Adaptive λ-tracking for a class of infinite-dimensional systems [*]

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Abstract: For a class of high-gain stabilizable multivariable linear infinite-dimensional systems we present an adaptive control law which achieves approximate asymptotic tracking in the sense that the tracking error tends asymptotically to a ball centred at 0 and of arbitrary prescribed radius $\lambda > 0$. This control strategy, called λ-tracking, combines proportional error feedback with a simple nonlinear adaptation of the feedback gain. It does not involve any parameter estimation algorithms, nor is it based on the internal model principle. The class of reference signals is $W^{1,\infty}$, the Sobolev space of absolutely continuous functions which are bounded and have essentially bounded derivative. The control strategy is robust with respect to output measurement noise in $W^{1,\infty}$ and bounded input disturbances. We apply our results to retarded systems and integrodifferential systems.

Keywords: Adaptive control; tracking; high-gain control; infinite-dimensional systems; functional differential equations; integrodifferential equations.

[*] This work was supported by the Human Capital and Mobility programme (Project number CHRX-CT93-0402).

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1. Introduction

We present an adaptive regulator for the class of infinite-dimensional m-input-m-output systems described by

$$\boxed{\dot{y}(t) = \mathcal{H}(y)(t) + Gu(t) + w(t)\,, \quad y(0) = y_0 \in \mathbb{R}^m\,,} \tag{1.1}$$

where as usual $u(\cdot)$ and $y(\cdot)$ denote the plant input and output, respectively, G is a real matrix $m \times m$-matrix whose eigenvalues have positive real parts, and $\mathcal{H}$ is a causal linear operator, which is input-output stable in a certain sense, see Section 2 for details. In applications $\mathcal{H}$ will be the input-output operator of a state-space system or a system described by a functional or partial differential equation. The function $w(\cdot)$ then models the effect of non-zero inital conditions. Our class covers retarded and integrodifferential systems which satisfy a generalized minimum phase condition and have a generalized high-frequency gain whose eigenvalues have positive real parts, and are hence stabilizable by static high-gain output feedback (for a detailed discussion see Section 4).

To the above class of systems we apply a simple adaptive controller of the form

$$\boxed{\begin{array}{rcl} e(t) & = & y_r(t) - y(t) \\ u(t) & = & k(t)\, e(t), \\ \dot{k}(t) & = & \left\{ \begin{array}{cl} (\|e(t)\| - \lambda)^2\,, & \text{if } \|e(t)\| \geq \lambda \\ 0\,, & \text{if } \|e(t)\| < \lambda \end{array} \right. \,, \quad k(0) = k_0\,. \end{array}} \tag{1.2}$$

This control strategy, called λ-*tracking*, is similar to a control law introduced by Ilchmann and Ryan [7] in a finite-dimensional context. In (1.2), $y_r(\cdot)$ is a reference signal which is assumed to belong to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$, i.e. y_r is absolutely continuous and y_r and $\dot{y}_r$ are essentially bounded. The constant $\lambda > 0$ is an upper bound for the asymptotic tracking error and is chosen by the designer. Setting $D(e) = \|e\| - \lambda$ if $\|e\| \geq \lambda$ and $D(e) = 0$ otherwise, the gain adaptation in (1.2) can be written as $\dot{k} = D^2(e)$. The closed-loop system given by (1.1) and (1.2) is shown in Figure 1.1.

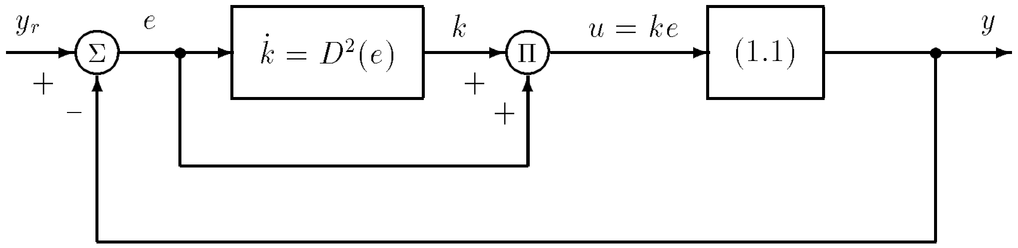

Figure 1.1: Closed-loop system

Our main result shows that for all initial conditions y_0 and k_0 and all bounded functions $w(\cdot)$, the controller (1.2) achieves convergence of the feedback gain $k(t)$, and the output

$y(t)$ will approach the ball $B_\lambda(y_r(t))$ of radius λ centred at $y_r(t)$ as $t \to \infty$, i.e.

$$\limsup_{t\to\infty} \|e(t)\| \leq \lambda\,. \tag{1.3}$$

Moreover, we show that closed-loop system is robust with respect to measurement noise in $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ and bounded input disturbances. Note that the control law (1.2) is of striking simplicity – it combines proportional error feedback with a simple nonlinear adaptation of the feedback gain. It does not invoke any parameter estimation, nor is it based on the internal model principle. We mention that for finite-dimensional problems, the concept of λ-tracking has been successfully applied to the control of industrial plants and processes such as: continuous stirred tank reactors for methanol synthesis [1], binary distillation columns [2], biogas tower reactors [6] and anaerobic fermentation processes [8].

The technique of representing a large class of high-gain stabilizable infinite-dimensional linear systems by an abstract Volterra integrodifferential equation of the form (1.1) was introduced by Logemann and Owens [10]. It has been exploited for adaptive asymptotic tracking by Logemann and Ilchmann [9]. However, in [9] we needed to invoke the internal model principle and the class of reference signals was essentially restricted to finite sums of sinusoids. In this paper we overcome this drawback by weakening the control objective slightly: instead of exact asymptotic tracking the design goal is approximate asymptotic tracking in the sense of (1.3).

