FOCUSING BLOW-UP FOR QUASILINEAR PARABOLIC EQUATIONS

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Abstract

We study the behaviour of the nonnegative blowing up solutions to the quasilinear parabolic equation with a typical reaction-diffusion right-hand side and with a singularity in the space variable which takes the form

$$x^q u_t = (u^m)_{xx} + u^p, \quad x \in (0,1), \;\; t > 0,$$

where $m \geq 1$, $p > 1$ are arbitrary constants, in the *critical* exponent case $q = (p-1)/m > 0$. We impose zero Dirichlet boundary conditions at the singular point $x = 0$ and at $x = 1$, and take large initial data. For a class of 'concave' initial functions we prove focusing at the origin of the solutions as t approaches the blow-up time T in the sense that $x = 0$ belongs to the blow-up set. The proof is based on an application of the intersection comparison method with an explicit 'separable' solution which has the same blow-up time as u. The method has a natural generalization to the case of more general nonlinearities in the equation. A description of different fine structures of blow-up patterns in the semilinear case $m = 1$ and in the quasilinear one $m > 1$ is also presented. A numerical study of the semilinear equation is also made using an adaptive collocation method. This is shown to give very close agreement with the fine structure predicted and allows us to make some conjectures about the general behaviour.

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1. Introduction

1.1. Statement of the problem and recent results. Studies of nonlinear phenomena associated with the quasilinear diffusion operators in an inhomogeneous media of the type $\rho(x)u_t - \Delta\phi(u)$ with the parabolicity condition $\phi' \geq 0$, such that the spatially dependent coefficient $\rho(x)$ vanishes somewhere on the boundary of the given bounded or unbounded domain, are popular in mathematical literature and are motivated by several applications in physics and mechanics. The condition on $\rho(x)$ implies that in the domain of uniform ellipticity of $\Delta\phi(u)$, the parabolic operator is uniformly parabolic from below (and no degeneracy on the boundary occurs) but not from above. This kind of singularity of the operator provides interesting new effects. For the porous medium (PM) operator with $\phi(u) = u^m$, $m > 1$, a well-known example comes from the plasma physics, where the inhomogeneity in $\mathbf{R}^N$ is approximately given by a power function $\rho(x) \sim |x|^q$, $q < 0$, for large $|x|$ (so that the density $\rho(x)$ vanishes at infinity), see [KR] and references therein. Another example is given by channel flows which are described by a one-dimensional operator with a lower-order term $xu_t - u_{xx} - e^u$ on the bounded interval $(0, 1)$ [O], see details below. In the later case the density $\rho(x)$ vanishes at a finite boundary point.

In the present paper we consider the blow-up behaviour of nonnegative solutions to the one-dimensional quasilinear parabolic equations with a singularity at the origin in the space variable and with an extra lower-order combustion term of the form

$$x^q u_t = (\phi(u))_{xx} + \psi(u) \quad \text{for } x \in (0, 1), \ \ t > 0, \tag{1.1}$$

where $q > 0$ is a fixed constant and ϕ and ψ are given functions with sufficient regularity (say, $\phi, \ \psi \in C^\infty((0, \infty)) \cap C^1([0, \infty)))$ to give local existence and uniqueness satisfying $\phi'(u) > 0$, $\phi''(u) \geq 0$, $\psi(u) > 0$ for $u > 0$, $\phi(0) = 0$, $\psi(0) \geq 0$, with the inverse function satisfying $\phi^{-1} : [0, \infty) \to [0, \infty)$. A typical example of such a problem is

$$\phi(u) = u^m, \quad \psi(u) = u^p, \ \text{ with } m \geq 1, \ \ p > 1. \tag{1.2}$$

We impose the following boundary and initial conditions

$$u = 0 \ \text{ for } x = 0 \text{ and } 1, \, t > 0; \ \ u(x, 0) = u_0(x) > 0 \text{ on } (0, 1), \tag{1.3}$$

where the function u_0 is assumed to be sufficiently smooth on $[0, 1]$, $u_0 = 0$ for $x = 0$ and $x = 1$. For instance, in the case of the degenerate equation with coefficients (1.2) we assume that $|(u_0^{m-1})_x| \leq C$.

The physical motivation for studying the general equation (1.1) comes from the mathematical model of the flow in a channel of a fluid whose viscosity is temperature-dependent:

$$xu_t = u_{xx} + e^u \quad \text{in } (0, 1) \times (0, \infty), \tag{1.4}$$

derived in [O]. The superlinear term e^u (which can be replaced by the power function u^p with $p > 1$ [L]) implies that sufficiently large solutions blow-up (in the L^∞-norm) in a finite time $T = T(u_0) > 0$. The singularity of the linear parabolic operator at $x = 0$ means that the origin plays an important role in the asymptotic behaviour of large solutions. The results in [F] (see also a formal analysis of [L]) proved for the equation

$$xu_t = u_{xx} + u^p, \quad p > 1, \tag{1.5}$$

say that in the exponential range $1 < p \leq 2$ a large solution with initial data u_0 satisfying the condition $(u_0(x)/x)' \leq 0$ in $(0, 1)$ blows up at the origin only. More precisely, the blow-up set $B = B(u_0)$ defined in the standard way [FM]

$$B = \{x \in [0, 1] : \ \exists \text{ sequences } \{x_n\} \to x, \ \ \{t_n\} \to T^- \text{ such that } u(x_n, t_n) \to \infty\},$$

satisfies $B = \{0\}$. It is worth mentioning that under the above hypothesis on u_0 the initial position of the maxima of the initial function $u_0(x)$ on the interval $(0, 1)$ plays no role in the ultimate location of the blow-up point.

We emphasise this important property: there exists a *critical* exponent, $p = 2$ for equation (1.5), such that due to the *singularity* of the parabolic operator at $x = 0$ we observe *focusing* of the blowing up solutions at the origin provided that the initial data is sufficiently concave.

1.2. Main results. In this paper we extend the results of [F], [L] by studying the phenomenon of focusing for the general quasilinear equation (1.1) in the most delicate critical case. For the equation with power nonlinearities (1.1), (1.2) the critical case arises when $q = (p - 1)/m$. We show that the effect of focusing to the singular point depends on the mutual (geometric) location of the initial data $u_0(x)$ and the initial function of an explicit 'separable' solution of the form $v(x, t) = \phi^{-1}(x \cdot l(t))$, where $l(t)$ is a function which has the same blow-up time $T = T(u_0)$. In terms of the flux representation we have that $\phi(v) = x \cdot l(t)$ so that this solutions is linear in x and therefore the diffusion term vanishes identically. Moreover, we prove that focusing to the singular point happens under the following condition:

(FOC) the number of intersections $J_T(0)$ of the functions $u_0(x)$ and $v(x, 0)$ is exactly one,

so that $J_T(0)$ is minimally possible. It follows from the Maximum Principle that we cannot have $J_T(0) = 0$ for solutions which have the same blow-up time (a fixed singularity in time). See Sections 2 and 3.

This geometric intersection condition of blow-up focusing applies to general equations (1.1) and, in the case of (1.5), is much weaker than that in [F] which is essentially pointwise (a differential inequality). The focusing assumption is sharp. If (FOC) is violated then in principle blow-up can occur at an interior point. We give some numerical evidence to demonstrate this.

In the next sections we study the fine structure of focusing blow-up for (1.1) with nonlinearities (1.2) in the critical case. We show that it is quite different in the semilinear, $m = 1$ (non-self-similar), and quasilinear, $m > 1$ (self-similar), cases. In Section 4 we give a description of focusing blow-up for the semilinear equation

$$x^2 u_t = u_{xx} + u^3, \tag{1.6}$$

for which some preliminary formal and numerical investigations have been made in [BCH]. In this case there exists three matching regions and the asymptotic behaviour is governed by a certain singularly perturbed semilinear Hamilton-Jacobi equation.

In Section 5 we describe a sophisticated numerical method for solving semilinear parabolic equations with blow-up in time and singularity in the space variable. We then use this method to investigate the solutions of (1.6) and demonstrate that it accurately reproduces the asymptotic profiles *in all three matching regions.* We also derive some further conjectures about its behaviour related to initial data and the critical exponents.

In contrast, in Sections 6 we show that in the quasilinear case $m > 1$,

$$x^2 u_t = (u^m)_{xx} + u^p, \tag{1.7}$$

with $p = 2m + 1$ (if we put the density function x^q into the left-hand side then the critical exponent is $q = (p-1)/m$) the asymptotics in the inner region is purely parabolic, i.e., there exists a nontrivial self-similar solution which is stable as $t \to T$ and $x \to 0$. In this case in order to describe a stable blow-up pattern, only two matching regions are involved.

2. Local existence and an explicit family of spatially linear solutions

2.1. Preliminaries. Observing that the quasilinear equation (1.1) is uniformly parabolic from below in any domain where the solution is bounded away from zero, we conclude that the construction of a (weak) solution provides no difficulties. One can construct the local solution u as the limit as $\varepsilon \to 0$ of the (monotone) sequence $\{u_\varepsilon\}$ of weak solutions to (1.1) in the domain $(\varepsilon, 0) \times (0, t_0]$, where no singularity occurs, with $u_\varepsilon(\varepsilon, t) \equiv 0$ and naturally truncated initial data. For (1.5) such a construction was performed in [F].

On the other hand, if we consider $\{u_\varepsilon\}$ to be the weak solutions in $(0,1) \times (0, t_0]$ of the regularized equation

$$(x+\varepsilon)^q u_t = (\phi(u))_{xx} + \psi(u), \tag{2.1}$$

one can derive standard interior Bernstein-type estimates on $(\delta, 1) \times (\delta, t_0]$, $\delta > 0$ small. For the coefficients (1.2), when (2.1) is a degenerate equation and an extra well-known regularization is necessary, this means that $|(u_\varepsilon^{m-1})_x| \le C(\delta)$. This estimate is known to be optimal for the weak solutions compactly supported on the interval $(0, 1]$. See the survey [K]. Since both the equations (1.1) and (2.1) admit the same set of stationary solutions $\{U\}$ satisfying

$$\phi(U)_{xx} + \psi(U) = 0, \tag{2.2}$$

a boundary gradient uniform estimate at the singular point $x = 0$ is derived by the standard comparison with $U(x)$ such that $U(0) = 0$ and $U'(0) \gg 1$. Then on $(0, t_0]$ we have that locally bounded solutions u_ε satisfy $u_\varepsilon(x,t) \le U(x)$ for $x \in (0, \delta)$ whence a gradient estimate on the boundary. For (2.1), (1.2) we obtain that $(u_\varepsilon^m)_x(0,t) \le C$. Finally, combining the boundary estimate with the interior one we derive the uniform estimate

$|(u_\varepsilon^m)_x| \le C$ for $t \le t_0$. The rest of the regularity properties, uniqueness and comparison results are proved by standard techniques from the theory of the porous-medium equation. See references in the survey [K] and in the book [SGK]. The general nonlinearities in (1.1) are treated in a similar way. The gradient estimate is proved in terms of the flux function $\phi(u)$. Observe that *continuity* of the boundary flux $(\phi(u))_x(0,t)$ at a fixed $t_1 \in [0,t_0]$ can be proved by local in x comparison with the stationary solutions $U_\pm(x)$ satisfying $U_\pm(0)=0$ and $\phi(U_\pm)_x(0) = \phi(u)_x(0,t_1) \pm \delta$, $\delta > 0$ being small. Then assuming that the derivative has a jump in time at $t = t_1^+$, by the usual comparison we arrive at a contradiction.

