Shooting methods and topological transversality

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Abstract. We show that shooting methods for homoclinic or heteroclinic orbits in dynamical systems may automatically guarantee the topological transversality of the stable and unstable manifolds. The interest of such kind of results is twofold. First these orbits persist under perturbations which destroy the structure allowing the shooting method and, second, topological transversality is often sufficient when some kind of transversality is required to obtain chaotic dynamics. We shall focus on heteroclinic solutions in the Extended Fisher-Kolmogorov equation.

1 Introduction

Shooting methods are quite efficient to study numerically homoclinic and heteroclinic solutions of ODEs (see e.g. [CS]). Often, some kind of transversality of the stable and unstable manifolds is required to prove that a homoclinic or heteroclinic orbit leads to chaotic behaviour. However, as observed by Mischaikov and Mrozek [MM3], estimates on the derivative along the solution are needed to establish transversality and they may demand a considerable amount of computing resources. Mischaikov and Mrozek introduce in [MM1,2,3] a topological condition that is weaker than transversality and more amenable to computation. Their theory is based on the Conley index and is applied to the Lorenz equation.

Alternatively, for some special ODEs, a shooting method has been developed that not only proves the existence of a connecting orbit, but also gives an infinite family of chaotic solutions. A precise and complete description of the dynamics of those ODEs is possible in this way, and many physically relevant equations have been successfully investigated (see e.g. [T1,Ha,CT,BT,Bu,PT1-3]).

For Hamiltonian and Lagrangian systems, the Calculus of Variations has been the main ingredient of a third approach, which has raised a lot of interest in recent years (see e.g. [Bo,Se,CR,Be]). Here again, a first connecting orbit is found and a much weaker assumption than transversality has been proved to be sufficient in many cases for the system to be chaotic. Some results are available for PDEs [CR].

Thanks to collaboration with A.R. Champneys, J.F. Toland and É. Séré on the following fourth-order equation occurring in water-wave theory

$$u^{(iv)}(x) + Pu''(x) + u(x) - u^2(x) = 0, \;\; x \in \mathbb{R}, \tag{1}$$

the author got acquainted with the shooting method and the variational approach [BCT,BGT,BS,BT]. The aim of this paper is to show that shooting methods ensure a weak form of transversality, called here "topological transversality", that is defined in the Appendix.

In Section 2, we relax the classical transversality condition in the Smale-Birkhoff Theorem for discrete dynamical systems and, in Section 3, in the theory of Devaney for autonomous fourth-order Hamiltonian systems with a saddle-focus equilibrium. In both cases, we prove that a topological transversality assumption is sufficient to ensure the dynamics to be chaotic, without requiring the analyticity of the dynamical system. Although we only deal with homoclinic orbits, our results extend to heteroclinic orbits without additional difficulties. The statements and their proofs are inspired by those developed in the Calculus of Variations.

Instead of equation (1), we shall focus in Sections 4-6 on the Extended Fisher-Kolmogorov equation (EFK)

$$-\gamma u^{(iv)}(x) + u''(x) + u(x) - u^3(x) = 0, \;\; x \in \mathbb{R},$$

which has been analysed by shooting methods by Peletier and Troy [PT1-3] and by variational methods by Kalies, Kwapisz and VanderVorst [KV,KKV]. In contrast with (1) where homoclinic solutions to 0 describe travelling waves, the objects of consideration in (EFK) are heteroclinic solutions (also called kinks) connecting the equilibria $u \equiv -1$ and $u \equiv 1$. Heteroclinic orbits being easier to deal with in the present context, this is why (EFK), rather than equation (1), is chosen as example to illustrate the ideas of this paper. The existence of a kink for (EFK) has been established in [PT2,PT3].

In Section 5, we show the existence of a kink for (EFK) when $0 < \gamma \leq 1/8$ that is the topologically transverse intersection of the stable and unstable manifolds. The techniques used stem from the work of Hofer and Toland [HT] and they are reminiscent of the index for monotonous Hamiltonian systems devised by Toland in [T2].

In Section 6, we consider (EFK) in the case $\gamma > 1/8$ and deduce the known fact [PT1,KV,KKV] that the Extended Fisher-Kolmogorov equation is chaotic for this range of values of the parameter γ. The proofs can be adapted to equation (1) and partially to higher-dimensional problems of the kind introduced in [HT].

This paper was mainly written when the author visited the Department of Mathematics of the University of Leiden (September 1996), whose hospitality is gratefully acknowledged. There, Prof. L.A. Peletier asked whether topological transversality may be used in proving the existence of slow travelling kinks for the Extended Fisher-Kolmogorov equation (see Section 7).

2 The Smale-Birkhoff Homoclinic Theorem

This theorem reads as follows (version taken from [GH]).

Theorem 1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that p is a hyperbolic fixed point and there exists a point $q \neq p$ of (algebraic) transversal intersection between $W^s(p)$ and $W^u(p)$. Then f has a hyperbolic invariant set on which f is topologically equivalent to a subshift of finite type.*

$W^s(p)$ and $W^u(p)$ denote respectively the stable and unstable manifolds of p. Instead of algebraic transversality, we now suppose topological transversality. See the Appendix for the meaning of "compact piece of manifold" and for the definition of the topological intersection number $I(M_1, M_2)$ of two compact pieces of manifolds M_1 and M_2.

Theorem 2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that p is a hyperbolic fixed point and there exist a compact piece M_u of $W^u(p)$ and a compact piece*

M_s of $W^s(p)$ such that $p \notin N := M_u \cap M_s$, $N \subset$ int $(M_u) \cap$ int (M_s) and $I(M_u, M_s) \neq 0$. Then there exist a bounded neighbourhood U_N of N, a small neighbourhood U_p of p whose closure does not intersect $\overline{U_N}$, and an integer $\tilde{m} \geq 2$ such that for each choice of $k \geq 1$ and

$$(m_1, \ldots, m_k) \in \{m \in \mathbb{N} : m \geq \tilde{m}\}^k$$

1. *there exists $(q_1, \ldots, q_{k+1}) \in U_N^{k+1}$ with $q_1 \in M_u$, $q_{i+1} = f^{m_i}(q_i)$ for $1 \leq i \leq k$, $q_{k+1} \in M_s$ and $f^{[i/2]}(q_i) \in U_p$ for $1 \leq i \leq k$;*

2. *there exists $(q_1, \ldots, q_{k+1}) \in U_N^{k+1}$ with $q_{i+1} = f^{m_i}(q_i)$ for $1 \leq i \leq k$, $q_{k+1} = q_1$ and $f^{[i/2]}(q_i) \in U_p$ for $1 \leq i \leq k$;*

moreover for each bi-infinite sequence $(m_i, i \in \mathbb{Z}) \subset \{m \in \mathbb{N} : m \geq \tilde{m}\}$ there exists $(q_i, i \in \mathbb{Z}) \subset \overline{U_N}$ with $q_{i+1} = f^{m_i}(q_i)$ and $f^{[i/2]}(q_i) \in \overline{U_p}$ for all $i \in \mathbb{Z}$.

Note that N can be a very complicated set. This result yields a rich family of homoclinic and periodic orbits, as well as an uncountable set of wild orbits. In particular, there exists a compact invariant set on which f has positive topological entropy [GH, def. 5.8.3]. This theorem does not ensure any uniqueness result. For example there may be many homoclinic orbits corresponding to a given sequence $(m_1, \ldots, m_k)$. This particular concept of chaos has been introduced by É. Séré [Se].