The paper is organized as follows. Section 2 contains some preliminaries on abstract Volterra integrodifferential systems which are needed to establish existence and uniqueness of solutions for the closed-loop system given by (1.1) and (1.2). Moreover, we give a frequency-domain interpretation for our class of systems in the case of shift-invariant $\mathcal{H}$. Section 3 contains the main result on adaptive λ-tracking, whilst Section 4 is devoted to the application of this result to retarded and integrodifferential systems.

Nomenclature: As usual, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$, where $\alpha \in \mathbb{R}$. The Euclidean norm on $\mathbb{R}^n$ and $\mathbb{C}^n$ and the matrix norm induced by the Euclidean norm will be denoted by $\|\cdot\|$. We will make use of the following function spaces, where $I \subset \mathbb{R}$ denotes an interval and $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

$C(I, \mathbb{K}^n)$	:=	vector space of all continuous functions on I with values in $\mathbb{K}^n$.
$L^p(I, \mathbb{K}^n)$	:=	vector space of all p-integrable functions on I with values in $\mathbb{K}^n$, $p \geq 1$.
$L^\infty(I, \mathbb{K}^n)$	:=	vector space of all essentially bounded functions on I with values in $\mathbb{K}^n$.
$L^p_{loc}(I, \mathbb{K}^n)$	:=	vector space of all locally p-integrable functions on I with values in $\mathbb{K}^n$, $p \geq 1$.
$W^{1,\infty}(\mathbb{R}_+, \mathbb{K}^n)$	:=	vector space of all absolutely continuous $\mathbb{K}^n$-valued functions f defined on $\mathbb{R}_+$ such that f and $\dot{f}$ are essentially bounded.

$BV(I, \mathbb{K}^{n\times n})$	:=	vector space of $\mathbb{K}^{n\times n}$-valued functions of bounded variation defined on I.
$M(\mathbb{R}_+, \mathbb{K}^{n\times n})$	:=	vector space of bounded Borel measures on $\mathbb{R}_+$ with values in $\mathbb{K}^{n\times n}$.
$H^\infty(\mathbb{C}^{n\times n})$	:=	algebra of bounded holomorphic functions defined on $\mathbb{C}_0$ with values in $\mathbb{C}^{n\times n}$.
$H^2(\mathbb{C}^n)$	:=	the usual Hardy-Lebesgue space of order 2 of holomorphic functions defined on $\mathbb{C}_0$ with values in $\mathbb{C}^n$.

Let f be a function defined on $[0, a)$, where $0 < a \leq \infty$. Then for all $\tau \in [0, a)$

$$(\mathcal{P}_\tau f)(t) := \begin{cases} f(t) & , \quad 0 \leq t \leq \tau \\ 0 & , \quad t > \tau. \end{cases}$$

If X is a normed space, then $B(X)$ denotes the set of linear bounded operators from X into X. If $T \in B(X)$, then $\sigma(T)$ denotes the spectrum of T.

$\mathfrak{L}$ denotes the Laplace transform. The superscript $\hat{}$ is used to denote Laplace transformed or Laplace-Stieltjes transformed functions.

2. Preliminaries and system description

The plant to be controlled is given by (1.1), where we assume that

(A1) $G \in \mathbb{R}^{m\times m}$ with $\sigma(G) \subset \mathbb{C}_0$,

(A2) $\mathcal{H}$ is causal and $\mathcal{H} \in B(L^2(\mathbb{R}_+, \mathbb{R}^m)) \cap B(L^\infty(\mathbb{R}_+, \mathbb{R}^m))$.

Recall that $\mathcal{H}$ is called shift-invariant if $\mathcal{S}_t\mathcal{H} = \mathcal{H}\mathcal{S}_t$ for all $t \geq 0$, where $\mathcal{S}_t$ denotes the operator of right-shift by t. Since shift-invariance implies causality (see Vidyasagar [13]), assumption (A2) is implied by

(A3) $\mathcal{H}$ is shift-invariant and $\mathcal{H} \in B(L^2(\mathbb{R}_+, \mathbb{R}^m)) \cap B(L^\infty(\mathbb{R}_+, \mathbb{R}^m))$.

Assumption (A3) is usually satisfied for systems given by linear autonomous exponentially stable differential equations (ODEs, PDEs as well as FDEs). In particular, it is satisfied for the classes of retarded and integrodifferential systems considered in Section 4.

If (A3) holds, then there exists $\mathbf{H} \in H^\infty(\mathbb{C}^{m\times m})$ such that

$$\mathcal{H} = \mathfrak{L}^{-1}\, \mathcal{M}_{\mathbf{H}}\, \mathfrak{L}\,, \tag{2.1}$$

where $\mathcal{M}_{\mathbf{H}}$ denotes the multiplication by $\mathbf{H}$ on the Hardy space $H^2(\mathbb{C}^m)$ (see for example Weiss [14]). Therfore, if $y_0 = 0$ and $w \equiv 0$, Laplace transformation of (1.1) gives

$$s\hat{y}(s) = \mathbf{H}(s)\hat{y}(s) + G\hat{u}(s)\,.$$

Setting

$$\mathbf{G}(s) = (sI - \mathbf{H}(s))^{-1} G\,, \tag{2.2}$$

it follows that $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$. Using (A1) and the fact that $\mathbf{H}$ is in $H^\infty(\mathbb{C}^{m\times m})$, it is not difficult to show that for all sufficiently large $\gamma > 0$

$$\mathbf{G}(I + \gamma\mathbf{G})^{-1} \in H^\infty(\mathbb{C}^{m\times m})\,,$$

i.e. high-gain feedback leads to a L^2-stable closed-loop system. This observation is the motivation for applying the high-gain adaptive control law (1.2) to the system (1.1).