Finally we arrive at the local continuous weak solution $u(x,t)$. It follows from the standard parabolic theory that if it is essentially local in time then there exists a finite time $T > 0$ such that

$$\sup_{x\in(0,1)} u(x,t) \to \infty \quad \text{as } t \to T^-. \tag{2.3}$$

2.2. A family of spatially 'linear' solutions. Let us look for a particular blowing up solution $v(x,t)$ to (1.1) such that the diffusion operator vanishes identically on v: $(\phi(v))_{xx} = 0$ a.e. Taking into account the boundary condition at the singularity $x=0$, this means that the function $\phi(v)$ is linear in x so that there exists a function $l(t)$ with $\phi(v) = x\,l(t)$ or

$$v(x,t) = \phi^{-1}(x\,l(t)). \tag{2.4}$$

Substituting (2.4) into (1.1) we obtain the identity

$$x^{q+1}\frac{dl}{dt} = (\psi\phi')(\phi^{-1}(xl)). \tag{2.5}$$

Therefore, such a solution exists if

$$\psi(s)\phi'(s) = \nu\phi^{q+1}(s), \quad s>0; \quad \nu = \text{const} > 0, \tag{2.6}$$

and then the function $l(t)$ satisfies the ODE

$$\frac{dl}{dt} = \nu l^{q+1}, \quad t > 0. \tag{2.7}$$

Finally, under hypothesis (2.6) we arrive at the following family of explicit blowing-up solutions

$$v(x,t;T) = \phi^{-1}[x(\nu q(T-t))^{-1/q}], \quad t < T, \tag{2.8}$$

where the blow-up time $T \in \mathbf{R}$ is a free parameter. For equation (1.1), (1.2), the condition (2.6) holds for the critical exponent $q = (p-1)/m$ with $\nu = m$. The linear solutions are given by

$$v(x,t;T) = kx^{1/m}(T-t)^{-1/(p-1)}, \quad k = (p-1)^{-1/(p-1)}. \tag{2.9}$$

Remark. Looking for a more general separable solution to (1.1), (1.2) of the form

$$\tilde{v}(x,t) = x^{\alpha} n(t) \quad (\alpha > 0) \tag{2.10}$$

satisfying the boundary condition at the singular point, we find that there are the two different possibilities: (i) $\alpha m = 1$ already studied, and (ii)

$$\alpha = 2/(m-p), \quad q = 2(p-1)/(m-p) \quad (p < m). \tag{2.11}$$

In this case the PME-term does not vanish and $n(t)$ solves

$$n' = \frac{2m(p+m)}{(m-p)^2}\, n^m + n^p. \tag{2.12}$$

Such a solution exists only if $p < m$. By comparison with stationary solutions of (2.2), (1.2) with $U(0) = 0$ and $(U^m)_x(0) \gg 1$ one can prove that in this range $1 < p < m$ the solution is uniformly bounded, $u(x,t) \le U(x)$ in $(0,1) \times \mathbf{R}_+$, i.e., no blow-up occurs. On the other hand $\tilde{v}$ plays an important role in the problem stated in the half-line $x \in (0,\infty)$. Let us note that in this case blow-up is indeed *global* , i.e., $B = [0,\infty)$. See a similar proof in [SGK, p. 418].

3. Focusing: intersection comparison

In this section we establish a sufficient condition of the focusing blow-up when $0 \in B(u_0)$. The main ingredients of the proof are an intersection comparison technique via the Sturmian argument and a form of the Hopf Boundary Lemma for a linear degenerate parabolic equation.

3.1. A Sturmian argument. Consider the two solutions $u(x,t)$ and $v(x,t)$ (defined above) to (2.2), (1.2) on $(0,1)\times(0,T)$. For a fixed $t \in [0,T)$ denote by $J_T(t)$ the number of intersections in x of the profiles, or the number of sign changes of the difference $w(x,t) = u(x,t) - v(x,t)$. Using the construction of the solutions via the approximating problems (which are uniformly parabolic on the positivity domains of both the solutions) and taking into account the fact that no new intersections appear on the regular end point $x = 1$, where $v > 0 = u$, we conclude that the Sturmian property is valid such that if $J_T(0) < \infty$ then

$$J_T(t) \text{ is nonincreasing on } (0,T), \text{ and in particular } J_T(t) \le J_T(0). \tag{3.1}$$

See the list of references in [SGK, Chapter 4].

3.2. Intersection comparison of solution with coinciding blow-up times. In order to compare solutions $u(x,t)$ and $v(x,t) \equiv v(x,t;T)$ having the same blow-up time $T = T(u_0) > 0$, we need an extra intersection comparison property which is treated as a lower bound on the number of intersections unlike the upper one in (3.1).

Lemma 3.1. *Let (2.6) hold and assume that $u(x,t)$ and $v(x,t;T)$ have the same blow-up time T. Then*

$$J_T(t) \geq 1 \quad \textit{for all } t \in [0,T). \tag{3.2}$$

Proof. We argue by contradiction. Assume that there exists $t_1 \in (0,T)$ such that

$$J_T(t_1) = 0, \quad (\text{i.e., } u(x,t_1) \leq v(x,t_1) \text{ on } [0,1]). \tag{3.3}$$

Then by comparison

$$u \leq v \;\; (u \not\equiv v) \quad \text{on } (0,1) \times [t_1, T). \tag{3.4}$$

We now prove that the solutions become 'separated' at the singular point $x = 0$ immediately at $t = t_1^+$ in the sense of the fluxes on the boundary. This is a result similar to the Hopf Boundary Lemma for the (possibly degenerate) equation (1.1). We need to consider the most delicate case where $u(x,t_1) > 0$ for all small $x > 0$ and moreover the fluxes coincide at the boundary. The functions $W = \phi(u)$ and $w = \phi(v)$ solves the equation

$$x^q w_t = a(w) w_{xx} + f(w), \quad a(w) = \phi'(\phi^{-1}(w)), \;\; f(w) = (\psi\phi')(\phi^{-1}(w)). \tag{3.5}$$

Subtracting both the equations and using the fact that $w_{xx} \equiv 0$ and $f'(w) \geq 0$ by (2.6) we obtain that the difference $z = w - W \geq 0$ satisfies the inequality

$$x^q z_t = a(W) z_{xx} + f(w) - f(W) \geq a(W) z_{xx} \tag{3.6}$$

in the classical sense for small $x > 0$ and $t - t_1 > 0$. Assume for contradiction that

$$z_x(0,t) = 0 \quad \text{for all } t - t_1 > 0 \text{ small}. \tag{3.7}$$

Recall that due to the continuity of the fluxes, which is proved by comparison, see Section 2 (see also general regularity properties for the PME's in [K]), we have that $(c_1 - \varepsilon)x \leq W(x,t) \leq (c_1 + \varepsilon)x$ for both $x > 0$ and $t - t_1 > 0$ small, where $c_1 = w_x(0,t_1) > 0$.

We now construct a subsolution of the parabolic differential inequality (3.6). Namely, since by assumption the function $a(W)$ is increasing, a subsolution $\underline{z}$ satisfies the equation

$$x^q \underline{z}_t = a((c_1 - \varepsilon)x) \underline{z}_{xx},$$

provided that $\underline{z}(x,\cdot)$ is a convex function, $\underline{z}_{xx} \geq 0$. Indeed, we have

$$x^q \underline{z}_t - a(W) \underline{z}_{xx} = [a((c_1 - \varepsilon)x) - a(W)] \underline{z}_{xx} \leq 0.$$

Therefore, using rescaling in x we need to prove Hopf's Boundary Lemma for the linear singular parabolic equation

$$z_t = g(x) z_{xx} \quad \text{for small } x > 0, \, t > t_1, \text{ where } \quad g(x) = \frac{a(x)}{x^q}. \tag{3.8}$$

We begin with a simpler case which reduces to the classical result for the heat equation.

(i) *(3.8) is uniformly parabolic.* This means that $g(x)$ does not vanish at the singular point, and $g(x) \geq \nu_1 > 0$ for all small $x > 0$. In view of (2.6) this means that

$$\limsup_{s \to 0} \frac{\psi(s)}{\phi(s)} < \infty$$

(for (1.7) it is true if $p \geq m$). By (3.4) we may suppose that $z(x, t_1) \geq 0$ and $\not\equiv 0$ for small $x > 0$. Therefore, fixing an arbitrary (local in x) solution $h(x,t) > 0$ of the heat equation

$$h_t = \nu_1 h_{xx} \quad \text{in } (0,\delta) \times (t_1, t_1 + \delta), \quad h(x,t_1) \geq 0 \, (\not\equiv 0), \;\; h(0,t) \equiv 0, \tag{3.9}$$

such that $h_t \geq 0$ (then $h_{xx} \geq 0$ which is important for comparison with (3.8)) one can derive that $z \geq h$. Since $h > 0$ and $h_x(0,t) > 0$, by the Strong Maximum Principle for uniformly parabolic equations (or by Hopf's Boundary Lemma) this contradicts (3.7).

(ii) *(3.8) degenerates at the origin.* It follows from the hypothesis (2.6) and the above estimate of W near the singular point, $t \approx t_1$, that we may suppose that in (3.8) $g(x) \geq \nu_2 \, x$ for small $x > 0$, i.e., the degeneracy in x of equation (3.8) near the singular point is at least linear. (The justification is performed by comparison similar to that given below). More precisely, we have that as $x \to 0$

$$g(x) = \frac{\phi'(\phi^{-1}(x))}{x^q} = \nu \, \frac{x}{\psi(\phi^{-1}(x))} = \nu x(1 + o(1)),$$

provided that $\psi(0) > 0$. If $\psi(0) = 0$ then the type of degeneracy of the function $g(x)$ is superlinear. For instance, $g(x) \sim x^{1-p/m} \gg x$ as $x \to 0$ for equation (1.7). Therefore, we need to prove Hopf's Boundary Lemma for the degenerate equation

$$z_t = x z_{xx} \quad \text{for } x \in (0,1), \;\; t > 0; \quad z(0,t) \equiv 0. \tag{3.10}$$

The result is local in time and does not depend on the boundary condition at the right-hand end point $x = 1$. For instance, we put $z(1,t) \equiv 0$.

