HYPERCOMPLEX STRUCTURES ASSOCIATED TO QUATERNIONIC MANIFOLDS

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ABSTRACT. If M is a quaternionic manifold and P is an S^1-instanton over M, then Joyce constructed a hypercomplex manifold we call $\mathcal{P}(M)$ over M. These hypercomplex manifolds admit a $U(2)$-action of a special type permuting the complex structures. We show that up to double covers, all such hypercomplex manifolds arise in this way. Examples, including that of a hypercomplex structure on $SU(3)$, show the necessity of including double covers of $\mathcal{P}(M)$.

1. INTRODUCTION

In [13], it was shown that over any quaternionic Kähler manifold there is a hyperKähler manifold with an isometric action of $SO(3)$ rotating the complex structures. The hyperKähler manifolds which arise this way were characterised by the properties of this action and it was shown how to recover the original quaternionic Kähler manifold via a moment map construction. The purpose of this article is to study similar constructions and results for quaternionic and hypercomplex structures—geometries that do not involve Riemannian metrics.

Salamon [11], in studying twistor spaces of quaternionic manifolds, showed that the total spaces of certain vector bundles admit hypercomplex structures. Joyce [6] proved that, in some cases, these structures have an action of $\mathbb{R}$ preserving the hypercomplex structure and that quotienting by $\mathbb{Z} \subset \mathbb{R}$ one may obtain examples of compact hypercomplex manifolds. The simplest of these are straightforward generalisations of Hopf surfaces, but good choices of vector bundles lead him to the first examples of compact simply-connected hypercomplex manifolds not built out of $K3$-surfaces.

Joyce's examples all admit actions of $U(2)$. Our first result shows how knowledge of certain $U(2)$-actions on hypercomplex manifolds is enough to construct a quaternionic manifold. Our approach gives a good geometric understanding of the associated family of torsion-free quaternionic connections in terms of the Obata connection of the hypercomplex structure. The curvature of these hypercomplex

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manifolds has various special properties which we describe; in particular, the $U(2)$-orbits are totally-geodesic flat submanifolds.

The main aim of the paper is to classify these hypercomplex manifolds with this type of $U(2)$-action. This involves obtaining explicit knowledge on how to construct hypercomplex manifolds over a quaternionic base. The first part is in the spirit of [13] and [9], constructing a standard hypercomplex structure explicitly from the quaternionic manifold and specifying precisely a constant that appears in [11]. This standard hypercomplex structure can then be twisted by a quaternionic instanton as in [6], but our constructions make this twisting particularly straightforward. We are then able to show that, up to double covers, all the hypercomplex manifolds discussed in §2 arise from the twisting construction. In §7, we give two examples of this construction: one over $\mathbb{C}P(2)$ and the other over $\mathbb{C}P(2) \# \mathbb{C}P(2)$. A detailed study of the case of $\mathbb{C}P(2)$, shows that $SU(3)$ admits a hypercomplex structure with $U(2)$-action as above, but that this is not obtained by the twisting construction, which only gives the quotient $SU(3)/(\mathbb{Z}/2)$, contrary to the claim in [6].

As an application of the above results, we discuss in §5 non-Riemannian analogues of a result of Gray [5] for submanifolds of hyperKähler and quaternionic Kähler manifolds. We show that an almost hypercomplex submanifold of a hypercomplex manifold is hypercomplex and totally geodesic and prove the corresponding result for quaternionic manifolds.

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2. Hypercomplex Manifolds

A *hypercomplex manifold* is a manifold N with three globally-defined, integrable complex structures I, J, K satisfying the quaternion identities

$$I^2 = J^2 = K^2 = -1 \qquad \text{and} \qquad IJ = K = -JI. \tag{2.1}$$

Obata [8] proved that a hypercomplex manifold admits a unique torsion-free connection ∇ such that $\nabla I = \nabla J = \nabla K = 0$.

In contrast, an *almost quaternionic manifold* is a $4n$-dimensional manifold M together with a rank-three bundle $\mathcal{G}$ of endomorphisms of the tangent bundle TM of M, such that $\mathcal{G}$ locally has a basis $\{I, J, K\}$ satisfying (2.1). Note that I, J, K need not be integrable nor globally defined. A torsion-free connection ∇^α on M is said to be *compatible* or *quaternionic* if $\nabla^\alpha \mathcal{G} \subset T^*M \otimes \mathcal{G}$. For a local basis $\{I, J, K\}$ as above, this is equivalent to the existence of local one-forms a_{IJ}, a_{JK}, etc., such that $\nabla^\alpha I = a_{IJ} \otimes J + a_{IK} \otimes K$, etc. If such a connection ∇^α exists, M is said to be *quaternionic*. Quaternionic connections on a quaternionic manifold are not unique but rather are parameterised by the space $\Omega^1(M)$ of one-forms on M [8, 11]. This parameterisation will become particularly clear in the proof of the following result.

Theorem 2.1. *Let N be a hypercomplex manifold. Suppose there is an action of $U(2)$ on N such that $N/U(2)$ is a manifold and the quotient map is a submersion. Let V be the vector field generated by the central $U(1) \subset U(2)$ and suppose that*

(1) *V preserves the hypercomplex structure, that is, $L_V I = 0 = L_V J = L_V K$,*
(2) *IV, JV, KV generate the action of $SU(2) \subset U(2)$,*
(3) *$L_{IV} J = K$ and $L_{JV} K = I$.*

Then $M = N/U(2)$ admits a quaternionic structure and all the compatible connections on M may be obtained from the Obata connection on N.

Proof. Choose an invariant one-form $\alpha \in \Omega^1(N)^{U(2)}$ such that $\alpha(V) = 1$ and $\alpha(IV) = 0$. (This may be done for example by making a choice of invariant hyperHermitian metric.) Note that the $U(2)$-invariance of α and condition (3) imply $\alpha(JV) = 0 = \alpha(KV)$.

We shall first use the one-form α to define a quaternionic structure on M. Then we shall show that this structure is independent of α and that the choices of α parameterise the compatible torsion-free connections.

Define a distribution $\mathcal{H}^\alpha \subset TN$ by

$$\mathcal{H}^\alpha = \ker \alpha \cap I \ker \alpha \cap J \ker \alpha \cap K \ker \alpha.$$

This distribution is $U(2)$-invariant, preserved by I, J, K and complementary to $\langle V \rangle_{\mathbb{H}} = \langle V, IV, JV, KV \rangle$, so the differential of the projection $\pi \colon N \to M$ identifies $\mathcal{H}^\alpha$ with TM.

For a vector field $X \in TM$, let $h^\alpha X \in \mathcal{H}^\alpha$ denote the horizontal lift. Given a local section $s \colon M \to N$, we define local endomorphisms I, J, K on $T_x M$ by

$$IX_x = \pi_*(I h^\alpha X_{s(x)}), \quad \text{etc.}$$

If we change the section s, then $s(x)$ changes via an element g of $U(2)$. But on N we have $g^* I = aI + bJ + cK$ for some constants a, b, c with $a^2 + b^2 + c^2 = 1$, so on M the space $\mathcal{G}_x = \langle I, J, K \rangle$ is independent of s and M is an almost quaternionic manifold.

We now claim that the connection ∇^α given by

$$\nabla^\alpha_X Y = \pi_*(\nabla_{h^\alpha X} h^\alpha Y), \tag{2.2}$$

where ∇ is the Obata connection on N, is a well-defined, torsion-free, quaternionic connection on M.

The connection ∇^α is well-defined, since ∇ is $U(2)$-invariant. This may be seen as follows. If g is an element of $U(2)$ then we obtain a new connection ∇^g on N given by $\nabla^g_X Y = g_*^{-1} \nabla_{g_* X} g_* Y$. This is again torsion-free and $\nabla^g g^* I = 0 = \nabla^g g^* J$. However, $g^* I$, $g^* J$ and $g^* K$ span $\mathcal{G}_x$ and hence ∇^g is the Obata connection on N, i.e. $\nabla^g = \nabla$.

The connection ∇^α is torsion-free, since

$$\begin{aligned}\nabla^\alpha_X Y - \nabla^\alpha_Y X &= \pi_*\pi^\alpha_{\mathcal{H}}(\nabla_{h^\alpha X} h^\alpha Y - \nabla_{h^\alpha Y} h^\alpha X)\\ &= \pi_* h^\alpha[X,Y]\\ &= [X,Y],\end{aligned}$$

where $\pi^\alpha_{\mathcal{H}}$ is the projection to $\mathcal{H}^\alpha$ in the splitting $TN = \langle V\rangle_{\mathbb{H}} \oplus \mathcal{H}^\alpha$.

To show that ∇^α preserves the almost quaternionic structure on M, choose a local basis $\{I, J, K\}$ of $\mathcal{G}$. We have

$$IX = \pi_*\big((aI + bJ + cK)(h^\alpha X)\big),$$

where a, b, c are functions on N such that $a^2 + b^2 + c^2 = 1$ and $aI + bJ + cK$ is $U(2)$-invariant. Now

$$\begin{aligned}(\nabla^\alpha_X I)Y &= \pi_*(\nabla_{h^\alpha X} h^\alpha(IY)) - I\pi_*\nabla_{h^\alpha X} h^\alpha Y\\ &= \pi_*\big((da(h^\alpha X)I + db(h^\alpha X)J + dc(h^\alpha X)K)(h^\alpha Y)\\ &\qquad + (aI + bJ + cK)\nabla_{h^\alpha X} h^\alpha Y\big) - I\pi_*\nabla_{h^\alpha X} h^\alpha Y\\ &= \pi_*\big((da\otimes I + db\otimes J + dc\otimes K)(h^\alpha X, h^\alpha Y)\big).\end{aligned} \tag{2.3}$$

This says that $\nabla^\alpha_X I$ is a linear combination of I, J and K and thus ∇^α is quaternionic.