Conversely, if a system is externally described by a transfer function $\mathbf{G}(s)$, and there exists $\mathbf{H} \in H^\infty(\mathbb{C}^{m\times m})$ and $G \in \mathbb{R}^{m\times m}$ with $\sigma(G) \subset \mathbb{C}_0$ such that (2.2) holds[†], then the system can be represented by (1.1), where $\mathcal{H}$ is given by (2.1). Whilst in this case the operator $\mathcal{H}$ is shift-invariant and in $B(L^2(\mathbb{R}_+, \mathbb{R}^m))$, it is not true in general that $\mathcal{H} \in B(L^\infty(\mathbb{R}_+, \mathbb{R}^m))$. However, under the extra assumption that the inverse Laplace transform of $\mathbf{H}(s)$ is in $M(\mathbb{R}_+, \mathbb{R}^{m\times m})$ (which is usually satisfied in applications), it follows that $\mathcal{H}$ given by (2.1) is in $B(L^\infty(\mathbb{R}_+, \mathbb{R}^m))$.

To prove existence and uniqueness of the solution of the closed-loop system given by (1.1) and (1.2), we first consider the following more general initial-value problem

$$\dot{x}(t) = (\mathcal{A}x)(t) + f(t, x(t)) + g(t)\,, \quad t \geq 0\,, \tag{2.3a}$$

$$x(0) = x_0 \in \mathbb{R}^n\,. \tag{2.3b}$$

Here we assume:

(i) $\mathcal{A} : L^2_{loc}(\mathbb{R}_+, \mathbb{R}^n) \to L^2_{loc}(\mathbb{R}_+, \mathbb{R}^n)$, $\mathcal{A}(0) = 0$ and there exists $\kappa > 0$ such that

$$\|\mathcal{P}_t(\mathcal{A}x - \mathcal{A}x')\|_{L^2} \leq \kappa \|\mathcal{P}_t(x - x')\|_{L^2}\,, \quad \text{for all } x, x' \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^n),\ t \geq 0\,,$$

i.e. $\mathcal{A}$ is unbiased, causal and of finite incremental gain;

(ii) $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a function with $f(t, x)$ being continuous in t and locally Lipschitz continuous in x, uniformly in t on bounded intervals;

(iii) g is in $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$.

In order to define what we mean by a solution of the initial value problem (2.3) on $[0, T)$, where $0 < T \leq \infty$, we have to give a meaning to $\mathcal{A}x$ if $x \in C([0, T), \mathbb{R}^n)$ (remember that $\mathcal{A}$ operates on functions whose domain of definition is $\mathbb{R}_+$). We set $(\mathcal{A}x)(t) = (\mathcal{A}\mathcal{P}_\tau x)(t)$ for $0 \leq t \leq \tau < T$. Since $\mathcal{A}$ is causal, this definition does not depend on the choice of τ. By a *solution* of (2.3) on $[0, T)$ we mean an absolutely continuous function x on $[0, T)$ such that the differential equation (2.3a) holds a.e. in $[0, T)$ and the initial condition (2.3b) is satisfied.

[†] If $\mathbf{G}$ is meromorphic in a region $\Omega \supset \mathbb{C}_0^{cl}$, then this will be the case if and only if $s\mathbf{G}(s) - G = \mathrm{O}(s)$ as $|s| \to \infty$ in $\mathbb{C}_0$ and $\mathbf{G}(s)$ has no zeros in $\mathbb{C}_0^{cl}$, see Logemann and Zwart [11].

Theorem 2.1 *The initial-value problem (2.3) has a unique solution defined on a maximal interval of existence* $[0,\omega)$, *where* $0 < \omega \le \infty$. *If* $\omega < \infty$, *then there exists a sequence* $t_i \in (0,\omega)$ *with* $\lim_{i\to\infty} t_i = \omega$ *and such that* $\lim_{i\to\infty} \|x(t_i)\| = \infty$.

The above theorem has been proved in Logemann and Owens [10]. Similar results can be found in Gripenberg et al. [3], pp. 359 and in Hinrichsen and Pritchard [4]. Theorem 2.1 implies that the initial value problem (1.1) has a unique solution for all $u, w \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ and $y_0 \in \mathbb{R}^m$. Moreover, it shows that the closed-loop system given by (1.1) and (1.2) has a unique solution.

For the proof of the main result in Section 3 we also need a simple lemma on the asymptotic behaviour of the convolution of L^p-functions. Although this result is certainly well-known, we could not find a suitable reference. However, implicitly, a proof was given by Vidyasagar [12], p. 242. For completeness we include a proof of the lemma which is based on [12].