Lemma 3.2. *The Hopf Boundary Lemma is true for classical solutions to (3.10), i.e., if initial data satisfy $z(x,0) \geq 0, \; \not\equiv 0$, then*

$$z_x(0,t) > 0 \quad \text{for } t > 0. \tag{3.11}$$

End of proof of Lemma 3.1. Thus, we have from (3.4) that for any $t_2 - t_1 > 0$ small

$$u < v \quad \text{on } (0,1) \times [t_2, T), \text{ and } \phi(u)_x < \phi(v)_x \text{ at } x = 0, \; t \in [t_2, T). \tag{3.12}$$

Indeed, this contradicts the fact that both solutions have the same blow-up time $T(u_0)$. It follows from (3.12) and from the continuity of the explicit solutions (2.8) upon T that there exists a small $\tau > 0$ such that the inequalities in (3.12) at $t = t_2$ are still true if in the solution v we replace T by $T + \tau$. Then by comparison $u(x,t) \leq v(x,t;T+\tau)$ in

$(0,1)\times(0,T)$. Setting here $t=T^-$ we then obtain $u(x,T^-)\le v(x,T;T+\tau)<\infty$, i.e. $u(x,T^-)$ is uniformly bounded, whence the contradiction with (2.3). $\square$

Proof of Lemma 3.2. It is easy to see by comparison that the result holds if there exists a classical, nontrivial for $t>0$, convex in x solution to the degenerate equation (3.10) for small $x>0$ such that

$$z(x,0)\equiv 0 \quad \text{but } z(x,t)>0 \text{ and } z_x(0,t)>0 \text{ for } t>0. \tag{3.13}$$

Fix an increasing nonnegative function $p(t)\in C^\infty([0,1])$. Then one can construct a formal solution to equation (3.10) in the form of the series

$$\tilde z(x,t)=\sum_{k=0}^{\infty} c_{k+1}(t)x^{k+1}, \quad c_{k+1}(t)=\frac{k+1}{[(k+1)!]^2}p^{(k)}(t). \tag{3.14}$$

Setting $p(t)$ to be a maximally flat function which has the form

$$p(t)=\exp(-t^{-s}), \quad \text{with an integer } s>1, \tag{3.15}$$

we now prove that the series (3.14) defines a smooth solution satisfying (3.13). Moreover, $\tilde z_t\ge 0$ and $\tilde z_{xx}\ge 0$. One can see that

$$p^{(k)}(t)=\mathcal{P}_{s+k}(t^{-1})\,p(t), \tag{3.16}$$

where $\mathcal{P}_n(r)$ denotes a polynomial of the order n. We have that for $t>0$

$$|\mathcal{P}_{s+k}(t^{-1})|\le b_0 k\frac{(s+k-1)!}{(s-1)!}t^{-(s+k)}, \;\; b_0>0. \tag{3.17}$$

Finally, we arrive at the estimate

$$|c_{k+1}(t)|\le M_k t^{-(s+k)}p(t), \quad M_k=b_0 k\frac{k+1}{[(k+1)!]^2}\frac{(s+k-1)!}{(s-1)!}. \tag{3.18}$$

In particular, this implies that for a fixed $t>0$, the radius of convergence of the power series (3.14) is equal to infinity, so that $\tilde z\in C_x^\infty$. On the other hand one can check directly (or from equation (3.10)) that $\tilde z\in C_t^\infty$. Let us prove that $\tilde z(x,0)=0$ for $x>0$ small. One can calculate that uniformly for $t>0$

$$t^{-(s+k)}p(t)\equiv t\cdot t^{-(s+k+1)}p(t)\le t\,N_k, \quad N_k=\left(\frac{s+k+1}{s}\right)^{(s+k+1)/s} e^{-(s+k+1)/s}. \tag{3.19}$$

Therefore, from (3.18) and (3.19) one derives the estimate

$$|c_{k+1}(t)|\le M_k N_k\,t, \quad \text{with } M_kN_k=o(k^{-k(s-1)/2s}), \tag{3.20}$$

provided that $s>1$. Using (3.20) in (3.14) we deduce that $\tilde z(x,t)\le c\,t$ uniformly on compact subsets so that $\tilde z(x,0)=0$.

Finally, the result (3.11) then follows by comparison of classical solutions performed locally in x and t. Indeed, we have that $z(x,0)\ge\tilde z(x,0)$ on a certain interval $x\in(0,\delta)$ where $z(\delta,t)>0$ for small $t\ge 0$. Hence, $z>\tilde z$ for $x\in(0,\delta)$ for all small $t>0$, and hence

$$z_x(0,t)\ge\tilde z_x(0,t)\equiv\exp(-t^{-s})>0 \quad \text{for } t>0 \text{ small}, \tag{3.21}$$

which completes the proof. $\square$

Remark. Let us derive a *minimal* positive behaviour for the heat equation without degeneracy at $x = 0$,

$$z_t = z_{xx}, \quad x > 0, \ t > 0; \quad z(0, t) \equiv 0. \tag{3.22}$$

In order to determine the minimal behaviour we take the step-like initial function which is compactly supported from the left

$$\tilde{z}(x, 0) = \{0 \text{ on } (0, 1), \ 1 \text{ on } [1, \infty)\}. \tag{3.23}$$

The solution to (3.22), (3.23) is given by the heat potential

$$\tilde{z}(x, t) = \frac{1}{2\sqrt{\pi t}} \left(\int_{1-x}^{1+x} e^{-\eta^2/4t} d\eta \right), \tag{3.24}$$

whence the minimal behaviour of the derivative

$$\tilde{z}_x(0, t) = \frac{1}{\sqrt{\pi t}} e^{-1/4t}(1 + o(1)) \quad \text{as } t \to 0. \tag{3.25}$$

In fact, (3.25) approximately corresponds to (3.15) with $s = 1$. It would be interesting to evaluate an optimal minimal behaviour for the degenerate problem (3.10).

3.3. Focusing at the origin. We now prove the main result of this section based on the intersection comparison.

Theorem 3.3. *Let (2.6) hold and $T = T(u_0)$ be the blow-up time of the solution $u(x, t)$. Assume that*

$$J_T(0) = 1. \tag{3.26}$$

Then

$$0 \in B(u_0). \tag{3.27}$$

Proof. It follows from (3.1), (3.26) and Lemma 3.1 that

$$J_T(t) \equiv 1. \tag{3.28}$$

Therefore $u(x, t) \geq v(x, t)$ for small $x > 0$, and hence the fluxes at $x = 0$ satisfy

$$\phi(u)_x \geq \phi(v)_x = (\nu q(T - t))^{-1/q} \to \infty \quad \text{as } t \to T. \tag{3.29}$$

Hence, $u(x, t)$ is not bounded on $(0, \delta)$ as $t \to T^-$. Otherwise, if it is not true, by comparison with stationary solutions as in Section 2, one can derive a uniform upper flux bound at the origin contradicting the unboundedness conclusion (3.29). By the definition of the blow-up set, this implies (3.27). $\square$

Remark. There are at least two different types of blow-up with (3.27) being valid. For (1.1), (1.2) this is *single point blow-up* when $p > m$ and $B = \{0\}$, and *regional blow-up* where $p = m$ (then $q = (m-1)/m$) and $\text{meas}\,(B) > 0$. *Global* blow-up for $p \in (1, m)$ occurs in the half-line $x > 0$. Thus, a general classification of blow-up solutions for the singular equation (1.1), (1.2) stays the same as for the quasilinear equation

$$\text{(QHE)} \qquad u_t = (u^m)_{xx} + u^p,$$

see [SGK, Chapter 4].

4. The fine structure of focusing for the semilinear case

In this section we consider the semilinear equation (1.6). We assume that the solution $u(x,t)$ blows up at a finite $t = T < \infty$ and moreover, due to Theorem 3.3, (3.27) holds.

4.1. Self-similar rescaling in the inner region. We now perform a standard self-similar rescaling setting

$$\text{(4.1)} \qquad u(x,t) = (T-t)^{-1/4} w(y,s), \quad y = x/(T-t)^{1/4}, \quad s = -\log(T-t).$$

Then the rescaled function w solves the equation

$$\text{(4.2)} \qquad w_s = \mathbf{B}(w) = \frac{1}{y^2}\left[w_{yy} - \frac{y^2}{4}(yw_y + w) + w^3\right]$$

in the domain $Q_* = \{0 < y < e^{s/4},\ s_0 < s < \infty\}$, $s_0 = -\log T$, with the corresponding zero Dirichlet boundary conditions and initial data $w_0(y)$.

We study the ω-limit set of the orbit $\{w(\cdot, s),\ s > s_0\}$. Observe that the stationary equation

$$\text{(4.3)} \qquad \mathbf{B}(w) = 0 \quad \text{for } y > 0,\ w(0) = 0,$$

admits the linear solution

$$\text{(4.4)} \qquad w_l(y) = y/\sqrt{2}.$$

In the evolution stability analysis of the linear solution (4.4) we follow the proof given in [GP1], [GP2] where the semilinear heat equation

$$\text{(SHE)} \qquad u_t = u_{xx} + u^p, \quad p > 1,$$

was considered. We begin with first preliminary bounds.