It remains to prove that the quaternionic structure is independent of α. We first have

Lemma 2.2. *The vector field V and the Obata connection ∇ satisfy*

$$\nabla V = -\tfrac{1}{2}\,\mathrm{Id}\,.$$

Proof. For any vector field X, we have

$$\begin{aligned}0 = (L_{IV} I)X &= [IV, IX] - I[IV, X]\\ &= -\nabla_{IX}(IV) + I\nabla_X(IV) = -I\nabla_{IX}V - \nabla_X V.\end{aligned}$$

Thus $\nabla_{IX}V = I\nabla_X V$, etc. However

$$KX = (L_{IV}J)X = -\nabla_{JX}(IV) + J\nabla_X(IV) = -2\nabla_{KX}V,$$

giving the result. ∎

Let β be another $U(2)$-invariant one-form with $\beta(V) = 1$ and $\beta(IV) = 0$. Then

$$\beta = \alpha + \pi^*\lambda,$$

for some $\lambda \in \Omega^1(M)$ and all such β are given this way. Now

$$\begin{aligned} IX_x = \pi_*(Ih^\alpha X_{s(x)}) &= \pi_*\Big(I\big(h^\beta X + \lambda(X)V - (\pi^*\lambda)(Ih^\beta X)IV \\ &\qquad - (\pi^*\lambda)(Jh^\beta X)JV - (\pi^*\lambda)(Kh^\beta X)KV\big)_{s(x)}\Big) \\ &= \pi_*(Ih^\beta X_{s(x)}), \end{aligned}$$

so I is independent of α. This also gives

$$\nabla^\beta = \nabla^\alpha + (\lambda \vee \mathrm{Id} + I\lambda \vee I + J\lambda \vee J + K\lambda \vee K), \tag{2.4}$$

for a local basis $\{I, J, K\}$ of $\mathcal{G}$, where $I\lambda(X) = -\lambda(IX)$ and $A \vee B = \frac{1}{2}(A \otimes B + B \otimes A)$. Thus $\nabla^\beta = \nabla^\alpha$ if and only if $\lambda = 0$ and we obtain all possible compatible connections. This completes the proof of the theorem. □

Note that formula (2.4) was obtained by Alekseevskiĭ & Marchiafava [1] using other techniques.

The requirement that a hypercomplex manifold N admits a $U(2)$-action as in the above Theorem, places some curvature restrictions on the Obata connection.

Proposition 2.3. *Let N be a hypercomplex manifold with vector field V as in Theorem 2.1. Then all components of the curvature tensor R of the Obata connection involving the quaternionic span of V vanish.*

Proof. It is enough to prove that $R_{X,Y}V = 0$, $R_{V,X}Y = 0$ and $R_{IV,X}Y = 0$. The first of these follows directly from the Lemma. To prove the second, it is enough to take X and Y to be local V-invariant vector fields. Then $[V, X] = 0 = [V, Y]$ and $[V, \nabla_X Y] = 0$, since ∇ is $U(2)$-invariant. So

$$\begin{aligned} R_{V,X}Y &= \nabla_V \nabla_X Y - \nabla_X \nabla_V Y \\ &= \nabla_{\nabla_X Y} V - \nabla_X \nabla_Y V = -\tfrac{1}{2}\nabla_X Y + \tfrac{1}{2}\nabla_X Y \\ &= 0. \end{aligned}$$

The identity $R_{IV,X}Y = 0$ may be obtained in a similar way. □

These results imply that the $U(2)$-orbits are totally geodesic and flat. Also, since $\nabla(IV) = -\frac{1}{2}I$, etc., we may rewrite the hypotheses on the manifold N purely in terms of the connection and the vector fields generated by the $U(2)$-action.

3. Associated Bundles

The definition of a quaternionic manifold is equivalent to requiring that the holonomy group lies in $GL(n, \mathbb{H})\, GL(1, \mathbb{H}) = (\mathbb{R}_{>0} \times SL(n, \mathbb{H}) \times Sp(1))/\{\pm 1\}$. Let F be the bundle of frames on M compatible with the almost quaternionic structure. Over a point $x \in M$, such a frame $u \in F_x$ is a linear isomorphism $u\colon \mathbb{H}^n \to T_xM$ such that the elements of $u^{-1}\mathcal{G}_x u$ act on $\mathbb{H}^n$ by right-multiplication by imaginary quaternions. Locally, F has a double-cover $\tilde{F}$, which is a principal

$(\mathbb{R}_{>0} \times SL(n,\mathbb{H}) \times Sp(1))$-bundle. Given a representation W of $\mathbb{R}_{>0} \times SL(n,\mathbb{H}) \times Sp(1)$, we define a local vector bundle W on M as the quotient of $\tilde{F} \times W$ by the $(\mathbb{R}_{>0} \times SL(n,\mathbb{H}) \times Sp(1))$-action $(f,w) \mapsto (f \cdot g, g^{-1} \cdot w)$. The bundle W is globally defined if $(1,-1,-1)$ acts as the identity on the representation W. Defining representations L, E^* and H of $\mathbb{R}_{>0} \times SL(n,\mathbb{H}) \times Sp(1)$ on $\mathbb{C}$, $\mathbb{C}^{2n}$ and $\mathbb{C}^2$ by

$$(\lambda, A, q) \cdot \zeta = \lambda\zeta, \quad (\lambda, A, q) \cdot \xi = A\xi \quad \text{and} \quad (\lambda, A, q) \cdot \eta = \eta\overline{q},$$

respectively, we have $T_{\mathbb{C}}M \cong LE^*H := L \otimes_{\mathbb{C}} E^* \otimes_{\mathbb{C}} H$ and $T^*_{\mathbb{C}}M \cong L^{-1}EH$. (As H is a quaternionic representation of $SU(2)$, we have $H \cong H^*$.) Note that our notation is slightly different from that in [11], where E denotes a representation of $GL(n,\mathbb{H})$ instead of $SL(n,\mathbb{H})$.

Choose a quaternionic connection ∇ on M. Then the ($\mathbb{H}^n$-valued) canonical one-form θ on F, defined by $\theta_u(v) = u^{-1}(\pi_* v)$, satisfies

$$d\theta = -\omega_{\mathbb{R}} \wedge \theta - \omega_{\mathfrak{sl}} \wedge \theta - \theta \wedge \omega_{\mathfrak{sp}}, \tag{3.1}$$

where $\omega = \omega_{\mathbb{R}} \oplus \omega_{\mathfrak{sl}} \oplus \omega_{\mathfrak{sp}} \in \Omega^1(F, \mathbb{R} \oplus \mathfrak{sl}(n,\mathbb{H}) \oplus \mathfrak{sp}(1)) \subset \Omega^1(F,\mathbb{R}) \oplus \Omega^1(F, M_n(\mathbb{H})) \oplus \Omega^1(F, \operatorname{Im}\mathbb{H})$ is the connection one-form (cf. [13, 9]).

Given a real number ℓ, let

$$\mathcal{U}_\ell(M) = (L^\ell H \setminus 0)/\{\pm 1\}.$$

This is a globally defined principal $(\mathbb{H}^*/\{\pm 1\})$-bundle on M and our immediate aim is to study almost hypercomplex structures on $\mathcal{U}_\ell(M)$. Note that locally $L^\ell H$ is defined as a quotient of $\tilde{F} \times \mathbb{H}$. Let x denote the projection onto the second factor and define, omitting pull-back symbols,

$$\boldsymbol{\alpha} := dx + \ell x \omega_{\mathbb{R}} - x\omega_{\mathfrak{sp}}.$$

Denoting the action of $\mathbb{R}_{>0} \times SL(n,\mathbb{H}) \times Sp(1)$ on $\tilde{F} \times \mathbb{H}$ by $R_{(\lambda,A,q)}$, we have

$$R^*_{(\lambda,A,q)}\theta = \lambda^{-1}A^{-1}\theta q, \qquad R^*_{(\lambda,A,q)}\omega_{\mathbb{R}} = \omega_{\mathbb{R}}, \tag{3.2}$$

$$R^*_{(\lambda,A,q)}x = \lambda^\ell x q, \qquad R^*_{(\lambda,A,q)}\omega_{\mathfrak{sp}} = \overline{q}\omega_{\mathfrak{sp}} q. \tag{3.3}$$

Proposition 3.1. *Let ∇ be a compatible torsion-free connection on a quaternionic manifold M. The manifold $\mathcal{U}_\ell(M)$ carries a well-defined almost hypercomplex structure given by*

$$I\boldsymbol{\alpha} = i\boldsymbol{\alpha}, \qquad J\boldsymbol{\alpha} = j\boldsymbol{\alpha}, \qquad K\boldsymbol{\alpha} = k\boldsymbol{\alpha},$$
$$I(\theta\overline{x}) = -\theta\overline{x}i, \quad J(\theta\overline{x}) = -\theta\overline{x}j, \quad K(\theta\overline{x}) = -\theta\overline{x}k.$$

Proof. Exactly as in [13, Proposition 3.2] we have that the one-forms $\boldsymbol{\alpha}$ and $\theta\overline{x}$ vanish on vectors tangent to the action of $\mathbb{R}_{>0} \times SL(n,\mathbb{H}) \times Sp(1)$. If $s\colon L^\ell H \to \tilde{F} \times \mathbb{H}$ is a local section, then the components of $s^*\boldsymbol{\alpha}$ and $s^*(\theta\overline{x})$ are a basis for $T^*(L^\ell H)$. Since I, J and K are specified on this basis, it is enough to check that their definitions are independent of the local section s. However, $R^*_{(\lambda,A,q)}\boldsymbol{\alpha} = \boldsymbol{\alpha}\lambda^\ell q$ and $R^*_{(\lambda,A,q)}(\theta\overline{x}) = \lambda^{\ell-1}A^{-1}(\theta\overline{x})$, which commute with the definitions. As the definitions also commute with the action of $\{\pm 1\}$, the structures I, J and K descend to $\mathcal{U}_\ell(M)$. □

Theorem 3.2. *Let M be a $4n$-dimensional quaternionic manifold. Then the space $\mathcal{U}_{n/(n+1)}(M)$ is a hypercomplex manifold with hypercomplex structure defined as in the previous proposition. Furthermore, this hypercomplex structure is independent of the choice of compatible connection on M.*

We break the proof up into the following two propositions.