Lemma 2.2 *If* $f \in L^p(\mathbb{R}_+, \mathbb{R})$ *and* $g \in L^q(\mathbb{R}_+, \mathbb{R})$ *for* $p, q \in (1,\infty)$ *with* $1/p + 1/q = 1$, *then*

$$\lim_{t\to\infty} (f \star g)(t) \; = \lim_{t\to\infty} \int_0^t f(t-\tau) g(\tau)\, d\tau \;=\; 0\,.$$

Proof: An application of Hölder's inequality yields

$$\begin{aligned}(f\star g)(t) &= \int_0^{t/2} f(t-\tau)g(\tau)\,d\tau + \int_{t/2}^t f(t-\tau)g(\tau)\,d\tau \\ &\le \|f\|_{L^p(\frac{t}{2},t)}\|g\|_{L^q(0,\frac{t}{2})} + \|f\|_{L^p(0,\frac{t}{2})}\|g\|_{L^q(\frac{t}{2},t)} \\ &\le \|f\|_{L^p(\frac{t}{2},\infty)}\|g\|_{L^q(0,\infty)} + \|f\|_{L^p(0,\infty)}\|g\|_{L^q(\frac{t}{2},\infty)}\,.\end{aligned}$$

This proves the claim, since by assumption the right-hand side of the last inequality converges to 0 as $t \to \infty$. $\square$

3. Adaptive λ-tracking

The following theorem is the main result of this paper.

Theorem 3.1 *Let the assumptions (A1) and (A2) be satisfied and let* $\lambda > 0$ *be given. Then, for all initial conditions* $y_0 \in \mathbb{R}^m$, $k_0 \in \mathbb{R}$, *all* $w \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ *and all reference signals* $y_r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$, *the closed-loop system given by (1.1) and (1.2) has the following properties:*

(i) *The unique solution* $(y(\cdot), k(\cdot))$ *exists on* $[0,\infty)$,

(ii) $\lim_{t\to\infty} k(t)$ *exists and is finite,*

(iii) $\limsup_{t\to\infty} \|y_r(t) - y(t)\| \le \lambda\,.$

Proof: To reduce the technical effort, we assume without loss of generality that $k_0 \geq 0$ in (1.2). By (A1) there exists a positive definite real $m \times m$-matrix $P = P^T$ satisfying the Lyapunov equation

$$G^T P + PG = 2I\,. \tag{3.1}$$

We define the P-induced norm $\|\cdot\|_P$ on $\mathbb{R}^m$ by setting $\|x\|_P = \sqrt{\langle x, Px\rangle}$. Let $\varrho > 0$ denote the square root of the smallest eigenvalue of P and set $\varrho_P = 1/\sqrt{\|P\|}$. Then,

$$\varrho\|x\| \leq \|x\|_P\,, \quad \varrho_P\|x\|_P \leq \|x\|\,, \qquad \text{for all } x \in \mathbb{R}^m. \tag{3.2}$$

We introduce the following distance functions which will be used throughout the proof.

$$\begin{aligned}
D &: \mathbb{R}^m \to \mathbb{R}_+, \quad x \mapsto \quad D(x) &&= \begin{cases} \|x\| - \lambda\,, & \text{if } \|x\| \geq \lambda \\ 0\,, & \text{if } \|x\| < \lambda\,, \end{cases} \\
d &: \mathbb{R}^m \to \mathbb{R}^m, \quad x \mapsto \quad d(x) &&= \begin{cases} \frac{\|x\|-\lambda}{\|x\|}\, x\,, & \text{if } \|x\| \geq \lambda \\ 0\,, & \text{if } \|x\| < \lambda\,, \end{cases} \\
D_P &: \mathbb{R}^m \to \mathbb{R}\,, \quad x \mapsto \quad D_P(x) &&= \begin{cases} \|x\|_P - \lambda\varrho\,, & \text{if } \|x\|_P \geq \lambda\varrho \\ 0\,, & \text{if } \|x\|_P < \lambda\varrho\,, \end{cases} \\
d_P &: \mathbb{R}^m \to \mathbb{R}^m, \quad x \mapsto \quad d_P(x) &&= \begin{cases} \frac{\|x\|_P-\lambda\varrho}{\|x\|_P}\, x\,, & \text{if } \|x\|_P \geq \lambda\varrho \\ 0\,, & \text{if } \|x\|_P < \lambda\varrho\,. \end{cases}
\end{aligned}$$

We proceed in five steps.

STEP 1 (Existence and uniqueness of a maximal solution): Setting

$$e = y_r - y \quad \text{and} \quad \tilde{w} = \dot{y}_r - w - \mathcal{H}(y_r)$$

the closed-loop system given by (1.1) and (1.2) can be written in the following form

$$\dot{e}(t) = -k(t)Ge(t) + \mathcal{H}(e)(t) + \tilde{w}(t)\,, \quad e(0) = e_0 := y_r(0) - y_0\,, \tag{3.3a}$$
$$\dot{k}(t) = D^2(e(t))\,, \quad k(0) = k_0\,. \tag{3.3b}$$

By assumption $y_r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$, and so $\dot{y}_r \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and, by (A2), $\mathcal{H}(y_r) \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. Moreover, $w \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$, and thus

$$\tilde{w} \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)\,. \tag{3.4}$$

The system (3.3) is of the form (2.3), and hence we can apply Theorem 2.1 to conclude that (3.3) has a unique maximal solution $(e(\cdot), k(\cdot))$ defined on $[0, \omega)$, the maximal interval of existence. Using the linearity of $\mathcal{H}$ and (3.3a), we see that e satisfies

$$\dot{e}(t) = -k(t)Ge(t) + \mathcal{H}(d_P(e))(t) + h(t)\,, \tag{3.5}$$

where

$$h(t) := \mathcal{H}(e - d_P(e))(t) + \tilde{w}(t)\,, \quad t \in [0, \omega)\,.$$