Proposition 4.1. *Under the hypothesis (3.26)*

$$\text{(4.5)} \qquad 1/\sqrt{2} < w_y(0,s) < C_1 \quad \textit{for all } s > s_0.$$

Proof. The lower bound immediately follows from (3.28). The upper one is proved by comparison with a self-similar profile $f(y;\mu)$ satisfying equation (4.3) with the conditions

$$f = 0, \ f' = \mu > 0 \quad \text{at } y = 0. \tag{4.6}$$

Using the standard rescaling procedure one can see that for $\mu \gg 1$ the function $f(y;\mu)$ has a unique maxima and vanishes at a point $y = y_\mu$. In fact, f is approximately given by the solution to the equation without lower order linear terms: $w_{yy} + w^3 = 0$, with the same boundary conditions satisfying the given properties. Hence $y_\mu \sim \mu^{-1/2} \to 0$ as $\mu \to 0$. See a similar justification in [SGK, p. 185]. Then the profile $f(y;\mu)$ with a fixed $\mu \gg 1$ intersects (sufficiently smooth) initial data $w_0(y)$ on $I_\mu = (0, y_\mu)$ exactly once, $J_T(s_0) = 1$. Then taking into account that $f \equiv 0$ at the right-hand end point of I_μ, exactly as in the proof of Lemma 3.1 we conclude that $J_T(s) \equiv 1$. This means that $w_y(0,s) < f'(0) = \mu, \ \ s > s_0$, whence the result. $\square$

Proposition 4.2. *The rescaled function w is bounded on any set $(0,l) \times (s_1, \infty)$, with arbitrary $l > 0$.*

Proof. The proof is similar to the previous one. We compare $w(y,s)$ with a family of self-similar profiles $f(y;\mu,a)$, $a \in \mathbf{R}$, satisfying (4.6) for $a = 0$, and for $a > 0$ the conditions

$$f(a;\mu,a) = 0, \quad f_y'(a;\mu,a) = \mu. \tag{4.7}$$

We again check that for $\mu \gg 1$ the functions f have a unique maxima and vanish for all a not very large. Then given $l > 0$ one can see that for a fixed $\mu \gg 1$ the functions f are sufficiently 'steep' for all $a \le l$ to satisfy the inequality $J_T(s_0) \le 2$. Then by the Sturmian argument $J_T(s) \le 2$ for all $s > s_0$. This inequality which is true for all $a \le l$ makes it possible to prove that the solution $w(y,s)$ cannot be larger than any function $f(y;\mu,a)$ on $(0,l)$ (otherwise, these solutions to (1.6) have different blow-up times). See a similar comparison in [SGK, p. 261]. Thus,

$$w \le C = \sup_{y,\, a \le l} f(y;\mu,a)$$

on $(0,l) \times (s_0, \infty)$. $\square$

Using a standard regularity theory for uniformly parabolic equations, from Propositions 4.1 and 4.2 we conclude that

$$|w_y| \le C \quad \text{on } (0,l) \times (s_0, \infty). \tag{4.8}$$

We now introduce the ω-limit set

$$\omega(w_0) = \{g \in C : \ \exists \ \{s_k\} \to \infty \ \text{ such that } w(\cdot, s_k) \to g(\cdot) \text{ uniformly on any } (0,l)\},$$

and then $\omega(w_0) \neq \emptyset$ by the above estimates. In the first approximation, the asymptotic behaviour in the inner self-similar region is given by the following theorem. For the (SHE), a similar stabilization result was proved first in [GP1] (see also [GP2]). A more general stabilization analysis in [GK1] for the N-dimensional equation (SHE) was essentially based on compactness estimates on the rescaled orbits which were proved later in [GK2].

Theorem 4.3. *As* $s \to \infty$

$$w(y,s) \to y/\sqrt{2} \quad \textit{uniformly on compact subsets in } y. \tag{4.9}$$

We begin with the following nonexistence of a nontrivial solution which is proved in Appendix A.

Lemma 4.4. *w_l is the unique nontrivial solution to (4.3) on* $\mathbf{R}_+$.

A similar uniqueness result for the (SHE) was established first in [AL] by construction of a family of subsolutions to the corresponding ordinary differential equation. For the N-dimensional (SHE) for the subcritical range $p \le (N+2)/(N-2)_+$ uniqueness was proved later in [GK1] by means of a Pohozaev-type identity (cf. Appendix A).

Proof of Theorem 4.3. Equation (4.2) can be written in the divergence form

$$w_s = \mathbf{B}(w) \equiv \frac{1}{y^2}\left[\frac{1}{\rho}(\rho w_y)_y - \frac{y^2}{4}\,w + w^3\right], \quad \rho(y) = e^{-y^4/16}. \tag{4.10}$$

Observe that due to the zero boundary condition at $y = e^{s/4}$, the problem does not admit a natural exact Lyapunov function. Nevertheless, we show that there exists an approximate Lyapunov function which makes it possible to pass to the limit in (4.10). Using a natural extension of the technique from [FM] to the current problem (see e.g. [F]) we prove that the blow-up set is a compact subset of $[0,1)$. Therefore, $u(x,t)$ is uniformly bounded together with derivatives in a neighbourhood of $x = 1$, which means that

$$|w_y(e^{s/4}, s)| \le Ce^{-s/2}. \tag{4.11}$$

By the classical Bernstein estimates (see e.g. the survey [K]), it follows from (4.5) and (4.11) that we have uniform bounds of the form

$$|w_y| \le Ce^{s/4}, \quad |w_s| \le Ce^{s/2} \tag{4.12}$$

in Q_*. Multiplying (4.10) by $\tilde{\rho} = y^2\rho$, integrating by parts over $(0, e^{s/4}) \times (s_0, \infty)$ and taking into account the boundary and uniform bounds on the derivatives, we derive the convergence of the integral

$$\int_{s_0}^{\infty}\int_0^{e^{s/4}} \tilde{\rho}(w_s)^2 dyds < \infty. \tag{4.13}$$

This estimate of the time derivative w_s in $L^2_{\tilde{\rho}}(Q_*)$ makes it possible to pass to the limit in (4.10) as $s \to \infty$ (see e.g. a similar analysis in a quasilinear blow-up problem in [BBG]), and to prove that the ω-limit set of the orbit, $\omega(w_0)$, consists of stationary profiles only:

$$\omega(w_0) \subseteq \{f > 0,\ f \in C:\ \mathbf{B}(f) = 0,\ f(0) = 0\}. \tag{4.14}$$

From Lemma 4.4 we then conclude that $\omega(w_0) = w_l$ whence (4.9). $\square$

4.2. Exact rate of decay in the inner region. Using the ideas from [FK] and [HV], we now give a perturbation analysis of the stability of the stationary self-similar profile. Setting

$$w(y,s) = w_l(y) + Y(y,s), \tag{4.15}$$

where by Theorem 4.3 $Y(\cdot,s) \to 0$ as $s \to \infty$ uniformly on compact subsets, we arrive at the semilinear equation

$$Y_s = -\mathcal{A}Y + \frac{3}{\sqrt{2}y}Y^2 + \frac{1}{y^2}Y^3, \tag{4.16}$$

where $\mathcal{A}$ is the linear symmetric operator

$$\mathcal{A}Y = -\frac{1}{y^2}\,\frac{1}{\rho}(\rho Y_y)_y - \frac{5}{4}Y \equiv -\frac{1}{y^2}\left(Y_{yy} - \frac{y^3}{4}Y_y\right) - \frac{5}{4}Y, \tag{4.17}$$

with the corresponding domain from $L^2_{\tilde\rho}(\mathbf{R}_+)$. We denote by $(\cdot,\cdot)$ the scalar product in this weighted L^2-space. Both the end points of the interval $y \in (0,\infty)$ are singular for the operator $\mathcal{A}$. Consider first the right-hand end point $y=\infty$. Solving the equation $\mathcal{A}Y=0$ for $y \gg 1$ one can find two linearly independent solutions

$$Y_1(y) \sim y^5 \in L^2_{\tilde\rho}([1,\infty)), \quad Y_2(y) \sim y^{-5}e^{y^2/16} \notin L^2_{\tilde\rho}([1,\infty)). \tag{4.18}$$

Therefore, the end point $y=\infty$ corresponds to the limit point case of a singular endpoint [R, p. 205]. Similarly, the same is true for the left-hand end point $y=0$. Consider now the operator $\bar{\mathcal{A}} = \mathcal{A} + \frac{5}{4}I > 0$. Solving the equation $\bar{\mathcal{A}}Y = 0,\ y>0,\ Y(0)=0$, we obtain the unique solution

$$\bar Y(y) = \int_0^y \frac{d\eta}{\rho(\eta)} \notin L^2_{\tilde\rho},$$

so that $\ker\bar{\mathcal{A}} = 0$. Therefore $\bar{\mathcal{A}}^{-1}$ is well defined. Moreover, it is a Hilbert-Schmidt operator with a symmetric kernel $C \in L^2_{\tilde\rho}$. Then there exists $\{\bar\lambda_k\}$, a nondecreasing sequence of the eigenvalues of $\bar{\mathcal{A}}$, and the corresponding eigenfunctions $\{\bar\psi_k\}$ form an orthonormal basis in $L^2_{\tilde\rho}$. See e.g. [R].

According to the stability result given in Theorem 4.3, we consider the sequence of nonnegative eigenvalues $\{\lambda_i \ge 0,\ i=0,1,...\}$ of the operator $\mathcal{A}$ and the corresponding orthonormal sequence of eigenfunctions $\{\psi_i(y),\ i=0,1,...\}$. From this it follows that the first nonnegative eigenvalue satisfies $\lambda_0 = 0$ and it is simple (so that $\lambda_i > 0$ for all $i \ge 1$). The first eigenfunction has the form

$$\psi_0(y) = d_0\left(y - \frac{1}{20}y^5\right), \tag{4.19}$$

where the constant d_0 is chosen such that ψ_0 has the unit norm in $L^2_{\tilde\rho}(\mathbf{R}_+)$:

$$d_0 = \left(y - \frac{y^5}{20},\ y - \frac{y^5}{20}\right)^{-1/2} = \sqrt{\frac{5}{8\Gamma(1/4)}}, \tag{4.20}$$

$\Gamma(\cdot)$ being Euler's gamma function. Hence, the linearized problem (4.16) has a one-dimensional centre manifold given by the linear span $\mathcal{L}\{\psi_0\}$. We show that this centre manifold is attractive for the nonlinear perturbed equation (4.16). Looking for the solution to (4.16) and using the Hilbert-Schmidt theorem, we may suppose that the function $Y \to 0$ as $s \to \infty$ is given by the series

$$Y(y,s) = \sum_{i=0}^{\infty} a_i(s)\psi_k(y). \tag{4.21}$$

In order to predict the stable behaviour of the orbit $\{Y(\cdot,s)\}$ for $s \gg 1$ near the centre manifold, we perform a formal perturbation analysis similar to that in [VGH] for the (SHE) which was justified in [FK], [HV], etc. We impose an assumption which is a natural one for such problems that the modes on the stable manifold $\mathcal{L}\{\psi_1,\psi_2,...\}$ are exponentially small (due to the inequality $\lambda_i \geq \lambda_1 > 0$ for all $i \geq 2$ proved above). Then substituting (4.21) into equation (4.16), via the standard Fourier procedure we arrive at the equation on the first coefficient

$$a_0' = -\gamma_0 a_0^2 + O(a_0^3), \quad s \gg 1, \tag{4.22}$$

where

$$\gamma_0 = -\frac{3}{\sqrt{2}}(\psi_0^2/y, \psi_0) = \frac{3 \cdot 2^6}{25\sqrt{2}}\, d_0^3\, \Gamma\left(\frac{1}{4}\right). \tag{4.23}$$

Equation (4.22) implies that if $a_0(s)$ is not exponentially small (cf. a similar conclusion in the main result of [FK]) then

$$a_0(s) = \frac{1}{\gamma_0}\frac{1}{s} + ... \quad \text{as } s \to \infty. \tag{4.24}$$