Proposition 3.3. *The almost hypercomplex structure on $\mathcal{U}_\ell(M^{4n})$ is independent of the choice of connection if and only if $\ell = n/(n+1)$.*

Proof. We start by investigating how the connection one-forms change when we change the connection via a one-form λ as in (2.4). Writing $\nabla^\beta_X Y = \nabla^\alpha_X Y + a_X Y$, we have

$$\begin{aligned} a_X Y = {} & \tfrac12 \lambda(Y)\,\mathrm{Id}(X) && \text{(3.4a)}\\ & - \tfrac12(\lambda(IY)I + \lambda(JY)J + \lambda(KY)K)(X) && \text{(3.4b)}\\ & + \tfrac12 \gamma_Y(X), && \text{(3.4c)} \end{aligned}$$

where

$$\gamma_Y(X) = \lambda(X)Y - \lambda(IX)IY - \lambda(JX)JY - \lambda(KX)KY.$$

In (3.4), we have that the operator in (3.4a) lies in $\mathbb{R}$, that in (3.4b) lies in $\mathfrak{sp}(1)$ and that in (3.4c) lies in $\mathfrak{gl}(n,\mathbb{H})$. With respect to the decomposition $\mathfrak{gl}(n,\mathbb{H}) = \mathbb{R} \oplus \mathfrak{sl}(n,\mathbb{H})$ we write the last operator as $\delta_Y \mathrm{Id} + \epsilon_Y$ and compute $n\delta_Y = \lambda(Y)$. Thus, we have

$$\omega_{\mathbb{R}} \mapsto \omega_{\mathbb{R}} + \tfrac12(1 + \tfrac1n)\lambda \quad \text{and} \quad \omega_{\mathfrak{sp}} \mapsto \omega_{\mathfrak{sp}} - \tfrac12(\lambda\circ Ii + \lambda\circ Jj + \lambda\circ Kk). \tag{3.5}$$

The one-form $\boldsymbol{\alpha}$ now changes as

$$\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} + \tfrac12 \ell(1+\tfrac1n)\pi^*\lambda x + \tfrac12 x(\pi^*\lambda\circ Ii + \pi^*\lambda\circ Jj + \pi^*\lambda\circ Kk).$$

Fixing I, we may write $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 j$, where $\boldsymbol{\alpha}_a$ are complex-valued one-forms of type $(1,0)$. For this type decomposition to be invariant under the above change of connection, we need $\ell(1+\frac1n) = 1$, giving the result. □

Proposition 3.4. *The almost complex structures on $\mathcal{U}_{n/(n+1)}(M)$ are integrable.*

The proof splits into two parts: the first is essentially integrability of the complex structure on the twistor space; the second then proves integrability in the remaining directions. (See Remark 3.7 for an explanation of the relationship with the twistor space.)

For the first part write $\boldsymbol{\alpha} = \boldsymbol{\alpha}_I^1 + \boldsymbol{\alpha}_I^2 j$, $x\overline{\theta} = \theta_I^1 + \theta_I^2 j$, where $\boldsymbol{\alpha}_I^a \in \Omega^1(\tilde F \times \mathbb{H}, \mathbb{C})$ and $\theta_I^a \in \Omega^1(\tilde F \times \mathbb{H}, \mathbb{C}^n)$, $a = 1,2$. The complex components of $s^*\boldsymbol{\alpha}_I^a$, $s^*\theta_I^a$ then form a basis of $(1,0)$-forms on $\mathcal{U}_{n/(n+1)}(M)$ and integrability is equivalent to the ideal $\mathcal{D}^{(1,0)}$ generated by these forms being closed under exterior differentiation.

Now we may choose the section $s\colon L^{n/(n+1)}H \to \tilde{F} \times \mathbb{H}$, so that at the point in question $s^*\omega_{\mathbb{R}}$, $s^*\omega_{\mathfrak{sp}}$ and $s^*\omega_{\mathfrak{sl}}$ vanish and $s^*x = 1$. (The first three conditions are just that s is horizontal at the point). Then, at this point, we have

$$\begin{aligned} s^*d(x\overline{\theta}) &= s^*(dx \wedge \overline{\theta} + x\,d\overline{\theta}) = s^*(\boldsymbol{\alpha} \wedge x\overline{\theta}) \\ &= s^*\big(\boldsymbol{\alpha}_I^1 \wedge \theta_I^1 + \overline{\theta}_I^2 \wedge \boldsymbol{\alpha}_I^2 + (\boldsymbol{\alpha}_I^1 \wedge \theta_I^2 - \overline{\theta}_I^1 \wedge \boldsymbol{\alpha}_I^2)j\big), \end{aligned}$$

so $d(s^*\theta_I^1)$ and $d(s^*\theta_I^2)$ are in $\mathcal{D}^{(1,0)}$. Also

$$s^*d\boldsymbol{\alpha} = s^*(\frac{n}{n+1}x d\omega_{\mathbb{R}} - x d\omega_{\mathfrak{sp}}) = s^*(\frac{n}{n+1}\Omega_{\mathbb{R}} - \Omega_{\mathfrak{sp}}),$$

where $\Omega_\bullet = d\omega_\bullet + \omega_\bullet \wedge \omega_\bullet$ is the curvature of $\omega_\bullet$. We would therefore like to show that this has no $(0,2)$-component.

Taking the exterior derivative of (3.1) one has

$$\begin{aligned} 0 = s^*d^2\theta &= s^*\big(-\Omega_{\mathbb{R}} \wedge \theta - \Omega_{\mathfrak{sl}} \wedge \theta - \theta \wedge \Omega_{\mathfrak{sp}}\big) \\ &= s^*\big(-\Omega_{\mathbb{R}} \wedge \overline{\theta_I^1} + j\Omega_{\mathbb{R}} \wedge \overline{\theta_I^2} \\ &\qquad - {\Omega_{\mathfrak{sl},I}}^1 \wedge \overline{\theta_I^1} - {\Omega_{\mathfrak{sl},I}}^2 \wedge \overline{\theta_I^2} + j\overline{{\Omega_{\mathfrak{sl},I}}^1} \wedge \overline{\theta_I^2} - j\overline{{\Omega_{\mathfrak{sl},I}}^2} \wedge \overline{\theta_I^1} \\ &\qquad - \overline{\theta_I^1} \wedge {\Omega_{\mathfrak{sp},I}}^1 - \theta_I^2 \wedge \overline{{\Omega_{\mathfrak{sp},I}}^2} - j\theta_I^1 \wedge \overline{{\Omega_{\mathfrak{sp},I}}^2} + j\overline{\theta_I^2} \wedge {\Omega_{\mathfrak{sp},I}}^1\big), \end{aligned}$$

where $\Omega_{\mathfrak{sl}} = {\Omega_{\mathfrak{sl},I}}^1 + {\Omega_{\mathfrak{sl},I}}^2 j$, etc. As $s^*\theta_I^a$ is of type $(1,0)$, we get

$$\big({\Omega_{\mathfrak{sp},I}}^2\big)^{(0,2)} = 0.$$

Thus $s^*d\boldsymbol{\alpha}_I^2$ has no $(0,2)$-component. This completes the first part of the proof.

The second part of the proof requires a deeper understanding of the curvature of the bundles over a quaternionic manifold in order to show that $(\frac{n}{n+1}\Omega_{\mathbb{R}} - {\Omega_{\mathfrak{sp},I}}^1)^{0,2} = 0$.

For any quaternionic connection ∇ on M we may consider the differential operator $D = p \circ \nabla$, where $p\colon T^* \otimes L^\ell H \cong L^{\ell-1}E(S^2H + \mathbb{R}) \to L^{\ell-1}ES^2H$ is the equivariant projection.