Proof. For $\tilde{m}$ sufficiently large and each $m \geq \tilde{m}$, the λ-lemma [PM] gives an injective immersion $\lambda_m \in C^1(M_s \times M_u, \mathbb{R}^{2n})$ such that

1. if $(c, d) \in \Lambda_m :=$ Image (λ_m), then $d = f^m(c)$ and $f^{[m/2]}(c) \in U_p$,

2. $\lim_{m \to \infty} \sup\{$ dist $(a, \lambda_m(a)) : a \in M_s \times M_u\} = 0$.

Let us give more details on how the mapping λ_m is obtained. We introduce two C^1-families of discs $(D_a^u, a \in M_s)$ and $(D_b^s, b \in M_u)$ transversal respectively to M_s and M_u and such that no disc meets another disc of the same family. By the λ-lemma, $f^{m_1}(D_a^u)$ and $f^{-m_2}(D_b^s)$ intersect each other transversally near p for large m_1 and m_2 at some point $q(a, b)$ which belongs to the same component of $f^{m_1}(D_a^u) \cap U_p$ as $f^{m_1}(a)$ and to the same component of $f^{-m_2}(D_b^s) \cap U_p$ as $f^{-m_2}(b)$. By compactness of M_s and M_u, this can be done uniformly in $(a, b) \in M_s \times M_u$. We set

$$\lambda_{m_1+m_2}(a, b) = \left(f^{-m_1}(q(a, b)), f^{m_2}(q(a, b))\right).$$

To prove the existence of the rich family of orbits, choose any $(m_1, \ldots, m_k) \in \{m \in \mathbb{N} : m \geq \tilde{m}\}^k$. First suppose that $k = 2l$ for some positive integer l. Homoclinic orbits corresponding to $(m_1, \ldots, m_k)$ are given by the points in the intersection of

$$\Lambda_{m_1} \times \ldots \times \Lambda_{m_{2i-1}} \times \ldots \times \Lambda_{m_{2l-1}} \times M_s$$

and

$$M_u \times \Lambda_{m_2} \times \ldots \times \Lambda_{m_{2i}} \times \ldots \times \Lambda_{m_{2l}}$$

in $\mathbb{R}^{(k+1)n}$. For $\tilde{m}$ large, these sets are admissible for intersection theory and a homotopy gives that their intersection number is equal to

$$I(\ (M_s \times M_u) \times \ldots \times (M_s \times M_u) \times M_s\ ,\ M_u \times (M_s \times M_u) \times \ldots \times (M_s \times M_u)\ )$$

$$= I(M_u, M_s)^{k+1} \neq 0.$$

Periodic orbits are obtained in the same way by considering the intersection of

$$\sigma(\Lambda_{m_1} \times \ldots \times \Lambda_{m_{2i-1}} \times \ldots \times \Lambda_{m_{2l-1}})$$

and

$$\Lambda_{m_2} \times \ldots \times \Lambda_{m_{2i}} \times \ldots \times \Lambda_{m_{2l}},$$

where σ is the permutation $\sigma(q_1, \ldots, q_r) = (q_2, \ldots, q_r, q_1)$, with $q_i \in \mathbb{R}^n$ for all i. Here, $r = 2l = k$.

Second suppose that $k = 2l + 1$ for some non-negative integer l. Homoclinic orbits corresponding to $(m_1, \ldots, m_k)$ are given by the points in the intersection of

$$\Lambda_{m_1} \times \ldots \times \Lambda_{m_{2i+1}} \times \ldots \times \Lambda_{m_{2l+1}}$$

and

$$M_u \times \Lambda_{m_2} \times \ldots \times \Lambda_{m_{2i}} \times \ldots \times \Lambda_{m_{2l}} \times M_s$$

in $\mathbb{R}^{(k+1)n}$. For $\tilde{m}$ large, these sets are admissible for intersection theory and a homotopy gives that their intersection number is equal to

$$I((M_s \times M_u) \times \ldots \times (M_s \times M_u), M_u \times (M_s \times M_u) \times \ldots \times (M_s \times M_u) \times M_s)$$

$$= I(M_u, M_s)^{k+1} \neq 0.$$

Periodic orbits are obtained by considering the intersection of

$$\sigma(\Lambda_{m_1} \times \ldots \times \Lambda_{m_{2i+1}} \times \ldots \times \Lambda_{m_{2l+1}})$$

and

$$\Lambda_{m_2} \times \ldots \times \Lambda_{m_{2i}} \times \ldots \times \Lambda_{m_{2l}} \times \Delta_{R^{2n}},$$

where $\Delta_{R^{2n}} \subset \mathbb{R}^{2n}$ is the diagonal.

The final assertion of the theorem is proved by taking limits of homoclinic orbits, as follows. Consider a bi-infinite sequence $(m_i, i \in \mathbb{Z}) \subset \{m \in \mathbb{N} : m \geq \tilde{m}\}$. For $r \geq 1$ and $(m_{-r}, \ldots, m_{r-1})$, let $(q_{-r}^{(r)}, \ldots, q_r^{(r)}) \in U_N^{2r+1}$ be given by the first part of the theorem. Considering a subsequence if necessary, the limit $q_0 := \lim_{r\to\infty} q_0^{(r)}$ exists. Set $q_i := f^{m_i}(q_0) = \lim_{r\to\infty} q_i^{(r)}$ for $i \in \mathbb{Z}$. ∎

3 Devaney's Theorem

We consider the Hamiltonian system

$$z' = J\nabla H(z), \ z \in \mathbb{R}^4, \tag{2}$$

where J is the standard symplectic matrix and $H \in C^5(\mathbb{R}^4;\mathbb{R})$. We suppose that $H(0) = 0$, $\nabla H(0) = 0$ and $0 \in \mathbb{R}^4$ to be a saddle-focus, i.e. the spectrum of $JH''(0)$ consists of $\pm\lambda \pm i\omega$ for some positive λ and ω. The following result is due to Devaney [D].

Theorem 3 *If H is analytic and $W^u(0)$ and $W^s(0)$ intersect each other transversally in $H_0 := \{z \in \mathbb{R}^4 : H(z) = 0\}$ along a (non-trivial) homoclinic orbit, then a Smale horseshoe is "contained" in the dynamics.*

Again our aim is to weaken the transversality assumption. It is known that the analyticity of H is not necessary [W] and we shall only require H to be of class C^5.

We suppose that some bounded open set $0 \notin U \subset \mathbb{R}^4$ admits a parametrisation $\psi : Y \times I_1 \times I_2 \to U$ such that

1. I_1 and I_2 are open intervals containing 0;

2. Y is a boundaryless and orientable C^1-manifold of dimension 2;

3. ψ is a diffeomorphism;

4. $(H \circ \psi)(y, h, s) = h$ on $Y \times I_1 \times I_2$;

5. under the flow, $ds/dt > 0$ on U;

6. there exist two compact pieces of manifolds, M_u and M_s, that are connected, included in Y, of dimension 1, admissible for intersection theory, and such that $\{\psi(y,0,0) : y \in M_{u/s}\} \subset W^{u/s}$, $M_u \neq M_s$ and $I(M_u, M_s) \neq 0$.

The parameter s describes a family of Poincaré sections of the flow. In what follows, we shall restrict attention to the Poincaré section $\{\psi(y,h,s) : s = 0\}$. Moreover $\Gamma(t, z_0)$ will denote the solution of (2) such that $\Gamma(0, z_0) = z_0$, and which is defined on some maximal open time-interval containing $t = 0$.