Finally, we arrive at the following decay rate of the solutions in the inner self-similar domain

$$Y(y,s) = \frac{1}{\gamma_0 s}\psi_0(y) + ... \equiv c_0\left(y - \frac{y^5}{20}\right)\frac{1}{s} + ..., \quad c_0 = \frac{d_0}{\gamma_0}, \tag{4.25}$$

uniformly on compact subsets in y. In view of the change (4.15) we obtain for $s \gg 1$

$$w(y,s) = \frac{y}{\sqrt{2}}\left(1 + \sqrt{2}c_0\frac{1}{s} - \frac{\sqrt{2}c_0}{20}\frac{y^4}{s}\right) + ... \,. \tag{4.26}$$

4.3. The intermediate Hamilton-Jacobi region. The last term on the right-hand side of (4.26) implies that there exists an external (intermediate) region governed by the extended spatial variable (cf. ignition kernel)

$$z = y/s^{1/4}. \tag{4.27}$$

Then we introduce the new rescaled function (cf. [BCH])

$$f(z,s)=\frac{1}{s^{1/4}}w(zs^{1/4},s), \tag{4.28}$$

so that $f_z \equiv w_y$ (that well-corresponds to convergence (4.9)), which solves the nonautonomous equation

$$f_s=\mathcal{H}(f)+\frac{1}{s}\left(\frac{1}{z^2}f_{zz}+\frac{1}{4}f_z z-\frac{1}{4}f\right), \tag{4.29}$$

where $\mathcal{H}$ is the first-order operator

$$\mathcal{H}(f)=\frac{1}{z^2}\left[-\frac{z^2}{4}(f_z z+f)+f^3\right]. \tag{4.30}$$

Observe that a hot-spot variable of the type (4.27) with a slow logarithmic factor for the (SHE) was introduced first in [HSS] in the case $p=3$ and later was used in [GP1], [GP2] for more general equation in order to prove an exact upper estimate of the blow-up solutions.

For $s \gg 1$ equation (4.29) is an asymptotically small singular (second-order) perturbation of the first-order Hamilton-Jacobi equation

$$f_s=\mathcal{H}(f). \tag{4.31}$$

The perturbation term in (4.29) is of order $1/s$, i.e. it is not integrable at $s=\infty$. A general approach to passing to the limit in singular perturbed dynamical systems of the type (4.29) governed by the Hamilton-Jacobi equations can be found in [GV]. Assuming that the passage to the limit as $s \to \infty$ is justified (e.g. all the three hypotheses in Section 3 of [GV] are valid including that on the uniform stability in a singular weighted metric space of the ω-limit set to the unperturbed equation (4.31), see a similar result in [GV, Section 3]), we conclude that $f(\cdot,s) \to g(\cdot)$ uniformly on compact subsets in z, where g solves the limit stationary equation

$$\mathcal{H}(g)=0, \ \ z>0; \ \ g(0)=0. \tag{4.32}$$

Solving (4.32) explicitly we deduce that there exists a family of possible stationary profiles (cf. [BCH])

$$g(z)=\frac{z}{\sqrt{2}}(1+\alpha z^4)^{-1/2}, \quad \alpha \geq 0 \text{ is arbitrary.} \tag{4.33}$$

Finally, in order to match (4.33) with (4.26) we derive the expansion of (4.33) for small z to get from (4.28) that as $s \to \infty$

$$w(y,s)=\frac{z}{\sqrt{2}}\left(1-\frac{\alpha}{2}z^4\right)+\ldots=\frac{y}{\sqrt{2}}\left(1-\frac{\alpha}{2}\frac{y^4}{s}\right)+\ldots. \tag{4.34}$$

Comparing the last terms in (4.34) and (4.26) we obtain that the constant α in (4.33) is uniquely determined:

$$\alpha = \frac{\sqrt{2}c_0}{10} = \frac{1}{24}. \tag{4.35}$$

4.4. Outer region. We now extend the asymptotic behaviour given by the limit profile (4.33) in the intermediate region to an outer one where x is assumed to be small. It follows from (4.33) that

$$g(z) = \frac{1}{\sqrt{2\alpha}}\,\frac{1}{z} + \ldots \quad \text{for } z \gg 1. \tag{4.36}$$

We fix $0 < x \ll 1$ and $t \approx T$ such that

$$\frac{z}{x} = [(T-t)|\log(T-t)|]^{-1/4} \gg 1 \quad \text{and } z \gg 1. \tag{4.37}$$

Then solving (4.37) with respect to $T-t$ we obtain that

$$T - t = \left(\frac{x}{z}\right)^4 \frac{1}{4|\log\, x|}\,(1+o(1)), \quad o(1) = O\left(\frac{\log\, x}{\log\, z}\right). \tag{4.38}$$

Finally, since by (4.1) and (4.28)

$$u(x,t) = (T-t)^{-1/4}|\log(T-t)|^{1/4} f(z,s), \tag{4.39}$$

using the convergence $f(\cdot,s) \to g(\cdot)$ and (4.36), (4.38) we conclude from (4.39) with $T-t$ taken from (4.38) that as $t \to T^-$

$$u(x,t) \to 4\sqrt{3}\, x^{-1}\, |\log\, x|^{1/2}(1+o(1)) \tag{4.40}$$

for any fixed $x > 0$ small.

Remark. We conjecture that the higher-order blow-up patterns with the behaviour in the inner region given by the expression $Y(y,s) \sim e^{-\lambda_i s}\psi_i(y),\ i \geq 1$ (cf. (4.25)) do occur. These patterns are expected to be unstable and play a border role between solutions with focusing and with interior blow-up. See also Section 5.

5. Numerical results via the moving mesh method

Our understanding of the behaviour of the solutions of (1.1), (1.2), when $m = 1$, can be enhanced by some careful numerical computations. We make these calculations both to verify the asymptotic profiles described in Section 4 and also to investigate the transient behaviour from a variety of initial data.

5.1. The numerical method. The approximation of the solution using a numerical method is a challenging task owing to the formation of the singularity as t approaches the blow-up time. To obtain an accurate solution for values of t close to the blow-up time for which the asymptotic formulae are expected to be sharp, it is necessary to use a method which adapts itself to the evolving solution. For this paper we consider an adaptation of the moving mesh method described in [BHR] which was employed to solve the non-singular blow-up problem $u_t = u_{xx} + u^p$.

To apply the moving mesh method, a sequence of $N+1$ mesh points $X_i(t)$ are placed in the interval $[0,1]$ so that

$$X_0(t) = 0, \quad X_i(t) < X_{i+1}(t), \quad X_N(t) = 1.$$

For all calculations we take $N = 41$. The function $u(x,t)$ is then approximated by a piecewise cubic interpolant with knots at the mesh points. The partial differential equation (1.1), (1.2) is discretised in space by using the collocation method to give a sequence of ordinary differential equations for a discrete approximation $U_i(t)$ to $u(X_i(t),t)$. The ordinary differential equations are then solved in time by using the adaptive backwards difference method (BDF) in the stiff ODE solving package DDASSL [Pet]. Simultaneously with this calculation the mesh points are moved so that a higher density of points are placed close to the singularity. This is achieved by solving a sequence of differential equations for the mesh points which are derived from a discretisation of the integral equation

$$\frac{dX_i}{dt} = -\frac{1}{\tau}\left[\int_0^{X_i} M dx - \frac{i}{N}\int_0^1 M dx\right], \quad i = 1,2,...,N-1, \tag{5.1}$$

where $M(u)$ is a suitable monitor function which is large where a greater density of mesh points is needed and τ is a small constant. A precise description of this procedure is given in [HRR] for which we use the method described there as MMPDE6. This method is a generalisation of the approach described first in [DD] which aims to equidistribute the function M between the mesh points. Crucial to the success of this method is the choice of M. It is shown in [BHR] that an effective choice is one which leaves the equation (5.1) invariant to the same group of transformations as the underlying partial differential equation. For solutions of (1.1) with focusing at the origin this gives

$$M(u) = u^{(q+2)(p-1)/2}. \tag{5.2}$$

This is what we will mainly use for our investigations. For solutions which do not focus at the origin, but instead blow-up at an interior point we take

$$M(u) = u^{(p-1)}. \tag{5.3}$$

Details of this construction are given in [BHR], [BCH]. Observe that both of the monitor functions concentrate mesh points where the solution is large.

The discretisation using the collocation algorithm, the discretisation of the equation (5.1) using a centred finite difference method (with subsequent smoothing) and the solution of the resulting ordinary differential equations was performed automatically using the code MOVCOL written by Huang and Russell [HR].

The results obtained by this method are in very close agreement with the asymptotic predictions. This both corroborates our asymptotic calculations and demonstrates the robustness and accuracy of the numerical method used. It is worth noting that a non-adaptive numerical method could only resolve the asymptotic structures with a very much larger number of mesh points.

5.2. Results for the critical problem. For our first investigation we consider the critical problem discussed in Section 4 looking both at initial data which leads to focusing and data which does not

5.2.1. Focusing. Following Section 4 we take $q = 2,\ p = 3$ and consider the concave initial data

$$u(0, x) = 20\sin(\pi x).$$

This data satisfies the intersection condition, intersecting the linear profile once in the interval $(0, 1)$, and necessarily leads to focusing behaviour. Rescaling the equation (4.26) we then have

$$u_x(0,t) = \frac{1}{\sqrt{2}(T-t)^{1/2}}\left(1+\sqrt{2}c_0\frac{1}{s}\right)+ ..., \quad s = -\log(T-t). \tag{5.4}$$

The value of c_0 is given by (4.20), (4.25) as $c_0 = \sqrt{2}/4.8$. Hence, evaluating the constants in the expression (5.4) we have asymptotically that as $s \to \infty$

$$u_x^{-2}(0,t) \sim 2(T-t)(1 - 0.8333333/s) + ... \tag{5.5}$$

In the numerical calculations, the value of $u_x(0, t)$ can be estimated accurately and in Figure 1 we plot this as a function of t. It is clear from this figure that $u_x^{-2}(0, t)$ is very close to being a linear function of t, with the values asymptotic to a straight line of gradient 2. (The very good agreement is surprising given the existence of the additional term $0.8333333/s$ in (5.5).) Using this result we estimate the blow-up time for this problem to be

$$T \approx 0.000148.$$

The agreement is in fact so good that we are justified in using the approximation

$$(T-t) \approx \frac{1}{2u_x^2(0,t)} \tag{5.6}$$

for all subsequent calculations.