Lemma 3.5. *The operator D on $L^\ell H$ is independent of the choice of connection if and only if $\ell = n/(n+1)$.*

Proof. We will use the same notation as in the proof of Proposition 3.3. Let $\{h, \tilde{h}\}$ be a local basis for H with $\tilde{h} = j_H h$, where j_H is the $Sp(1)$-invariant quaternionic structure, such that $h \wedge \tilde{h}$ is the symplectic structure on H. As λ is an element of $T^* = L^{-1}EH$, we may write $\lambda = \mu h + \nu\tilde{h}$ with $\mu, \nu \in L^{-1}E$. Let $a = xh + y\tilde{h}$ be an arbitrary element of $L^\ell H$. Then when the connection changes via λ, the contribution in the change of Da from $\omega_{\mathbb{R}}$ is

$$p(\tfrac{\ell}{2}\tfrac{n+1}{n}\lambda \otimes a) = \tfrac{\ell}{2}\tfrac{n+1}{n}(x\mu\, h^2 + y\nu\,\tilde{h}^2 + (y\mu + x\nu)h \vee \tilde{h}). \tag{3.6}$$

To find the contribution from the change in $\omega_{\mathfrak{sp}}$, we write $I = ih \vee \tilde{h}$, $J = h^2 + \tilde{h}^2$, $K = -i(h^2 - \tilde{h}^2)$, so that

$$\begin{aligned} Ia &= -ixh + iy\tilde{h}, & \lambda \circ I &= i\mu h - i\nu\tilde{h}, \\ Ja &= yh - x\tilde{h}, & \lambda \circ J &= -\nu h + \mu\tilde{h}, \\ Ka &= -iyh - ix\tilde{h}, & \lambda \circ K &= i\nu h + i\mu\tilde{h}. \end{aligned}$$

Thus the change in $\omega_{\mathfrak{sp}}$ adds

$$\begin{aligned} -\tfrac{1}{2}p\big((\lambda \circ I) \otimes Ia &+ (\lambda \circ J) \otimes Ja + (\lambda \circ K) \otimes Ka\big) \\ &= -\tfrac{1}{2}\big(x\mu\, h^2 + y\nu\, \tilde{h}^2 - (y\mu + x\nu) h \vee \tilde{h} \\ &\qquad - y\nu\, h^2 - x\mu\, \tilde{h}^2 + (y\mu + x\nu) h \vee \tilde{h} \\ &\qquad + y\nu\, h^2 + x\mu\tilde{h}^2 + (y\mu + x\nu) h \vee \tilde{h}\big) \\ &= -\tfrac{1}{2}\big(x\mu\, h^2 + y\nu\, \tilde{h}^2 + (y\mu + x\nu) h \vee \tilde{h}\big) \end{aligned} \tag{3.7}$$

to Da. Combining (3.6) and (3.7) shows that Da is invariant if and only if $(\ell/2)(n+1)/n - 1/2 = 0$, that is, $\ell = n/(n+1)$. □

Corollary 3.6. *Salamon's bundle H' in* [11] *is $L^{n/(n+1)}H$.* □

This enables us to apply results of [11] to study the curvature $\mathcal{R}$ of $L^{n/(n+1)}H$. Firstly, think of $\mathcal{R}$ as the map $\mathbf{a} \circ \nabla \circ \nabla \colon L^{n/(n+1)}H \to \Lambda^2T^* \otimes L^{n/(n+1)}H$, where $\mathbf{a}\colon T^* \otimes T^* \to \Lambda^2T^*$ is the alternation map. Now

$$\begin{aligned} \Lambda^2T^* \otimes L^{n/(n+1)}H &\cong L^{-2}(\Lambda^2ES^2H + S^2E) \otimes L^{n/(n+1)}H \\ &\cong L^{-2+n/(n+1)}(\Lambda^2ES^3H + \Lambda^2EH + S^2EH). \end{aligned}$$

Theorem 5.5 of [11] applied to $L^{n/(n+1)}H = H' = A^{0,1}$, implies that the part of the curvature in $L^{-2+n/(n+1)}\Lambda^2ES^3H$ is zero. Thus $\mathcal{R}$ lies in $L^{-2}(\Lambda^2E + S^2E)H \otimes H$.

If I is a compatible almost complex structure, then we can say something about type decomposition of $\mathcal{R}$. Choosing I is equivalent to choosing a direct sum splitting $H = H^{1,0} + H^{0,1}$ of H into two complex lines, so that $j_H H^{1,0} = H^{0,1}$, $T^{1,0}M = LE^*H^{1,0}$ and $T^{0,1} = LE^*H^{0,1}$. Now the inclusion $H \hookrightarrow S^2H \otimes H$ is given by $h \mapsto h^2 \otimes \tilde{h} - h \vee \tilde{h} \otimes h$. Thus $\mathcal{R}$ maps $H^{1,0}$ into $\Lambda^{2,0} \otimes H^{0,1} + \Lambda^{1,1} \otimes H^{1,0}$ and

$$\begin{aligned} \mathcal{R} \in \Lambda^{1,1} &\otimes (H^{1,0}H^{0,1} + H^{0,1}H^{1,0}) \\ &+ \Lambda^{2,0} \otimes H^{0,1}H^{0,1} + \Lambda^{0,2} \otimes H^{1,0}H^{1,0}. \end{aligned}$$

We have the splitting $H \otimes H = \mathbb{R} \oplus \mathfrak{sp}(1)$ with $\mathbb{R} = H^{1,0} \wedge H^{0,1}$ and $I \in H^{1,0} \vee H^{0,1}$. Thus, as $\mathcal{R} = \frac{n}{n+1}\Omega_{\mathbb{R}} - \Omega_{\mathfrak{sp}}$, we have

$$\frac{n}{n+1}\Omega_{\mathbb{R}} - \Omega_{\mathfrak{sp},I}^{\ 1} \in \Lambda^{1,1} \qquad \text{and} \qquad \Omega_{\mathfrak{sp},I}^{\ 2} \in \Lambda^{2,0} + \Lambda^{0,2}.$$

The first of these is the statement required to complete the proof of Proposition 3.4 and hence of Theorem 3.2. □

Remark 3.7. The twistor space Z of M is the sphere bundle of $\mathcal{G}$. The group $\mathbb{C}^*$ acts on $\mathcal{U}_{n/(n+1)}(M)$ by right-multiplication and the quotient is the twistor space Z. The action preserves I and so Z inherits an almost complex structure. The complex one-form $\boldsymbol{\alpha}_I^1$ on $\mathcal{U}_{n/(n+1)}(M)$ defines an I-invariant and $\mathbb{C}^*$-invariant horizontal distribution and the first part of the proof of Proposition 3.4, proves in a simple way, using only the structure equation (3.1), that the almost complex structure on Z is integrable. This should be compared with the proofs given in [11, 3].

4. Twisting and Compact Quotients

We may modify the construction in the previous section in two ways. Firstly we may twist by an instanton and secondly we may divide out by a fibrewise action of the integers in such a way that the fibres of the bundle become compact and the hypercomplex structure now admits an action of $U(2)$ as in Theorem 2.1. Both of these ideas are due to Joyce [6].

Let M be a quaternionic manifold and suppose that $P' \to M$ is an $\mathbb{R}_{>0}$-bundle which is self-dual, i.e. P' comes equipped with a connection ω, whose curvature $\Omega = d\omega$ lies in the subbundle $L^{-2}S^2E$ of Λ^2T^*M, see [11]. This implies that Ω is of type $(1,1)$ with respect to each of the compatible almost complex structures on M. Now we may construct the bundle

$$\mathcal{P}' = P' \times_{\mathbb{R}_{>0}} \mathcal{U}_{n/(n+1)}(M),$$

where $\mathbb{R}_{>0}$ acts on $\mathcal{U}_{n/(n+1)}(M)$ via scaling on the second factor of $\tilde{F} \times \mathbb{H}$. Writing

$$\boldsymbol{\alpha}' := dx + x\omega + \frac{n}{n+1} x\omega_{\mathbb{R}} - x\omega_{\mathfrak{sp}},$$

we may proceed exactly as in the previous section to construct an almost hypercomplex structure on $\mathcal{P}'$. We get integrability of the resulting complex structures since if we choose s as in Proposition 3.4 and with $s^*\omega = 0$, then $s^*d\boldsymbol{\alpha}' = s^*(\Omega + \frac{n}{n+1}\Omega_{\mathbb{R}} - \Omega_{\mathfrak{sp}})$ which still has no $(0,2)$-component. Thus we have proved:

Lemma 4.1. *The twist $\mathcal{P}'$ of $\mathcal{U}_{n/(n+1)}(M)$ by an $\mathbb{R}_{>0}$-instanton P' admits a hypercomplex structure.*

Note that $\mathcal{P}'$ admits an action of $\mathbb{H}^*$ by left-multiplication on $L^{n/(n+1)}H$. The action of $Sp(1) \leqslant \mathbb{H}^*$ permutes the complex structures I, J and K, whilst the action of $\mathbb{R}_{>0} \leqslant \mathbb{H}^*$ preserves them. Dividing out by the action of $\mathbb{Z} \leqslant \mathbb{R}_{>0} \leqslant \mathbb{H}^*$, gives a hypercomplex manifold which is a $S^1 \times SO(3)$-bundle over M (since $U(2)/\{\pm 1\} \cong S^1 \times SO(3)$). We define

$$\mathcal{V}(M) = \mathcal{U}_{n/(n+1)}(M)/\mathbb{Z}. \tag{4.1}$$

If P is a circle bundle over M which is also an instanton, we may twist $\mathcal{V}(M)$ by P, by locally passing to the universal cover P' of P and constructing $\mathcal{P}$ as the bundle $\mathcal{P}'/\mathbb{Z}$. Alternatively, the one-form $\tilde{\alpha} := \boldsymbol{\alpha}\overline{x}/|x|$ is well-defined on $\mathcal{V}(M)$ and we may define the hypercomplex structure on

$$\mathcal{P} = P \times_{S^1} \mathcal{V}(M)$$

via the one-forms $\tilde{\alpha} + \omega$.