Theorem 4 *There exist a constant $\tau_1 > 0$, a constant $\tau_2 \in (\tau_1, \tau_1 + 4\pi/\omega)$ and a small neighbourhood $V \subset \mathbb{R}^4$ of 0 whose closure does not intersect $\overline{U}$, such that for each choice of $k \in \mathbb{N}$ and $(m_1, \ldots, m_k) \in \mathbb{N}^k$*

1. there exists $(q_1, \ldots, q_{k+1}) \in Y^{k+1}$ with $q_1 \in M_u$, $q_{k+1} \in M_s$ and, for all $1 \leq i \leq k$,

$$\left.\begin{array}{l} \psi(q_{i+1}, h, 0) = \Gamma\left(T, \psi(q_i, h, 0)\right) \ \textit{and}\ \Gamma\left(T/2, \psi(q_i, h, 0)\right) \in V \\ \textit{for some } h \in I_1 \textit{ and } T \in \mathbb{R} \textit{ satisfying} \\ T \in \left[\tau_1 + (m_i - 1/2)\pi/\omega, \tau_2 + (m_i + 1/2)\pi/\omega\right]; \end{array}\right\} \qquad (3)$$

2. there exists $(q_1, \ldots, q_{k+1}) \in Y^{k+1}$ such that $q_1 = q_{k+1}$ and relation (3) holds for all $1 \leq i \leq k$.

Moreover for each bi-infinite sequence $(m_i, i \in \mathbb{Z}) \subset \mathbb{N}$ there exists a sequence $(q_i, i \in \mathbb{Z}) \subset Y$ such that relation (3) holds for all $i \in \mathbb{Z}$.

Finally, for each of these three families, all orbits are distinct.

Proof. For T_0 sufficiently large, the λ-lemma for continuous dynamical systems [PM, chapter 2, lemma 7.2] gives a C^1-map $\lambda :]T_0, \infty[\times M_s \times M_u \to Y^2 \times I_1$ such that

1. if $(c, d, h) \in$ Image $(\lambda(T, ., .))$, then $\Gamma(T, \psi(c, h, 0)) = \psi(d, h, 0)$;

2. $\lim_{T\to\infty} \sup\{ \text{ dist } (a,c), \text{ dist } (b,d), |h| : a \in M_s, b \in M_u, (c,d,h) = \lambda(T,a,b)\} = 0$.

As in the previous section, the map λ is obtained by introducing two C^1-families of 2-dimensional discs in $Y \times I_1$, $(D^u_a, a \in M_s)$ and $(D^s_b, b \in M_u)$, transversal respectively to $M_s \times \{0\}$ and $M_u \times \{0\}$ and such that no disc meets another disc of the same family. By the λ-lemma, $\Gamma\left(T_1, \psi(D^u_a, 0)\right)$ and $\Gamma\left(-T_2, \psi(D^s_b, 0)\right)$ intersect each other transversally near 0 for large $T_1, T_2 > 0$ at some point $q(a,b) \in \mathbb{R}^4$ which belongs to the same component of $\Gamma\left(T_1, \psi(D^u_a, 0)\right) \cap V$ as $\Gamma(T_1, \psi(a,0,0))$ and to the same component of $\Gamma\left(-T_2, \psi(D^s_b, 0)\right) \cap V$ as $\Gamma(-T_2, \psi(b,0,0))$. By compactness of M_s and M_u, this can be done uniformly in $(a,b) \in M_s \times M_u$. Then $\lambda(T_1+T_2, a, b) = (c,d,h)$ if and only if (by definition) $\psi(c,h,0) = \Gamma(-T_1, q(a,b))$ and $\psi(d,h,0) = \Gamma(T_2, q(a,b))$.

Set $\mathcal{H}(c,d,h) := h$. It is proved in [BS, lemma 11] that $(\mathcal{H} \circ \lambda)(T,a)$ oscillates around 0 as a function of $T >> 0$, for fixed $a \in M_s \times M_u$, roughly like

$$\exp(-\lambda T)\sin(\omega T - \omega\tau(a))$$

for some shift $\tau(a) \in \mathbb{R}$ continuous in $a \in M_s \times M_u$. Clearly, by adding some constant to the shift-function τ, we can assume, without loss of the generality, that $\tau(a) \in]T_0, \infty[$ for all $a \in M_s \times M_u$ and any given T_0. As a consequence, for all $\sigma \in \pi/(2\omega) + (\pi/\omega)\mathbb{Z}_+$,

$$\inf\{\sin(\omega\sigma) \times (\mathcal{H} \circ \lambda)(\tau(a) + \sigma, a) : a \in M_s \times M_u\} > 0$$

if $T_0 > 0$ is large enough. For $m \in \mathbb{N}$, we set

$$\Lambda_m = \{(c,h,d,0) : (c,d,h) = \lambda(\tau(a)+\sigma, a), a \in M_u \times M_s, |\sigma - m\pi/\omega| \leq \pi/(2\omega)\}.$$

If T_0 is large enough, $\Lambda_m \subset (Y \times I_1)^2$ is a compact piece of a C^1-submanifold of dimension 3. Note that the function τ and the constants τ_1, τ_2 appearing in the statement of the theorem are related by $\tau_1 = \min\{\tau(a) : a \in M_s \times M_u\}$ and $\tau_2 = \max\{\tau(a) : a \in M_s \times M_u\}$.

Choose $(m_1, \ldots, m_k) \in \mathbb{N}^k$. First suppose that $k = 2l$ for some positive integer l. Homoclinic orbits corresponding to $(m_1, \ldots, m_k)$ are given by the points in the intersection of

$$\Lambda_{m_1} \times \ldots \times \Lambda_{m_{2i-1}} \times \ldots \times \Lambda_{m_{2l-1}} \times (M_s \times [-1,1])$$

and

$$(M_u \times \{0\}) \times \Lambda_{m_2} \times \ldots \times \Lambda_{m_{2i}} \times \ldots \times \Lambda_{m_{2l}}$$

in $(Y \times I_1)^{k+1}$. These sets are admissible for intersection theory and a homotopy gives that their intersection number is equal to the one of

$$(M_s \times [-1,1] \times M_u \times \{0\}) \times \ldots \times (M_s \times [-1,1] \times M_u \times \{0\}) \times (M_s \times [-1,1])$$

and

$$(M_u \times \{0\}) \times (M_s \times [-1,1] \times M_u \times \{0\}) \times \ldots \times (M_s \times [-1,1] \times M_u \times \{0\}),$$

which is equal to $\pm I(M_u, M_s)^{k+1} \neq 0$.

For k odd, for periodic orbits and for other wild orbits, the method is the same. See the preceding section. ■

4 The EFK equation

The *Extended Fisher-Kolmogorov equation* (EFK) is given by

$$-\gamma u^{(iv)}(t) + u''(t) + u(t) - u^3(t) = 0,$$

where $\gamma > 0$ is a parameter. Setting $V(u, u'') = (1/4)(u^2-1)^2 - (\gamma/2)(u'')^2$ and $T = \begin{pmatrix} -1 & \gamma \\ \gamma & 0 \end{pmatrix}$, it may be written as the Lagrangian system

$$Tq'' + \nabla V(q) = 0$$

with $q = (u, u'')$, referred to as (EFK'). This Lagrangian system belongs to the family studied in [HT]. Its corresponding Hamiltonian is

$$H(q, q') = \frac{1}{2}\langle Tq', q' \rangle + V(q).$$

Equation (EFK) admits three equilibria: $u \equiv -1$, $u \equiv 0$ and $u \equiv 1$. We shall be interested in heteroclinic connections (also called kinks) between -1 and 1. For such solutions, the Hamiltonian takes the value 0 and hence $H_0 := \{z \in \mathbb{R}^4 : H(z) = 0\}$ will play a particular role, as well as $V_0 := \{q \in \mathbb{R}^2 : V(q) = 0\}$.