It follows from the definition of d_P that $e - d_P(e) \in L^\infty(0,\omega;\mathbb{R}^m)$ and hence by assumption (A2) and (3.4) we conclude

$$h \in L^\infty(0,\omega;\mathbb{R}^m)\,. \tag{3.6}$$

Step 2 (An estimate for a Lyapunov-type function): We claim that there exist constants $\mu_1, \mu_2 > 0$ such that differentiation of the function

$$V : \mathbb{R}^m \to \mathbb{R}_+, \quad x \mapsto V(x) = \frac{1}{2} D_P^2(x)$$

along the solution of (3.5) yields, for almost all $t \in [0,\omega)$,

$$\frac{d}{dt} V(e(t)) \le -\mu_1 [k(t) - \mu_2] D_P(e(t)) \|e(t)\|_P + \mu_2 \|d_P(e(t))\| \|\mathcal{H}(d_P(e))(t)\|\,. \tag{3.7}$$

In the following the argument t will be omitted for brevity. By (3.1), (3.2), (3.5) and (3.6) and since $k(t) \ge k_0 \ge 0$, we have almost everywhere on $[0,\omega)$,

$$\begin{aligned}\frac{d}{dt} V(e) &= \frac{D_P(e)}{\|e\|_P} \langle e, P\dot{e}\rangle \\ &= -\frac{D_P(e)}{\|e\|_P} k \|e\|^2 + \frac{D_P(e)}{\|e\|_P} \langle e, P\mathcal{H}(d_P(e)) + Ph\rangle \\ &\le -\varrho_P^2 k D_P(e)\|e\|_P + \|P\|\|d_P(e)\|\|\mathcal{H}(d_P(e))\| + \|P\|\|h\|_{L^\infty(0,\omega)}\|d_P(e)\|\,.\end{aligned} \tag{3.8}$$

Using (3.2) and the definitions of D_P and d_P we obtain

$$\|d_P(x)\| \;\le\; D_P(x)\frac{1}{\varrho} \;\le D_P(x) \frac{1}{\varrho^2 \lambda} \|x\|_P\,, \qquad \text{for all } x \in \mathbb{R}^m\,. \tag{3.9}$$

Combining (3.8) and (3.9), we see that there exist positive constants μ_1 and μ_2 such that (3.7) holds on $[0,\omega)$.

Step 3 (Boundedness of k on $[0,\omega)$): We claim that $k \in L^\infty(0,\omega;\mathbb{R})$. Let $\kappa > 0$ be the L^2-induced operator norm of $\mathcal{H}$. It follows from Hölder's inequality that for all $f \in L^2_{loc}(\mathbb{R}_+,\mathbb{R}^m)$ and all $t \in \mathbb{R}_+$,

$$\int_0^t \|f\|\|\mathcal{H}(f)\| \le \kappa \int_0^t \|f\|^2\,. \tag{3.10}$$

Now either $k(t) \le \mu_2(1 + \kappa/\mu_1\varrho^2)$ for all $t \in [0,\omega)$, whence the claim follows, or there exists a $t_0 \in [0,\omega)$ such that $k(t_0) > \mu_2(1 + \kappa/\mu_1\varrho^2)$. Then

$$\mu_3 := \mu_1(k(t_0) - \mu_2) > \kappa\mu_2/\varrho^2 > 0\,. \tag{3.11}$$

Integration of (3.7) from t_0 to $t \in [0,\omega)$, using that $k(\cdot)$ is non-decreasing and an application of (3.10) yields

$$\begin{aligned} V(e(t)) &\le V(e(t_0)) - \mu_3 \int_{t_0}^t D_P(e)\|e\|_P + \mu_2 \int_{t_0}^t \|d_P(e)\|\|\mathcal{H}(d_P(e))\| \\ &\le V(e(t_0)) - \mu_3 \int_{t_0}^t D_P(e)\|e\|_P + \kappa\mu_2 \int_0^t \|d_P(e)\|^2\,. \end{aligned} \tag{3.12}$$

Clearly, for all $x \in \mathbb{R}^m$, we have

$$D_P(x)\|x\|_P = D_P(x)(\|x\|_P - \varrho\lambda) + \varrho\lambda D_P(x) \geq D_P^2(x)\,.$$

Combining this with (3.2) and (3.12) shows that

$$0 \leq V(e(t_0)) + \kappa\mu_2 \int_0^{t_0} \|d_P(e)\|^2 - \int_{t_0}^{t} (\mu_3 - \kappa\mu_2/\varrho^2) D_P^2(e)$$

Setting

$$\mu_4 := V(e(t_0)) + \kappa\mu_2 \int_0^{t_0} \|d_P(e)\|^2\,, \quad \mu_5 := 2(\mu_3 - \kappa\mu_2/\varrho^2)\,,$$

we obtain

$$0 \leq \mu_4 - \mu_5 \int_{t_0}^{t} V(e)\,, \quad \text{for all } t \in [t_0, \omega)\,.$$

By (3.11), $\mu_5 > 0$, and thus we may conclude

$$\int_{t_0}^{t} V(e) \;\leq \mu_4/\mu_5\,, \quad \text{for all } t \in [t_0, \omega)\,. \tag{3.13}$$

Now, using (3.2), it is easy to show that

$$D_P(x) \geq \varrho D(x)\,, \quad \text{for all } x \in \mathbb{R}^m\,.$$

Therefore, for all $t \in [t_0, \omega)$,

$$k(t) = k(t_0) + \int_{t_0}^{t} D^2(e) \;\leq k(t_0) + \frac{1}{\varrho^2}\int_{t_0}^{t} D_P^2(e)\,.$$

Appealing to (3.13), we obtain

$$k(t) \leq k(t_0) + \frac{2\mu_4}{\mu_5\varrho^2}\,, \quad \text{for all } t \in [t_0, \omega)\,,$$

which shows that $k(\cdot)$ is bounded on $[0, \omega)$.