The asymptotic formula presented in (4.33) predicts that the function $u(x, t)$ should have a maximum value $u^*(t)$ at the point $x^*(t)$ such that

$$u^*(t) \sim \frac{\alpha^{-1/4}}{2}(T-t)^{-1/4}|\log(T-t)|^{1/4} \tag{5.7}$$

and

$$x^*(t) \sim \alpha^{-1/4}(T-t)^{1/4}|\log(T-t)|^{1/4}. \tag{5.8}$$

Furthermore, if $z = x/x^*(t)$ and $v(z,t) = u(x^*(t)z,t)/u^*(t)$ then, as $t \to T$, $v(z,t)$ should converge to the asymptotic profile

$$v(z, T^-) = \frac{\sqrt{2}z}{\sqrt{1+z^4}}. \tag{5.9}$$

In Figure 2 we present the computed function $v(z,t)$ for times at which $u^*(t) = 1.2 \times 10^4, 6 \times 10^3, 3 \times 10^3, 1.5 \times 10^3, 7.5 \times 10^2$ and 3.75×10^2. These functions are all close to the predicted asymptotic profile, with particularly good agreement close to the origin. From (5.7) we have

$$\log(u^*(T-t)^{1/4}) \approx \log(\alpha^{-1/4}/2) + \frac{1}{4}\log|\log(T-t)|, \tag{5.10}$$

with a similar result for x^*. It follows from this that the value of α can be estimated from the numerical calculations by plotting $\log(u^*(T-t)^{1/4})$ as a function of $\log|\log(T-t)|$. This is presented in Figure 3 with a similar graph for x^* given in Figure 4. After an initial transient, both graphs asymptotically linear with gradient 1/4. Estimating the intercept in both cases gives the same value of

$$\alpha = 1/22.2.$$

This is close to the predicted value of $\alpha = 1/24$ in (4.35) and is a sharp test of the accuracy of the algorithm.

As a final calculation in this section we compute the asymptotic profile $u(x, T^-)$. From (4.40) we expect that for $x \neq 0$

$$u(x, T^-) \sim \frac{4\sqrt{3}}{x}|\log x|^{1/2}.$$

Accordingly, to check this result, we compute the scaled function

$$V(x,t) \equiv xu(x,t)/|\log x|^{1/2}$$

for a sequence of times approaching T. These show a rapid convergence to the asymptotic profile. In Figure 5 we present the results of this calculation for a time t such that $\|u(\cdot,t)\|_\infty = 10^4$. It is clear from this figure that as x increases the function $v(x)$ is asymptotic to the constant value of 7.2. This value is close to the predicted constant of $4\sqrt{3} \equiv 6.928$.

5.2.2. Non-focusing. For this section we take as initial data the function

$$u_0(x) = Ax^3(1-x). \tag{5.11}$$

Unlike the initial data in the previous section, this data intersects the linear solution twice provided that A is sufficiently large and thus violates the sufficient condition for focusing at the origin. The resulting location for the blow-up point is thus unclear. Let us mention, however, that the proof that focusing never occurs for all $A \gg 1$, i.e., $0 \notin B(u_0)$, presents no difficulties. Taking a series of values for A we find that blow-up does not occur if $A < 35$ and in this case the solution decays to zero. For larger values of A, blow-up does occur and, for sufficiently large A, the blow-up point x^* is not at the origin. The intersection condition thus appears to be sharp. The values of x^* as a function of A estimated when $\|u\|_\infty$ took the indicated values are shown in the following table.

A	x^*	$\|u\|_\infty$
40	0.0023	0.45×10^4
50	0.0023	0.47×10^4
56	0.002	0.58×10^4
56.95	0.01857	0.23×10^4
57	0.1452	0.83×10^4
60	0.4384	0.13×10^5
80	0.61245	0.20×10^5
100	0.64128	0.20×10^5

These results appear to indicate that if A is close to 35 then (to within the resolution of the numerical method) the blow-up point x^* is at the origin, but that the value of x^* increases rapidly as A passes through the interval $(56, 57)$. From this we infer that there exists a constant $A_* \approx 57$ such that for $A < A_*$ focusing blow-up occurs at the origin, while if $A > A_*$ then the blow-up occurs at a single point interior to the domain. Continuity arguments imply that the critical value $A = A_*$ also belongs to the focusing case, but in this case the asymptotic behaviour is described by a higher-mode pattern. For data (5.11) which have exactly two intersections with the corresponding linear solution, in the inner region this pattern is expected to be generated by the second eigenfunction, $\psi_1(y)$ $(\lambda_1 > 0)$ with two zeros for $y > 0$.

5.3. Results for the sub-critical problem. By way of contrast with the previous calculations, we make a brief numerical study of the problem (1.1), (1.2), $m = 1$, when $q < p - 1$. As in the last subsection, the sufficient condition for focusing is violated. We firstly investigate the sharpness of this condition by taking $q = 1$, $p = 3$ and looking at the evolution of the solution from three different initial functions, namely

$$u_0(x) = 100\sin(\pi x), \quad u_0(x) = 100x(1-x)e^{-x}, \quad u_0(x) = 100x(1-x)e^{-(1-x)}.$$

The solution was calculated until $\|u\|_\infty = 1.6 \times 10^4$ and the resulting scaled profile $u(x,t)/\|u\|_\infty$ is presented as a function of x in Figure 6. In these figures it is clear that the blow-up point is always located interior to the interval. Observe the dense location of the mesh points within the peak. Closer study shows that the profile of the peak is identical to that obtained by solving the related uniformly parabolic equation

$$x^* u_t = u_{xx} + u^3.$$

For a second calculation we take $q = 2$ and $p = 3 + \epsilon$ where $\epsilon > 0$ is small, and set $u_0(x) = 20\sin(\pi x)$. Of interest here is the dependence of the blow-up point x^* upon the value of ϵ. To examine this, the equation was integrated until the solution satisfied $\|u\|_\infty = 1 \times 10^4$ and the spatial location of the maximum taken to be an approximation for x^*. Results were as follows

ϵ	x^*
8×10^{-1}	0.1356
8×10^{-2}	0.0152
8×10^{-3}	0.0110
8×10^{-4}	0.0107
8×10^{-5}	0.0106
8×10^{-6}	0.0107

Observe that x^* does decrease initially, but does not appear to converge to zero in the limit of $\epsilon \to 0$. At this stage, it is unclear whether this is a real effect or a consequence of the numerical approximation.

For this case we expect that the proof of the result that $0 \notin B(u_0)$ for all data, can be performed via the exterior gradient estimates proposed in [FM] (a generalization to quasilinear equations can be found in [BBG]), so that in the subcritical case $q < (p-1)/m$ (as for $q = 0$) the blow-up set is always a compact subset of $(0, 1)$.

6. Self-similar focusing for the quasilinear equation

We consider the quasilinear equation (1.7), with $m > 1$, $p = 2m + 1$. This admits the blowing up self-similar solution

$$u_*(x,t) = (T-t)^{-1/(3m+1)}\theta(y), \quad y = x(T-t)^{-(m+1)/2(3m+1)}, \tag{6.1}$$

where $\theta \geq 0$ solves the nonlinear degenerate ODE

$$\mathbf{B}(\theta) \equiv (\theta^m)'' - y^2\left[\frac{m+1}{2(3m+1)}\theta' y + \frac{1}{3m+1}\theta\right] + \theta^{2m+1} = 0, \;\; y > 0; \;\; \theta(0) = 0. \tag{6.2}$$

Equation (6.2) admits the 'linear' solution

$$\theta_l(y) = k y^{1/m}, \quad k = (2m)^{-1/2m}. \tag{6.3}$$

Using the self-similar structure in (6.1), we introduce the rescaled variables

$$u(x,t) = (T-t)^{-1/(3m+1)} w(y,s), \quad s = -\log(T-t), \tag{6.4}$$

where w solves the parabolic equation

$$w_s = \frac{1}{y^2}\mathbf{B}(w) \quad \text{in } Q_*. \tag{6.5}$$

By Theorem 3.3 and by the same comparison techniques we prove that Propositions 4.1 (with $1/\sqrt{2}$ replaced by k in the lower bound) and 4.2 hold. Then by the Bernstein estimates we establish that

$$|(w^{m-1})_y| \leq C \quad \text{on } (0,l) \times (s_0, \infty), \tag{6.6}$$

and define the ω-limit set as in Section 4.

This part of the compactness orbit analysis looks quite similar to the semilinear one. Nevertheless, there is a fundamental difference. Namely, we have the following result to be proved in Appendix B (cf. Lemma 4.4).

Lemma 6.1. *Let $m > 1$. Then problem (6.2) admits a solution $\theta_*(y) > 0$ for $y > 0$ such that $\theta_*(y) \to 0$ as $y \to \infty$.*

The fact that under suitable hypotheses

$$\omega(w_0) \subseteq \{\theta \geq 0,\ \theta \not\equiv \theta_l : \ \ \mathbf{B}(\theta) = 0,\ \ \theta(0) = 0\} \tag{6.7}$$

can be proved via a Lyapunov-type construction as in [G], see also [BDG]. A geometric construction of a Lyapunov-type function similar to that in [GPe] is also expected to work in this case. More precisely, let $\theta_*(y)$ be the 'maximal' self-similar profile which has the maximal boundary derivative $(\theta^m)'(0)$ and has a single intersection for $y > 0$ with the profile $\theta_l(y)$. The existence of such a maximal solution to (5.2) (as well as of the corresponding 'minimal' one) is proved by standard arguments, see Appendix B. Recall that uniqueness of the profile θ_* having a single intersection with θ_l seems to be an open problem. Let T be the blow-up time of the solution $u(x,t)$ with a continuous, compactly supported initial function u_0. Assume that the number of intersections $I_T(t)$ of the solutions $u(x,t)$ and (6.1), with the maximal profile θ_* and the same existence time T satisfies

$$I_T(0) = 1,$$

and $u_0(x) > u_*(x,0)$ for small $x > 0$. In view of the uniqueness of the maximal profile θ_*, from (6.7) we then conclude that the rescaled solution $w(y,\tau)$ converges as $\tau \to \infty$ to the maximal profile $\theta_*(y)$ uniformly on compact subsets.

The above results imply that the inner behaviour is purely self-similar and nontrivial, and hence no intermediate region occurs. In order to get the outer expansion of the final-time profile, we observe that for $y \gg 1$ (see Appendix B)

$$\theta_*(y) = C_* y^{-2/(m+1)}(1 + o(1)). \tag{6.8}$$

Therefore, given a fixed small $x > 0$, assuming that $\omega(w_0) = \{\theta_*\}$ and passing to the limit in (6.4) as $t \to T^-$, we obtain that the final-time profile has the form (cf. (4.40))

$$u(x,T^-) = C_* x^{-2/(m+1)}(1 + o(1)) \quad \text{as } x \to 0. \tag{6.9}$$

See a similar techniques in [BBG, Section 7].