The Extended Fisher Kolmogorov equation enjoys many symmetries. If $(q(t), q'(t))$ is an orbit, then so are $(q(-t), -q'(-t))$, $(-q(t), -q'(t))$ and $(-q(-t), q'(-t))$. These symmetries are sometimes used in a essential way and sometimes only to keep at a low level the amount of notations or to shorten an argument.

For any orbit (q, q') lying in H_0, i.e. $\frac{1}{2}\langle Tq', q'\rangle + V(q) = 0$, the sign of $\langle Tq', q'\rangle$ changes only when q crosses the set V_0. Hence the vector q' stays in one of the following open cones as long as q remains away from V_0:

$$K_0 = \{v \in \mathbb{R}^2 : \langle Tv, v\rangle < 0, v_1 > 0\} = \{v \in \mathbb{R}^2 : -v_1 + 2\gamma v_2 < 0, v_1 > 0\},$$

$$K_1 = \{v \in \mathbb{R}^2 : \langle Tv, v\rangle > 0, v_1 > 0\} = \{v \in \mathbb{R}^2 : -v_1 + 2\gamma v_2 > 0, v_1 > 0\},$$

$$K_2 = \{v \in \mathbb{R}^2 : \langle Tv, v\rangle < 0, v_1 < 0\} = \{v \in \mathbb{R}^2 : -v_1 + 2\gamma v_2 > 0, v_1 < 0\},$$

$$K_3 = \{v \in \mathbb{R}^2 : \langle Tv, v\rangle > 0, v_1 < 0\} = \{v \in \mathbb{R}^2 : -v_1 + 2\gamma v_2 < 0, v_1 < 0\}.$$

Note that the slope of $\partial K_0 \cap \partial K_1$ is $1/(2\gamma)$ and that $\partial K_0 \cap \partial K_3$ is vertical. More precisely, if $V(q) > 0$, then q' stays in $K_0 \cup K_2$, and if $V(q) < 0$, then q' stays in $K_1 \cup K_3$. As a consequence, the *path* $q(t)$ of an *orbit* $(q(t), q'(t))$ in H_0 is monotonous as long as q does not meet V_0. This remark, due in a more general setting to Hofer and Toland [HT], is the basic ingredient of the shooting method. The following lemma describes the set V_0.

Lemma 5 *$V_0 \subset \mathbb{R}^2$ is the union of two C^∞-curves intersecting each other at $(-1, 0)$ and $(1, 0)$. The derivatives of these curves at the intersection points are $\pm\sqrt{2/\gamma}$. The set $\mathcal{C} := \{q \in \mathbb{R}^2 : V(q) > 0, -1 < q_1 < 1\}$ is strictly convex. Finally, if $0 < \gamma \leq 1/8$, then*

$$\mathcal{C} \subset ((-1, 0) + K_0) \cap ((1, 0) + K_2).$$

PROOF. From $V(q_1, q_2) = 0$, we find $q_2 = \pm(2\gamma)^{-1/2}(q_1^2 - 1)$. We deduce that V_0 is the union of two smooth curves which intersect each other at $(\pm 1, 0)$. It is also easy to compute their slopes at their intersection points. Let us show that $\langle V''\nabla V^\perp, \nabla V^\perp\rangle < 0$ at every $q \in \partial\mathcal{C}\backslash\{(1, 0), (-1, 0)\}$. We have $\nabla V(q) = (q_1(q_1^2 - 1), -\gamma q_2)$ and $\nabla V(q)^\perp = (\gamma q_2, q_1(q_1^2 - 1))$. Hence

$$\langle V''\nabla V^\perp, \nabla V^\perp\rangle = \langle \begin{pmatrix} 3q_1^2 - 1 & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} q_2\gamma \\ q_1(q_1^2 - 1) \end{pmatrix}, \begin{pmatrix} q_2\gamma \\ q_1(q_1^2 - 1) \end{pmatrix}\rangle$$

$$
\begin{aligned}
&= \gamma^2(3q_1^2-1)q_2^2 - \gamma q_1^2(q_1^2-1)^2 \\
&= \gamma^2(3q_1^2-1)q_2^2 - 2\gamma^2 q_1^2 q_2^2 \\
&= (q_1^2-1)\gamma^2 q_2^2 \\
&< 0.
\end{aligned}
$$

■

Setting $u = -1 + w$, (EFK) becomes

$$-\gamma w^{(iv)}(t) + w''(t) + w(t)(w(t)-1)(2-w(t)) = 0,$$

the linearisation of which is $-\gamma w^{(iv)}(t) + w''(t) - 2w(t) = 0$. Solutions of the form $w(t) = \exp \lambda t$ are given by $\lambda^2 = (1 \pm \sqrt{1-8\gamma})/(2\gamma)$. We deduce that $u \equiv -1$ is a real saddle if $0 < \gamma \leq 1/8$ and a saddle-focus if $\gamma > 1/8$. We denote by $W^u(-1)$ and $W^u_{loc}(-1)$ its global and local unstable manifolds. By symmetry, we obtain the same result for $u \equiv 1$ and we denote by $W^s(1)$ and $W^s_{loc}(1)$ its global and local stable manifolds.

5 The case $\gamma \leq 1/8$

Throughout this section, we assume that $0 < \gamma \leq 1/8$. As a consequence of the monotonicity property, if $(q(t), q'(t))$ is an orbit in H_0 such that $q(0) \in \mathcal{C}$ and $q'(0) \in K_0$, then either $lim_{t\to+\infty} q(t) = (1,0)$ or $q(t)$ has to leave $\mathcal{C}$ at some positive *exit time* $\tau^+(q(0), q'(0))$ continuous in $(q(0), q'(0))$. In the second case, $q'(\tau^+) \neq 0$ belongs to $\partial K_0 \cap \partial K_1$ or $\partial K_0 \cap \partial K_3$ if, respectively, $q_2(\tau^+) > 0$ or $q_2(\tau^+) < 0$. Analogously, either $lim_{t\to-\infty} q(t) = (-1,0)$ or $q(t)$ had to enter $\mathcal{C}$ at some negative *entrance time* $\tau^-(q(0), q'(0))$ continuous in $(q(0), q'(0))$. Also, in the second case, $q'(\tau^-) \neq 0$ belongs to $\partial K_0 \cap \partial K_1$ or $\partial K_0 \cap \partial K_3$ if, respectively, $q_2(\tau^-) < 0$ or $q_2(\tau^-) > 0$ (this is different from the previous situation!).

In the following lemma, $\phi(t, q, v) \in \mathbb{R}^2$ denotes the solution of (EFK') satisfying $\phi(0, q, v) = q$ and $\phi'(0, q, v) = v$ (derivative with respect to time). Note that the maximal time-interval where $\phi(t, v, q)$ is defined is not necessarily all $\mathbb{R}$.