STEP 4 (Boundedness of e on $[0,\omega)$): From (3.13) we see that $D_P(e) \in L^2(0,\omega;\mathbb{R})$, and thus $d_P(e) \in L^2(0,\omega;\mathbb{R}^m)$. Moreover, by the definition of the function $d_P(\cdot)$ we have that $e - d_P(e) \in L^\infty(0,\omega;\mathbb{R}^m)$. Hence Step 3, assumption (A2) and (3.6) yield

$$\begin{aligned} f_1 &:= (1-k)G d_P(e) + \mathcal{H}(d_P(e)) \in L^2(0,\omega;\mathbb{R}^m)\,, \\ f_2 &:= (1-k)G[e - d_P(e)] + h \in L^\infty(0,\omega;\mathbb{R}^m)\,. \end{aligned}$$

Trivially, (3.5) may be rewritten as

$$\dot{e}(t) = -Ge(t) + f_1(t) + f_2(t)\,.$$

An application of the variations-of-parameters formula to this equation yields

$$e(t) = e^{-Gt}e_0 + \int_0^t e^{-G(t-\tau)}[f_1(\tau) + f_2(\tau)]\,d\tau\,. \tag{3.14}$$

By (A1), e^{-Gt} is exponentially stable, and therefore we obtain from Lemma 2.2 that

$$e \in L^\infty(0,\omega;\mathbb{R}^m)\,. \tag{3.15}$$

Step 5 (Global existence and convergence): From Steps 3 and 4 we have boundedness of k and e on $[0,\omega)$. Therefore, combining the maximality of ω and Theorem 2.1, we obtain that $\omega = \infty$, and hence statement (i) follows. Consequently, using Steps 3 and 4, we conclude that $k(\cdot)$ and $e(\cdot)$ are bounded on $[0,\infty)$. Since $k(\cdot)$ is non-decreasing we obtain statement (ii). To prove statement (iii), note that by (A2), (3.5) and (3.6) and the boundedness of e on $[0,\infty)$,

$$\dot e \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)\,. \tag{3.16}$$

An easy calculation leads to

$$|\frac{d}{dt}D^2(e)| = 2\frac{D(e)}{\|e\|}|\langle e,\dot e\rangle| \le 2D(e)\|\dot e\|\,, \tag{3.17}$$

so that from (3.15)–(3.17) we obtain

$$\frac{d}{dt}D^2(e) \in L^\infty(\mathbb{R}_+,\mathbb{R})\,. \tag{3.18}$$

Moreover, since $k \in L^\infty(\mathbb{R}_+,\mathbb{R})$, we have

$$D^2(e) \in L^1(\mathbb{R}_+,\mathbb{R})\,. \tag{3.19}$$

Finally, (3.18) and (3.19) show that (see e.g. Lemma 2.1.7 in [5])

$$\lim_{t\to\infty} D^2(e(t)) = 0\,,$$

which in turn implies statement (iii). □

The following remark shows that the controller (1.2) is robust with respect to measurement noise and input disturbances.

Remark 3.2 (i) Suppose that the feedback system is subject to a bounded input disturbance δ and $W^{1,\infty}$-measurement noise ν, see Figure 3.1. This means we have to replace e and u in (1.2) by $e(t) = y_r(t) - y(t) - \nu(t)$ and $u(t) = k(t)e(t) + \delta(t)$, respectively. Then, by absorbing δ into the $\tilde w$ term (see Step 1 in the proof of Theorem 3.1), it follows that statements (i) and (ii) of Theorem 3.1 remain true and, moreover

$$\limsup_{t\to\infty} \|y_r(t) - y(t) - \nu(t)\| \le \lambda\,, \tag{3.20}$$

provided that (A1) and (A2) hold, $w,\delta \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)$ and $y_r,\nu \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^m)$. Setting $\lambda_\nu := \lambda + \|\nu\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^m)}$, (3.20) trivially implies that

$$\limsup_{t\to\infty} \|y_r(t) - y(t)\| \le \lambda_\nu\,.$$

(ii) In applications it is often useful to modify the gain adaptation law in (1.2) to

$$\dot{k}(t) = \begin{cases} \gamma(\|e(t)\| - \lambda)^2, & \text{if } \|e(t)\| \geq \lambda \\ 0, & \text{if } \|e(t)\| < \lambda \end{cases},$$

where $\gamma > 0$ is an additional design parameter. This modification with suitably chosen γ can be used to improve the transient response of the closed-loop system. $\diamond$