Appendix A

Proof of Lemma 4.4. By means of well-known arguments (see e.g. references in [SGK, Chapter 4]) we conclude that if $w(y) > 0$ is a solution of (4.3) then either $w(y)/y \to 1/\sqrt{2}$ as $y \to \infty$ or

$$w(y) \to 0 \quad \text{as } y \to \infty. \tag{A.1}$$

It follows from the local analysis of the equation at infinity that the linear solution $w_l = y/\sqrt{2}$ is unstable as $y \to \infty$. Therefore, we need to prove that (4.3) does not admit a nonnegative solution w satisfying (A.1). Then it is proved that

$$|w(y)| \leq \frac{c}{y}, \quad |w'(y)| \leq \frac{c}{y^2} \quad \text{for } y \gg 1, \tag{A.2}$$

see [SGK], [BG] and references therein. These estimates make it possible to derive the following family of Pohozaev-type identities. Denote the linear differential operator

$$D_C w = y^2 w' + Cy, \tag{A.3}$$

where C is an arbitrary fixed constant.

Lemma A. *Let w be a solution of (4.3) satisfying (A.2). Then for a fixed C*

$$\int_0^\infty (D_C w)^2 + P(C) \int_0^\infty y^2 w^2 + 3 \int_0^\infty w^4 = 0, \tag{A.4}$$

where $P(C)$ is the trinomial

$$P(C) = -C^2 + 3C - \frac{5}{4}.$$

Proof. We use the technique goes back to the original paper [P] (cf. [GK1]). We multiply equation (4.3) by $y^2 w$ and by $2y^3 w'$ (Pohozaev's multiplier) respectively and integrate over $(0, \infty)$ using (A.2). We obtain the equalities

$$\begin{aligned} -\int (w')^2 + \frac{1}{8} \int y^2 w^2 + \int w^4 = 0, \\ -\int (w')^2 + \frac{3}{4} \int y^2 w^2 - \frac{1}{2} \int w^4 - \frac{1}{2} \int y^4 (w')^2 = 0. \end{aligned} \tag{A.5}$$

Subtracting we arrive at

$$\frac{1}{2} \int y^4 (w')^2 - \frac{5}{8} \int y^2 w^2 + \frac{3}{2} \int w^4 = 0. \tag{A.6}$$

Integrating the obvious identity

$$y^4 (w')^2 = (D_C w)^2 - C^2 y^2 w^2 - 2C y^3 w w',$$

over $(0, \infty)$ and substituting into the first term on the left-hand side of (A.6) we obtain (A.4). $\square$

End of proof. Setting $C = 1$ in (A.4) so that $P(1) > 0$ we have that all three terms on the left-hand side of (A.4) are nonnegative. Therefore $w \equiv 0$ is the unique solution satisfying (A.2). $\square$

Observe that (A.4) gives a contradiction for a range of the parameter, $C \in [0.5, 2.5]$, so that there exists an uncountable set of Pohozaev's inequalities leading to the nonexistence of a nontrivial solution.

Appendix B

Proof of Lemma 6.1. We apply a standard scheme to prove the existence. See [SGK, p. 190], [BG] and references therein. We omit technical details. The main conclusion, which is not valid for the semilinear case $m = 1$, concerns the oscillating property of the solutions of equation (6.2) about the special 'linear" solution (6.3). Namely, we fix a $\theta(y)$ satisfying (5.2) and the second boundary condition $(\theta^m)'(0) = k^m + \varepsilon$, where $|\varepsilon| \ll 1$ is a parameter. Observe that the solution depends continuously upon ε on compact subsets in $y > 0$. Therefore, in order to study the behaviour of θ as $\varepsilon \to 0$ we set

$$\theta = \theta_l + \varepsilon Y, \quad |\varepsilon| \ll 1, \tag{B.1}$$

and using the standard linearization procedure, we arrive at the perturbed equation

$$\mathcal{B}Y = \Phi_\varepsilon(Y), \tag{B.2}$$

where $\mathcal{B}$ is the linear operator

$$\begin{gathered} \mathcal{B}Y = mk^{m-1}(y^{(m-1)/m}Y)'' + y^2(\mu Y - \nu Y' y) = 0, \\ \mu = \frac{6m^2+3m+1}{2m(3m+1)}, \quad \nu = \frac{m+1}{2(3m+1)}. \end{gathered} \tag{B.3}$$

The nonlinear perturbation Φ_ε satisfies

$$\Phi_\varepsilon(Y) = O(\varepsilon) \quad \text{as } \varepsilon \to 0$$

on compact subsets for smooth functions $y \in C^2$.

In view of the continuous dependence of solutions of linear second-order homogeneous equations upon small perturbations of the right-hand side, the oscillating properties of Y for $|\varepsilon| \ll 1$ about the trivial solution $Y \equiv 0$ are the same as of solutions of the non-perturbed linear equation

$$\mathcal{B}Y = 0, \quad y > 0. \tag{B.4}$$

Using the transformation

$$Y(y) = (bz)^{1/(3m+1)}W(z), \quad y = (bz)^{m/(3m+1)}, \quad b = 2k^{m-1}\frac{(3m+1)^2}{m+1}, \tag{B.5}$$

we reduce (B.4) to the degenerate hypergeometric equation

$$zW'' + (c-z)W' + aW = 0, \quad z > 0, \tag{B.6}$$

with the parameters

$$a = \frac{2m}{m+1} \in (1,2), \quad c = \frac{4m+1}{3m+1} > 0. \tag{B.7}$$

The solution to (B.6) is given by a convergent Kummer series, and moreover has exactly

$$K = -[-a] = 2 \tag{B.8}$$

zeros for $z > 0$ [BE].

In terms of representation (B.1) this means that for $|\varepsilon| \ll 1$ the function θ intersects the profile θ_l for $y > 0$ at least two times. According to the general scheme of construction, this makes it possible to prove existence of a positive solution in $\mathbf{R}_+$ such that $\theta_*(y) > \theta_l(y)$ for small $y > 0$ (more exactly, one can construct the maximal and minimal solutions which are expected to coincide).

The behaviour of such a $\theta_*(y)$ for large y follows from the local analysis of equation (6.2) for $y \gg 1$ and takes the form

$$\theta_*(y) = C_* y^{-2/(m+1)}(1 + o(1)) \quad \text{as } y \to \infty, \tag{B.9}$$

where $C_* = C_*(m) > 0$ is a constant. See [SGK, Chapter 4] and [BG]. $\square$

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Figure Captions

Figure 1. A plot of u_x^{-2} as a function of t.

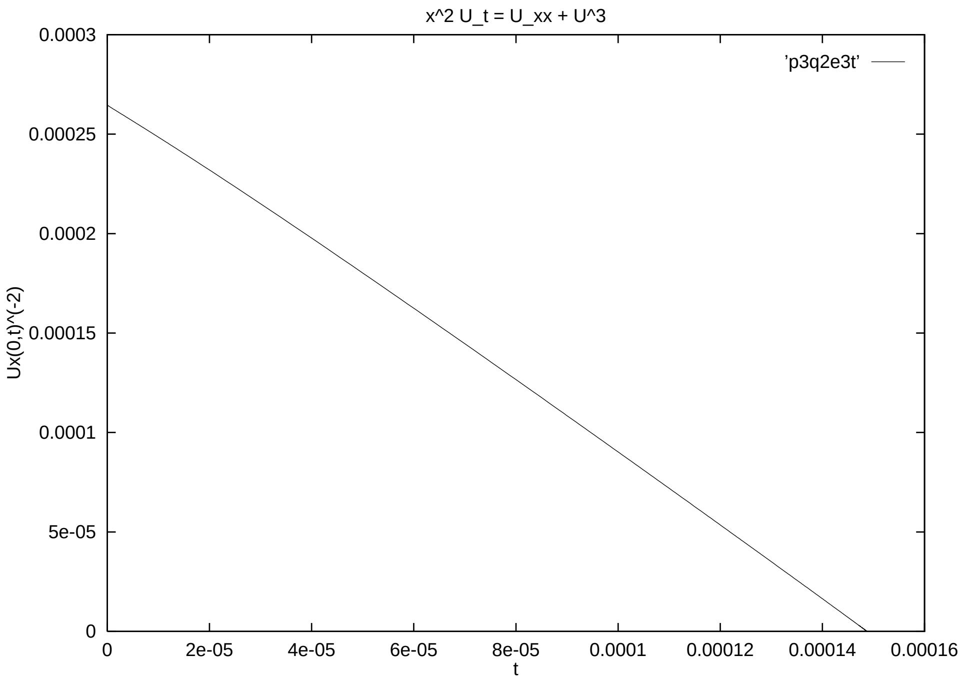
Figure 2. The scaled function $v(z,t)$ when $u^*(t) = 1.2 \times 10^4, 6 \times 10^3, 3 \times 10^3, 1.5 \times 10^3, 7.5 \times 10^2$ and 3.37×10^2.

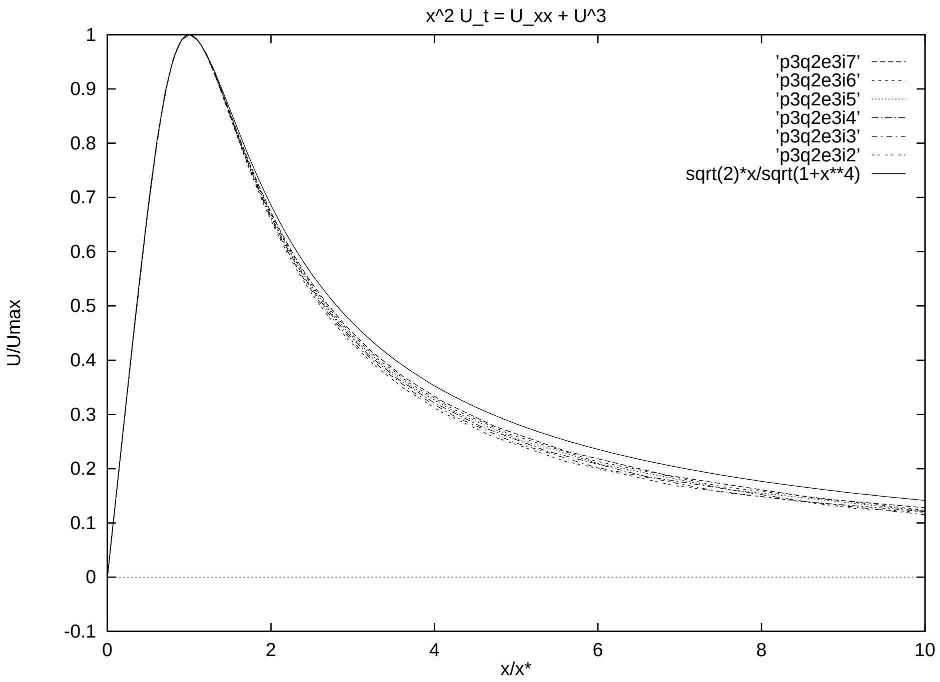
Figure 3. The value of $\log(u^*(T-t)^{1/4})$ plotted as a function of $\log|\log(T-t)|$.