Lemma 6 *Let $\gamma \in (0, 1/8]$ be fixed. For small $\epsilon > 0$ and all $(q, v) \in W^u_{loc}(-1)$ such that $q \in \overline{\mathcal{C}}$ and $q_1 = -1 + \epsilon$, we have $v \in \overline{K_0} \backslash \{0\}$ and $\phi(t, q, v) \in \mathcal{C}$*

for all negative t*. Moreover to each* $q \in \overline{\mathcal{C}}$ *with* $q_1 = -1 + \epsilon$ *corresponds a unique* $v \in \overline{K_0}$ *such that* $(q, v) \in W^u_{loc}(-1)$*. If, in addition,* $q \in \partial\mathcal{C}$*, then* $v \in \partial K_0 \cap \partial K_1$ *when* $q_2 > 0$ *and* $v \in \partial K_0 \cap \partial K_3$ *when* $q_2 < 0$*.*

PROOF. This lemma is a consequence of the facts that $W^u_{loc}(-1)$ is tangent to the unstable manifold of the linearised problem around $u = -1$, for which explicit computations can be made, and the monotonicity property of paths. For example, let us prove that $\tau^-(q, v) = -\infty$. Ad absurdum, suppose $\tau^-(q, v)$ finite and $\phi_2(\tau^-, q, v) > 0$, say. Then $\phi'(\tau^-, q, v) \in \partial K_0 \cap \partial K_3$ and $\phi'(t, q, v) \in K_3$ for $t < \tau^-$ near τ^-. We can suppose that $\tilde{q}_1 < 0$ for all $(\tilde{q}, \tilde{v}) \in W^u_{loc}(-1)$ and, in particular, that $\phi_1(t, q, v) < 0$ for all $t \leq 0$. A picture shows that $\phi(t, q, v) \notin V_0$ and $\phi'(t, q, v) \in K_3$ for all $t < \tau^-$. Hence $\lim_{t \to -\infty} |\phi(t, q, v)| = \infty$. Contradiction. ∎

We choose $\epsilon > 0$ small enough and set

$$\mathcal{A} := \{(q, v) \in W^u_{loc}(-1) : q \in \overline{\mathcal{C}}, q_1 = -1 + \epsilon\}.$$

Let $q(t)$ be a solution of (EFK') with zero Hamiltonian and such that $q(0) \in \overline{\mathcal{C}}$ and $q'(0) \in \overline{K_0}$. For $-1 < q_1(0) \leq x < 1$, $q(t)$ has to leave the region $\{w \in \overline{\mathcal{C}} : w_1 \leq x\}$ at some finite exit time $\tau^+_x(q(0), q'(0)) \geq 0$ that is continuous in $(q(0), q'(0))$. Using the notations $z = (q(0), q'(0))$ and

$$\Gamma_x(z) = (\phi, \phi')(\tau^+_x(z), z),$$

we set $M := \Gamma_0(\mathcal{A}) \subset W^u(-1)$. The image of M under the symmetry $S(q, v) = (-q, v) \in \mathbb{R}^4$ clearly satisfies $SM \subset W^s(1)$.

Theorem 7 *Let* $\gamma \in (0, 1/8]$ *be fixed. The sets* M *and* SM *have a compact intersection included in*

$$Y := \{(q, v) \in H_0 : q \in \mathcal{C}, v \in K_0, q_1 = 0\}.$$

Restricting M *and* SM *to an appropriate closed subset of* Y*, the restrictions are compact, smooth, orientable manifolds with boundary that are embedded in* Y *and admissible for intersection theory. Their orientations are given by the images by the flow of orientations on* $\mathcal{A}$ *and* $S\mathcal{A}$*. Their intersection number, denoted by* $I(M, SM)$*, is equal to* ± 1*.*

Remark. The symmetry has only been introduced for the sake of simplicity.

PROOF. Let $\tilde{S}:\mathbb{R}^4 \to \mathbb{R}^4$ be the symmetry $\tilde{S}(q_1,q_2,v_1,v_2)=(q_1,-q_2,v_1,v_2)$. Note that SM and $\tilde{S}M$ coincide on Y. Set

$$F_x = \{q \in \partial\mathcal{C} : -1+\epsilon \le q_1 \le x\} \cup \{q \in \mathcal{C} : q_1 = x\}.$$

F_x is homeomorphic to the segment $[-1,1]$. Let $\alpha_x : F_x \to [-1,1]$ be a homeomorphism. We may suppose α_x to be continuous both in x and q and differentiable if $q \in \mathcal{C}$. Finally, we define

$$\beta_x : \begin{cases} \left(F_x \times \overline{K_0}\right) \cap H_0 \to [-1,1] \times \mathbb{R} \\ \beta_x(q,v) = (\alpha_x(q), v_2) \end{cases},$$

which is also a homeomorphism and is differentiable in q and x if $q \in \mathcal{C}$. The domains of the maps $\beta_x \circ \Gamma_x$ and $\beta_x \circ \tilde{S} \circ \Gamma_x$ are taken to be $\mathcal{A}$ and their ranges are included in $\mathbb{R}^2$. We obtain

$$\begin{aligned} I(M,SM) &= I(\Gamma_0(\mathcal{A}), S\Gamma_0(\mathcal{A})) \\ &= I(\Gamma_0(\mathcal{A}), \tilde{S}\Gamma_0(\mathcal{A})) \\ &= \pm I(\beta_0 \circ \Gamma_0, \beta_0 \circ \tilde{S} \circ \Gamma_0) \\ &= \pm I(\beta_x \circ \Gamma_x, \beta_x \circ \tilde{S} \circ \Gamma_x) \end{aligned}$$

for all $x \in [-1+\epsilon, 0]$, because the parameter x describes a homotopy whose admissibility results from $\Gamma_x(\partial\mathcal{A}) = \partial\mathcal{A}$, $(\tilde{S} \circ \Gamma_x)(\partial\mathcal{A}) = \tilde{S}(\partial\mathcal{A})$ and $\partial A \cap \tilde{S}(\partial\mathcal{A}) = \emptyset$. Hence

$$\begin{aligned} I(M,SM) &= \pm I(\beta_{-1+\epsilon}, \beta_{-1+\epsilon} \circ \tilde{S}) \\ &= \pm I(\beta_{-1+\epsilon}(\mathcal{A}), \beta_{-1+\epsilon}(\tilde{S}\mathcal{A})) \\ (&= I(\mathcal{A}, \tilde{S}\mathcal{A}) \;\;). \end{aligned}$$

There is a smooth transformation of $\mathbb{R}^2$ which changes $\beta_{-1+\epsilon}(\mathcal{A})$ and $\beta_{-1+\epsilon}(\tilde{S}\mathcal{A})$ into $\{(a,a) : |a| \le 1\}$ and $\{(-a,a) : |a| \le 1\}$, without the boundaries being involved in the intersection during the process. We deduce $I(M,SM) = \pm 1$. Incidentally, we also obtain the relation $I(M,SM) = I(\mathcal{A}, \tilde{S}\mathcal{A})$, the ambient manifold for the left-hand member being Y and the one for the right-hand member $\{(q,v) \in H_0 : q_1 = -1+\epsilon\}$. ■

Corollary 8 *There exists $\epsilon > 0$ such that for all $1/8 < \gamma < 1/8 + \epsilon$, (EFK) is chaotic and admits infinitely many kinks connecting -1 to 1.*

PROOF. The sets M and SM still intersect each other in Y for $\gamma > 1/8$ near $1/8$ thanks to the topological transversality of the intersection for $\gamma = 1/8$. Thus (EFK) admits a kink from $u \equiv -1$ to $u \equiv 1$ and the reversibility of (EFK) gives another one from $u \equiv 1$ to $u \equiv -1$. Using the fact that $u \equiv -1$ and $u \equiv 1$ are saddle-foci for $\gamma > 1/8$ and arguing as in Section 3, we can prove that (EFK) is chaotic. ∎

This last corollary is of the same flavour of a similar result known for the water-wave equation (1) (see [CT]) and is a contribution to the understanding of the chaotic dynamics of (EFK) (see [PT1,KV,KKV]).