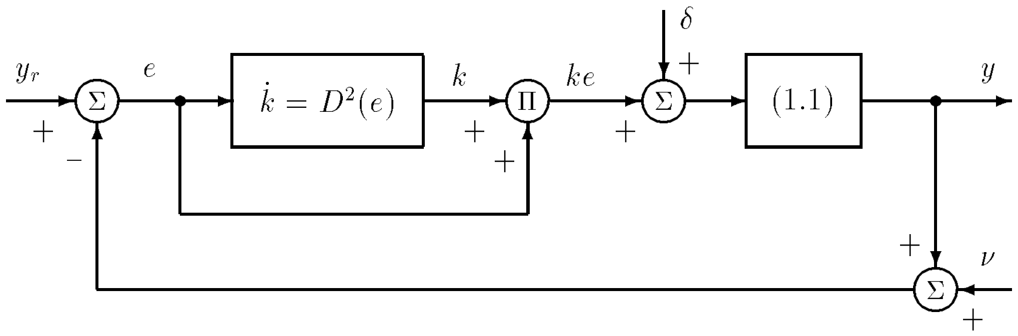

Figure 3.1: Closed-loop system with input disturbances and measurement noise

4. Applications to retarded systems and integrodifferential systems

In this section we show that Theorem 3.1 can be applied to retarded systems and integrodifferential systems. We solve the adaptive λ-tracking problem for these classes of systems under the assumptions that the plant is minimum-phase and that the high-frequency gain matrix has its spectrum in the open right-half plane. Moreover, it turns out that in both cases the internal variables of the controlled plant remain bounded.

4.1 Retarded systems

In the following we extend any function $F \in BV([a,b], \mathbb{R}^{n\times n})$ to the whole real axis by setting $F(t) = F(a)$ for $t < a$ and $F(t) = F(b)$ for $t > b$. Any measurable function $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}$, will be extended to the whole real axis by defining $f(t) = 0$ for $t \notin \Omega$. For $F = (F_{ij}) \in BV([0,h], \mathbb{R}^{n\times n})$ and $f = (f_1, \ldots, f_n)^T$, $f_i \in L^1_{loc}(\mathbb{R}, \mathbb{R})$, $1 \leq i \leq n$, we define

$$dF * f := \begin{pmatrix} \sum_{j=1}^n dF_{1j} * f_j \\ \vdots \\ \sum_{j=1}^n dF_{nj} * f_j \end{pmatrix},$$

where dF_{ij} denotes the Borel measure on $\mathbb{R}$ induced by F_{ij} and $dF_{ij} * f_j$ denotes the convolution of the measure dF_{ij} and the function f_j (on the whole real line). If f is continuous on $[-h,\infty)$, then, of course, $dF * f$ can be expressed as a Riemann-Stieltjes integral

$$(dF * f)(t) = \int_0^h dF(\tau)f(t-\tau)\,, \qquad \text{for } t \geq 0\,.$$

Consider the retarded system

$$\dot{x} = dA * x + Bu\,, \qquad x|_{[-h,0]} = x_0 \in C([-h,0],\mathbb{R}^n)\,, \tag{4.1a}$$
$$y = Cx\,, \tag{4.1b}$$

where $A \in BV([0,h],\mathbb{R}^{n\times n})$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. We assume that

$$\sigma(CB) \subset \mathbb{C}_0 \tag{4.2}$$

and

$$\det \begin{pmatrix} sI - \hat{A}(s) & -B \\ C & 0 \end{pmatrix} \neq 0\,, \qquad \text{for all } s \in \mathbb{C}_0^{cl}\,, \tag{4.3}$$

where $\hat{A}(s) := \int_0^h \exp(-s\tau)dA(\tau)$ denotes the Laplace-Stieltjes transform of A. The transfer function matrix $\mathbf{G}(s)$ of (4.1) is given by $\mathbf{G}(s) = C(sI-\hat{A}(s))^{-1}B$.

Remark 4.1 Condition (4.3) is the so-called minumum-phase condition. As in the finite-dimensional case, see Ilchmann [5], it can be shown that (4.3) holds if and only if the following three conditions hold:

(i) The transfer function matrix $\mathbf{G}(s)$ has no zeros in $\mathbb{C}_0^{cl}$,

(ii) $\operatorname{rk}(sI-\hat{A}(s), B) = n\,, \quad \text{for all } \mathbb{C}_0^{cl}$,

(iii) $\operatorname{rk}\begin{pmatrix} sI-\hat{A}(s) \\ C \end{pmatrix} = n\,, \quad \text{for all } \mathbb{C}_0^{cl}$. $\diamond$

We show that if (4.2) and (4.3) are satisfied, then the retarded system (4.1) can be written in the form (1.1) with (A1) and (A2) being satisfied. The condition (4.2) means in particular that CB is invertible. Therefore, $m \leq n$, $\dim\ker C = n-m$, $\dim\operatorname{im} B = m$ and

$$\ker C \cap \operatorname{im} B = \{0\}\,.$$

Let $v_1,\ldots,v_{n-m} \in \mathbb{R}^n$ be a basis for $\ker C$, then the matrix

$$Q := (B(CB)^{-1}, v_1, \ldots, v_{n-m})$$

is invertible, and moreover

$$Q^{-1}B = \begin{pmatrix} CB \\ 0 \end{pmatrix}, \qquad CQ = (I_m,\, 0)\,.$$

It is useful to partition the matrix $Q^{-1}A(\cdot)Q$ as follows

$$Q^{-1}A(\cdot)Q = \begin{pmatrix} A_{11}(\cdot) & A_{12}(\cdot) \\ A_{21}(\cdot) & A_{22}(\cdot) \end{pmatrix},$$

where $A_{11}(\cdot)$, $A_{12}(\cdot)$, $A_{21}(\cdot)$ and $A_{22}(\cdot)$ are matrices with entries in $BV([0,h],\mathbb{R})$ of size $m \times m$, $m \times (n-m)$, $(n-m) \times m$ and $(n-m) \times (n-m)$, respectively. Noticing that $Q^{-1}x$ is of the form