Proposition 4.2. *If P is an S^1-instanton over M then $\mathcal{P}$ admits a hypercomplex structure and an action of $U(2)$ satisfying the hypotheses of Theorem* 2.1. □

Remark 4.3. This is a special case of [6, Theorem 2.2]. However, it is worth recording here that Joyce's theorem really only applies to *Abelian* groups G: the proof requires the Δ-action of G to commute with the Φ-action of G; this forces G to be Abelian whenever the $(\Phi \times \Psi)$-action of $G \times G$ is effective.

Let us have a brief look at the topology, and in particular the fundamental group, of $\mathcal{P}$. Topologically, $\mathcal{V}(M)$ is $S^1 \times \mathcal{S}$, where $\mathcal{S}$ is the $\mathbb{R}\mathrm{P}(3)$-bundle

$$\mathcal{S} = \tilde{F}/(\mathbb{R}_{>0} \times SL(n, \mathbb{H})).$$

The bundle $\mathcal{S}$ is globally defined on M and has the geometrical interpretation as the bundle of triples $\{I, J, K\} \subset \mathcal{G}$ satisfying the quaternion identities (2.1). When we twist with an instanton P, we have $\mathcal{P} = P \times \mathcal{S}$, as a fibrewise product.

Let us now assume that M is simply-connected, which is the case for example when M is a compact quaternionic Kähler manifold of positive scalar curvature. The obstruction to lifting $\mathcal{S}$ to an S^3-bundle is the class $\varepsilon \in H^2(M, \mathbb{Z}/2)$ defined by Marchiafava & Romani [7]. The exact homotopy sequence gives

$$\pi_2(M) \xrightarrow{\ \varepsilon\ } \pi_1(\mathbb{R}\mathrm{P}(3)) = \mathbb{Z}/2 \longrightarrow \pi_1(\mathcal{S}) \longrightarrow 1 = \pi_1(M),$$

where ε is evaluation of the class ε on elements of $\pi_2(M)$. In particular, the fundamental group of $\mathcal{S}$ satisfies

$$\pi_1(\mathcal{S}) = \begin{cases} 1, & \text{if } \varepsilon \neq 0, \\ \mathbb{Z}/2, & \text{if } \varepsilon = 0. \end{cases}$$

The fundamental group of $\mathcal{P}$ may be computed from the sequence

$$\pi_2(M) \xrightarrow{(c_1(P), \varepsilon)} \pi_1(S^1 \times \mathbb{R}\mathrm{P}(3)) = \mathbb{Z} \oplus \mathbb{Z}/2 \longrightarrow \pi_1(\mathcal{P}) \longrightarrow 1 = \pi_1(M), \tag{4.2}$$

where $c_1(P)$ is the connecting homomorphism in the sequence

$$\pi_2(M) \xrightarrow{c_1(P)} \pi_1(S^1) = \mathbb{Z} \longrightarrow \pi_1(P) \longrightarrow 1 = \pi_1(M) \tag{4.3}$$

given by evaluating the first Chern class of P on elements of $\pi_2(M)$ (see [12]).

The circle bundle P is *primitive* if the map $c_1(P)$ is surjective. The sequence (4.3) shows P is primitive if and only if P is simply-connected. From (4.2), for P primitive and M simply-connected, the twist $\mathcal{P}$ is simply-connected if and only if the map

$$(c_1(P), \varepsilon)\colon \pi_2(M) \to \mathbb{Z} \oplus \mathbb{Z}/2$$

is surjective.

Example 4.4. Let M be $\mathbb{C}\mathrm{P}(2)$, which is self-dual and hence quaternionic. Then $\pi_2(M) = \mathbb{Z}$ and $\varepsilon = w_2 \neq 0$. Let P be the primitive instanton bundle, i.e. a primitive bundle with self-dual connection, for which the map $c_1(P)$ is the identity; for one such choice, the total space of P is S^5. Then $(c_1(P), \varepsilon)$ is the map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2$ such that $1 \mapsto (1, 1)$. This is not surjective and $\pi_1(\mathcal{P}) = \mathbb{Z}/2$. We see later that $\mathcal{P}$ is a $\mathbb{Z}/2$-quotient of $SU(3)$, which corrects a statement in [6].

Example 4.5. Let M be $2\,\mathbb{C}\mathrm{P}(2) = \mathbb{C}\mathrm{P}(2)\#\,\mathbb{C}\mathrm{P}(2)$. This is self-dual [10] and has $\pi_2(M) = \mathbb{Z}^2$. The map ε is given by $(a, b) \mapsto a + b \pmod 2$. Suppose $c_1(P)$ maps (a, b) to $n_1a + n_2b$. Then P is primitive if and only if n_1 and n_2 are coprime.

For $(n_1, n_2) = (1, 0)$, the map $(c_1(P), \varepsilon)\colon \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2$ is surjective and the resulting space $\mathcal{P}$ is simply-connected.

For $(n_1, n_2) = (1, 1)$, the map $(c_1(P), \varepsilon)\colon \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2$ is $(a, b) \mapsto (a + b, a + b)$, which is not surjective and $\pi_1(\mathcal{P})$ is $\mathbb{Z}/2$.

Note that the self-dual part of $H^2(2\,\mathbb{C}\mathrm{P}(2))$ satisfies $H^2(2\,\mathbb{C}\mathrm{P}(2))_+ = \mathbb{R}^2$, so all these circle bundles carry instanton connections. Thus different primitive instantons over the same base space, can give rise to twisted bundles with different fundamental groups.

5. Almost Quaternionic Submanifolds

Before proceeding to the classification result, we discuss an application of the results obtained so far.

If $\overline{M}$ is an almost quaternionic manifold, we say that a submanifold $M \subset \overline{M}$ is *almost quaternionic* if TM is preserved by the elements of $\overline{\mathcal{G}}$, that is if it is preserved by compatible local almost complex structures.

Proposition 5.1. *Let N be an almost quaternionic submanifold of a hypercomplex manifold $\overline{N}$. Then N is totally geodesic with respect to the Obata connection $\overline{\nabla}$ on $\overline{N}$.*

Proof. Note that for each $\overline{I}$ on $\overline{N}$, the restriction I of $\overline{I}$ to N is integrable, so N is itself hypercomplex. Let ∇ be the Obata connection on N and define $a(X, Y) = \overline{\nabla}_XY - \nabla_XY$ on N. As $\overline{\nabla}$ and ∇ are torsion-free, a is a symmetric tensor with values in $i^*T\overline{N}$, where $i\colon N \to \overline{N}$ is the inclusion. Now

$$\begin{aligned} a(X, IY) &= \overline{\nabla}_X(IY) - \nabla_X(IY) = \overline{\nabla}_X(\overline{I}Y) - I\nabla_X(Y) = \overline{I}(\overline{\nabla}_XY - \nabla_XY) \\ &= \overline{I}a(X, Y). \end{aligned}$$

So Lemma 5.2, below, implies that a is zero and N is totally geodesic. $\square$

Lemma 5.2. *If $a \in S^2T^* \otimes T$ is a tensor with*

$$a(A, IB) = Ia(A, B), \quad a(A, JB) = Ja(A, B) \\ \text{and} \quad a(A, KB) = Ka(A, B),$$

then a is zero.

Proof. We have $a(IA, IB) = -a(A, B)$ and $a(IA, IB) = a(JKA, JKB) = a(A, B)$. □

Theorem 5.3. *Let M be an almost quaternionic submanifold of a quaternionic manifold $\overline{M}$. Then M is quaternionic and each quaternionic connection $\overline{\nabla}^{\alpha}$ on $\overline{M}$ defines a quaternionic connection ∇^{α} on M such that (M, ∇^{α}) is a totally geodesic submanifold of $(\overline{M}, \overline{\nabla}^{\alpha})$. Moreover, every quaternionic connection on M arises in this way.*

Proof. Let $i\colon M \hookrightarrow \overline{M}$ denote the inclusion. Then by the Proposition, $i^*\mathcal{V}(\overline{M})$ is a totally geodesic hypercomplex submanifold of $\mathcal{V}(\overline{M})$. Moreover the $U(2)$-action on $\mathcal{V}(\overline{M})$ restricts to a $U(2)$-action on $i^*\mathcal{V}(\overline{M})$ of the type required in Theorem 2.1. Now $i^*\mathcal{V}(\overline{M})/\,U(2)$ is precisely M and the almost quaternionic structure on the quotient agrees with that on the submanifold. Since each compatible connection on the quotient is obtained from the Obata connection on $i^*\mathcal{V}(\overline{M})$, the result follows. □

Note that the corresponding result in the metric case, for quaternionic Kähler manifolds was proved by Gray [5].

6. Classification Result

The aim of this section is to show that the above construction of twisting by an instanton essentially gives all hypercomplex manifolds N admitting an action of $U(2)$ as in Theorem 2.1.

Suppose N is as in Theorem 2.1. Choose a one-form $\alpha \in \Omega^1(N)^{U(2)}$ such that $\alpha(V) = 1$ and $\alpha(IV) = 0$. This defines a quaternionic structure on $M = N/\,U(2)$ and hence we may construct the hypercomplex manifold $\mathcal{V}(M)$ of (4.1).

First we need a way of relating the curvatures of M and N.