6 The case $\gamma > 1/8$

Throughout this section, we assume that $\gamma > 1/8$. An infinite family of kinks is found in [PT3] by a shooting method, starting from the invariant set $S_0 := \{(q, v) \in \mathbb{R}^4 : q = 0\}$ of the symmetry $S(q, v) = (-q, v)$. There are two difficulties when trying to prove some kind of transversality. First, the stable and unstable manifolds do not play an explicit role in the shooting method and, second, the kinks are no more monotonous. To cope with these difficulties, we shall use the analyticity of (EFK) and rely on the Hopf Degree Theorem. More specifically, we shall need the following result.

Proposition 9 *Let M_1 and M_2 be two compact pieces of manifolds embedded in an orientable and boundaryless manifold Y such that $\dim M_1 = \dim M_2 = \dim Y/2$ and $M_1 \cap M_2 = \{y\}$. We denote the inclusion mapping of M_1 in Y by $j \in C^1(M_1, Y)$. If M_1 and M_2 are admissible for intersection theory and $I(M_1, M_2) = 0$, then there exists an embedding $f \in C^1(M_1, Y)$ such that $f^{-1}(M_2) = \emptyset$, f may be chosen to be as near to j in $C(M_1, Y)$ as necessary and to be equal to j outside an open set of M_1 that is as small as necessary.*

PROOF. Let $\dim Y = 2m$. Restricting ourselves to smaller compact pieces of M_1 and M_2 and using local coordinates $(y_1, y_2) \in \mathbb{R}^{2m}$ with $y_i \in \mathbb{R}^m$ for $i = 1, 2$, we may suppose without loss of generality that $Y = \mathbb{R}^{2m}$, $y = 0$, $M_1 = \{(y_1, y_2) : y_1 \in \bar{B}_\epsilon, y_2 = 0\}$ and $M_2 = \{(y_1, g(y_1)) : y_1 \in \bar{B}_\epsilon\}$ for some small $\epsilon > 0$ and some map $g \in C^1(\bar{B}_\epsilon, \mathbb{R}^m)$ such that $g(0) = 0$. We use the notation $\bar{B}_\epsilon := \{y \in \mathbb{R}^m : ||y|| \leq \epsilon\}$.

The definition of the topological degree of a map gives

$$\deg(g, 0, \bar{B}_\epsilon) = \pm I(M_1, M_2) = 0.$$

By the Hopf Degree Theorem [FG,PM], there exists a map $\tilde{g} \in C^1(\bar{B}_\epsilon, \mathbb{R}^m)$ which never takes the value 0, which is as close as necessary to g in $C(\bar{B}_\epsilon, \mathbb{R}^m)$ and which is equal to g outside an open set that is as small as necessary. ∎

Following [PT3], for $\alpha \neq 0$, we denote by $u(., \alpha)$ the solution of (EFK) such that $u(0, \alpha) = u''(0, \alpha) = 0$, $u'(0, \alpha) = \alpha$ and $u'''(0, \alpha) = (2\alpha\gamma)^{-1}(\alpha^2 - 1/2)$. Clearly $(u, u', u'', u''')(0, \alpha)$ is in $S_0 \cap H_0$. We use the notation $z = (u, u', u'', u''')$. Inspecting [PT3], we obtain the following result, which is based on a shooting method. Let us recall that $W^s_T(1)$ is the set of points in $\mathbb{R}^4$ that are sent by the flow in $W^s_{loc}(1)$ after a period of time T.

Theorem 10 *There exist $T > 0$ and a sequence of compact intervals $([a_i, b_i], i \in \mathbb{N})$ such that $0 < a_i < a_{i+1} < b_{i+1} < b_i$ $(i \in N)$ and which satisfy the following properties.*

1. *Kink: for all $\alpha \in \cap_{i \in \mathbb{N}}[a_i, b_i]$, the solution $z(0, \alpha)$ is in $W^s_T(1)$.*
2. *Periodic solutions: in each interval $[a_i, b_i]$, there exists α such that $z(., \alpha)$ is a periodic orbit.*
3. *Persistence under perturbation: if $\gamma \in C([a_i, b_i], \mathbb{R}^4)$ is such that $\gamma(a_i) = z(0, a_i)$, $\gamma(b_i) = z(0, b_i)$ and γ is sufficiently close to $z(0, .)$ in $C([a_i, b_i], \mathbb{R}^4)$, then there exists $\alpha \in [a_i, b_i]$ such that $\gamma(\alpha) \in W^s_T(1)$.*

We now prove the following theorem.

Theorem 11 *We have $\lim a_i = \lim b_i := \alpha_\infty$ and $z(0, \alpha_\infty)$ is a topological transverse intersection point of $W^s_T(1)$ and S_0 in*

$$Y := \{(q, v) \in H_0 : q \in \mathcal{C}, v \in K_0, q_1 = 0\}.$$

Proof. Ad absurdum, suppose that $\lim a_i \neq \lim b_i$. Denote the limits by, respectively, α_∞ and β_∞. By the first property of the preceding theorem, $S_0 \cap W^s_T(1) \cap Y$ contains the 1-manifold $\{z(0, \alpha) : \alpha_\infty \leq \alpha \leq \beta_\infty\}$. As

$S_0 \cap Y$ and $W^s_T(1) \cap Y$ are analytical 1-manifolds embedded in the analytical 2-manifold Y, there exists an interval (a,b) containing $[\alpha_\infty, \beta_\infty]$ and such that $z(0,\alpha) \in S_0 \cap W^s_T(1)$ for all $\alpha \in (a,b)$. As $[a_i, b_i] \subset (a,b)$ for large $i \in \mathbb{N}$, this contradicts the second property of the preceding theorem. Therefore $\alpha_\infty = \beta_\infty$ and, again by analyticity, $z(0,\alpha_\infty)$ is an isolated point of $S_0 \cap W^s_T(1) \cap Y$.

Suppose now that the intersection is not topologically transverse and let us find a contradiction. By Proposition 9, given some small neighbourhood of $z(0,\alpha_\infty)$ in Y, it is possible to perturb slightly $S_0 \cap Y$ inside this neighbourhood in such a way that the perturbed C^1-manifold of dimension 1 does not intersect $W^s_T(1) \cap Y$ (in this neighbourhood). Hence if we start shooting from the perturbed manifold, orbits do not reach $W^s_{loc}(1)$ by time T. This contradicts the third property of the preceding theorem. ∎

The next task is to deduce the topological transversality of $W^u(-1)$ and $W^s(1)$ in Y.

Lemma 12 *Let M_1 and M_2 be two compact pieces of 1-dimensional manifolds that are embedded in $\mathbb{R}^2$, admissible for intersection theory and such that $M_1 \cap M_2 = \{0\}$. Let $\overline{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear symmetry $\overline{S}(y_1, y_2) = (y_1, -y_2)$, $y_1, y_2 \in \mathbb{R}$, and let $\overline{S}_0 = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = 0\}$ be its invariant set. If $M_2 = \overline{S} M_1$ and $I(M_1, \overline{S}_0) \neq 0$, then $I(M_1, M_2) \neq 0$.*

This lemma is inspired from Section 4.2 of [BCT].