$$Q^{-1}x = \begin{pmatrix} Cx \\ \xi \end{pmatrix},$$

(4.1a) can be re-written as

$$\dot{y} = dA_{11} * y + dA_{12} * \xi + CBu\,, \quad y|_{[-h,0]} = y_0\,, \tag{4.4a}$$

$$\dot{\xi} = dA_{21} * y + dA_{22} * \xi\,, \quad \xi|_{[-h,0]} = \xi_0\,, \tag{4.4b}$$

where

$$\begin{pmatrix} y_0 \\ \xi_0 \end{pmatrix} = Q^{-1}x_0\,.$$

Consider (4.4b) as differential equation in ξ with forcing term $dA_{21} * y$ and suppose for a moment that y can be chosen arbitrarily. For given initial functions ξ_0 and y_0 and given $y \in L^1_{loc}([-h,\infty),\mathbb{R}^m)$, let $\xi(t;\xi_0,y_0,y)$ denote the solution of the initial-value problem

$$\dot{\xi} = dA_{22} * \xi + dA_{21} * y\,; \qquad \xi|_{[-h,0]} = \xi_0\,, \quad y|_{[-h,0]} = y_0\,.$$

Setting

$$\mathcal{H}(y) = dA_{12} * \xi(\cdot\,;0,0,y) + dA_{11} * y\,,$$
$$w = dA_{12} * \xi(\cdot\,;\xi_0,y_0,0) + dA_{11} * y_0\,,$$

(4.4a) can be expressed as

$$\dot{y} = \mathcal{H}(y) + CBu + w\,, \qquad y(0) = Cx_0(0)\,.$$

Defining $\hat{A}_{22}(s) := \int_0^h e^{-s\tau}\, dA_{22}(\tau)$, the minimum-phase assumption (4.3) implies that

$$\det(sI - \hat{A}_{22}(s)) \neq 0\,, \qquad \text{for all } s \in \mathbb{C}_0^{cl}\,,$$

see [9]. This means that the zero solution of the retarded equation $\dot{\xi} = dA_{22} * \xi$ is exponentially stable, and hence the linear operator $y(\cdot) \mapsto \xi(\cdot\,;0,0,y)$ maps the space $L^p(\mathbb{R}_+,\mathbb{R}^m)$ boundedly into itself for all $p \in [1,\infty]$, and moreover, $\xi(\cdot\,;\xi_0,y_0,0) \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)$. Consequently, $\mathcal{H}$ satisfies (A2) and $w \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)$. Also notice that, by (4.4b), if $y(\cdot)$ is bounded, then $\xi(\cdot)$, and hence $x(\cdot)$, are bounded.

Combining Theorem 3.1 with the above findings yields the following corollary.

Corollary 4.2 *Assume that (4.2) and (4.3) are satisfied and let $\lambda > 0$ be given. Then, for all initial conditions $x_0 \in C([-h,0],\mathbb{R}^n)$, $k_0 \in \mathbb{R}$, all $w \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)$ and all reference signals $y_r \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^m)$, the closed-loop system given by (4.1) and (1.2) has the following properties:*

(i) *The solution $(x(\cdot),k(\cdot))$ exists on $[0,\infty)$ and is unique,*

(ii) *$x(\cdot)$ is bounded and $\lim_{t\to\infty} k(t)$ exists and is finite,*

(iii) $\limsup_{t\to\infty} \|y_r(t) - y(t)\| \le \lambda$.

4.2 Integrodifferential systems

Another interesting class of systems covered by Theorem 3.1 is a class of integrodifferential systems. Consider the system

$$\dot{x} = A * x + Bu\,, \qquad x(0) = x_0 \in \mathbb{R}^n\,, \tag{4.5a}$$

$$y = Cx \tag{4.5b}$$

where $A \in M(\mathbb{R}_+,\mathbb{R}^{n\times n})$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. The Volterra integrodifferential system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \int_0^t A_1(t-\tau)x(\tau)\,d\tau + Bu(t)\,, \qquad x(0) = x_0 \in \mathbb{R}^n\,, \\ y(t) &= Cx(t) \end{aligned}$$

where $A_0 \in \mathbb{R}^{n\times n}$ and $A_1 \in L^1(\mathbb{R}_+,\mathbb{R}^{n\times n})$, is obviously a special case of (4.5).

As in Subsection 4.1 we assume that

$$\sigma(CB) \subset \mathbb{C}_0 \tag{4.6}$$

and

$$\det \begin{pmatrix} sI - \hat{A}(s) & -B \\ C & 0 \end{pmatrix} \neq 0\,, \qquad \text{for all } s \in \mathbb{C}_0^{cl}\,, \tag{4.7}$$

where $\hat{A}(s) := \int_0^\infty \exp(-s\tau)dA(\tau)$.

Combining standard results from the theory of integrodifferential equations (see [3]) with ideas similar to those in Subsection 4.1, the following analogue of Corollary 4.2 for integrodifferential systems can be proved.

Corollary 4.3 *Assume that (4.6) and (4.7) are satisfied and let $\lambda > 0$ be given. Then, for all initial conditions $x_0 \in \mathbb{R}^n$, $k_0 \in \mathbb{R}$, all $w \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)$ and all reference signals $y_r \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^m)$, the closed-loop system given by (4.5) and (1.2) has the following properties:*

(i) *The solution $(x(\cdot),k(\cdot))$ exists on $[0,\infty)$ and is unique,*

(ii) *$x(\cdot)$ is bounded and $\lim_{t\to\infty} k(t)$ exists and is finite,*

(iii) $\limsup_{t\to\infty} \|y_r(t) - y(t)\| \le \lambda$.

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