Lemma 6.1. *The curvature R^{α} of the quaternionic connection ∇^{α} on M is related to the curvature R of the Obata connection ∇ on N by*

$$R^{\alpha}_{X,Y}Z = \pi_*\Big(R_{A,B}C + \tfrac{1}{2}\big\{\alpha^{\mathbb{H}}(\nabla_B C)A - \alpha^{\mathbb{H}}(\nabla_A C)B - \alpha^{\mathbb{H}}([A,B])C\big\}\Big),$$

where $A = h^{\alpha}X$, $B = h^{\alpha}Y$ and $C = h^{\alpha}Z$ are the horizontal lifts, and

$$\alpha^{\mathbb{H}}(X)Y := \alpha(X)Y - \alpha(IX)IY - \alpha(JX)JY - \alpha(KX)KY.$$

Proof. Recall that $R^\alpha_{X,Y}Z = [\nabla^\alpha_X, \nabla^\alpha_Y]Z - \nabla^\alpha_{[X,Y]}Z$, and that ∇^α is defined by equation (2.2). Hence $h^\alpha\nabla^\alpha_X Y = \pi^\alpha_{\mathcal{H}}(\nabla_{h^\alpha X} h^\alpha Y)$, where

$$\pi^\alpha_{\mathcal{H}}(W) = W - \alpha^{\mathbb{H}}(W)V.$$

It is now straightforward to calculate $h^\alpha R^\alpha$ and hence obtain the desired result. □

Corollary 6.2. *The lift to N of the curvature of ∇^α acting on the bundle of volume forms on M is*

$$\tfrac{n+1}{2}d\alpha + \operatorname{Re}\operatorname{Tr}^{\mathbb{H}} R.$$

Proof. Let $\{e_1, \dots, e_n\}$ be a quaternionic basis for T_xM. Then the horizontal lift of the required curvature is

$$\begin{aligned}\operatorname{Re}\operatorname{Tr}^{\mathbb{H}}(h^\alpha R^\alpha_{e_a,e_b}) &= \sum_{i=1}^n \operatorname{Re}(h^\alpha R^\alpha_{e_a,e_b}e_i)_{e_i}\\ &= \operatorname{Re}\operatorname{Tr}^{\mathbb{H}} R_{e_a,e_b} - (R_{e_a,e_b}V)_V\\ &\qquad + \tfrac{n}{2}d\alpha(e_a,e_b) + \tfrac12\alpha(\nabla_{e_b}e_a - \nabla_{e_a}e_b)\\ &= \operatorname{Re}\operatorname{Tr}^{\mathbb{H}} R_{e_a,e_b} + \tfrac{n+1}{2}d\alpha(e_a,e_b),\end{aligned}$$

where $(\cdot)_W$ denotes the W-component and we use Proposition 2.3. □

Let $F_{\mathbb{R}}$ be the principal $\mathbb{R}_{>0}$-bundle $F/\,SL(n,\mathbb{H})\,Sp(1)$ over M. As the fibres are contractible, this bundle has global sections.

Let $s\colon M \to F_{\mathbb{R}}$ be a section. Write $\mathcal{V}$ for $\mathcal{V}(M)$, the compact hypercomplex manifold assoctiated to M in (4.1). We denote the complex structures on $\mathcal{V}$ by $I_{\mathcal{V}}$, $J_{\mathcal{V}}$ and $K_{\mathcal{V}}$ and write $V_{\mathcal{V}}$ for the central generator of the $U(2)$-action on $\mathcal{V}$. We claim that there is a one-form $\beta \in \Omega^1(\mathcal{V})^{U(2)}$ with $\beta(V_{\mathcal{V}}) = 1$ and $\beta(I_{\mathcal{V}}V_{\mathcal{V}}) = 0$ such that the induced quaternionic connection ∇^β has $s^*\omega_{\mathbb{R}} = 0$, i.e. such that s is parallel. Choose any $\beta_0 \in \Omega^1(\mathcal{V})^{U(2)}$ with $\beta_0(V_{\mathcal{V}}) = 1$ and $\beta_0(I_{\mathcal{V}}V_{\mathcal{V}}) = 0$ and let $\omega^0_{\mathbb{R}}$ be the connection one-form on $F_{\mathbb{R}}$. Then by (3.5), we have $s^*\omega_{\mathbb{R}} = 0$ if we set $\beta = \beta_0 - (2n/(n+1))\pi^*s^*\omega^0_{\mathbb{R}}$. Note that choosing such a section s is equivalent to fixing a volume form on M and making a choice of reduction of the holonomy group to $SL(n,\mathbb{H})\,Sp(1)$.

Now modify the choice of α on N so that $\nabla^\alpha = \nabla^\beta$. Then s is parallel with respect to ∇^α. Our aim is to show that α induces an instanton connection on a circle bundle over M.

Lemma 6.3. *All the $U(2)$-orbits on N are isomorphic.*

Proof. The hypothesis that $L_{IV}J = K$ guarantees that V is nowhere vanishing. The tangent space to each orbit is spanned by V, IV, JV and KV, so each $U(2)$-orbit is four-dimensional and the stabilisers are finite. Let $\mathcal{O} = U(2)/H$ be an orbit.

Fix $x \in \mathcal{O}$ with stabiliser H and let W be the fibre $T_xN/T_x\mathcal{O}$ of the normal bundle. Then $\mathcal{O}$ has a tubular neighbourhood U equivariantly isomorphic to $U(2)\times_H W$ and $M = N/\,U(2)$ is locally diffeomorphic to W/H. However, the quotient map is a submersion. We claim that this implies that the action of H on W is trivial.

As H is compact, the representation $W \otimes \mathbb{C}$ is unitary for an appropriate choice of Hermitian inner product. Thus $h \in H$ acts on $W \otimes \mathbb{C}$ via a diagonalisable linear tranformation. Let $w \in W \otimes \mathbb{C}$ be an eigenvector with eigenvalue λ. Then w and λw have the same image in $(W/H) \otimes \mathbb{C}$. However the map $W \to W/H$ is a linear isomorphism, since H is finite and the quotient map $N \to M$ is a submersion. This implies $\lambda = 1$ and hence h acts as the identity on W.

As the action of H on W is trivial, we have all the orbit types in U are $U(2)/H$, which implies $\mathcal{O}$ is a principal orbit. Thus all $U(2)$-orbits on N are principal and there is only one orbit type. □

Proposition 6.4. *If α on N defines a quaternionic connection ∇^α on M with a parallel volume form, then α induces a connection on $P = N/\,SU(2)$ which is self-dual.*

Proof. That P is a circle bundle over M, follows from Lemma 6.3. Invariance of α under the action of $SU(2)$, together the with properties of α on V and IV imply that α pushes forward to a connection one-form for the S^1-bundle $N/\,SU(2) \to M$. It is now sufficient to show that $d\alpha$ is of type $(1,1)$ with respect to each complex structure on N. This follows from Corollary 6.2 and the following lemma. □

Lemma 6.5. *The curvature R of a hypercomplex manifold X satisfies*

$$R_{IA,IB} = R_{A,B},$$

for all tangent vectors A, B and each compatible complex structure I.

Proof. Write $T_{\mathbb{C}}X \cong \boldsymbol{E}^* \otimes \boldsymbol{H}$ as we did for a quaternionic manifold, but now with $\boldsymbol{E}$ a representation of $GL(n,\mathbb{H})$ instead of $SL(n,\mathbb{H})$ (as in [11]). The first prolongation $\mathfrak{gl}(n,\mathbb{H})^{(1)}$ is zero, so the curvature R lies in the kernel of the map

$$\mathfrak{gl}(n,\mathbb{H}) \otimes \Lambda^2 T^* \xrightarrow{\partial} T \otimes \Lambda^3 T^*,$$

where ∂ is the inclusion $\mathfrak{gl}(n,\mathbb{H}) \cong \boldsymbol{E}^* \otimes \boldsymbol{E} \hookrightarrow \mathrm{End}(TX) \cong T \otimes T^*$ followed by the alternation map. Salamon [11] shows that $\ker \partial \cong S^2\boldsymbol{E} \oplus \boldsymbol{U}$, where $\boldsymbol{U}$ is the irreducible representation satisfying $\boldsymbol{E}^* \otimes S^3\boldsymbol{E} \cong S^2\boldsymbol{E} + \boldsymbol{U}$. These summands only occur in the subspace $\boldsymbol{E}^* \otimes \boldsymbol{E} \otimes S^2\boldsymbol{E}$ of $\mathfrak{gl}(n,\mathbb{H}) \otimes \Lambda^2 T^* \cong \mathfrak{gl}(n,\mathbb{H}) \otimes (S^2\boldsymbol{E} + \Lambda^2\boldsymbol{E}S^2\boldsymbol{H})$. But $S^2\boldsymbol{E} \subset \Lambda^2 T^*$ consists precisely of all those two-forms of type $(1,1)$ with respect to each compatible complex structure I, so $R_{IA,IB} = R_{A,B}$. □

Having examined $P = N/\,SU(2)$, let us now look at $N/\,U(1)$. Over M we have the $\mathbb{R}\mathrm{P}(3)$-bundle $\mathcal{S}$ defined in (4). This is a principal $SO(3)$-bundle and comes equipped with the connection one-form $\omega_{\mathfrak{sp}}$. We define a map $\phi_\alpha \colon N \to \mathcal{S}$ by

$$\phi_\alpha(n) = (\pi_* I h_n^\alpha, \pi_* J h_n^\alpha, \pi_* K h_n^\alpha).$$

The map ϕ_α is smooth, surjective and $U(1)$-invariant, so induces a diffeomorphism $\phi\colon N/\,U(1) \to \mathcal{S}$.

