive tracking problem with a control input constraint, *Automatica* **29** (1993), 877–887.

[22] D.E. Miller and E.J. Davison. The self-tuning robust servomechanism problem, *IEEE Trans. Auto. Control* **34** (1989), 511–523.

[23] M. Morari. Robust stability of systems with integral control, *IEEE Trans. Auto. Control* **30** (1985), 574–577.

[24] R. Pallu de la Barrière. *Optimal Control Theory*, Dover Publications, New York, 1980.

[25] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[26] S. Pohjolainen. Robust controller for systems with exponentially stable strongly continuous semigroups, *J. Math. Analysis and Applications* **111** (1985), 622–636.

[27] S. Pohjolainen. Robust multivariable PI-controllers for infinite-dimensional systems, *IEEE Trans. Auto. Control* **27** (1982), 17–30.

[28] S. Pohjolainen and I. Lätti. Robust controller for boundary control systems, *Int. J. Control* **38** (1983), 1189–1197.

[29] A.J. Pritchard and S. Townley. Robustness of linear systems, *J. Differential Eqns.* **77** (1989), 254–286.

[30] A.J. Pritchard and S. Townley. Robustness optimization for uncertain infinite-dimensional systems with unbounded inputs, *IMA J. Math. Control & Information* **8** (1991), 121–133.

[31] R. Rebarber. Conditions for the equivalence of internal and external stability for distributed parameter systems, *IEEE Trans. Auto. Control* **38** (1993), 994–998.

[32] D. Salamon. Realization theory in Hilbert space, *Math. Systems Theory* **21** (1989), 147–164.

[33] S. Townley and M. Kamstra. Integral control with saturating input, pp. 805–808 in *Proceedings of UKACC Control '96.*

[34] G. Weiss. Two conjectures on the admissibility of control operators, pp. 367–378 in *Control and Estimation of Distributed Parameter Systems* edited by F. Kappel, K. Kunisch and W. Schappacher, Birkhäuser Verlag, Basel, 1991.

[35] G. Weiss. Transfer functions of regular linear systems, part I: Characterization of regularity, *Trans. Amer. Math. Soc.* **342** (1994), 827–854.

[36] G. Weiss. Admissibility of unbounded control operators, *SIAM J. Control & Opt.* **27** (1989), 527–545.

[37] G. Weiss. Admissible observation operators for linear semigroups, *Israel J. Math.* **65** (1989), 17–43.

[38] G. Weiss. The representation of regular linear systems on Hilbert spaces, pp. 401–416 in *Distributed Parameter System* edited by F. Kappel, K. Kunisch and W. Schappacher, Birkhäuser Verlag, Basel, 1989.