Proof. Let us first suppose M_1 tangent to $\overline{S}_0$. We can find $h \in C^1([-\epsilon,\epsilon],\mathbb{R})$ such that $M_1 = \{(y_1, h(y_1)) : |y_1| \leq \epsilon\}$ (restricting M_1 to a smaller piece if necessary). We obtain $M_2 = \{(y_1, -h(y_1)) : |y_1| \leq \epsilon\}$ and

$$I(M_1, M_2) = \pm \deg(2h, [-\epsilon,\epsilon], 0) = \pm \deg(h, [-\epsilon,\epsilon], 0) = \pm I(M_1, \overline{S}_0) \neq 0.$$

Second, suppose that M_1 is not tangent to $\overline{S}_0$. Then we can find $h \in C^1([-\epsilon,\epsilon],\mathbb{R})$ such that $M_1 = \{(h(y_2), y_2) : |y_2| \leq \epsilon\}$ (restricting M_1 to a smaller piece if necessary). We obtain $M_2 = \{(h(-y_2), y_2) : |y_2| \leq \epsilon\}$ and $I(M_1, M_2) = \deg(h(x) - h(-x), [-\epsilon,\epsilon], 0) = \pm 1$ (the map is odd). ∎

Theorem 13 *There exists a kink which is a topologically transverse intersection point of $W^u(-1) \cap Y$ and $W^s(1) \cap Y$ in*

$$Y = \{(q,v) \in H_0 : q \in \mathcal{C}, v \in K_0, q_1 = 0\}.$$

SKETCH OF PROOF. The idea is to start shooting from $W^u_{loc}(-1)$. We thus manage to obtain some topologically transverse intersection point of $W^u(-1) \cap Y$ and $W^s(1) \cap Y$ or we can prove that $z(0, \alpha_\infty)$ is the limit of points $z_n \in W^u_T(-1) \cap W^s_{\tau_n}(-1) \cap Y$ with $\lim_{n\to\infty} \tau_n = \infty$. Every point z_n belongs to a homoclinic orbit to -1 that spends a period of time near 1 that tends to infinity with n.

In the second case, by the analyticity of $W^u_T(-1) \cap Y$, $W^s_T(1) \cap Y$ and Y, $z(0, \alpha_\infty)$ is an isolated intersection point of $W^u_T(-1) \cap Y$ and $W^s_T(1) \cap Y$, otherwise $W^u_T(-1) \cap W^s_T(1) \cap Y$ would include a 1-manifold that contains $z(0, \alpha_\infty)$, in contradiction with the existence of the sequence $(z_n, n \in \mathbb{N})$. We conclude by applying proposition 12 (via some local chart) to $W^s_T(1) = SW^u_T(1)$, $W^u_T(1)$ and S_0 restricted to the two dimensional manifold Y. ■

This gives a new proof that the Extended Fisher-Kolmogorov equation is chaotic for all values of $\gamma > 1/8$ (see [PT1,KV,KKV]).

7 Travelling kinks

A more general form of the EFK equation reads as follows (see e.g. [PT2]):

$$\frac{\partial}{\partial t} u(t,x) = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + (u-a)(1-u^2), \ -1 < a < 1, \ \gamma > 0.$$

Stationary travelling waves propagating at speed $c \in \mathbb{R}$ are described by the equation

$$cu'(t) - \gamma u^{(iv)}(t) + u''(t) + (u-a)(1-u^2) = 0, \ -1 < a < 1, \ \gamma > 0 \qquad (4)$$

(we denote the independent variable by t instead of x). When $a = c = 0$, equation (4) is simply the EFK equation introduced in Section 4. Our aim is to show that topologically transverse kinks when $a = c = 0$ persist for $c \neq 0$ small.

Theorem 14 *Let $\gamma > 0$, $a = c = 0$,*

$$Y = \{(q,v) \in H_0 : q \in \mathcal{C}, v \in K_0, q_1 = 0\},$$

and $M_s \subset W^s_T(1) \cap Y$ and $M_u \subset W^u_T(-1) \cap Y$ be two compact pieces of manifolds that are admissible for intersection theory and such that $I(M_u, M_s) \neq 0$

in the manifold Y (for some $T > 0$). Then there exist $a_0 \in (0,1)$ and $c_0 > 0$ such that, for all $a \in (-a_0, a_0)$, it is possible to find $c \in (-c_0, c_0)$ and $u \in C^\infty(\mathbb{R})$ such that (4) holds and $\lim_{t\to\pm\infty} u(t) = \pm 1$. Moreover $c \neq 0$ if $a \neq 0$.

PROOF. Note that

$$H(u, u', u'', u''') := 2\gamma u'u''' - \gamma(u'')^2 - (u')^2 + \frac{1}{2}(1-u^2)^2 + 2a(u - u^3/3)$$

is an invariant for $c = 0$ and a Lyapunov function for $c \neq 0$; indeed

$$\begin{aligned} &(d/dt)H(u, u', u'', u''') \\ &= 2\left\{\gamma u^{(iv)} - u'' - (u-a)(1-u^2) - cu'\right\} u' + 2cu'(t)^2 \\ &= 2cu'(t)^2 \end{aligned} \tag{5}$$

has the same sign as c for all t and any solution $u \in C^4(\mathbb{R})$. Let Z be the 3-dimensional manifold $Z = \{(q,v) \in \mathbb{R}^4 : q \in \mathcal{C}, v \in K_0, q_1 = 0\}$, $W^s(a,c)$ be a compact piece of the manifold $W^s_T(1)$ for equation (4) and $W^u(a,c)$ be a compact piece of the manifold $W^u_T(-1)$ for equation (4) with a and c arbitrary (but small). We assume that $W^u(0,0) \cap Z = M_u$ and $W^s(0,0) \cap Z = M_s$. We also suppose that the sets $\cup_{|c|\leq c_0} W^{u/s}(a,c) \times \{c\}$ are two compact pieces of manifolds for $|a| < a_0$ and $c_0, a_0 > 0$ small enough.

As a consequence, for $|a| < a_0$, the sets

$$N_{u/s}(a) = \cup_{|c|\leq c_0} \left(W^{u/s}(a,c) \cap Z\right) \times \{c\} \subset Z \times \mathbb{R}$$

are two compact pieces of 2-dimensional manifolds. Let us show that, for $a = 0$, they are admissible for intersection theory and $I(N_u(0), N_s(0)) \neq 0$ in $Z \times \mathbb{R}$.

By equation (5), for $c > 0$, the value of H is positive on $W^u(0,c) \cap Z$ and negative on $W^s(0,c) \cap Z$. For $c < 0$, the value of H is negative on $W^u(0,c) \cap Z$ and positive on $W^s(0,c) \cap Z$. In short, the value of cH is positive on $W^u(0,c) \cap Z$ and negative on $W^s(0,c) \cap Z$ if $c \neq 0$. As a consequence, $N_u(0)$ and $N_s(0)$ intersect each other in $Z \times \{0\}$. Therefore the admissibility of $N_u(0)$ and $N_s(0)$ results from the one of M_u and M_s.