The manifold $N/\,U(1)$ carries a connection induced by $\alpha^{\mathbb{H}}$. This agrees with the connection one-form $\omega_{\mathfrak{sp}}$ coming from ∇^α, because for a local section I of $\mathcal{G}$ coming from $aI + bJ + cK$ on N, we have

$$(\nabla^\alpha_X I)Y = \pi_*\big((da \otimes I + db \otimes J + dc \otimes K)(\tilde{X} - \alpha^{\mathbb{H}}(\tilde{X}), h^\alpha Y)\big),$$

by equation (2.3), for any lift $\tilde{X}$ of X. Moreover, ϕ_α preserves the almost complex structures on the horizontal distributions of $N/\,U(1)$ and $\mathcal{S}$, where the latter are defined tautologically. The vector fields IV, JV, KV are mapped under ϕ_α to generators of the principal $SO(3)$-action on $\mathcal{S}$. The images $V(I)$, $V(J)$, $V(K)$ are characterised by being vertical vector fields satisfying the relations $L_{V(I)}I = 0$, $L_{V(I)}J = K$, etc., for the almost complex structures on the horizontal distribution of $\mathcal{S}$.

We now have a surjection from N to the fibrewise product $P \times \mathcal{S}$ given by $n \mapsto (\psi(n), \phi_\alpha(n))$, where $\psi\colon N \to P = N/\,U(1)$ is the projection. This map is a local diffeomorphism and $U(2)$-equivariant if we let $U(2)$ act on $P \times \mathcal{S}$ via the action of $U(1) \times SO(3)$. The space $P \times \mathcal{S}$ has an almost hypercomplex structure defined tautologically on the horizontal distribution $\ker(\omega_{\mathfrak{sp}} + \omega)$ and via the canonical generators (V on P and $V(I), V(J), V(K)$ on $\mathcal{S}$) on the orbits of the $U(2)$-action (by $IV = V(I)$, $IV(J) = V(K)$, etc.). The map from N to $P \times \mathcal{S}$ preserves the almost hypercomplex structure. Examining this map on the $U(2)$-orbits shows that it is either one-to-one or is bijective.

We may replace N in the above argument by $\mathcal{P}$ and obtain a hypercomplex diffeomorphism with $P \times \mathcal{S}$. This proves:

Theorem 6.6. *If N is a connected hypercomplex manifold with an action of $U(2)$ satisfying the hypotheses of Theorem* 2.1*, then up to a double cover N is the twist $\mathcal{P}$ of $\mathcal{V}(N/\,U(2))$ by an instanton P.* □

7. Examples

In the following, we shall describe algebraically the topological twists over $\mathbb{CP}(2)$ and over the connected sum $\mathbb{CP}(2) \# \mathbb{CP}(2)$ by primitive instantons. The first case is partly included as preparation for the second and partly to show that the twisting construction can only construct examples up to double covers.

7.1. **Complex Projective Space.** Let $V = \mathbb{C}^{n+1}$ be a complex $(n+1)$-dimensional vector space with a Hermitian inner product and suppose that $n \geqslant 2$. The Grassmannian $\mathrm{Gr}_2(V)$ of two-planes in V is a quaternionic Kähler manifold [14]. Its twistor space Z is a divisor of bi-degree $(1,1)$ in $\mathbb{P}(V) \times \mathbb{P}(V^*)$. Algebraically,

$$Z = \{\, (v, \ell) \in \mathbb{P}(V) \times \mathbb{P}(V^*) : \ell(v) = 0 \,\}.$$

The twistor projection $Z \to \mathrm{Gr}_2(V)$ maps (v, ℓ) to the linear span of v and the orthogonal complement of the kernel of the linear functional ℓ. We claim that the associated bundle $\mathcal{U}(\mathrm{Gr}_2(V))$ of $\mathrm{Gr}_2(V)$ is

$$\mathcal{U}(\mathrm{Gr}_2(V)) = \{\, v \otimes \ell \in V \otimes V^* : v \otimes \ell \neq 0,\ \ell(v) = 0 \,\}.$$

This may be seen as follows. The total space of the associated bundle is the $(n+1)$st root $K^{1/(n+1)}$ of the canonical bundle of the twistor space Z with the zero-section removed. In this case, this bundle is the bundle with bidegree $(-1, -1)$. It is the pull-back of the tautological line bundle on the projectivisation of $V \otimes V^*$ by the Segré embedding. With the zero section removed, the total space of the tautological line bundle is $V \otimes V^*$ with the origin removed, therefore, $\mathcal{U}(\mathrm{Gr}_2(V))$ is indeed as described. In [13], this is described as the minimal nilpotent orbit in $\mathfrak{sl}(n+1, \mathbb{C})$.

Up to orientation, there is only one primitive S^1-instanton bundle over $\mathrm{Gr}_2(V)$. Let P denote the pull-back of this bundle to the twistor space. We claim that the corresponding holomorphic $\mathbb{C}^*$-bundle P^c is the principal bundle of $\mathcal{O}(1, -1)$ or equivalently $\mathcal{O}(-1, 1)$. The bundle $\mathcal{O}(1, -1)$ is trivial on twistor lines, the fibres of the projection $Z \to \mathrm{Gr}_2(V)$. So by the Ward correspondence [2, 4], $\mathcal{O}(1, -1)$ defines an instanton on $\mathrm{Gr}_2(V)$. This instanton is primitive, since the restriction of $\mathcal{O}(1, -1)$ to a line $\mathbb{C}\mathrm{P}(1) \times \{*\} \subset Z \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$ is isomorphic to $\mathcal{O}(1)$, which has no roots. Now the total space of $\mathcal{O}(1, -1)$ is $S \setminus (\mathbb{P}(V) \cup \mathbb{P}(V^*))$, where

$$S = \{\, [v, \ell] \in \mathbb{P}(V \oplus V^*) : \ell(v) = 0 \,\}.$$

We may thus identify P as

$$P = \{\, [v, \ell] \in \mathbb{P}(V \oplus V^*) : \ell(v) = 0,\ |v| = |\ell| \,\}.$$

Let r be a positive number, define a $\mathbb{Z}$-action on $\mathcal{U}(\mathrm{Gr}_2(V))$ by $(n, v \otimes \ell) \mapsto \exp(rn) v \otimes \ell$ and write $\mathcal{V}$ for the quotient of $\mathcal{U}(\mathrm{Gr}_2(V))$ by this action. Then the $\mathbb{R}$-action $(\theta, v \otimes \ell) \to \exp(r\theta/2\pi) v \otimes \ell$ on $\mathcal{U}(\mathrm{Gr}_2(V))$ descends to define a $U(1)$-action on $\mathcal{V}$. Define

$$\mathcal{D} = \{\, v \otimes \ell \in \mathcal{U} : 1 < |v \otimes \ell| \leqslant e^r \,\},$$

which is a fundamental domain for the $\mathbb{Z}$-action. If ϖ is the projection from $\mathcal{V}$ to Z, then $\varpi^* P$ is diffeomorphic to

$$\big\{\, ([v, \ell], w) \in \mathbb{P}(V \oplus V^*) \times \mathcal{P} : v \in \operatorname{Im} w,\ \ker \ell = \ker w,\ \ell(v) = 0,\ |v| = |\ell| \,\big\}.$$

The twist $\mathcal{P}$ is $(\varpi^* P)/\Delta$, where the $U(1)$-action Δ is defined by

$$\Delta(e^{i\theta}, ([v, \ell], w)) \longmapsto \big([e^{i\theta} v, \ell], e^{r\theta/2\pi} w\big).$$

The group $SU(n+1)$ acts on $\varpi^* P$ via

$$A \cdot v = Av, \qquad A \cdot \ell = \ell A^*, \qquad A \cdot w = AwA^*,$$

for $A \in SU(n+1)$. This action is transitive on $(\varpi^*P)/\Delta$.

Let $n=2$. Consider the map $\mathbb{P}(V\oplus V^*)\times\mathcal{D}\to U(3)$ defined by

$$\big(w,[v,\ell]\big)\longmapsto\left(\frac{v}{|v|},\frac{\overline{\ell}}{|\ell|},\frac{v\times\overline{\ell}}{|v\times\overline{\ell}|}\right).$$

Under the Δ-action of $U(1)$, each orbit contains an element of $SU(3)$. The group $U(1)$ acts on the right-hand side by $(a,b,c)\mapsto(e^{i\theta}a,b,e^{i\theta}c)$. Thus each $U(1)$-orbit meets $SU(3)$ in precisely two points. We now have an $SU(3)$-equivariant surjection $(\varpi^*P)/\Delta\to U(3)/\,U(1)=SU(3)/(\mathbb{Z}/2)$. Counting dimensions and noting that $(\varpi^*P)/\Delta$ has fundamental group $\mathbb{Z}/2$, as calculated in Example 4.4, shows that the map $(\varpi^*P)/\Delta\to SU(3)/(\mathbb{Z}/2)$ is a diffeomorphism.