Figure 4. The value of $\log(x^*(T-t)^{1/4})$ plotted as a function of $\log|\log(T-t)|$.

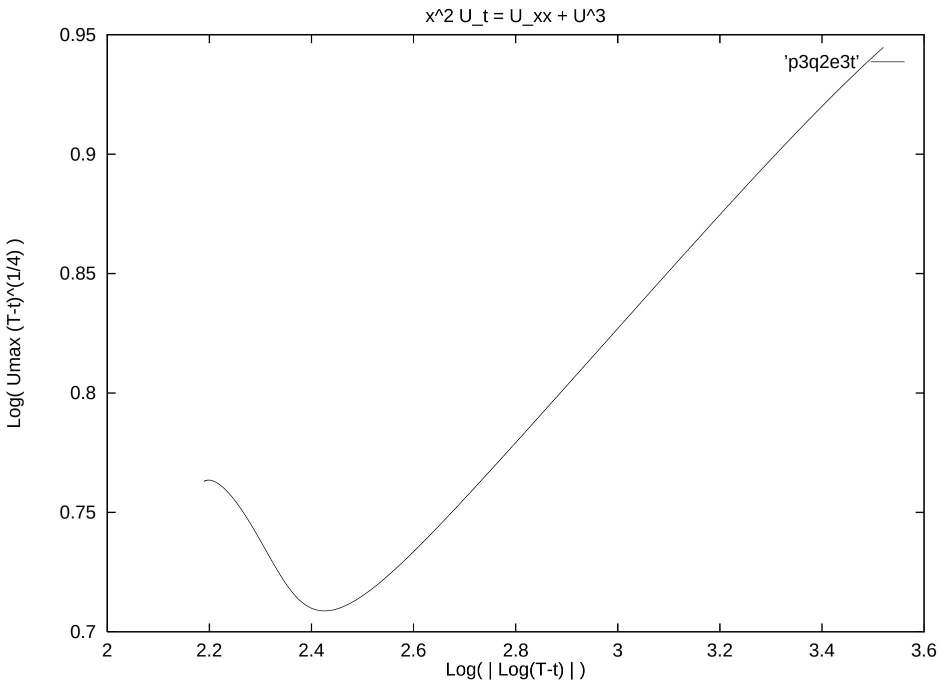
Figure 5. The asymptotic profile $xu(x,t)/|\log x|^{1/2}$ plotted when $\|u\|_\infty = 1 \times 10^4$.

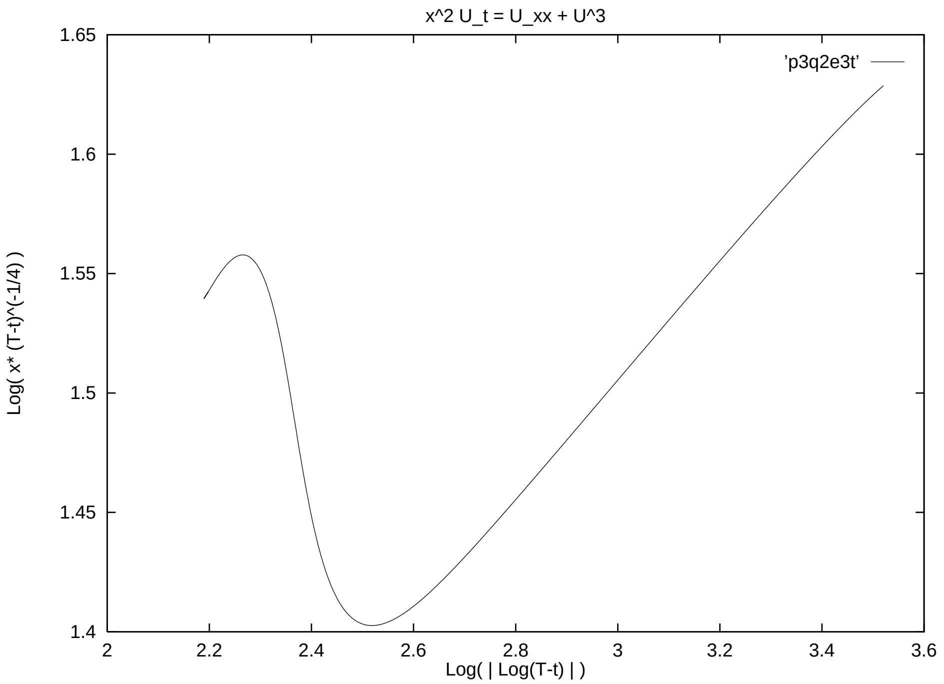
Figure 6. Scaled solutions of $xu_t = u_{xx} + u^3$ for a variety of initial data. Here $S1$ corresponds to $u_0 = 100\sin(\pi x)$, $S2$ to $100x(1-x)e^{-x}$ and $S3$ to $100x(1-x)e^{-(1-x)}$.


x^2 U_t = U_xx + U^3
'p3q2e3t'
Ux(0,t)^(-2)
t
0.0003
0.00025
0.0002
0.00015
0.0001
5e-05
0
2e-05
4e-05
6e-05
8e-05
0.0001
0.00012
0.00014
0.00016

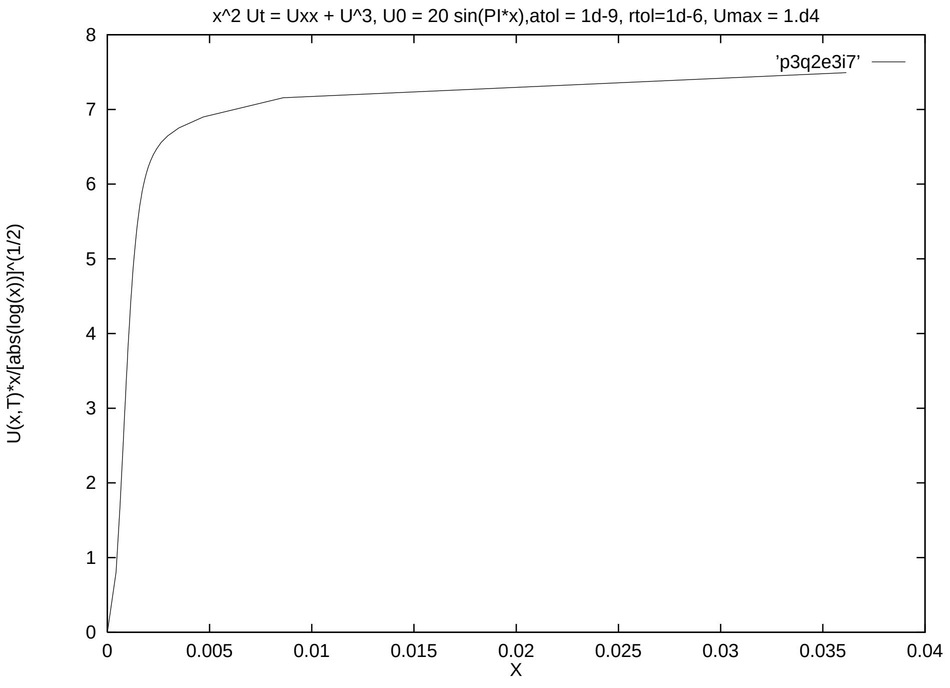


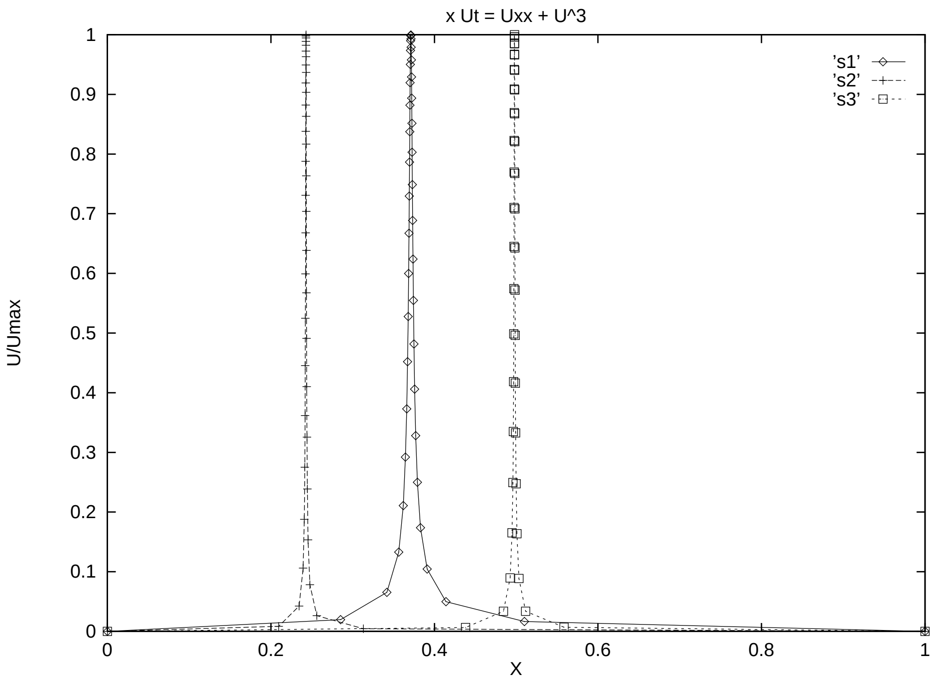
x^2 U_t = U_xx + U^3
'p3q2e3i7'
'p3q2e3i6'
'p3q2e3i5'
'p3q2e3i4'
'p3q2e3i3'
'p3q2e3i2'
sqrt(2)*x/sqrt(1+x**4)
U/Umax
x/x*
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
2
4
6
8
10


x^2 U_t = U_xx + U^3
'p3q2e3t'
Log(Umax (T-t)^(1/4))
Log(| Log(T-t) |)
0.95
0.9
0.85
0.8
0.75
0.7
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6



x^2 U_t = U_xx + U^3
'p3q2e3t'
1.65
1.6
1.55
1.5
1.45
1.4
Log(x* (T-t)^(-1/4))
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
Log(| Log(T-t) |)


x^2 Ut = Uxx + U^3, U0 = 20 sin(PI*x),atol = 1d-9, rtol=1d-6, Umax = 1.d4
'p3q2e3i7'
U(x,T)*x/[abs(log(x))]^(1/2)
X
0
1
2
3
4
5
6
7
8
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04


x Ut = Uxx + U^3
U/Umax
X
's1'
's2'
's3'
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1