The manifold Z admits a global chard $\phi : Z \to \mathbb{R}^3$ given by $\phi(q,v,c) = (q_2, v_2, H(q,v))$. We now embed $(\phi \times I)(N_u(0))$ and $(\phi \times I)(N_s(0))$ in a continuous family of 2-manifolds in $\mathbb{R}^4$ in such a way to obtain an admissible homotopy, which we parametrise by $\theta \in [0,1]$. We set

$$\tilde{N}_u(\theta) = \{(q_2, v_2, h + \theta c, c) : (q_2, v_2, h) \in \phi\left(W^u(0, c - \theta c) \cap Z\right), |c| \le c_0\},$$

$$\tilde{N}_s(\theta) = \{(q_2, v_2, h - \theta c, c) : (q_2, v_2, h) \in \phi\left(W^s(0, c - \theta c) \cap Z\right), |c| \le c_0\}.$$

Clearly $\tilde{N}_{u/s}(0) = (\phi \times I)(N_{u/s}(0))$ and

$$\tilde{N}_u(1) = \{(q_2, v_2, c, c) : (q_2, v_2, 0) \in \phi(W^u(0,0) \cap Z), |c| \le c_0\},$$

$$\tilde{N}_s(1) = \{(q_2, v_2, -c, c) : (q_2, v_2, 0) \in \phi(W^s(0,0) \cap Z), |c| \le c_0\}.$$

We now check the admissibility of the homotopy. Let

$$(q_2^u, v_2^u, h^u + \theta c^u, c^u) \in \tilde{N}_u(\theta)$$

and

$$(q_2^s, v_2^s, h^s - \theta c^s, c^s) \in \tilde{N}_s(\theta)$$

be equal. We get $c^u = c^s := c$ and $c(h^s - h^u) = 2c^2\theta \ge 0$. As $c(h^s - h^u) < 0$ if $c \ne 0$, we deduce that $c = 0$, $h^u = h^s = 0$,

$$(q_2^u, v_2^u, 0) \in \phi(W^u(0,0) \cap Z)$$

and

$$(q_2^s, v_2^s, 0) \in \phi(W^s(0,0) \cap Z).$$

Thus the admissibility of $\tilde{N}_u(\theta)$ and $\tilde{N}_s(\theta)$ follows from the one of M_u and M_s. By homotopy,

$$\begin{aligned} & I(N_u(0), N_s(0)) = \pm I\left(\tilde{N}_u(0), \tilde{N}_s(0)\right) = \pm I\left(\tilde{N}_u(1), \tilde{N}_s(1)\right) \\ = \quad & \pm I(\ M_u \times \{(-c,c) : |c| \le c_0\}\ ,\ M_s \times \{(c,c) : |c| \le c_0\}\ ) \\ = \quad & \pm I(M_u, M_s) \ne 0. \end{aligned}$$

This persists under small perturbations, and hence there exists $a_0 \in (0,1)$ such that $I(N_u(a), N_s(a)) \ne 0$ in $Z \times \mathbb{R}$ for all $|a| < a_0$. As a consequence, for all $|a| < a_0$, there exists $c \in (-c_0, c_0)$ such that the manifolds $W^u(a,c)$ and $W^s(a,c)$ intersect each other in Z.

If $a \ne 0$ and $c = 0$, the invariant H takes two different values at $u \equiv 1$ and $u \equiv -1$ and, therefore, no kink can connect these two stationary solutions. ■

8 Appendix: topological transversality

We call X a *compact piece of manifold* if X is a non-empty compact subset of some boundaryless and orientable C^1-manifold $\tilde{X}$ of finite dimension such that $X = \overline{\text{int } X}$ in $\tilde{X}$. We endow X with the induced topology.

Let X be a compact piece of manifold and let Z and Y be boundaryless and orientable C^1-manifolds of finite dimension, such that $\dim X + \dim Z = \dim Y$ and Z is a closed submanifold of Y. A map $f \in C(X, Y)$ is called *admissible* if $f^{-1}(Z) \subset \text{ int } (X)$. Two admissible maps $f_0, f_1 \in C(X, Y)$ are *homotopic* if there exists $F \in C([0,1] \times X, Y)$ such that $F(s, .)$ is admissible for all $s \in [0,1]$, $F(0, .) = f_0$ and $F(1, .) = f_1$. An admissible map f is *transversal* if f is C^1 and, for all $x \in f^{-1}(Z)$, $\text{Image } (df_x) + T_z(Z) = T_z(Y)$, where $z = f(x)$. As $\dim X + \dim Z = \dim Y$, the sum is direct. The orientation of $T_x(X)$ and the map f define a natural orientation of $\text{Image } (df_x)$. If completed with the orientation of $T_z(Z)$, it gives the orientation of $T_z(Y)$, then the *orientation number* at $x \in f^{-1}(Z)$ is $+1$, and -1 otherwise. Any admissible map f is homotopic to a transversal one g that can be chosen as uniformly closed to f as necessary (Y may be embedded in some Euclidean space and is then endowed with the induced metric). The *intersection number* of an admissible map f and Z is an integer $I(f, Z) \in \mathbb{Z}$ which satisfies the following properties.

1. If f is transversal, then $f^{-1}(Z)$ is a finite collection of points and $I(f, Z)$ is the sum of the orientation numbers over $x \in f^{-1}(Z)$.

2. If f_0 and f_1 are homotopic, then $I(f_0, Z) = I(f_1, Z)$.

3. If $I(f, Z) \neq 0$, then $f^{-1}(Z) \neq \emptyset$.

4. If $\beta \in C^1(Y, Y')$ is a diffeomorphism, then $I(\beta \circ f, \beta(Z)) = \pm I(f, Z)$ depending whether β preserves orientation or not.

5. Let X_i, Z_i, Y_i, f_i be admissible for intersection theory ($1 \leq i \leq k$). Then $X_1 \times \ldots \times X_k$, $Z_1 \times \ldots \times Z_k$, $Y_1 \times \ldots \times Y_k$, $f_1 \times \ldots f_k$ are also admissible for intersection theory and
$$I(f_1 \times \ldots \times f_k, Z_1 \times \ldots \times Z_k) = \Pi_{i=1}^{k} I(f_i, Z_i).$$

Let X_1 and X_2 be compact pieces of manifolds and let Y be a boundaryless and orientable C^1-manifold of finite dimension such that $\dim X_1 + \dim X_2 = \dim Y$. A pair of maps $f_i \in C(X_i, Y)$, $i = 1, 2$, is called *admissible* if $f_1(x_1) = f_2(x_2)$ implies $x_1 \in \text{ int } (X_1)$ and $x_2 \in \text{ int } (X_2)$. In such a case, the intersection number of f_1 and f_2 is defined by

$$I(f_1, f_2) = (-1)^{\dim X_2} I(f_1 \times f_2, \Delta_{Y^2}),$$

Δ_{Y^2} denoting the diagonal in Y^2 endowed with the obvious orientation induced from the one of Y. The factor $(-1)^{\dim X_2}$ ensures that if X_2 is a compact and boundaryless submanifold of Y and f_2 is the inclusion mapping, then the relation $I(f_1, f_2) = I(f_1, X_2)$ holds.

When both f_1 and f_2 are C^1-embeddings and $M_i := \text{ Image } (f_i)$ $(i = 1, 2)$, the notation

$$I(M_1, M_2) := I(f_1, f_2)$$

is used. The assumption that (f_1, f_2) is an admissible pair is then equivalent to $M_1 \cap M_2 \subset \text{ int } (M_1) \cap \text{ int } (M_2)$. We mean by *topological transversality* of M_1 and M_2 that $I(M_1, M_2) \neq 0$ if $M_1 \cap M_2 \neq \emptyset$. To distinguish topological transversality from the classical notion of transversality, we sometimes call the last one *algebraic transversality*.

An introduction to intersection theory is provided by [GP]. There, the manifolds X, Z, Y are smooth and boundaryless and the maps are smooth. It is simple to extend their results to compact pieces of smooth manifolds, because, by definition of admissible maps, boundaries are not involved in intersections. Then, using the dense inclusion $C^\infty(X, Y) \subset C(X, Y)$ (see e.g. [PM], chapter 1, prop. 2.7), continuous maps can be considered too. Finally, by a theorem of Whitney (see e.g. [PM], chapter 1, prop. 0.19), any boundaryless C^1-manifold of dimension m can be C^1-embedded in $\mathbb{R}^{2m+1}$ as a closed C^∞-submanifold. This allows one to deal with C^1-manifolds.

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