Thus $SU(3)/(\mathbb{Z}/2)$ admits a hypercomplex structure. This lifts to the universal cover $SU(3)$. Moreover, $SU(3)$ fibres over $\mathbb{C}\mathrm{P}(2)$ with fibre $U(2)$. The projection is the natural quotient map from $SU(3)$ to $SU(3)/\,U(2)$. In terms of the above coordinates, it is the projection

$$\left(\frac{v}{|v|},\frac{\overline{\ell}}{|\ell|},\frac{v\times\overline{\ell}}{|v\times\overline{\ell}|}\right)\mapsto[v\times\overline{\ell}].$$

Thus $SU(3)$ carries a hypercomplex structure with $U(2)$-action, but only the quotient $SU(3)/(\mathbb{Z}/2)$ can be constructed via the twisting construction.

7.2. **The Connected Sum of Two Projective Spaces.** When $M=2\,\mathbb{C}\mathrm{P}(2)$, it is also possible to describe the twist of $\mathcal{V}$ by a primitive instanton.

To construct $\mathcal{U}(2\,\mathbb{C}\mathrm{P}(2))$, recall that the twistor space Z of $2\,\mathbb{C}\mathrm{P}(2)$ is a small resolution of an intersection Z^0 of two quadrics in $\mathbb{C}\mathrm{P}(5)$ [10]. Let $z=[z_0,\dots,z_5]$ be homogeneous coordinates on $\mathbb{C}\mathrm{P}(5)$, then $Z^0=Q_{-1}\cap Q_{-2}$, where

$$\begin{aligned} Q_{-1}&=\left\{\, z\in\mathbb{C}\mathrm{P}(5): q_{-1}=z_0^2+z_1^2+(\lambda-1)z_2^2+\tfrac12 z_3^2=0\,\right\},\\ Q_{-2}&=\left\{\, z\in\mathbb{C}\mathrm{P}(5): q_{-2}=(2-\lambda)z_2^2+\tfrac12 z_3^2+z_4^2+z_5^2=0\,\right\}\end{aligned}\tag{7.1}$$

and λ is any real number between $3/2$ and 2. The real structure is $\sigma(z)=[\overline{z}_0,\overline{z}_1,\overline{z}_2,-\overline{z}_3,-\overline{z}_4,-\overline{z}_5]$. The intersection Z^0 is singular at four ordinary double points.

Let $s\in(0,1)$ be the number such that $\lambda=(2+s^2)/(1+s^2)$. We consider the following collection of homogeneous functions of degree 1 on $\mathbb{C}\mathrm{P}(5)$:

$$w_0=z_0+iz_1,\qquad w_1=z_0-iz_1,\qquad w_2=z_4+iz_5,\qquad w_3=-z_4+iz_5.$$

$$u_4=-i\frac{s}{\sqrt{1+s^2}}z_2+\frac{1}{\sqrt2}z_3,\qquad u_5=i\frac{s}{\sqrt{1+s^2}}z_2+\frac{1}{\sqrt2}z_3,$$

$$v_4=i\frac{1}{\sqrt{1+s^2}}z_2+\frac{1}{\sqrt2}z_3,\qquad v_5=i\frac{1}{\sqrt{1+s^2}}z_2-\frac{1}{\sqrt2}z_3.$$

The real small resolution from Z^0 to the twistor space Z is a subspace of $Z^0\times\mathbb{C}\mathrm{P}(1)\times\mathbb{C}\mathrm{P}(1)\times\mathbb{C}\mathrm{P}(1)\times\mathbb{C}\mathrm{P}(1)$. With respect to homogeneous coordinates $(z,\alpha,\beta,$

$\gamma, \delta)$, where $\alpha = [\alpha_1, \alpha_2]$, etc., the resolution is given by

$$\begin{aligned} \alpha_1 u_4 = \alpha_2 w_2, \quad \alpha_1 w_3 = \alpha_2 u_5, \quad \beta_1 u_4 = \beta_2 w_3, \quad \beta_1 w_2 = \beta_2 u_5, \\ \gamma_1 v_4 = \gamma_2 w_0, \quad \gamma_1 w_1 = \gamma_2 v_5, \quad \delta_1 v_4 = \delta_2 w_1, \quad \delta_1 w_0 = \delta_2 v_5. \end{aligned} \tag{7.2}$$

Recall that the associated bundle $\mathcal{U}$ is the bundle $K^{\frac{1}{2}}$ with the zero-section removed. Since the associated meromorphic map defined by section of the bundle $K^{-\frac{1}{2}}$ on the twistor space Z is precisely the resolution of the singular intersection $Q_{-1} \cap Q_{-2}$ [10], the bundle $K^{\frac{1}{2}}$ is the pull-back of the tautological line bundle on $\mathbb{C}\mathrm{P}(5)$. As the total space of the tautological line bundle with the zero-section removed is $\mathbb{C}^6 \setminus \{0\}$, the total space of $\mathcal{U}$ can be described as the subspace of $\mathbb{C}^6 \setminus \{0\} \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1)$ defined by (7.1) and (7.2). The $\mathbb{C}^*$-action on $\mathcal{U}$ is induced by the action of scalar multiplication on $\mathbb{C}^6$. For any positive real number r, define maps $\mathbb{Z} \to \mathbb{C}^*$ and $U(1) \to \mathbb{C}^*$ by $n \mapsto e^{rn}$ and $e^{i\theta} \mapsto e^{r\theta/2\pi}$ respectively. Then $U(1)$ acts on $\mathcal{V} = \mathcal{U}/\mathbb{Z}$ via the $\mathbb{C}^*$-action.

Note that $w_2 = u_4 = 0$ and $w_3 = u_5 = 0$ define two elements in a pencil of effective divisors on Z. Denote the line bundle associated to these divisors by $\mathcal{D}_{11}$. Let $\mathcal{D}_{-1-1}$ be its conjugate bundle. Then $\mathcal{D}_{11}$ and $\mathcal{D}_{-1-1}$ are the tautological bundles for the first and second $\mathbb{C}\mathrm{P}(1)$-factors in the small resolution. The tautological bundles for the third and fourth $\mathbb{C}\mathrm{P}(1)$-factors in the small resolution are denoted by $\mathcal{D}_{1-1}$ and $\mathcal{D}_{-11}$ respectively. In particular,

$$\begin{aligned} \mathcal{D}_{11} &= \{ (z, \alpha, \beta, \gamma, \delta) \times \zeta : \alpha_1 \zeta_2 = \alpha_2 \zeta_1 \}, \\ \mathcal{D}_{1-1} &= \{ (z, \alpha, \beta, \gamma, \delta) \times \xi : \gamma_1 \xi_2 = \gamma_2 \xi_1 \}. \end{aligned}$$

From [10], one sees that $\mathcal{D}_{11}^{-1} \otimes \mathcal{D}_{1-1}$ is the associated line bundle of the pull-back of a primitive $U(1)$-instanton bundle on M. Let $[\xi_1, \xi_2, \zeta_1, \zeta_2]$ be homogeneous coordinates on $\mathbb{C}\mathrm{P}(3)$. Let W be $\mathbb{C}\mathrm{P}(3)$ with the lines $\zeta_1 = \zeta_2 = 0$ and $\xi_1 = \xi_2 = 0$ removed. Then the total space of the principal $\mathbb{C}^*$-bundle of $\mathcal{D}_{11}^{-1} \otimes \mathcal{D}_{1-1}$ is the subspace of $Z \times W$ defined by the equations

$$\alpha_1 \zeta_2 = \alpha_2 \zeta_1, \qquad \gamma_1 \xi_2 = \gamma_2 \xi_1. \tag{7.3}$$

The total space of the principal $U(1)$-bundle P is defined by the additional equation

$$|\xi_1|^2 + |\xi_2|^2 = |\zeta_1|^2 + |\zeta_2|^2. \tag{7.4}$$

Let π be the projection from $\mathcal{U}$ onto the twistor space Z. By definition, $\pi^* P$ is a subspace of $\mathcal{U} \times W$ defined by the equations (7.3) and (7.4). The given $\mathbb{Z}$-action defines an isomorphism from $(\mathbb{C}^6 \setminus \{0\})/\mathbb{Z}$ to $S^1 \times S^{11}$. Then $\mathcal{V}$ is a subspace of $S^1 \times S^{11} \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1)$ and the $U(1)$-action on $\mathcal{V}$ is rotation of the S^1-factor by the angle θ.

Note that $\pi^* P$ descends to a bundle on $\mathcal{V}$. Then the total space of $\pi^* P$ over $\mathcal{V}$ is a subspace of $\mathcal{V} \times W$. For the diagonal action Δ on $\pi^* P$ as a bundle over $\mathcal{V}$, the action of $e^{i\theta} \in U(1)$ is

$$(e^{i(t+\theta)}, \frac{z}{|z|}, \alpha, \beta, \gamma, \delta, [e^{i\theta}\xi, \zeta]).$$

The projection $(e^{it}, [\xi, \zeta]) \mapsto [e^{-it}\xi, \zeta]$ is a surjection from $S^1 \times W$ onto W with fibre $(e^{i(t+\theta)}, [e^{i\theta}\xi, \zeta])$ for all $\theta \in [0, 2\pi)$. Therefore, the topological twist $\mathcal{P} = \pi^* P/\Delta$ is a subspace of

$$S^{11} \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1) \times \mathbb{C}\mathrm{P}(1) \times W.$$

In the coordinates $(z, \alpha, \beta, \gamma, \delta, [\xi, \zeta])$, it is defined by the equations

(7.1), the equations of intersection of two quadrics;
(7.2), the equations defining the small resolutions;
(7.3), the equations defining the principal $\mathbb{C}^*$-bundle of a primitive bundle;
(7.4), the equations defining the principal $U(1)$-bundle.

Note that this describes one of the complex structures on $\mathcal{P}$.

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